## INSTRUCTIONS <br> PHYSICS DEPARTMENT WRITTEN EXAM

## PART I

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to attempt two problems. Each question will be graded on a scale of zero to ten points. Circle the number of each of the two problems that you wish to be graded.

| SECTION : | CLASSICAL <br> MECHANICS |  | ELECTRO- <br> MAGNETISM |  | MATHEMATICAL <br> AND GENERAL |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PROBLEMS : | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ |

## SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens \& pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, etc.) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
a. Write the problem number and your ID number on each white paper sheet;
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## \#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM:
A uniform disc of mass $M$ and radius $R$ turns in the horizontal plane about its center, on a frictionless axle. Its initial angular velocity is $\omega$. At the center of the disc a spherical ball of mass $M$ and radius $a$ is placed with zero initial center of mass velocity or angular velocity. The ball rolls without slipping across the disc to the edge.

Find the angular velocity of the disc when the ball reaches its edge. (Hint: the moment of inertia of the ball about its center is $I_{\text {ball }}=2 M a^{2} / 5$, and the moment of inertia of the disc about its center is $I_{\text {disc }}=M R^{2} / 2$.)

## \#2 : GRADUATE CLASSICAL MECHANICS

PROBLEM:
The pivot of a simple pendulum of length $l$ and mass $m$ is oscillated in the $y$ (vertical) direction according to $y(t)=a \cos \Omega t, a \ll l$. Assuming that $\Omega^{2} \gg g / l$, where $g$ is the acceleration of gravity, find a criterion for $\Omega$ such that $\theta=0$ is a stable equilibrium (see the diagram).


## \#3 : GRADUATE CLASSICAL MECHANICS

PROBLEM:
Consider the following Hamiltonian

$$
\begin{equation*}
H(q, p)=\frac{p^{2}}{\alpha q^{6}}+\alpha q^{4} \tag{1}
\end{equation*}
$$

with $p$ the momentum, $q$ the generalized coordinate, and $\alpha$ a constant.
Find the equations of motion $q=q(t)$, and $p=p(t)$ using the method of Hamilton-Jacobi.

## \#4 : UNDERGRADUATE CLASSICAL ELECTRODYNAMICS

## PROBLEM:

Consider a charged sphere of total radius $b$. The charge density is given by $\rho$ up to radius $a=b / 2^{1 / 3}$, and by $-\rho$ for $a<r<b$. Both $\rho$ and $-\rho$ are uniform charge densities.

Calculate the electric field $\vec{E}(\vec{r})$ in the three regions $r<a=b / 2^{1 / 3}, a<$ $r<b$, and $r>b$, and find the total electrostatic energy of the charge configuration. Is the energy positive, negative, or zero? Why?

## \#5 : GRADUATE CLASSICAL ELECTRODYNAMICS

PROBLEM:
A non-magnetic sphere of dielectric permittivity $\varepsilon$ and radius $a$ moves at constant velocity $\mathbf{v}=v \hat{\mathbf{z}}$ in a uniform magnetic field $\mathbf{B}_{0}=-B_{0} \hat{\mathbf{y}}$. Find the electric field everywhere to the leading order in $v / c \ll 1$.

Hint: the relativistic transformation formulas of the parallel and perpendicular field components are (SI units)

$$
\begin{aligned}
B_{\|}^{\prime} & =B_{\|}, & c \mathbf{B}_{\perp}^{\prime} & =\gamma\left(c \mathbf{B}_{\perp}-\boldsymbol{\beta} \times \mathbf{E}\right), \\
E_{\|}^{\prime} & =E_{\|}, & \mathbf{E}_{\perp}^{\prime} & =\gamma\left(\mathbf{E}_{\perp}+\boldsymbol{\beta} \times \mathbf{B}\right)
\end{aligned}
$$

where $\boldsymbol{\beta}=\mathbf{v} / c$ and $\gamma=1 / \sqrt{1-\beta^{2}}$.

## \#6 : GRADUATE CLASSICAL ELECTRODYNAMICS

PROBLEM:
Find the magnetic field distribution created by a charged solid sphere rotating with a constant angular velocity $\omega$. The sphere has radius $a$ and a uniform charge density $\rho_{0}$.

Hint: Assume that the vector potential is a linear combination of terms

$$
\mathbf{A}(r, \theta, \phi)=r^{l} \sin \theta \hat{\boldsymbol{\phi}}
$$

where $r, \theta, \phi$ are spherical coordinates and $l=-2,1$, or 3 . The vector Laplacian and the magnetic field corresponding to this $\mathbf{A}$ are

$$
\begin{aligned}
\nabla^{2} \mathbf{A} & =(l+2)(l-1) r^{l-2} \sin \theta \hat{\boldsymbol{\phi}} \\
\mathbf{B}=\nabla \times \mathbf{A} & =2 r^{l-1} \cos \theta \hat{\boldsymbol{r}}-(l+1) r^{l-1} \sin \theta \hat{\boldsymbol{\theta}}
\end{aligned}
$$

## \#7 : UNDERGRADUATE MATHEMATICS AND GENERAL PHYSICS

 PROBLEM:A solid sphere with a hydrophobic surface and a density twice that of water is placed on top of a large pool of still water, and is found to float due to surface tension of the water such that the water level comes up to the sphere's "equator." Assuming that the bowl formed by the water follows the shape of the sphere closely, estimate what is the radius of the sphere that results in equilibrium support. The surface tension for water is $\gamma=0.07 \mathrm{~N} / \mathrm{m}$.

## \#8 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:
Consider the following ODE

$$
x \frac{d^{2} y(x)}{d x^{2}}+\frac{d y(x)}{d x}=y(x) .
$$

By using methods of asymptotic expansion and dominant balance, find the dominant asymptotic behavior as $x \rightarrow \infty$.

## \#9 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

 PROBLEM:Consider the integral ( $a$ positive)

$$
I=\int_{0}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)^{2}}
$$

Calculate its value by a proper contour in the complex plane.

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PROBLEM:
A uniform disc of mass $M$ and radius $R$ turns in the horizontal plane about its center, on a frictionless axle. Its initial angular velocity is $\omega$. At the center of the disc a spherical ball of mass $M$ and radius $a$ is placed with zero initial center of mass velocity or angular velocity. The ball rolls without slipping across the disc to the edge.

Find the angular velocity of the disc when the ball reaches its edge. (Hint: the moment of inertia of the ball about its center is $I_{\text {ball }}=2 M a^{2} / 5$, and the moment of inertia of the disc about its center is $I_{d i s c}=M R^{2} / 2$.)

SOLUTION: The Lagrangian of the system consists entirely of kinetic energy and is given by

$$
L=\frac{1}{2} I_{\text {disc }} \dot{\phi}^{2}+\frac{1}{2} I_{\text {ball }} \Omega^{2}+\frac{1}{2} M v_{\text {ball }}^{2}
$$

where $\phi$ is the angle the disc makes with the x axis of a frame fixed in the lab, $I_{\text {disc }}=M R^{2} / 2$ is the moment of inertia of the disc, $\mathbf{v}_{\text {ball }}$ is the center of mass velocity of the ball as seen in the lab frame, $\Omega$ is the angular rotation frequency of the ball about its own center of mass, and $I_{\text {ball }}=2 M a^{2} / 5$ is the moment of inertia of the ball.

In a frame rotating with the disc, let the location of the ball be $\mathbf{r}_{r o t}=$ $r\left(\cos \theta \hat{x}_{\text {rot }}+\sin \theta \hat{y}_{\text {rot }}\right)$ where the unit vectors here are fixed to the disc. As seen in the lab frame, the location is $\mathbf{r}=r(\cos (\theta+\phi) \hat{x}+\sin (\theta+\phi) \hat{y})$.

