#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: Two material points P_1 and P_2 with mass m_1 and m_2, respectively, are allowed to slide, without friction, in a vertical plane along the y axis of a reference frame $(O; x, y)$ fixed with respect to the ground. The y axis is oriented towards the ground. The point P_1 is attached to the origin by a (massless) spring with elastic constant $k_1 > 0$, and P_2 is attached to P_1 by a (massless) spring with elastic constant $k_2 > 0$.

3 pt (a) Determine the Lagrangian of the system.

3 pt (b) Determine the Lagrange equations of motion.

4pt (c) Find the equilibrium positions (fixed points) of the system and determine if they are stable or not.

SOLUTION:

Let us call y_1 and y_2 the ordinates of the two points, respectively. The kinetic energy is then:

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2,$$

while the potential energy due to gravity and elastic forces is (g is the acceleration of gravity):

$$V = -m_1 g y_1 - m_2 g y_2 + \frac{1}{2}k_1 y_1^2 + \frac{1}{2}k_2(y_2 - y_1)^2.$$

The Lagrangian is then

$$L(y_1, \dot{y}_1, y_2, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_1 g y_1 + m_2 g y_2 - \frac{1}{2}k_1 y_1^2 - \frac{1}{2}k_2(y_2 - y_1)^2$$

The Lagrange equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = 0,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} - \frac{\partial L}{\partial y_2} = 0,$$

or

$$m_1\ddot{y}_1 = m_1 g - k_1 y_1 + k_2(y_2 - y_1),$$
$$m_2\ddot{y}_2 = m_2 g - k_2(y_2 - y_1),$$

The fixed points correspond to those positions that render the derivative of the potential energy zero:

$$\frac{\partial V}{\partial y_1} = 0 = -m_1 g + k_1 y_1 - k_2(y_2 - y_1),$$
$$\frac{\partial V}{\partial y_2} = 0 = -m_2 g + k_2(y_2 - y_1).$$

Therefore we find the fixed point of the system:

$$y_1 = \frac{(m_1 + m_2)g}{k_1}; \qquad y_2 = \frac{(m_1 + m_2)g}{k_1} + \frac{m_2 g}{k_2}.$$

To determine whether this fixed point is stable or not we need to determine the Hessian of the potential energy V:

$$\mathcal{H}(V(y_1, y_2)) = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Since the trace of the Hessian is positive ($\mathrm{Tr}(\mathcal{H}) = k_1 + 2k_2$) and its determinant is positive ($\det(\mathcal{H}) = k_1 k_2$) then the equilibrium position is stable.

#2 : GRADUATE CLASSICAL MECHANICS

PROBLEM:

Consider the following Hamiltonian

$$H(q,p) = -p\sin(2q),$$

with p the momentum, and q the generalized coordinate.

Apply the following canonical transformation

$$P = -2\sqrt{p}\sin q; \quad Q = \sqrt{p}\cos q$$

to the above Hamiltonian.

5 pt (a) Solve the Hamilton equations of motion of the transformed Hamiltonian.

5 pt (b) Determine the generating function in the form $F = F(q, P)$.

SOLUTION:

From the canonical transformation we easily find that the new Hamiltonian $K(Q, P) = PQ$. Therefore, the new Hamilton equations of motion are:

$$\dot{P} = -P; \qquad \dot{Q} = Q.$$

From $\dot{P} + P = 0$ we get the solution

$$P(t) = Ae^{-t+\alpha},$$

while from $\dot{Q} - Q = 0$ we get

$$Q(t) = Be^{t+\beta},$$

with A, B, α, β constants determined by the initial conditions.

Since we are looking for a generating function of the transformation of the form $F = F(q, P)$ we get:

$$p = \frac{P^2}{4\sin^2 q} = \frac{\partial F}{\partial q}$$

and

$$Q = -\frac{P\cos q}{2\sin q} = \frac{\partial F}{\partial P}.$$

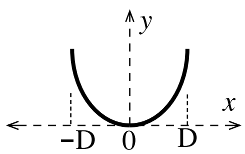
Therefore, we get

$$F(q, P) = -\frac{P^2\cos q}{4\sin q}.$$

#3 : GRADUATE CLASSICAL MECHANICS

PROBLEM:

A uniform string , with mass per unit length ρ and fixed length, hangs in uniform gravity g between two identical posts located at $x = -D$ and $x = D$ respectively. The string hangs down to $(x, y) = (0, 0)$.



5 pt (a)Use the Calculus of Variations and Lagrangian mechanics (and in particular, constants of the motion) to find a *first*-order differential equation for the equilibrium shape $y(x)$ of the string.

3 pt (b)Show by substitution that your ODE is satisfied by $y(x) = \alpha\text{cosh}(x/\alpha) - \alpha$ (the Catenary equation), where $\alpha > 0$ is a parameter related to the other parameters in your ODE; and find this relation.

2 pt (c)Find the length of the string in terms of α and D.

SOLUTION:

(a) The equilibrium is a state of minimum potential energy subject to the constraint that the end locations and the length L are fixed. Therefore we minimize the functional $A[y] = E[y] + \lambda L[y]$ where λ is a Lagrange multiplier, $E[y]$ is the string potential energy and $L[y]$ is the string length, each given by
$E = \int_{-D}^{D} dx\sqrt{1 + \dot{y}^2}(\rho g y)$ and $L = \int_{-D}^{D} dx\sqrt{1 + \dot{y}^2}$ where $\dot{y} = dy/dx$.

Thus, we minimize an "action" $A = \int dx\sqrt{1 + \dot{y}^2}(\rho g y + \lambda)$and we can identify the "Lagrangian" as

$$\mathcal{L} = \sqrt{1 + \dot{y}^2}(\rho g y + \lambda).$$

One could apply the Euler-Lagrange equations to this Lagrangian, but that would result in a (complicated) second-order ODE for y. Instead, note that this Lagrangian is independent of the "time" x so the energy $\mathcal{E}$ is conserved, where $\mathcal{E} = p\dot{y} - \mathcal{L}$. The momentum p is

$$p = \partial\mathcal{L}\,/\partial\dot{y}\, = (\rho g y + \lambda)\dot{y}/\sqrt{1+\dot{y}^2}.$$

Therefore,

$$\mathcal{E} = \frac{(\rho g y + \lambda)\dot{y}^2}{\sqrt{1+\dot{y}^2}} - \sqrt{1+\dot{y}^2}(\rho g y + \lambda) = -\frac{(\rho g y + \lambda)}{\sqrt{1+\dot{y}^2}} = \text{constant}.$$

Physically, the constant $\mathcal{E}$ is the $x-$component of the string tension force.