Then rolling without slipping implies that $a^{2} \Omega^{2}=\left|\dot{\mathbf{r}}_{\text {rot }}\right|^{2}=\dot{r}^{2}+r^{2} \dot{\theta}^{2}$. On the other hand, the center of mass energy of the ball as seen in the lab frame is $\frac{1}{2} M v_{\text {ball }}^{2}=\frac{1}{2} M|\dot{\mathbf{r}}|^{2}=\frac{1}{2} M\left[\dot{r}^{2}+r^{2}(\dot{\theta}+\dot{\phi})^{2}\right]$. Putting this together we obtain

$$
L=\frac{1}{4} M R^{2} \dot{\phi}^{2}++\frac{1}{5} M\left(\dot{r}^{2}+r^{2} \dot{\theta}^{2}\right)+\frac{1}{2} M\left(\dot{r}^{2}+r^{2}(\dot{\theta}+\dot{\phi})^{2}\right)
$$

There are three constants of the motion for this system, and three degrees of freedom, so the system is integrable. One of the constants is kinetic
energy, which we will not need to solve this problem. Another is

$$
p_{\theta}=\partial L / \partial \dot{\theta}=M r^{2}(\dot{\theta}+\dot{\phi})+\frac{2}{5} M r^{2} \dot{\theta}=\text { constant }=0
$$

since $r=0$ initially. This implies that

$$
\dot{\theta}=-\frac{5}{7} \dot{\phi}
$$

Lastly,

$$
p_{\phi}=\partial L / \partial \dot{\phi}=\frac{1}{2} M R^{2} \dot{\phi}+M r^{2}(\dot{\theta}+\dot{\phi})=\text { constant }=\frac{1}{2} M R^{2} \omega,
$$

This implies that

$$
\dot{\phi}=\frac{R^{2}}{R^{2}+4 r^{2} / 7} \omega
$$

Therefore when $r=R$ the angular speed of the disc is $\dot{\phi}=7 \omega / 13$.

## \#2 : GRADUATE CLASSICAL MECHANICS

PROBLEM:
The pivot of a simple pendulum of length $l$ and mass $m$ is oscillated in the $y$ (vertical) direction according to $y(t)=a \cos \Omega t, a \ll l$. Assuming that $\Omega^{2} \gg g / l$, where $g$ is the acceleration of gravity, find a criterion for $\Omega$ such that $\theta=0$ is a stable equilibrium (see the diagram).


## SOLUTION:

The position of the pendulum bob is $\mathbf{r}=l \sin \theta \hat{x}+(l \cos \theta+y(t)) \hat{y}$. This implies that the potential energy of the pendulum is $V=m g l \cos \theta+m g y(t)$. The kinetic energy is

$$
T=\frac{1}{2} m|\dot{\mathbf{r}}|^{2}=\frac{1}{2} m\left[l^{2} \dot{\theta}^{2}-2 l \dot{y}(t) \dot{\theta} \sin \theta+\dot{y}(t)^{2}\right]
$$

so the Lagrangian is

$$
L=\frac{1}{2} m\left[l^{2} \dot{\theta}^{2}-2 l \dot{y}(t) \dot{\theta} \theta\right]+m g l \theta^{2} / 2
$$

where we have dropped constant terms and terms that depend only on time, since they do not affect the dynamics, and have expanded in small $\theta$ anticipating that we will only need to consider this regime to determine stability at $\theta=0$.

The Euler-Lagrange equation then yields

$$
\ddot{\theta}=\frac{g+\ddot{y}(t)}{l} \theta=\frac{g-\Omega^{2} a \cos \Omega t}{l} \theta
$$

(This result can also be obtained directly by considering the apparent acceleration of gravity in the noninertial frame of the moving pivot.)

Solve this problem with two-timescale analysis to obtain the ponderomotive potential. Take $\theta=\bar{\theta}+\theta_{1}$ where $\theta_{1}$ is rapidly varying and small, and $\bar{\theta}$ is relatively-slowly varying. Using these in the Euler-Lagrange equation, the fast variation equation is

$$
\ddot{\theta}_{1}=\frac{g}{l} \theta_{1}-\frac{\Omega^{2} a}{l} \bar{\theta} \cos \Omega t
$$

Treating $\bar{\theta}$ as a constant the driven solution is

$$
\theta_{1}=\frac{\Omega^{2} a}{\Omega^{2} l+g} \bar{\theta} \cos \Omega t \approx \frac{a}{l} \bar{\theta} \cos \Omega t
$$

where we applied $\Omega^{2} \gg g / l$.
The slow time equation is given by the fast-time average of the EulerLagrange equation,

$$
\ddot{\bar{\theta}}=\frac{g}{l} \bar{\theta}-\frac{\Omega^{2} a}{l} \overline{\cos \Omega t \theta_{1}}=\frac{g}{l} \bar{\theta}-\frac{\Omega^{2} a^{2}}{2 l^{2}} \bar{\theta}
$$

Thus, slow motion is stable near $\bar{\theta}=0$ provided that $g / l-\Omega^{2} a^{2} /\left(2 l^{2}\right)<0$, or $\Omega^{2}>2 g l / a^{2}$.

## \#3 : GRADUATE CLASSICAL MECHANICS

PROBLEM:
Consider the following Hamiltonian

$$
\begin{equation*}
H(q, p)=\frac{p^{2}}{\alpha q^{6}}+\alpha q^{4} \tag{1}
\end{equation*}
$$

with $p$ the momentum, $q$ the generalized coordinate, and $\alpha$ a constant.
Find the equations of motion $q=q(t)$, and $p=p(t)$ using the method of Hamilton-Jacobi.

## SOLUTION:

Since the Hamiltonian does not depend explicitly on time, Hamilton's principle function is $S=W(q, E)-E t$, with $E$ a constant. Therefore, the Hamilton-Jacobi equation is

$$
\begin{equation*}
H\left(q, \frac{\partial W}{\partial q}\right)=E \tag{2}
\end{equation*}
$$

This then becomes:

$$
\begin{equation*}
\frac{1}{\alpha q^{6}}\left(\frac{\partial W}{\partial q}\right)^{2}+\alpha q^{4}=E \tag{3}
\end{equation*}
$$

From this we get:

$$
\begin{equation*}
\frac{\partial W}{\partial q}=\sqrt{\alpha q^{6}\left(E-\alpha q^{4}\right)} \tag{4}
\end{equation*}
$$

or

$$
\begin{equation*}
W=\int \sqrt{\alpha q^{6}\left(E-\alpha q^{4}\right)} d q \tag{5}
\end{equation*}
$$

Hamilton's principle function then assumes the form:

$$
\begin{equation*}
S=\int \sqrt{\alpha q^{6}\left(E-\alpha q^{4}\right)} d q-E t \tag{6}
\end{equation*}
$$

Set $\beta=\partial S / \partial E$, then

$$
\begin{equation*}
\beta=\int \frac{\alpha q^{6}}{2 \sqrt{\alpha q^{6}\left(E-\alpha q^{4}\right)}} d q-t \tag{7}
\end{equation*}
$$

namely

$$
\begin{equation*}
\beta+t=\sqrt{\alpha} \int \frac{q^{3}}{2 \sqrt{E-\alpha q^{4}}} d q \tag{8}
\end{equation*}
$$

We can solve this integral by making the substitution $x=E-\alpha q^{4}$ to obtain:

$$
\begin{equation*}
\beta+t=-\frac{1}{4 \sqrt{\alpha}} \sqrt{E-\alpha q^{4}} \tag{9}
\end{equation*}
$$

Inverting this equation we finally get:

$$
\begin{equation*}
q(t)=\left[\frac{1}{\alpha}\left[E-16 \alpha(t+\beta)^{2}\right]\right]^{1 / 4} \tag{10}
\end{equation*}
$$

Since $p=\partial W / \partial q$, the equation $p=p(t)$ can be obtained by replacing (??) into (??).

## \#4 : UNDERGRADUATE CLASSICAL ELECTRODYNAMICS

## PROBLEM:

Consider a charged sphere of total radius $b$. The charge density is given by $\rho$ up to radius $a=b / 2^{1 / 3}$, and by $-\rho$ for $a<r<b$. Both $\rho$ and $-\rho$ are uniform charge densities.

Calculate the electric field $\vec{E}(\vec{r})$ in the three regions $r<a=b / 2^{1 / 3}, a<$ $r<b$, and $r>b$, and find the total electrostatic energy of the charge configuration. Is the energy positive, negative, or zero? Why?

SOLUTION:
Gauss' Law:

$$
\oint \vec{E} \cdot d \vec{a}=4 \pi Q_{\mathrm{enclosed}}
$$

By symmetry $\vec{E}=|\vec{E}| \hat{r}$.
$r<a$

$$
|\vec{E}|\left(4 \pi r^{2}\right)=\frac{1}{\epsilon_{0}} \rho\left(\frac{4 \pi r^{3}}{3}\right) \Rightarrow \vec{E}(r)=\frac{4 \pi}{3} \rho r \hat{r}
$$

$a<r<b$

$$
|\vec{E}|\left(4 \pi r^{2}\right)=4 \pi \rho\left[\left(\frac{4 \pi a^{3}}{3}\right)-\left(\frac{4 \pi r^{3}}{3}-\frac{4 \pi a^{3}}{3}\right)\right] \Rightarrow \vec{E}(r)=\frac{4 \pi}{3} \rho\left(\frac{b^{3}}{r^{2}}-r\right) \hat{r}
$$

$r>b$
$|\vec{E}|\left(4 \pi r^{2}\right)=4 \pi \rho\left[\left(\frac{4 \pi a^{3}}{3}\right)-\left(\frac{4 \pi b^{3}}{3}-\frac{4 \pi a^{3}}{3}\right)\right] \Rightarrow \vec{E}(r)=\frac{4 \pi}{3} \rho\left(\frac{2 a^{3}-b^{3}}{r^{2}}\right) \hat{r}=0$

The electrostatic energy is given by $\epsilon=\int d^{3} r|\vec{E}(r)|^{2} / 8 \pi$. By definition this energy must be positive-definite if $E$ is nonzero. Performing the integral
over the two regions for which $\vec{E}$ is nonzero gives

$$
\begin{aligned}
\epsilon & =\frac{1}{8 \pi} \int_{0}^{a} 4 \pi r^{2} d r\left(\frac{4 \pi \rho}{3} r\right)^{2}+\frac{1}{8 \pi} \int_{a}^{b} 4 \pi r^{2} d r\left(\frac{4 \pi \rho}{3}\left(\frac{b^{3}}{r^{2}}-r\right)\right)^{2} \\
& =\frac{1}{2}\left(\frac{4 \pi \rho}{3}\right)^{2}\left(\frac{a^{5}}{5}+\int_{a}^{b} d r\left(\frac{b^{6}}{r^{2}}-2 b^{3} r+r^{4}\right)\right) \\
& =\frac{1}{2}\left(\frac{4 \pi \rho}{3}\right)^{2}\left(\frac{b^{6}}{a}-b^{5}-b^{3}\left(b^{2}-a^{2}\right)+\frac{b^{5}}{5}\right) \\
& =\frac{1}{2}\left(\frac{4 \pi \rho}{3}\right)^{2}\left(\frac{b^{6}}{a}-\frac{9}{5} b^{5}+b^{3} a^{2}\right) \\
& =\frac{1}{2}\left(\frac{4 \pi \rho}{3}\right)^{2} b^{5}\left(2^{1 / 3}+2^{-2 / 3}-\frac{9}{5}\right) .
\end{aligned}
$$

The coefficient $2^{1 / 3}+2^{-2 / 3}-9 / 5=0.0899$ is positive, so the electrostatic energy is positive, as expected.

## \#5 : GRADUATE CLASSICAL ELECTRODYNAMICS

PROBLEM:
A non-magnetic sphere of dielectric permittivity $\varepsilon$ and radius $a$ moves at constant velocity $\mathbf{v}=v \hat{\mathbf{z}}$ in a uniform magnetic field $\mathbf{B}_{0}=-B_{0} \hat{\mathbf{y}}$. Find the electric field everywhere to the leading order in $v / c \ll 1$.

Hint: the relativistic transformation formulas of the parallel and perpendicular field components are (SI units)

$$
\begin{aligned}
B_{\|}^{\prime} & =B_{\|}, & c \mathbf{B}_{\perp}^{\prime} & =\gamma\left(c \mathbf{B}_{\perp}-\boldsymbol{\beta} \times \mathbf{E}\right), \\
E_{\|}^{\prime} & =E_{\|}, & \mathbf{E}_{\perp}^{\prime} & =\gamma\left(\mathbf{E}_{\perp}+\boldsymbol{\beta} \times \mathbf{B}\right) .
\end{aligned}
$$

where $\boldsymbol{\beta}=\mathbf{v} / c$ and $\gamma=1 / \sqrt{1-\beta^{2}}$.

SOLUTION: Go to the frame co-moving with the sphere. Since $v \ll c$, the field transformation formulas simplify to

$$
\mathbf{B}_{0}^{\prime}=\mathbf{B}_{0}, \quad \mathbf{E}_{0}^{\prime}=\mathbf{E}_{0}+\mathbf{v} \times \mathbf{B}_{0}=\mathbf{v} \times \mathbf{B}_{0}=v B_{0} \hat{\mathbf{x}}
$$

The magnetic field has no effect on the dielectric sphere. The uniform electric field $\mathbf{E}_{0}^{\prime}$ polarizes the (stationary) sphere, which creates the additional electric field

$$
\Delta \mathbf{E}^{\prime}=\frac{\varepsilon-\varepsilon_{0}}{\varepsilon+2 \varepsilon_{0}} v B_{0}\left\{\begin{array}{cl}
-\hat{\mathbf{x}}, & r<a \\
\frac{3(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}) \hat{\mathbf{x}}-\hat{\mathbf{x}}}{r^{3}} a^{3}, & r>a
\end{array}\right.
$$

The derivation of this expression can be found in literally any E\&M textbook, so it is omitted. Finally, transformation back to the lab frame gives, to the same order in $v / c$

$$
\mathbf{E}=\mathbf{E}^{\prime}-\mathbf{v} \times \mathbf{B}_{0}=\left(\mathbf{v} \times \mathbf{B}_{0}+\Delta \mathbf{E}^{\prime}\right)-\mathbf{v} \times \mathbf{B}_{0}=\Delta \mathbf{E}^{\prime}
$$

so the answer is simply the preceding formula.

## \#6 : GRADUATE CLASSICAL ELECTRODYNAMICS

## PROBLEM:

Find the magnetic field distribution created by a charged solid sphere rotating with a constant angular velocity $\omega$. The sphere has radius $a$ and a uniform charge density $\rho_{0}$.

Hint: Assume that the vector potential is a linear combination of terms

$$
\mathbf{A}(r, \theta, \phi)=r^{l} \sin \theta \hat{\boldsymbol{\phi}}
$$

where $r, \theta, \phi$ are spherical coordinates and $l=-2,1$, or 3 . The vector Laplacian and the magnetic field corresponding to this $\mathbf{A}$ are

$$
\begin{aligned}
\nabla^{2} \mathbf{A} & =(l+2)(l-1) r^{l-2} \sin \theta \hat{\boldsymbol{\phi}} \\
\mathbf{B}=\nabla \times \mathbf{A} & =2 r^{l-1} \cos \theta \hat{\boldsymbol{r}}-(l+1) r^{l-1} \sin \theta \hat{\boldsymbol{\theta}}
\end{aligned}
$$

SOLUTION: Choose the $z$-axis along the rotation axis of the sphere. The rotation creates the electric current density

$$
\mathbf{j}(\mathbf{r})=\omega \hat{\mathbf{z}} \times \rho_{0} \mathbf{r}=\rho_{0} \omega r \sin \theta \hat{\boldsymbol{\phi}}, \quad r<a
$$