This is the first order ODE. It can be rewritten as

$$1 + \dot{y}^2 = \frac{(\rho g y + \lambda)^2}{\mathcal{E}^2}.$$

The constant $\mathcal{E}$ can be replaced by λ since $\dot{y} = 0$ at $y = 0$.

(b) To find the relation between α and these parameters, substitute $y(x) = \alpha\text{cosh}(x/\alpha) - \alpha$ into the ODE, noting that $\dot{y} = \text{sinh}(x/\alpha)$, so we have

$$1 + \sinh^2 = \frac{(\lambda - \rho g\alpha + \rho g\alpha\cosh)^2}{\lambda^2}.$$

Without doing any hard algebra, note that there is only one possible way to make this work, and that is if $\lambda = \rho g\alpha$. In that case the equation simplifies to $1 + \sinh^2 = \cosh^2$, which is of course true. Therefore $\alpha = \lambda/\rho g$.

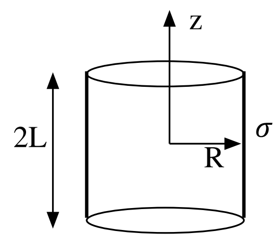
(c)The length of the string is given by $L = \int_{-D}^{D} dx\sqrt{1+\dot{y}^2}$. Again substituting in the solution for y yields

$$L = 2\int_0^D dx\,\text{cosh}(x/\alpha) = 2\alpha\text{sinh}(D/\alpha).$$

#4 : UNDERGRADUATE CLASSICAL ELECTRODYNAMICS

PROBLEM:

A cylindrical shell of charge, with its central axis along the $\hat{z}$ axis, has finite length $2L$ extending from $z = -L$ to $z = L$. The shell has a uniform surface charge density σ with cylindrical radius R.



5 pt (a) Find the electric field along the $\hat{z}$ axis directly by integrating over the charge distribution for the cases $z > L$ and for $0 < z < L$.

3 pt(b)Check that your answer reduces to the expected result $\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{Q_{\rm TOT}}{z^2}\hat{z}$, where $Q_{\rm TOT}$ is the total charge on the shell, in the limit $z \gg L, R$.

2 pt (c)Find $\vec{E}$ at the origin $r = z = 0$.

SOLUTION:

(a)

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')da'}{|\vec{r}-\vec{r}'|^2}\cos\alpha$$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\sigma(2\pi R)\int_{-L}^{L} dz' \frac{(z-z')}{[(z-z')^2 + R^2]^{3/2}}$$

where

$$\vec{r} = (0,0,z)$$

$$\vec{r}' = (R,0,z')$$

$$|\vec{r}-\vec{r}'| = \left[(z-z')^2 + R^2\right]^{1/2}$$

$$\cos\alpha = \frac{(\vec{r}-\vec{r}')\cdot\vec{z}}{|\vec{r}-\vec{r}'|z} = \frac{(z-z')}{|\vec{r}-\vec{r}'|}$$

Notice that $\cos\alpha$ changes sign at $z = z'$, so for $0 < z < L$, $\int_z^L$ contributes in $-\hat{z}$ direction while $\int_{-L}^{z}$ contributes in $+\hat{z}$ direction. For $z > L$, the entire integration $\int_{-L}^{L}$ contributes in the $+\hat{z}$ direction.

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\sigma(2\pi R)\left[\frac{1}{[(z-z')^2+R^2]^{1/2}}\right]\Bigg|_{-L}^{L}$$

$z > L$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\sigma(2\pi R)\left(\frac{1}{[(z-L)^2+R^2]^{1/2}} - \frac{1}{[(z+L)^2+R^2]^{1/2}}\right)$$

$0 < z < L$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\sigma(2\pi R)\left(\frac{1}{[(L-z)^2+R^2]^{1/2}} - \frac{1}{[(z+L)^2+R^2]^{1/2}}\right)$$

(b) Check that $z > L$ result reduces correctly in limit $z >> L, R$.

$$Q_{\mathrm{TOT}} = \sigma(2\pi R)(2L)$$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R)}{z}\left(\frac{1}{\left[\left(1-\frac{L}{z}\right)^2+\left(\frac{R}{z}\right)^2\right]^{1/2}} - \frac{1}{\left[\left(1+\frac{L}{z}\right)^2+\left(\frac{R}{z}\right)^2\right]^{1/2}}\right)$$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R)}{z}\left(\frac{1}{\left[\left(1-2\frac{L}{z}+\frac{(R^2+L^2)}{z^2}\right)\right]^{1/2}} - \frac{1}{\left[\left(1+2\frac{L}{z}+\frac{(R^2+L^2)}{z^2}\right)\right]^{1/2}}\right)$$

$$E_z(z) = \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R)}{z}\left(\left(1+\frac{L}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right) - \left(1-\frac{L}{z}+\mathcal{O}\left(\frac{1}{z^2}\right)\right)\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi R)(2L)}{z^2} + \cdots = \frac{1}{4\pi\epsilon_0}\frac{Q_{\mathrm{TOT}}}{z^2} + \cdots$$

(c) Find $\vec{E}$ at $r = z = 0$: $E_x = E_y = 0$ by symmetry. E_z is also zero by symmetry. This can also be shown by taking $z \to 0$ limit of the $0 < z < L$ result.

$$E_z(0) = \frac{1}{4\pi\epsilon_0}\sigma(2\pi R)\left(\frac{1}{[L^2 + R^2]^{1/2}} - \frac{1}{[L^2 + R^2]^{1/2}}\right) = 0$$

This answer makes sense since the two halves of the charged cylindrical shell contribute equally and oppositely to $E_z(0)$. Thus, $E_z(-z) = -E_z(z)$.