The Maxwell equation to solve is

$$
\nabla \times \mathbf{B}=\nabla \times(\nabla \times \mathbf{A})=\mu_{0} \mathbf{j}
$$

with the boundary conditions (BC) $\mathbf{B}(\mathbf{r}) \rightarrow 0$ as $r \rightarrow \infty$ and $|\mathbf{B}(\mathbf{r})|<\infty$ as $r \rightarrow 0$. The suggested choice of the vector potential obeys the Coulomb gauge condition $\nabla \cdot \mathbf{A}=0$, which allows us to bring the above equation to

$$
-\nabla^{2} \mathbf{A}=\mu_{0} \mathbf{j}
$$

Let $B_{0}=\mu_{0} \rho_{0} \omega a^{2}$, then

$$
\begin{array}{ll}
\nabla^{2} \mathbf{A}=-\frac{B_{0}}{a^{2}} r \sin \theta \hat{\phi}, & \\
\nabla^{2} \mathbf{A}=0, & r>a
\end{array}
$$

The second equation, for points outside the sphere, is satisfied for an arbitrary combination of $l=-2$ or $l=1$ terms. However, only $l=-2$ term, which describes a magnetic dipole, obeys the BC at infinity. The second equation, for points inside the sphere, can be satisfied taking $l=3$. We can also add a term with $l=1$ (but not $l=-2$ ). Adjusting the coefficients of these three possible terms to make A and its derivatives continuous at $r=a$, we find

$$
\begin{array}{ll}
A_{\phi}=\left(\frac{1}{6} \frac{r}{a}-\frac{1}{10} \frac{r^{3}}{a^{3}}\right) B_{0} a \sin \theta, & r<a \\
A_{\phi}=\frac{1}{15} \frac{a^{2}}{r^{2}} B_{0} a \sin \theta, & r>a
\end{array}
$$

Hence, the magnetic field is

$$
\begin{array}{rlr}
\frac{\mathbf{B}}{B_{0}}=\left(\frac{1}{3}-\frac{1}{5} \frac{r^{2}}{a^{2}}\right) \cos \theta \hat{\boldsymbol{r}}-\left(\frac{1}{3}-\frac{2}{5} \frac{r^{2}}{a^{2}}\right) \sin \theta \hat{\boldsymbol{\theta}}, & r<a \\
\frac{\mathbf{B}}{B_{0}}=\frac{1}{15} \frac{a^{3}}{r^{3}}(2 \cos \theta \hat{\boldsymbol{r}}+\sin \theta \hat{\boldsymbol{\theta}}), & r>a
\end{array}
$$

This field is plotted in the Figure below.


## \#7 : UNDERGRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:
A solid sphere with a hydrophobic surface and a density twice that of water is placed on top of a large pool of still water, and is found to float due to surface tension of the water such that the water level comes up to the sphere's "equator." Assuming that the bowl formed by the water follows the shape of the sphere closely, estimate what is the radius of the sphere that results in equilibrium support. The surface tension for water is $\gamma=0.07 \mathrm{~N} / \mathrm{m}$.

SOLUTION:
This is a balance between net potential energy and the energy in surface tension. The latter is (dimensionally, for instance) $E_{\text {surf }}=\gamma A_{\text {wet }}$, where $A_{\text {wet }}$ is the "wetted" surface area.

For net potential energy change, the sphere displaces water (overall water level rises as the sphere enters the water). The average distance the water is displaced is $\frac{3}{8} R$, which can be calculated as the average of $r \cos \theta$ through a hemisphere, where $\theta$ is the usual polar angle. A sphere of density $\rho_{\mathrm{s}}$ therefore loses potential energy $\frac{4}{3} \pi R^{3} \rho_{\mathrm{s}} g R$ when sinking by a height equal to its radius. Meanwhile, the water in the displaced hemisphere gains potential energy $\frac{2}{3} \pi R^{3} \rho_{\mathrm{w}} g \cdot \frac{3}{8} R$ for a net loss of $\left(\frac{4}{3} \rho_{\mathrm{s}}-\frac{1}{4} \rho_{\mathrm{w}}\right) \pi g R^{4}$. When $\rho_{\mathrm{s}}=2 \rho_{\mathrm{w}}$, we end up with a net potential energy change of $\frac{29}{12} \rho_{\mathrm{w}} \pi g R^{4}$.

The surface tension acts over the area of the hemisphere in "contact" with the water, by an amount corresponding to the increase in surface area: from $\pi R^{2}$ to $2 \pi R^{2}$, so that $E_{\text {surf }}=\gamma \pi R^{2}$. Equating the potential energy change to the surface tension contribution, we find that

$$
R=\sqrt{\frac{12}{29} \frac{\gamma}{\rho_{\mathrm{w}} g}} .
$$

This evaluates to a little less than 2 mm in radius.

## \#8 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:
Consider the following ODE

$$
x \frac{d^{2} y(x)}{d x^{2}}+\frac{d y(x)}{d x}=y(x) .
$$

By using methods of asymptotic expansion and dominant balance, find the dominant asymptotic behavior as $x \rightarrow \infty$.

SOLUTION: By using the ansatz $y \sim e^{S(x)}$, we obtain the equation

$$
\begin{equation*}
x S^{\prime \prime}+x{S^{\prime}}^{2}+S^{\prime}=1 \tag{1}
\end{equation*}
$$

Dominant balance for large $x$ shows then that $x S^{\prime 2} \simeq 1$, i.e. $S \simeq \pm 2 \sqrt{x}+$ $S_{1}(x)$. Inserting this expression into Eq. (??) and using again dominant balance, we obtain $S_{1}=-\frac{1}{4} \ln x$ and one can check that the next order is subdominant. We conclude that the dominant behavior is $y(x) \propto \frac{e^{ \pm 2 \sqrt{x}}}{x^{1 / 4}}$.

## \#9 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:
Consider the integral ( $a$ positive)

$$
I=\int_{0}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)^{2}}
$$

Calculate its value by a proper contour in the complex plane.

SOLUTION:
The integral $I=\frac{1}{2} \int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)^{2}}$. The integral $\int_{-\infty}^{\infty} \frac{d x}{\left(x^{2}+a^{2}\right)^{2}}$ can be calculated by closing the contour in the upper plane by a big circle of radius $R \rightarrow \infty$ (which gives a vanishing contribution) and noting that the integrand has a pole at $i a$. Its residue is easy to calculate Res $=\frac{1}{4 i a^{3}}$, which yields $I=\frac{\pi}{4 a^{3}}$.

## INSTRUCTIONS PHYSICS DEPARTMENT WRITTEN EXAM PART II

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to attempt two problems from each section.

The questions are grouped in two sections: quantum mechanics and statistical physics. You must attempt two problems from each of these sections, for a total of four problems. Credit will be assigned for four (4) questions only. Each question will be graded on a scale of zero to ten points. Circle the number of each of the four problems you wish to be graded.

|  | QUANTUM |  | STATISTICAL <br> PHYSICS |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SECTION : | MECHANICS |  | $\mathbf{3}$ | $\mathbf{1 4}$ | $\mathbf{1 4}$ | $\mathbf{1 5}$ |

## SPECIAL INSTRUCTIONS DURING EXAM

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$\qquad$

## \#10 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM:
Suppose that a system has $H=\frac{1}{2 I} \vec{L}^{2}+\alpha L_{z}$, where $I$ and $\alpha$ are constants.
(a)(2pt) Find $\left[H, L_{x}\right],\left[H, L_{y}\right]$, and $\left[H, L_{z}\right]$. Which $L_{i}$ are $t$ independent in the Heisenberg picture?
(b)(3 pt)At time $t=0$, the system is in a superposition of $|\ell, m\rangle$ states:

$$
|\psi(t=0)\rangle=\frac{1}{\sqrt{5}}(|2,0\rangle+2 i|2,-1\rangle)
$$

Evaluate the wave-function $|\psi(t)\rangle$ at time $t>0$ by solving the time-dependent Schrodinger equation.
(c)(3 pt) What energies can be measured, and with what probabilities, for the above state for general $t$ ? Also, what is $\langle H\rangle$ for general $t$ ?
(d)(2 pt) Find $\left\langle L_{x}\right\rangle$ and $\left\langle L_{z}\right\rangle$ for general $t$.
$\qquad$
$\qquad$