#5 : GRADUATE CLASSICAL ELECTRODYNAMICS

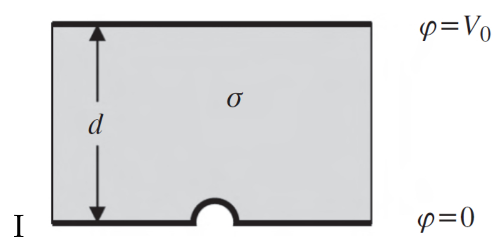
`PROBLEM:`

A voltage difference V_0 drives a steady current from one perfect conductor to another through an Ohmic medium of conductivity σ whose shape is depicted in the Figure. A small hemispherical bump (radius a) in the bottom conductor perturbs the simple uniform current flow that would exist in the absence of the bump (i.e., in a slab of the uniform thickness d).

10 pt Calculate the corresponding change in the effective conductance $1/R$ of the medium assuming $d \gg a$.

Hint: Find how the presence of the bump modifies the total Joule heating

$$P = \int d^3r(\mathbf{j} \cdot \mathbf{E}) \equiv \frac{V_0^2}{R}\,.$$



`SOLUTION:` We take the center of the bump as the origin of the coordinates with the z-axis pointing up. The field $\mathbf{E} = -\nabla\varphi$ can be found from the potential φ. Since the field is related to the current via Ohm's law $\mathbf{j} = \sigma\mathbf{E}$, the potential must satisfy the Laplace equation

$$\nabla^2\varphi = -\nabla \cdot \mathbf{E} = -\sigma\nabla \cdot \mathbf{j} = 0\,.$$

The potential must also obey the boundary conditions (BC) shown in the Figure. To make the problem more familiar we extend φ to negative z by the relation $\varphi(z) \equiv -\varphi(-z)$, $-d < z < 0$, so that the BC at $z = 0$ is automatically satisfied. We arrive at the electrostatic problem of a conducting sphere positioned inside a parallel-plate capacitor with plates at $z = \pm d$

maintained at fixed potentials $\varphi = \pm V_0$. When $d \gg a$, this problem is equivalent to that of a sphere subject to the uniform field $E_0\hat{\mathbf{z}}$,

$$E_0 = \frac{V_0}{d}\,.$$

The solution is well known:

$$\varphi = -E_0 z + E_0 a^3 \frac{z}{r^3}\,.$$

Calculating the gradient, we find the electric field components

$$E_r = E_0 \cos\theta \left(1 + 2\frac{a^3}{r^3}\right), \qquad E_\theta = E_0 \sin\theta \left(-1 + \frac{a^3}{r^3}\right),$$

and the square of the field,

$$E^2 = E_0^2 \left[\cos^2\theta \left(1 + 2\frac{a^3}{r^3}\right)^2 + \sin^2\theta \left(-1 + \frac{a^3}{r^3}\right)^2\right].$$

The angular average of the latter quantity is

$$\langle E^2 \rangle = \int_0^{\pi/2} \sin\theta d\theta E^2 = E_0^2 \left[\frac{1}{3}\left(2\frac{a^3}{r^3} + 1\right)^2 + \frac{2}{3}\left(\frac{a^3}{r^3} - 1\right)^2\right] = E_0^2 \left(1 + \frac{2a^6}{r^6}\right).$$

This result is valid for $a < r \ll d$. At $r < a$, inside the bump, the field is zero. The excess Joule heating caused by the bump is

$$\Delta P = \sigma \int d^3r (E^2 - E_0^2) = 2\pi\sigma E_0^2 \int_0^a dr r^2 (-1) + 2\pi\sigma E_0^2 \int_a^\infty dr r^2 \frac{2a^6}{r^6} = \frac{2\pi}{3}\sigma E_0^2 a^3\,.$$

Therefore,

$$\Delta\left(\frac{1}{R}\right) = \frac{\Delta P}{V_0^2} = \frac{2\pi}{3}\frac{\sigma a^3}{d^2}\,.$$

#6 : GRADUATE CLASSICAL ELECTRODYNAMICS

`PROBLEM`: Two airplanes are flying at a distance d apart at a height h above a plane water surface. One sends radio signals to the other, both sending and receiving antennas being vertical wires of length much shorter than the radio wavelength λ.

5 pt a) Show that the Fresnel reflection coefficient of a p-polarized plane wave at the air-water interface is

$$\frac{\epsilon_1 \cos\theta_0 - \epsilon_0\sqrt{\frac{\epsilon_1}{\epsilon_0} - \sin^2\theta_0}}{\epsilon_1 \cos\theta_0 + \epsilon_0\sqrt{\frac{\epsilon_1}{\epsilon_0} - \sin^2\theta_0}}\,,$$

where ϵ_1, ϵ_0 are the permittivities of water and air and θ_0 is the angle of incidence.

5 pt b) Find the ratio of intensity of the signal received by water reflection to that of the direct signal, assuming $d \gg \lambda$, $h \gg \lambda$.

`SOLUTION`: a. In a p-polarized wave the incident $\mathbf{E}_I$, the reflected $\mathbf{E}_R$, and the transmitted $\mathbf{E}_T$ electric field amplitudes are all parallel to the plane of incidence. The magnetic field amplitudes $\mathbf{H}_i$'s are related to the corresponding $\mathbf{E}_i$'s via the impedance factors Z_i:

$$Z_i \mathbf{H}_i = \widehat{\mathbf{k}}_i \times \mathbf{E}_i\,, \qquad Z_i = \sqrt{\frac{\mu_i}{\epsilon_i}}\,.$$

The angle of incidence θ_0 and the angle of refraction θ_1 are linked by the Snell law,

$$\sqrt{\mu_0\epsilon_0}\,\sin\theta_0 = \sqrt{\mu_1\epsilon_1}\,\sin\theta_1\,.$$

The matching conditions for the total E- and H-fields give

$$E_I\cos\theta_0 - E_R\cos\theta_0 = E_T\cos\theta_1\,, \qquad H_I + H_R = H_T\,.$$

The second equation entails

$$Z_2(E_I + E_R) = Z_1 E_T\,.$$

It follows that the reflection coefficient is

$$r_p \equiv \frac{E_R}{E_I} = \frac{Z_0 \cos\theta_0 - Z_1 \cos\theta_1}{Z_0 \cos\theta_0 + Z_1 \cos\theta_1} \,.$$

In the present case $\mu_1 = \mu_0$, and so we arrive at the desired formula:

$$r_p = \frac{\epsilon_1 \cos\theta_0 - \epsilon_0 \sqrt{\frac{\epsilon_1}{\epsilon_0} - \sin^2\theta_0}}{\epsilon_1 \cos\theta_0 + \epsilon_0 \sqrt{\frac{\epsilon_1}{\epsilon_0} - \sin^2\theta_0}} \,.$$

b. We treat the sending antenna as a dipole which is polarized in the $\hat{\mathbf{z}}$-direction and has the radiation pattern

$$\mathbf{E} = -\frac{A}{r}\, e^{ikr - i\omega t}\, \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}}) \,, \qquad A = \text{const} \,.$$

The receiving antenna responds to the $\hat{\mathbf{z}}$ component of the field,

$$E_z = \mathbf{E} \cdot \hat{\mathbf{z}} = \frac{A}{r}\, e^{ikr - i\omega t}\, |\hat{\mathbf{r}} \times \hat{\mathbf{z}}|^2.$$

This formula gives the "direct" signal

$$|E_z^{\text{dir}}| = \frac{|A|}{d}$$

upon substitution $\mathbf{r} \to \hat{\mathbf{x}} d$, the vector from the sender to the receiver. To compute the signal due to reflection off water we employ the method of images. Namely, we set $\mathbf{r} \to \hat{\mathbf{x}} d + \hat{\mathbf{z}} \cdot 2h$, the vector from the position of the image (which is distance h under water) to the receiver. Additionally, we multiply $\mathbf{E}$ by the reflection coefficient r_p at the air-water interface. We can use the Fresnel formula for the p-polarization because the relevant ($\hat{\mathbf{z}}$) component of the field is in the incidence plane, i.e., the plane defined by the propagation wavevector $\mathbf{k} = \hat{\mathbf{r}} k$ and the interface normal $\hat{\mathbf{z}}$. The angle of incidence θ_0 is the angle between the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$:

$$\sin\theta_0 = |\hat{\mathbf{r}} \times \hat{\mathbf{z}}| = \sqrt{\frac{d^2}{d^2 + 4h^2}} \,.$$

Since $r = d/\sin\theta$, the "reflected" signal is $|E_z^{\mathrm{ref}}| = |(A/d)r_p|\sin^3\theta_0$. The ratio of the intensities of the two signals is

$$\left|\frac{E_z^{\mathrm{ref}}}{E_z^{\mathrm{dir}}}\right|^2 = |r_p|^2 \sin^6\theta_0 = \left|\frac{2h\frac{\epsilon_1}{\epsilon_0} - \sqrt{\frac{\epsilon_1}{\epsilon_0}(d^2+4h^2) - d^2}}{2h\frac{\epsilon_1}{\epsilon_0} + \sqrt{\frac{\epsilon_1}{\epsilon_0}(d^2+4h^2) - d^2}}\right|^2 \left(\frac{d^2}{d^2+4h^2}\right)^3.$$

#7 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:

3 pt a) Consider the integral (x positive)

$$I(x) = \int_0^\infty e^{-xt-\frac{1}{t}}\, dt\,.$$

Find its dominant asymptotic behavior as $x \to \infty$.

3 pt b) Find the first dominant term in the asymptotic expansion of $I(x)$ as $x \to 0$.

4 pt c)The differential equation satisfied by $F(x) \equiv x^{1/2} I(x)$ is:

$$\left[x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} - \left(\frac{1}{4} + x\right)\right] F(x) = 0\,.$$

Perform a local analysis as

i) $x \to 0$ and

ii) $x \to \infty$

of this differential equation , and use the results to obtain the asymptotic behaviors of $I(x)$.

SOLUTION:

(a) The asymptotic behavior is obtained by the Laplace method. The integral has a moving maximum that scales as $1/\sqrt{x}$. Defining $t \equiv u/\sqrt{x}$, the integral becomes

$$I(x) = \int_0^\infty e^{-\sqrt{x}(u+1/u)}\, \frac{du}{\sqrt{x}}\,.$$

The argument of the exponential has a maximum at $u = 1$, with second derivative at that point equal to -2. The final result is then

$$I(x) \sim \frac{\sqrt{\pi}}{x^{3/4}}\; e^{-2\sqrt{x}}\,.$$

(b) The integral for $x = 0$ diverges at infinity. The asymptotic behavior can be obtained integrating by parts. We write $e^{-xt} = -\frac{1}{x}\frac{de^{-xt}}{dt}$ and integrate by parts to obtain

$$I(x) = \frac{1}{x}\int_0^\infty e^{-tx-1/t}\frac{1}{t^2}\,dt\,,$$

where we can now safely put $x = 0$ to obtain $I(x) \sim 1/x$ as $x \to 0$.

(c) Let us consider first $x \to 0$. Inserting $F(x) \sim x^\gamma$ into the equation for $F(x)$, we obtain $\left(\gamma\,(\gamma-1)+\gamma-\frac{1}{4}\right)x^\gamma \simeq 0$, where we have neglected the subdominant term $-x^{\gamma+1}$. It follows that $\gamma^2 = 1/4$ and the most singular term is $\gamma = -1/2$. Since $F(x) = x^{1/2}I(x) \sim x^{-1/2}$ we obtain again $I(x) \sim 1/x$.

For the behavior at infinity, we make the transformation $F(x) = e^{-S(x)}$, which gives the equation for S:

$$x^2\left(S'^2 - S''\right) - xS' - \left(\frac{1}{4}+x\right) = 0\,. \tag{1}$$

Assuming $S'^2 \gg S''$ (which is then verified self-consistently) and applying dominant-balance arguments, we see that $x^2S'^2 \sim x$, which gives $S' \sim 1/\sqrt{x}$ (the negative sign is not acceptable as the solution would blow up at infinity). This gives $S \sim 2\sqrt{x}$, which is the same exponential factor found in question (1). It is verified that indeed $S'^2 \gg S''$ and $xS' \ll x^2S'^2$.

To find the next subdominant terms, we write $S \sim 2\sqrt{x} + S_1(x)$ and insert this expression into Eq. (1), which yields:

$$2x^{3/2}S_1' + \frac{1}{2}x^{1/2} - x^{1/2} - \frac{1}{4} = 0\,.$$

The last term is obviously subdominant, so that $S_1' \simeq \frac{1}{4x}$, i.e. $S_1 \simeq \frac{1}{4}\ln x$. Combining $F(x) = x^{1/2}I(x)$ and $F(x) = e^{-S(x)}$, we obtain $I(x) \sim e^{-2\sqrt{x}}x^{-3/4}$, in agreement with question (1).