## \#11 : GRADUATE QUANTUM MECHANICS

PROBLEM: Let $L_{ \pm}=L_{x} \pm i L_{y}$ and $L_{z}$ be orbital angular momentum operators. Consider an operator $V_{+}$which satisfies

$$
\left[L_{+}, V_{+}\right]=0, \quad\left[L_{z}, V_{+}\right]=\hbar V_{+}
$$

(a)(3pt) Let $|\ell, m\rangle$ be a simultaneous eigenfunction of $L^{2}$ and $L_{z}$ with eigenvalues $\ell(\ell+1) \hbar^{2}$ and $m \hbar$, respectively. Show that

$$
V_{+}|\ell, \ell\rangle=\text { const }|\ell+1, \ell+1\rangle
$$

(b)(3pt) Demonstrate, for the case of orbital angular momenta, that

$$
V_{+}=e^{i \phi} \sin \theta
$$

satisfies the commutation relations of $V_{+}$with $L_{+}$and $L_{z}$ given above. Recall that the operators $L_{+}$and $L_{z}$ are given by the differential operators

$$
\begin{gathered}
L_{+}=\hbar e^{i \phi}\left(\frac{\partial}{\partial \theta}+i \cot \theta \frac{\partial}{\partial \phi}\right) \\
L_{z}=-i \hbar \frac{\partial}{\partial \phi}
\end{gathered}
$$

(c)( $4 \mathbf{p t}$ ) Assume that $|0,0\rangle=$ const $\equiv 1 / \sqrt{4 \pi}$. Using the equations in parts (a) and (b), determine the functions $|\ell, \ell\rangle$ for arbitrary $\ell$, normalized such that

$$
\langle\ell, \ell \mid \ell, \ell\rangle=1 .
$$

A useful integral is

$$
\int_{0}^{\pi / 2} d \theta \sin ^{2 \ell+1} \theta=\frac{2^{\ell} \ell!}{(2 \ell+1)!!}
$$

where $(2 \ell+1)!!\equiv(2 \ell+1)(2 \ell-1) \cdots 5 \cdot 3 \cdot 1$.
$\qquad$

## \#12 : GRADUATE QUANTUM MECHANICS

PROBLEM:
Consider $N$ identical, non-interacting spinless fermions in a potential $V(x)$ where $V(x)=0$ for $0 \leq x \leq L$, and $V(x)=\infty$ otherwise. Recall that the single-particle eigenstates for this potential are given by $\phi_{n}(x)=\langle x \mid n\rangle=\sqrt{\frac{2}{L}} \sin \left(\frac{n \pi x}{L}\right)$ where $n$ runs from 1 to $\infty$.

2pts (i) Denoting the second-quantized operator that creates a fermion in the singleparticle level $|n\rangle$ as $a^{\dagger}(n)$, write down the ground state $\left|\psi_{0}\right\rangle$ of this system in terms of the operators $a^{\dagger}(n)$ acting on the vacuum $|0\rangle$ (= the state with no particles). It is OK to leave the wavefunction unnormalized.

3pts (ii) Calculate the density of fermions $\rho(x)$ at location $x$ in the real-space for the ground state $\left|\psi_{0}\right\rangle$. You may express your answer in terms of the function $F\left(N, y_{1}, y_{2}\right)=\sum_{n=1}^{N} \sin \left(n y_{1}\right) \sin \left(n y_{2}\right)$ which you are not required to evaluate explicitly in terms of $N, y_{1}, y_{2}$.

5 pts (iii) Next, calculate the correlation function $\left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a^{\dagger}\left(x_{2}\right) a\left(x_{2}\right) a\left(x_{1}\right)\left|\psi_{0}\right\rangle$ where $a^{\dagger}(x)$ is the operator that creates a fermion at location $x$ in the real-space. Assume $0 \leq x_{1}, x_{2} \leq L$. Again, just express the answer in terms of the function $F$ defined in part (ii) above.
$\qquad$
$\qquad$

## \#13 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider a system of $N$ identical bosons with spin zero in two dimensional space of area $A$ at temperature $T$. The total number of bosons is conserved, and the energy of single-particle levels is given by $\epsilon(\vec{p})=c|\vec{p}|$ where $\vec{p}$ is the corresponding momentum. Assuming $\frac{h c}{T} \gg \rho^{-1 / 2}$ where $\rho$ is the particle density (= number of particles per unit area), calculate the number of particles in the ground state as a function of $T, A$, and $N$.

You may use the integral $\int_{0}^{\infty} \frac{x d x}{e^{x}-1}=\pi^{2} / 6$.
$\qquad$ SCORE: $\qquad$

## \#14 : GRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider the modified Blume-Capel Hamiltonian,

$$
\hat{H}=-\frac{1}{2} J \sum_{\langle i j\rangle} S_{i} S_{j}+\Delta \sum_{i} S_{i}^{2}
$$

on a regular lattice of coordination number $z$. The first sum is over all links of the lattice, and the second sum is over all sites. At each site, the spin variable $S_{i}$ may be in one of four states: $S_{i} \in\{-1,0,0,+1\}$. Note that the state with $S_{i}=0$ is doubly degenerate; this is important.. The coupling $J$ is positive, but $\Delta$ may be of either sign.
(a)(3pt) Making the mean field approximation in the first term of $\hat{H}$ only, find the mean field Hamiltonian $\hat{H}_{\mathrm{MF}}$.
(b)(4pt) Adimensionalize by writing $\theta \equiv k_{\mathrm{B}} T / z J$ and $\omega \equiv \Delta / z J$, and find the dimensionless free energy per site $f \equiv F / N z J$ as a function of $\theta, \omega$, and the local magnetization $m=\left\langle S_{i}\right\rangle$ (same for all sites).
(c) $(3 \mathbf{p t})$ Find the self-consistent mean field equation for $m$.
$\qquad$ SCORE: $\qquad$

## \#15 : GRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider two systems in thermal contact, so that energy (and only energy) can be exchanged between them. Let the density of states of system 1 be $D_{1}\left(E_{1}\right)$ and that of system 2 be $D_{2}\left(E_{2}\right)$. The interfacial energy between the two systems is negligible in comparison with the bulk energies.
(a)(2pt) Suppose the total energy is fixed at $E=E_{1}+E_{2}$. Find an expression for the total density of states $D(E)$.
 that the total energy is $E$.
(c)(2pt) Show that under the condition that $P_{1}\left(E_{1}\right)$ is maximized the temperatures satisfy $T_{1}=T_{2}$.
(d) $(4 \mathbf{p t})$ Expanding $E_{1}=E_{1}^{*}+\Delta E_{1}$ about the maximum, show, in the thermodynamic limit, that

$$
P_{1}\left(E_{1}^{*}+\Delta E_{1}\right)=P_{1}\left(E_{1}^{*}\right) \exp \left(-\frac{\left(\Delta E_{1}\right)^{2}}{2 k_{\mathrm{B}} T^{2} \bar{C}_{V}}\right)
$$

where $T$ is the common temperature. Find $\bar{C}_{V}$ in terms of the heat capacities $C_{V, 1}$ and $C_{V, 2}$ of the individual systems.

## INSTRUCTIONS PHYSICS DEPARTMENT WRITTEN EXAM PART II

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| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
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(a)(2pt) Find $\left[H, L_{x}\right],\left[H, L_{y}\right]$, and $\left[H, L_{z}\right]$. Which $L_{i}$ are $t$ independent in the Heisenberg picture?
(b)(3 pt)At time $t=0$, the system is in a superposition of $|\ell, m\rangle$ states:

$$
|\psi(t=0)\rangle=\frac{1}{\sqrt{5}}(|2,0\rangle+2 i|2,-1\rangle)
$$

Evaluate the wave-function $|\psi(t)\rangle$ at time $t>0$ by solving the time-dependent Schrodinger equation.
(c)(3 pt) What energies can be measured, and with what probabilities, for the above state for general $t$ ? Also, what is $\langle H\rangle$ for general $t$ ?
(d)(2 pt) Find $\left\langle L_{x}\right\rangle$ and $\left\langle L_{z}\right\rangle$ for general $t$.