By writing $S(x) \sim 2\sqrt{x} + \frac{1}{4}\ln x + S_2(x)$, one can verify that $S_2(x)$ is a constant (which cannot be fixed by local analysis as the original equation is linear).

#8 : GRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:

Consider the ordinary differential equation

$$\frac{d^2z(t)}{dt^2} + z(t) - \varepsilon \left[\frac{dz}{dt} - \frac{1}{3} \left(\frac{dz}{dt} \right)^3 \right] = 0 \,, \tag{1}$$

where ε is a positive constant. This is known as Rayleigh nonlinear oscillator.

Using the transformation $w = \frac{dz}{dt}$ the Rayleigh oscillator is converted into the van der Pol oscillator

$$\frac{d^2w(t)}{dt^2} + w(t) - \varepsilon \left(1 - w^2\right) \frac{dw}{dt} = 0 \,. \tag{2}$$

3 pt a) Show that the origin $\left(w, \frac{dw}{dt}\right) = (0, 0)$ is an unstable fixed point of Eq. (2).

7 pt b) For the sake of the following calculations, we shall find it convenient to use Eq. (1). We consider it in the limit of small ε and use multiple-scale analysis to find the approach of the solution to its limit cycle. The slow time scale $\tau \equiv \varepsilon t$ and the solution is sought as $Z_0(t, \tau) + \varepsilon Z_1(t, \tau) + \ldots$. Apply multiple-scale analysis at the first order in ε and write the solution $z(t)$ at that order for the initial conditions $z(0) = 2\gamma$ (with $\gamma > 1$) and $dz/dt(0) = 0$. It will be convenient to write $Z_0(t, \tau)$ as $Z_0(t, \tau) = 2A(\tau) \cos(t + \phi(\tau)) = A(\tau) \left[e^{i(t+\phi(\tau))} + e^{-i(t+\phi(\tau))}\right]$, with the equations for the amplitude $A(\tau)$ and the phase $\phi(\tau)$ to be determined. Identify period and amplitude of the asymptotic limit cycle.

SOLUTION:

(a) The linearized stability equations are

$$\frac{dw}{dt} = \dot{w} \,; \quad \frac{d\dot{w}}{dt} = -w + \varepsilon \dot{w} \,.$$

The eigenvalues are $\frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$. Both of them have their real part positive for any $\varepsilon > 0$.

(b) By using the multiple-scales method, we have $\partial_t \mapsto \partial_t + \varepsilon\partial_\tau$ and $z(t) \mapsto Z_0(t,\tau) + \varepsilon Z_1(t,\tau) + \ldots$, which leads to the equations

$$\begin{aligned} \frac{d^2 Z_0(t,\tau)}{dt^2} + Z_0(t,\tau) &= 0\,; && (3) \\ \frac{d^2 Z_1(t,\tau)}{dt^2} + Z_1(t,\tau) &= -2\partial_t\partial_\tau Z_0 + \partial_t Z_0 - \frac{1}{3}\left(\partial_t Z_0\right)^3 . && (4) \end{aligned}$$

at the orders ε^0 and ε^1. The general solution to the first equation is written in the suggested form $Z_0(t,\tau) = A(\tau)\left[e^{i(t+\phi(\tau))} + e^{-i(t+\phi(\tau))}\right]$ for future convenience.

Let us now consider Eq. (4). To avoid resonances, the r.h.s. should be orthogonal to the elements in the kernel of the adjoint of the l.h.s., i.e. terms proportional to e^{it} and e^{-it} should vanish. That leads to the equations

$$2\left(\dot{A}(\tau) + iA(\tau)\dot{\phi}(\tau)\right) - A(\tau) + A^3(\tau) = 0\,.$$

Separating real and imaginary parts, we obtain $\dot{\phi}(\tau) = 0$ and $2\dot{A}(\tau) = A(\tau) - A^3(\tau)$, which is easily integrated to yield $\frac{A(\tau)}{\sqrt{A^2-1}} = Ke^{\tau/2}$, where K is a free constant to be fixed by the initial conditions and we have anticipated that $A^2 > 1$. Initial values impose $K = \gamma/\sqrt{\gamma^2-1}$ and $\phi(\tau) = \phi(0) = 0$, which brings the final expression

$$z(t) \simeq \frac{2\gamma\cos(t)}{\sqrt{\gamma^2 - e^{-\varepsilon t}\left(\gamma^2-1\right)}}\,,$$

where we have replaced $\tau = \varepsilon t$.

We conclude that the asymptotic limit cycle has period 2π and amplitude 2.

#9 : UNDERGRADUATE MATHEMATICS AND GENERAL PHYSICS

PROBLEM:

The drag force on a sphere moving through a medium can take two forms depending on how important viscosity is relative to the size and speed of the object.

7 pt a) Ignoring gravity (e.g., horizontal motion, or space station), use dimensional and physical arguments to create two expressions for the drag force: one that includes kinematic viscosity, ν (1.5×10^{-5} $\mathrm{m}^2\,\mathrm{s}^{-1}$ for air), and one that does not (inertial drag regime).

3 pt b) For modest, everyday speeds in air, how small is the sphere where the regimes cross (are comparable)?

SOLUTION:

Relevant paramaters would be the radius and velocity of the sphere, R and v, the density of the medium, ρ, and the viscosity (in one case), ν. One may be tempted to include the mass (or density) of the sphere, but absent gravity there are no forces—including buoyancy—acting on the sphere. Furthermore, while the acceleration/deceleration of said object would depend on its mass, the magnitude of the force is just an interaction of the geometry pushing through the fluid, independent of what is within the exterior surface.

Dimensionally, we aim for a force in Newtons, or kg m s^{-2}. Density of the medium should be important in both regimes, and in any case is the only variable that contributes kilograms. When viscosity is important, presumably the drag force is proportional to some positive power, preferably $n = 1$, of viscosity. Putting these two together, we find that we still need m^2 s^{-1} to form Newtons, which we get by multiplying radius and velocity to form $F = \rho\nu Rv$. Note that we could also achive the correct units via $\rho\nu^2$, or $\rho R^2 v^2$. We would expect any valid result to depend on the size and speed of the object, so can reject the first of these. The second, however, is interesting for the case where viscosity is not important.

So at a minimum, we have two possible regimes: $F \propto \rho\nu Rv$, and $F \propto$

$\rho R^2 v^2$. We might appreciate that the second form relates to disrupted kinetic energy in a tube of oncoming air (in the frame of the sphere): the sphere sweeps up a cylindrical volume $V = \pi R^2 v \Delta t$ in time Δt so that the kinetic energy per time interval is $\frac{1}{2}\rho V v^2/\Delta t$. This is power, which is force times velocity. Thus after combining, $F = \frac{1}{2}\rho\pi R^2 v^2$. In practice, a coefficient of drag is added to the expresion, which for a sphere is 0.3–0.5.

In its most simple form, ignoring numerical factors, we can ask where the two drag regimes become comparable, when $\rho\nu R v = \rho R^2 v^2$. The answer is when $Rv \approx \nu$ (same as saying Reynolds Number equals unity). Picking a mundane speed of 1 m/s, we find a crossing when $R = 15\ \mu$m—a small grain of dust. At that size, perhaps our speed pick was ambitiously high; 0.1 m/s would allow a more typical large dust grain. As the size and speed diminish, the R^2v^2 term vanishes so that viscosity dominates for the small and slow.

Done correctly, Stokes drag is $6\pi\rho\nu R v$ and inertial drag is $\frac{1}{2}\rho c_{\mathrm{D}} A v^2$, where c_{D} is the coefficient of drag and A is the cross-sectional area. For a sphere having $c_{\mathrm{D}} \sim 0.4$, we find the crossing at $Rv = 12\nu/c_{\mathrm{D}}$ (or Reynolds number, $Rv/\nu = 12/c_{\mathrm{D}} \sim 30$).

#10 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM:

Consider a spinless particle of mass m moving nonrelativstically in one dimension x under the influence of a δ-function potential

$$V(x) = -\alpha\delta(x),$$

where $\alpha > 0$ has units of energy times length.

5 pt (a) Show that there is a single bound state, and find the energy E and wavefunction $\psi(x)$ of this state.

5 pt (b) Find an approximate value E_{app} for the bound state energy E using a variational wavefunction

$$\psi_b(x) = B\cos(\pi x/(2b)), -b < x < b,$$

and $\psi_b(x) = 0$ otherwise.

(Hint: To obtain E_{app} find the best value of the variational parameter b.)

SOLUTION:

(a) The energy eigenstates satisfy the 1-dimensional time-independent Shroedinger equation

$$-\tfrac{\hbar^2}{2m}\tfrac{d^2\psi}{dx^2} - \alpha\delta(x)\psi(x) = E\psi(x).$$

This equation can be solved by matching across the δ-function. For a bound state, $E < 0$. Solutions for $x \neq 0$ therefore satisfy

$$-\tfrac{\hbar^2}{2m}\tfrac{d^2\psi}{dx^2} = -|E|\psi(x),$$

which has exponential general solution $\psi = Ae^{kx} + Be^{-kx}$ where $k = \sqrt{2m|E|}/\hbar$. For a bound state we need $\psi \to 0$ as $x \to \pm\infty$. This implies $\psi = Ae^{kx}, x < 0$ and $\psi = Be^{-kx}, x > 0$. To find A, B and E, note that ψ must be continuous at $x = 0$ so $A = B$. Integrating the Schroedinger equation across $x = 0$ yields the second condition:

$$-\tfrac{\hbar^2}{2m}\tfrac{d\psi}{dx}\Big|_{x=0^-}^{x=0^+} - \alpha\psi(0) = 0,$$

which then implies

$$2A\tfrac{\hbar^2 k}{2m} - A\alpha = 0.$$

Thus, $k = m\alpha/\hbar^2$, so $E = -m\alpha^2/2\hbar^2$, and the wavefunction is

$$\psi = Ae^{-k|x|}.$$

The constant A can be obtained by normalization, $\int |\psi|^2 dx = 1$, implying $A = \sqrt{k}$.

(b) The constant B can be obtained by normalization, $\int |\psi|^2 dx = 1$, implying $B = 1/\sqrt{b}$. Then the energy is determined by minimizing

$$E = \left\langle \psi_v \left| H \right| \psi_v \right\rangle = \left\langle \psi_v \left| -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x) \right| \psi_v \right\rangle$$

with respect to b. Substituting for ψ_v and performing the integral yields

$$E = B^2 \left(\frac{\hbar^2}{2m}\left(\frac{\pi}{2b}\right)^2 b - \alpha\right) = \frac{\hbar^2\pi^2}{8mb^2} - \frac{\alpha}{b}$$

Setting $dE/db = 0$ then yields

$$b = \frac{\hbar^2\pi^2}{4m\alpha}$$

and using this above yields

$$E_{app} = -\frac{2m\alpha^2}{\pi^2\hbar^2}.$$

As expected, this approximate energy is less negative than the exact ground state energy given in (a) because $2/\pi^2 = 0.203 < 1/2$.

#11 : GRADUATE QUANTUM MECHANICS

PROBLEM:

6 pt (a) Consider elastic scattering of two identical spinless non-relativistic fermions where the potential for their relative coordinate is a hard-sphere potential, i.e. $V(\vec{r}) = \infty$ for $\vec{r} < a$, and $V(\vec{r}) = 0$ otherwise. Assuming that the energy of each fermion is $k^2/2m$ in the center of mass frame, which angular momentum channel contributes to the scattering cross-section at the leading order when $ka \ll 1$? What is the scattering phase shift δ in this channel while still assuming $ka \ll 1$?

4 pt (b) Now, instead consider the elastic scattering of two identical spinless non-relativistic *bosons*. Which angular momentum channel contributes to the scattering cross-section at the leading order when $ka \ll 1$? What is the scattering phase shift in this channel while still assuming $ka \ll 1$?

Hint:

As $x \to 0$, the spherical Bessel functions behave as $j_l(x) \approx x^l/(2l+1)!!$ while $n_l(x) \approx -(2l-1)!!/x^{l+1}$.