SOLUTION:
(a) $\left[H, L_{x}\right]=\alpha\left[L_{z}, L_{x}\right]=i \hbar \alpha L_{y},\left[H, L_{y}\right]=-i \alpha \hbar L_{x},\left[H, L_{z}\right]=0$. So only $L_{z}$ is $t$-independent in the Heisenberg picture.
(b) $|\psi(t)\rangle=e^{-i H t / \hbar}|\psi(t=0)\rangle$ :

$$
|\psi(t)\rangle=\frac{1}{\sqrt{5}} e^{-3 i \hbar t / I}\left(|2,0\rangle+2 i e^{i \alpha t}|2,-1\rangle\right) .
$$

(c) $E_{2,0}=3 \hbar^{2} / I$ with probability $1 / 5$ and $E_{2,-1}=3 \hbar^{2} / 2 I-\alpha \hbar$ with probability $4 / 5$. So $\langle H\rangle=3 \hbar^{2} / I-(4 / 5) \alpha$. These are all time independent.
(d) $\left\langle L_{x}\right\rangle=\frac{1}{2}\left\langle\left(L_{+}+L_{-}\right)\right\rangle=\frac{-2 \sqrt{6}}{5} \sin \alpha t,\left\langle L_{z}\right\rangle=-\frac{4}{5} \hbar$.
$\qquad$
$\qquad$

## \#11 : GRADUATE QUANTUM MECHANICS

PROBLEM: Let $L_{ \pm}=L_{x} \pm i L_{y}$ and $L_{z}$ be orbital angular momentum operators. Consider an operator $V_{+}$which satisfies

$$
\left[L_{+}, V_{+}\right]=0, \quad\left[L_{z}, V_{+}\right]=\hbar V_{+}
$$

(a)(3pt) Let $|\ell, m\rangle$ be a simultaneous eigenfunction of $L^{2}$ and $L_{z}$ with eigenvalues $\ell(\ell+1) \hbar^{2}$ and $m \hbar$, respectively. Show that

$$
V_{+}|\ell, \ell\rangle=\text { const }|\ell+1, \ell+1\rangle
$$

(b)(3pt) Demonstrate, for the case of orbital angular momenta, that

$$
V_{+}=e^{i \phi} \sin \theta
$$

satisfies the commutation relations of $V_{+}$with $L_{+}$and $L_{z}$ given above. Recall that the operators $L_{+}$and $L_{z}$ are given by the differential operators

$$
\begin{gathered}
L_{+}=\hbar e^{i \phi}\left(\frac{\partial}{\partial \theta}+i \cot \theta \frac{\partial}{\partial \phi}\right) \\
L_{z}=-i \hbar \frac{\partial}{\partial \phi}
\end{gathered}
$$

(c)(4pt) Assume that $|0,0\rangle=$ const $\equiv 1 / \sqrt{4 \pi}$. Using the equations in parts (a) and (b), determine the functions $|\ell, \ell\rangle$ for arbitrary $\ell$, normalized such that

$$
\langle\ell, \ell \mid \ell, \ell\rangle=1 .
$$

A useful integral is

$$
\int_{0}^{\pi / 2} d \theta \sin ^{2 \ell+1} \theta=\frac{2^{\ell} \ell!}{(2 \ell+1)!!}
$$

where $(2 \ell+1)!!\equiv(2 \ell+1)(2 \ell-1) \cdots 5 \cdot 3 \cdot 1$.

SOLUTION:
(a)

$$
\begin{gathered}
L^{2}|\ell, m\rangle=\ell(\ell+1) \hbar^{2}|\ell, m\rangle \\
L_{z}|\ell, m\rangle=m \hbar|\ell, m\rangle \\
{\left[L_{+}, V_{+}\right]=0 \quad \Rightarrow \quad L_{+} V_{+}|\ell, \ell\rangle=V_{+} L_{+}|\ell, \ell\rangle=0}
\end{gathered}
$$

$\qquad$
where $L_{+}|\ell, \ell\rangle=0$ since $m=\ell$ eigenstate is the eigenstate with the highest $L_{z}$ eigenvalue, for $L^{2}$ eigenvalue equal to $\ell(\ell+1) \hbar^{2}$.

$$
\left[L_{z}, V_{+}\right]=\hbar V_{+} \quad \Rightarrow \quad L_{z}\left(V_{+}|\ell, \ell\rangle\right)=\left(V_{+} L_{z}+V_{+} \hbar\right)|\ell, \ell\rangle=(\ell+1) \hbar\left(V_{+}|\ell, \ell\rangle\right)
$$

Thus, $\left(V_{+}|\ell, \ell\rangle\right)$ is an eigenstate (up to normalization) of $L_{z}$ with eigenvalue $m \hbar=$ $(\ell+1) \hbar$.

Since

$$
L_{+}\left(V_{+}|\ell, \ell\rangle\right)=0
$$

one concludes that it cannot be raised to an eigenstate of $L_{z}$ with eigenvalue $m \hbar=$ $(\ell+2) \hbar$. Thus,

$$
\left(V_{+}|\ell, \ell\rangle\right)=\text { const }|\ell+1, \ell+1\rangle
$$

(b)

$$
\begin{gathered}
V_{+}=e^{i \phi} \sin \theta \\
{\left[L_{+}, V_{+}\right]=\left[\hbar e^{i \phi}\left(\frac{\partial}{\partial \theta}+i \cot \theta \frac{\partial}{\partial \phi}\right), e^{i \phi} \sin \theta\right]} \\
=\hbar e^{i \phi}\left(\cos \theta e^{i \phi}+i \cot \theta\left(i e^{i \phi} \sin \theta\right)\right)=0 \\
{\left[L_{z}, V_{+}\right]=\left[-i \hbar \frac{\partial}{\partial \phi}, e^{i \phi} \sin \theta\right]=\hbar e^{i \phi} \sin \theta=V_{+} \hbar}
\end{gathered}
$$

(c)

$$
\begin{gathered}
|0,0\rangle=\frac{1}{\sqrt{4 \pi}} \\
\langle 0,0 \mid 0,0\rangle=\frac{1}{4 \pi} \int d \Omega=1
\end{gathered}
$$

For $\ell=1$,

$$
\begin{gathered}
|1,1\rangle=c_{1} V_{+}|0,0\rangle=\frac{1}{\sqrt{4 \pi}} c_{1} e^{i \phi} \sin \theta \\
\langle 1,1 \mid 1,1\rangle=\left|c_{1}\right|^{2}\langle 0,0| V_{+}^{\dagger} V_{+}|0,0\rangle=\frac{1}{4 \pi}\left|c_{1}\right|^{2} \int_{0}^{2 \pi} d \phi \int_{-1}^{+1} d(\cos \theta) \sin ^{2} \theta=\frac{2}{3}\left|c_{1}\right|^{2}=1 \\
\Rightarrow c_{1}=\sqrt{\frac{3}{2}}, \quad|1,1\rangle=\sqrt{\frac{3}{8 \pi}} e^{i \phi} \sin \theta
\end{gathered}
$$

For general $\ell$,

$$
|\ell, \ell\rangle=c_{\ell}\left(V_{+}\right)^{\ell}|0,0\rangle=\frac{1}{\sqrt{4 \pi}} c_{\ell} e^{i \ell \phi} \sin ^{\ell} \theta
$$

$$
\begin{gathered}
\langle\ell, \ell \mid \ell, \ell\rangle=\left|c_{\ell}\right|^{2} \frac{1}{4 \pi} \int_{0}^{2 \pi} d \phi \int_{-1}^{+1} d(\cos \theta) \sin ^{2 \ell} \theta=\frac{1}{2}\left|c_{\ell}\right|^{2} \int_{0}^{\pi} d \theta \sin ^{2 \ell+1} \theta \\
=\left|c_{\ell}\right|^{2} \int_{0}^{\pi / 2} d \theta \sin ^{2 \ell+1} \theta=\left|c_{\ell}\right|^{2} \frac{2^{\ell} \ell!}{(2 \ell+1)!!}=1 \\
\Rightarrow \quad c_{\ell}=\sqrt{\frac{(2 \ell+1)!!}{2^{\ell} \ell!}}, \quad(2 \ell+1)!!\equiv(2 \ell+1)(2 \ell-1) \cdots 5 \cdot 3 \cdot 1 \\
|\ell, \ell\rangle=\sqrt{\frac{(2 \ell+1)!!}{2^{\ell} \ell!}} \frac{1}{\sqrt{4 \pi}} e^{i \ell \phi} \sin ^{\ell} \theta
\end{gathered}
$$

$\qquad$

## \#12 : GRADUATE QUANTUM MECHANICS

PROBLEM:
Consider $N$ identical, non-interacting spinless fermions in a potential $V(x)$ where $V(x)=0$ for $0 \leq x \leq L$, and $V(x)=\infty$ otherwise. Recall that the single-particle eigenstates for this potential are given by $\phi_{n}(x)=\langle x \mid n\rangle=\sqrt{\frac{2}{L}} \sin \left(\frac{n \pi x}{L}\right)$ where $n$ runs from 1 to $\infty$.