SOLUTION:

(a) The wavefunction for the two fermions must be anti-symmeteric under exchange, therefore, the s-wave scattering amplitude is identically zero. Mathematically, the wavefunction is proportional to $P_l(\cos(\theta)) - P_l(-\cos(\theta)) = (1 - (-1)^l)P_l(\cos(\theta))$, where θ is the scattering angle in the center of mass frame, and P_l are Legendre polynomial. Therefore, the component of wavefunction for even l vanishes identically. When $ka \ll 1$, the leading order non-zero contribution to the cross-section therefore comes from angular momentum $l = 1$, i.e., p-wave scattering. Recall that the partial-wave component of a scattering state in the l'th angular momentum channel is proportional to $(\cos(\delta_l)j_l(kr) - \sin(\delta_l)n_l(kr))$. Since the wavefunction must vanish at $r = a$, $\tan(\delta_l) = j_l(ka)/n_l(ka)$. Expanding to the leading order in ka,

$$\begin{aligned}
\tan(\delta_{l=1}) &= j_{l=1}(ka)/n_{l=1}(ka) \\
&\approx \frac{\frac{ka}{3}}{\frac{-1}{(ka)^2}} = -(ka)^3/3 \\
\Rightarrow \delta_{l=1} &\approx -(ka)^3/3.
\end{aligned}$$

Thus, to the leading order, the phase shift is $-(ka)^3/3$.

(b) For bosons, the wavefunction is symmetric under exchange, and therefore, the leading non-zero contribution comes from $l = 0$, i.e., s-wave scattering. Repeating the same argument, to the leading order, the phase shift is $\delta_0 \approx j_{l=0}(ka)/n_{l=0}(ka) \approx -ka$.

#12 : GRADUATE QUANTUM MECHANICS

PROBLEM:

An electron moves in the presence of a uniform magnetic field in the $\hat{z}$ direction ($\vec{B} = B_z \hat{z}$).

3 pt (a) Evaluate $[\Pi_x, \Pi_y]$, where $\Pi_x \equiv p_x - \frac{eA_x}{c}$ and $\Pi_y \equiv p_y - \frac{eA_y}{c}$.

7 pt (b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional harmonic oscillator problem, show that one can write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB_z|\hbar}{mc}\right)\left(n + \frac{1}{2}\right),$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.

SOLUTION:

(a)

$$[\Pi_x, \Pi_y] = \left[p_x - \frac{eA_x}{c}, p_y - \frac{eA_y}{c}\right] = [p_x, p_y] - \frac{e}{c}[A_x, p_y] - \frac{e}{c}[p_x, A_y] + \frac{e^2}{c^2}[A_x, A_y]$$

$$= -\frac{e}{c}\left[A_x, \frac{\hbar}{i}\frac{\partial}{\partial y}\right] - \frac{e}{c}\left[\frac{\hbar}{i}\frac{\partial}{\partial x}, A_y\right] = \frac{i\hbar e}{c}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) = \frac{i\hbar e}{c}B_z$$

Uniform field $\vec{B}$ implies that B_z is a constant, independent of position.

(b)

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\vec{\Pi}^2$$

For concreteness, one can choose $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r} = \frac{1}{2}B_z(-y, x, 0)$, where B_z is a constant, independent of position. Then, the Hamiltonian is

$$H = \frac{1}{2m}p_z^2 + \frac{1}{2m}\Pi_x^2 + \frac{1}{2m}\Pi_y^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right)$$

where Π_x and Π_y satisfy the commutation relation in part (a), as well as $[\Pi_x, \Pi_x] = 0$ and $[\Pi_y, \Pi_y] = 0$. The part of the Hamiltonian $\frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right)$ can be transformed to the Hamiltonian of a one-dimensional harmonic oscillator $\hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2$, with $[Q, P] = i\hbar$ and $a = \sqrt{\frac{m\omega}{2\hbar}}\left(Q + i\frac{P}{m\omega}\right)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(Q - i\frac{P}{m\omega}\right)$ if one performs a rescaling. The rescaling is $Q \equiv \frac{|eB_z|}{c}\Pi_x$, $P \equiv \Pi_y$, and $\omega \equiv \frac{|eB_z|}{mc}$. Thus, the Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \left(\frac{|eB_z|\hbar}{mc}\right)\left(a^\dagger a + \frac{1}{2}\right)$$

with eigenvalues

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB_z|\hbar}{mc}\right)\left(n + \frac{1}{2}\right)$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and $n = 0, 1, 2, \cdots$ is the eigenvalue of the number operator $N = a^\dagger a$.

#13 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:

Consider a three-dimensional non-relativistic Fermi gas consisting of N spinless particles with mass m in a volume V at temperature T.

3 pts (a) First, express the Fermi wave-vector k_F in terms of N and V.

5 pts (b) Next, denoting the Fermi energy by $\epsilon_F (= k_F^2/(2m))$, **assume** that the temperature dependence of the chemical potential μ is given by:

$$\mu = \epsilon_F - \frac{b\,T^2}{\epsilon_F}$$

where b is a constant. Using this expression, determine the entropy S via an appropriate Maxwell relation. You may further assume that the entropy S vanishes at $N = 0$.

2 pts (c) Using the entropy S obtained in part (b), calculate $C_{V,N}$, the specific heat at constant N, V.

SOLUTION:

(a) The total number of fermions are given by

$$\begin{aligned} N &= \tfrac{V}{(2\pi)^3} \int_{|\vec{k}|<k_F} d^3\vec{k} \\ &= \tfrac{V}{(2\pi)^3} \tfrac{4}{3}\pi k_F^3 \\ \Rightarrow k_F &= \left(6\pi^2 \tfrac{N}{V}\right)^{1/3} \end{aligned}$$

(b) From part (a), $\epsilon_F = k_F^2/2m$ is independent of the temperature T. Since we are given $\mu(T)$, and we are interested in finding entropy S, via $d(E - TS) = -SdT - pdV + \mu dN$, a useful Maxwell relation is $-\frac{dS}{dN}\big|_{T,V} = \frac{d\mu}{dT}\big|_{N,V}$. Using this, one finds,

$$
\begin{aligned}
-\frac{dS}{dN}\Big|_{T,V} &= \frac{d\mu}{dT}\Big|_{N,V} \\
&= \frac{-2bT}{\epsilon_F} \\
&= -4bTm\left(\frac{V}{6\pi^2 N}\right)^{2/3}
\end{aligned}
$$

Integrating the above equation w.r.t. N, one obtains,

$$S = 12bTmN^{1/3}\left(\frac{V}{6\pi^2}\right)^{2/3}$$

where we have set the constant of integration to zero using the physical assumption given in the problem that the entropy vanishes at $N = 0$.

(c) Using part (b), the specific heat $C_{V,N} = T\frac{dS}{dT}\Big|_{N,V} = 12bTmN^{1/3}\left(\frac{V}{6\pi^2}\right)^{2/3}$.

#14 : GRADUATE STATISTICAL MECHANICS

PROBLEM:

Consider the Ising model,

$$\hat{H} = -J \sum_{\langle ij \rangle} S_i \, S_j - H \sum_i S_i \quad ,$$

where the individual spins range over the values $S_i \in \big\{ -\frac{3}{2}\,,\, -\frac{1}{2}\,,\, +\frac{1}{2}\,,\, +\frac{3}{2} \big\}$. The interactions are between all nearest neighbors on a regular lattice of coordination number z, with $J > 0$. The total number of lattice sites is N.

2 pt(a) Writing $m = \langle S_i \rangle$ as the uniform mean field value of the local magnetization, derive the mean field Hamiltonian.

2 pt (b) Writing $f = F/NzJ$, $\theta = k_{\rm B}T/zJ$ and $h = H/zJ$, write the dimensionless free energy per site, $f(m, \theta, h)$.

3 pt (c) Find the self-consistent mean field equation.

2 pt (d) Do you expect the zero field mean field transition to be first or second order? Why?

1 pt (e) What is the mean field critical temperature $\theta_{\rm c}^{\rm MF}$?

SOLUTION:

(a) Writing $S_i = m + (S_i - m)$ and neglecting terms of order $\delta S_i \, \delta S_j$ in $\hat{H}$, we arrive at

$$\hat{H}^{\rm MF} = \tfrac{1}{2} NzJm^2 - (H + zJm) \sum_i S_i \quad .$$

(b) Taking the trace of $\exp(-\hat{H}^{\rm MF}/k_{\rm B}T)$ over the allowed values of S_i and then the logarithm, we obtain

$$f(m, \theta, h) = \tfrac{1}{2} m^2 - \theta \ln \left[2 \cosh\left(\frac{m+h}{2\theta} \right) + 2 \cosh\left(\frac{3(m+h)}{2\theta} \right) \right] \quad .$$

(c) The mean field equation is obtained by setting $\partial f / \partial m = 0$. This gives

$$m = \frac{\sinh\left(\frac{m+h}{2\theta}\right) + 3 \, \sinh\left(\frac{3(m+h)}{2\theta}\right)}{2 \cosh\left(\frac{m+h}{2\theta}\right) + 2 \cosh\left(\frac{3(m+h)}{2\theta}\right)} \quad .$$

(d) We expect a second order transition due to the Ising symmetry $f(m, \theta, h = 0) = f(-m, \theta, h = 0)$.

(e) The simplest way to determine the critical temperature is to expand the RHS of the mean field equation at $h = 0$:

$$m = \frac{\frac{m}{2\theta} + \frac{9m}{2\theta} + \mathcal{O}(m^3)}{4 + \mathcal{O}(m^2)} = \frac{5m}{4\theta} + \mathcal{O}(m^3) \quad .$$

The critical temperature is identified by equating the slopes of the LHS and the RHS. Therefore $\theta_c^{\rm MF} = \frac{5}{4}$. One can also go through the tedium of expanding $f(m, \theta, h)$, obtaining

$$f = -\theta \ln 4 - \frac{5}{4\,\theta}\, hm + \frac{1}{2}\big(1 - \frac{5}{4\theta}\big) m^2 + \frac{17}{192\,\theta^3}\, m^4 + \mathcal{O}(m^6, h^2, hm^3) \quad .$$

Setting the coefficient of the term quadratic in m to zero yields the critical temperature. Note that $f(m, \theta, h = 0)$ is an even function of m at fixed θ.

#15 : GRADUATE STATISTICAL MECHANICS

PROBLEM:

Consider a monatomic gas of N identical particles of mass m in three space dimensions. The Hamiltonian of each particle is

$$\hat{h} = \frac{\boldsymbol{p}^2}{2m} + \hat{h}_{\text{el}} \quad ,$$

where $\hat{h}_{\text{el}}$ is an electronic Hamiltonian with $(g+1)$ levels: a nondegenerate ground state at energy $\varepsilon_0 = 0$ and a g-fold degenerate excited state at energy $\varepsilon_1 = \Delta$.

4 pt (a) What is the single particle partition function ζ. Assume the system is confined to a box of volume V.

3 pt (b) What is the Helmholtz free energy $F(T, V, N)$?

3 pt (c) What is the heat capacity at constant volume $C_V(T, V, N)$?

SOLUTION:

(a) Integrating over momentum and summing over electronic states,

$$\zeta(T, V) = \frac{V}{\lambda_T^3}\left(1 + g\, e^{-\Delta/k_{\rm B}T}\right) \quad ,$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_{\rm B}T}$ is the thermal de Broglie wavelength.

(b) We have $F = -k_{\rm B}T \ln Z(T, V, N)$ where $Z = \zeta^N/N!$. Thus,

$$F(T, V, N) = -Nk_{\rm B}T \ln\big(1 + g\, e^{-\Delta/k_{\rm B}T}\big) = \tfrac{3}{2}Nk_{\rm B}T \ln\left(\frac{mk_{\rm B}T}{2\pi\hbar^2}\right) - Nk_{\rm B}T \ln\left(\frac{eV}{N}\right) \quad .$$

where we have used Stirling's rule $\ln K! = K \ln K - K + \mathcal{O}(\ln K)$ for K large.

(c) The heat capacity is

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = -T\,\frac{\partial^2 F}{\partial T^2} = \frac{N\Delta^2}{k_{\rm B}T^2}\left(\frac{1}{g^{-1} + \exp(\Delta/k_{\rm B}T)}\right)^2 + \tfrac{3}{2}Nk_{\rm B} \quad .$$

This expression is a linear sum of the Schottky-like peak from the electronic degrees of freedom and the usual monatomic ideal gas heat capacity.