2pts (i) Denoting the second-quantized operator that creates a fermion in the singleparticle level $|n\rangle$ as $a^{\dagger}(n)$, write down the ground state $\left|\psi_{0}\right\rangle$ of this system in terms of the operators $a^{\dagger}(n)$ acting on the vacuum $|0\rangle$ (= the state with no particles). It is OK to leave the wavefunction unnormalized.

3pts (ii) Calculate the density of fermions $\rho(x)$ at location $x$ in the real-space for the ground state $\left|\psi_{0}\right\rangle$. You may express your answer in terms of the function $F\left(N, y_{1}, y_{2}\right)=\sum_{n=1}^{N} \sin \left(n y_{1}\right) \sin \left(n y_{2}\right)$ which you are not required to evaluate explicitly in terms of $N, y_{1}, y_{2}$.

5 pts (iii) Next, calculate the correlation function $\left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a^{\dagger}\left(x_{2}\right) a\left(x_{2}\right) a\left(x_{1}\right)\left|\psi_{0}\right\rangle$ where $a^{\dagger}(x)$ is the operator that creates a fermion at location $x$ in the real-space. Assume $0 \leq x_{1}, x_{2} \leq L$. Again, just express the answer in terms of the function $F$ defined in part (ii) above.

SOLUTION:
(i) $\left|\psi_{0}\right\rangle=\prod_{i=1}^{N} a^{\dagger}(i)|0\rangle$ (up to normalization).
(ii) $\rho(x)=\left\langle\psi_{0}\right| a^{\dagger}(x) a(x)\left|\psi_{0}\right\rangle=\sum_{m, n}\left\langle\psi_{0}\right| a^{\dagger}(n) a(m)\left|\psi_{0}\right\rangle\langle n \mid x\rangle\langle x \mid m\rangle$. Only $m=n$ term contributes since $\left|\psi_{0}\right\rangle$ is an eigenstate. Further, $\left\langle\psi_{0}\right| a^{\dagger}(n) a(n)\left|\psi_{0}\right\rangle=1$ for $1 \leq$ $n \leq N$, and $\left\langle\psi_{0}\right| a^{\dagger}(n) a(n)\left|\psi_{0}\right\rangle=0$ otherwise.

$$
\begin{align*}
\Rightarrow \rho(x) & =\sum_{n=1}^{N}|\langle n \mid x\rangle|^{2}  \tag{1}\\
& =\frac{2}{L} \sum_{n=1}^{N} \sin ^{2}(n \pi x / L)  \tag{2}\\
& =\frac{2}{L} F(N, \pi x / L, \pi x / L) \tag{3}
\end{align*}
$$

(iii)

$$
\begin{align*}
& \left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a^{\dagger}\left(x_{2}\right) a\left(x_{2}\right) a\left(x_{1}\right)\left|\psi_{0}\right\rangle  \tag{4}\\
= & \left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a\left(x_{1}\right)\left|\psi_{0}\right\rangle\left\langle\psi_{0}\right| a^{\dagger}\left(x_{2}\right) a\left(x_{2}\right)\left|\psi_{0}\right\rangle-\left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a\left(x_{2}\right)\left|\psi_{0}\right\rangle\left\langle\psi_{0}\right| a^{\dagger}\left(x_{2}\right) a\left(x_{1}\right) \mid  \tag{5}\\
= & \left.\rho\left(x_{1}\right) \rho\left(x_{2}\right)-\left|\sum_{m, n}\left\langle\psi_{0}\right| a^{\dagger}(n) a(m)\right| \psi_{0}\right\rangle\left.\left\langle n \mid x_{1}\right\rangle\left\langle x_{2} \mid m\right\rangle\right|^{2} \tag{6}
\end{align*}
$$

Again, only $m=n$ contributes in the above sum.

$$
\begin{align*}
\Rightarrow \quad & \left\langle\psi_{0}\right| a^{\dagger}\left(x_{1}\right) a^{\dagger}\left(x_{2}\right) a\left(x_{2}\right) a\left(x_{1}\right)\left|\psi_{0}\right\rangle  \tag{7}\\
= & \rho\left(x_{1}\right) \rho\left(x_{2}\right)-\left(\frac{2}{L} F\left(N, \pi x_{1} / L, \pi x_{2} / L\right)\right)^{2}  \tag{8}\\
= & \frac{4}{L^{2}}\left[F\left(N, \pi x_{1} / L, \pi x_{1} / L\right) F\left(N, \pi x_{2} / L, \pi x_{2} / L\right)-F^{2}\left(N, \pi x_{1} / L, \pi x_{2} / L\right)\right] \tag{9}
\end{align*}
$$

Note that this correlation function vanishes when $x_{1}=x_{2}$ as it should since $\left(a^{\dagger}(x)\right)^{2}=$ 0 .
$\qquad$

## \#13 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider a system of $N$ identical bosons with spin zero in two dimensional space of area $A$ at temperature $T$. The total number of bosons is conserved, and the energy of single-particle levels is given by $\epsilon(\vec{p})=c|\vec{p}|$ where $\vec{p}$ is the corresponding momentum. Assuming $\frac{h c}{T} \gg \rho^{-1 / 2}$ where $\rho$ is the particle density (= number of particles per unit area), calculate the number of particles in the ground state as a function of $T, A$, and $N$.

You may use the integral $\int_{0}^{\infty} \frac{x d x}{e^{x}-1}=\pi^{2} / 6$.

SOLUTION: When $\frac{h c}{T} \gg \rho^{-1 / 2}$, the particles will bose-condense, and therefore, the chemical potential $\mu=0$. Denoting the number of particles in the ground state as $N_{0}$, and the total number of particles in the excited states as $N_{e x}$, clearly $N_{0}+N_{e x}=$ $N . N_{e x}$ can be calculated by summing the occupation number of particles in the excited states:

$$
\begin{align*}
N_{e x} & =\frac{A}{(2 \pi)^{2}} \int_{0}^{\infty} \frac{2 \pi k d k}{e^{\beta \hbar c c k}-1}  \tag{10}\\
& =\frac{A T^{2}}{2 \pi(\hbar c)^{2}} \int_{0}^{\infty} \frac{x d x}{e^{x}-1}  \tag{11}\\
& =\frac{\pi A T^{2}}{12(\hbar c)^{2}}  \tag{12}\\
\Rightarrow N_{0} & =N-\frac{\pi A T^{2}}{12(\hbar c)^{2}} \tag{13}
\end{align*}
$$

$\qquad$
$\qquad$

## \#14 : GRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider the modified Blume-Capel Hamiltonian,

$$
\hat{H}=-\frac{1}{2} J \sum_{\langle i j\rangle} S_{i} S_{j}+\Delta \sum_{i} S_{i}^{2}
$$

on a regular lattice of coordination number $z$. The first sum is over all links of the lattice, and the second sum is over all sites. At each site, the spin variable $S_{i}$ may be in one of four states: $S_{i} \in\{-1,0,0,+1\}$. Note that the state with $S_{i}=0$ is doubly degenerate; this is important.. The coupling $J$ is positive, but $\Delta$ may be of either sign.
(a)(3pt) Making the mean field approximation in the first term of $\hat{H}$ only, find the mean field Hamiltonian $\hat{H}_{\mathrm{MF}}$.
(b)(4pt) Adimensionalize by writing $\theta \equiv k_{\mathrm{B}} T / z J$ and $\omega \equiv \Delta / z J$, and find the dimensionless free energy per site $f \equiv F / N z J$ as a function of $\theta, \omega$, and the local magnetization $m=\left\langle S_{i}\right\rangle$ (same for all sites).
(c) $(3 \mathbf{p t})$ Find the self-consistent mean field equation for $m$.

SOLUTION:
(a) Writting $S_{i}=m+\delta S_{i}$ and neglecting terms quadratic in the fluctuations $\delta S_{i}$ in the first term in $\hat{H}$, we obtain

$$
\hat{H}_{\mathrm{MF}}=\frac{1}{2} N z J m^{2}-z J m \sum_{i} S_{i}+\Delta \sum_{i} S_{i}^{2} .
$$

(b) The partition function is

$$
\begin{aligned}
Z_{\mathrm{MF}}=\operatorname{Tr} e^{-\hat{H}_{\mathrm{MF}} / k_{\mathrm{B}} T} & =e^{-N m^{2} / 2 \theta}\left(\sum_{S} e^{m S / \theta} e^{-\omega S^{2} / \theta}\right)^{N} \\
& =e^{-N m^{2} / 2 \theta}\left(2+2 e^{-\omega / \theta} \cosh (m / \theta)\right)^{N}=e^{-N f / \theta}
\end{aligned}
$$

Thus,

$$
f(m, \theta, \omega)=\frac{1}{2} m^{2}-\theta \ln \left(2+2 e^{-\omega / \theta} \cosh (m / \theta)\right)
$$

(c) Setting $\partial f / \partial m=0$, we obtain the mean field equation

$$
m=\frac{\sinh (m / \theta)}{e^{\omega / \theta}+\cosh (m / \theta)} .
$$

$\qquad$

## \#15 : GRADUATE STATISTICAL MECHANICS

PROBLEM:
Consider two systems in thermal contact, so that energy (and only energy) can be exchanged between them. Let the density of states of system 1 be $D_{1}\left(E_{1}\right)$ and that of system 2 be $D_{2}\left(E_{2}\right)$. The interfacial energy between the two systems is negligible in comparison with the bulk energies.
(a)(2pt) Suppose the total energy is fixed at $E=E_{1}+E_{2}$. Find an expression for the total density of states $D(E)$.
 that the total energy is $E$.
(c) $(2 \mathbf{p t})$ Show that under the condition that $P_{1}\left(E_{1}\right)$ is maximized the temperatures satisfy $T_{1}=T_{2}$.
(d) $(4 \mathbf{p t})$ Expanding $E_{1}=E_{1}^{*}+\Delta E_{1}$ about the maximum, show, in the thermodynamic limit, that

$$
P_{1}\left(E_{1}^{*}+\Delta E_{1}\right)=P_{1}\left(E_{1}^{*}\right) \exp \left(-\frac{\left(\Delta E_{1}\right)^{2}}{2 k_{\mathrm{B}} T^{2} \bar{C}_{V}}\right)
$$

where $T$ is the common temperature. Find $\bar{C}_{V}$ in terms of the heat capacities $C_{V, 1}$ and $C_{V, 2}$ of the individual systems.

## SOLUTION:

(a) The combined density of states is

$$
D(E)=\int_{-\infty}^{\infty} d E_{1} D_{1}\left(E_{1}\right) D_{2}\left(E-E_{1}\right)
$$

(b) The probability density for system 1 to have energy $E_{1}$ is then

$$
P_{1}\left(E_{1}\right)=\frac{D_{1}\left(E_{1}\right) D_{2}\left(E-E_{1}\right)}{D(E)}
$$

Note that $\int_{-\infty}^{\infty} d E_{1} P_{1}\left(E_{1}\right)=1$ is normalized.
(c) We now ask: what is the most probable value of $E_{1}$ ? We find out by differentiating $P_{1}\left(E_{1}\right)$ with respect to $E_{1}$ and setting the result to zero. This requires

$$
0=\frac{\partial}{\partial E_{1}} \ln P_{1}\left(E_{1}\right)=\frac{\partial}{\partial E_{1}} \ln D_{1}\left(E_{1}\right)+\frac{\partial}{\partial E_{1}} \ln D_{2}\left(E-E_{1}\right)
$$

$\qquad$

Since $S_{j}\left(E_{j}\right)=k_{\mathrm{B}} \ln D_{j}\left(E_{j}\right)$, we conclude that the maximally likely partition of energy between systems 1 and 2 is realized when

$$
\frac{\partial S_{1}}{\partial E_{1}}=\frac{\partial S_{2}}{\partial E_{2}}
$$

This guarantees that

$$
S\left(E, E_{1}\right)=S_{1}\left(E_{1}\right)+S_{2}\left(E-E_{1}\right)
$$

is a maximum with respect to the energy $E_{1}$, at fixed total energy $E$. As $T=$ $\partial S / \partial E$, we have $T_{1}=T_{2}$.
(d) We saw that the probability distribution $P_{1}\left(E_{1}\right)$ is maximized when $T_{1}=T_{2}$, but how sharp is the peak in the distribution? Let us write $E_{1}=E_{1}^{*}+\Delta E_{1}$, where $E_{1}^{*}$ is the location of the maximum of $P_{1}\left(E_{1}\right)$. We then have

$$
\ln P_{1}\left(E_{1}^{*}+\Delta E_{1}\right)=\ln P_{1}\left(E_{1}^{*}\right)+\left.\frac{1}{2 k_{\mathrm{B}}} \frac{\partial^{2} S_{1}}{\partial E_{1}^{2}}\right|_{E_{1}^{*}}\left(\Delta E_{1}\right)^{2}+\left.\frac{1}{2 k_{\mathrm{B}}} \frac{\partial^{2} S_{2}}{\partial E_{2}^{2}}\right|_{E_{2}^{*}}\left(\Delta E_{1}\right)^{2}+\ldots,
$$

where $E_{2}^{*}=E-E_{1}^{*}$. We must now evaluate

$$
\frac{\partial^{2} S}{\partial E^{2}}=\frac{\partial}{\partial E}\left(\frac{1}{T}\right)=-\frac{1}{T^{2}}\left(\frac{\partial T}{\partial E}\right)_{V, N}=-\frac{1}{T^{2} C_{V}}
$$

where $C_{V}=(\partial E / \partial T)_{V, N}$ is the heat capacity. Thus,

$$
P_{1}\left(E_{1}^{*}+\Delta E_{1}\right)=P_{1}\left(E_{1}^{*}\right) \exp \left(-\frac{\left(\Delta E_{1}\right)^{2}}{2 k_{\mathrm{B}} T^{2} \bar{C}_{V}}\right)
$$

where

$$
\bar{C}_{V}=\frac{C_{V, 1} C_{V, 2}}{C_{V, 1}+C_{V, 2}}
$$

The distribution is therefore a Gaussian, and the fluctuations in $\Delta E_{1}$ can now be computed:

$$
\left\langle\left(\Delta E_{1}\right)^{2}\right\rangle=k_{\mathrm{B}} T^{2} \bar{C}_{V} \quad \Longrightarrow \quad\left(\Delta E_{1}\right)_{\mathrm{RMS}}=k_{\mathrm{B}} T \sqrt{\bar{C}_{V} / k_{\mathrm{B}}}
$$

The individual heat capacities $C_{V, 1}$ and $C_{V, 2}$ scale with the volumes $V_{1}$ and $V_{2}$, respectively. If $V_{2} \gg V_{1}$, then $C_{V, 2} \gg C_{V, 1}$, in which case $\bar{C}_{V} \approx C_{V, 1}$. Therefore the RMS fluctuations in $\Delta E_{1}$ are proportional to the square root of the system size, whereas $E_{1}$ itself is extensive. Thus, the ratio $\left(\Delta E_{1}\right)_{\text {RMS }} / E_{1} \propto V^{-1 / 2}$ scales as the inverse square root of the volume. The distribution $P_{1}\left(E_{1}\right)$ is thus extremely sharp.

