INSTRUCTIONS
PHYSICS DEPARTMENT WRITTEN EXAM
PART I

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to attempt two problems. Each question will be graded on a scale of zero to ten points. **Circle the number of each of the two problems that you wish to be graded.**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:

   a. Write the problem number and your ID number on each white paper sheet;

   b. Write only on one side of the paper;

   c. Start each problem on the attached examination sheets;

   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and that your ID number is written on each sheet.

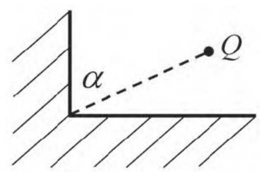
Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. On the top sheet, circle the problem numbers you will be submitting for grading.

Put everything back into the envelope that will be given to you at the start of the exam, and submit it to the proctor. Do not discard any paper.

#1 : CLASSICAL ELECTRODYNAMICS

PROBLEM:

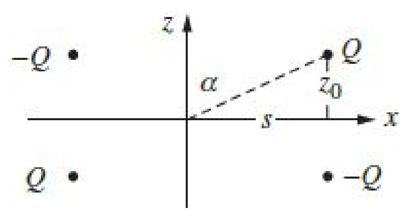
Two grounded conducting planes meet at a right angle. Find the charge induced on each plane when a point charge Q is introduced as shown in the diagram.



Hint:

$$\int \frac{dz}{(z^2+1)^{3/2}} = 2\,.$$

SOLUTION:



The diagram above shows an image system that ensures that both the $z = 0$ and $x = 0$ planes are grounded. These images can be used to compute the potential in the domain $x > 0$, $z > 0$. Moreover, all the field lines which leave Q end on one of the conducting surfaces because both are infinite in extent. This permits us to focus on the charge $Q_{z=0}$ induced on the $z = 0$ plane for $x > 0$; the charge on the $x = 0$ plane for $z > 0$ must be $-Q - Q_{z=0}$. The image solution for a charge q at the point $(0, 0, z_0)$ above the (grounded) $z = 0$ plane corresponds to the charge density induced on that plane equal to

$$\sigma(x, y) = -\frac{qz_0}{2\pi} \frac{1}{(x^2 + y^2 + z_0^2)^{3/2}}.$$

For our problem, we have a charge Q at $(s, 0, z_0)$ and a charge $-Q$ at $(-s, 0, z_0)$. Taking account of their images, both contribute to the charge density on the $z = 0$ plane. The total charge induced does not depend on the choice of origin for either charge. Therefore, the charge induced on the horizontal plate is

$$\begin{aligned}
Q_{z=0} &= -\frac{Qz_0}{2\pi} \int_{-s}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(y^2 + x^2 + z_0^2)^{3/2}} + \frac{Qz_0}{2\pi} \int_{+s}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(y^2 + x^2 + z_0^2)^{3/2}} \\
&= -\frac{Qz_0}{2\pi} \int_{-s}^{\infty} \frac{2dx}{x^2 + z_0^2} + \frac{Qz_0}{2\pi} \int_{+s}^{\infty} \frac{2dx}{x^2 + z_0^2} \\
&= \frac{Qz_0}{2\pi} \frac{2}{z_0} \left\{ \left[\arctan\left(\frac{x}{z_0}\right) \right] \Bigg|_{-s}^{\infty} - \left[\arctan\left(\frac{x}{z_0}\right) \right] \Bigg|_{+s}^{\infty} \right\} \\
&= -\frac{2Q}{\pi} \arctan\left(\frac{s}{z_0}\right) \\
&= -\frac{2\alpha}{\pi} Q \,.
\end{aligned}$$

The charge on the vertical plate is $Q_{x=0} = -Q - Q_{z=0} = Q[(2\alpha/\pi) - 1]$.

A more elegant derivation is via the Gauss theorem. According to it, the charge $Q_{z=0}$ induced on the horizontal half-plane is equal to the electric field flux through this half-plane multiplied by $(-\epsilon_0)$. The flux in question is the sum of the fluxes due to four point charges (the source and the images). For each of these charges $q = \pm Q$, the flux is the product of $q/(4\pi\epsilon_0)$ and the solid angle Ω subtended by the $z = 0$ half-plane when observed from the charge location. In turn, Ω is equal to twice the dihedral angle between the two half-planes passing through the charge: the horizontal one and the one extended from charge to the corner ($y = 0$ line). For example, for the source charge Q, the dihedral angle is $\frac{\pi}{2} + \alpha$. This argument reproduces the earlier result for $Q_{z=0}$ without any integration.

#2 : CLASSICAL ELECTRODYNAMICS

PROBLEM:

A coil carrying AC current $Ie^{-i\omega t}$ is tightly wound around a slab $-h < x < h$ of thickness $2h$, conductivity σ, and permeability μ. The turns of the coil are parallel to the z-axis and their density per unit length is n. Determine the magnetic field $\mathbf{H}(x)e^{-i\omega t}$ inside the slab using the quasi-static approximation and neglecting the fringing field. Derive the absolute value $|\mathbf{H}(x)|$ of the field amplitude (a real quantity) and its asymptotic behavior in the cases of strong ($\delta \ll h$) and weak ($\delta \gg h$) skin effect.

SOLUTION:

If the slab were nonconducting, the coil would produce a uniform field in the region $-h < x < h$:

$$\mathbf{H} = H_0 e^{-i\omega t}\hat{\mathbf{y}}, \qquad H_0 = \mu n I\,.$$

Outside the coil, $|x| > h$, the field would vanish (within the quasi-static approximation). On the other hand, inside a conducting slab, the field obeys the equation

$$\sigma\mu\,\frac{\partial}{\partial t}\mathbf{H} - \nabla^2\mathbf{H} = 0\,.$$

Neglecting fringing fields, $\mathbf{H}$ remains directed along the y-axis; however, H_0 is replaced by a function $H(x)$ such that

$$H''(x) + i\mu\sigma\omega H(x) = 0\,, \qquad H(\pm h) = H_0\,.$$

Introducing the skin depth δ, the solution of these equations is

$$H(x) = H_0\,\frac{\cosh\left(\frac{1-i}{\delta}\,x\right)}{\cosh\left(\frac{1-i}{\delta}\,h\right)}\,, \qquad \delta = \sqrt{\frac{2}{\mu\sigma\omega}}\,.$$

For the absolute value of $\mathbf{H}$, we get, after some trigonometry:

$$\begin{aligned}
|\mathbf{H}| = |H(x)| &= H_0 \sqrt{\frac{\sinh^2(x/\delta) + \cos^2(x/\delta)}{\sinh^2(h/\delta) + \cos^2(h/\delta)}} \\
&\simeq H_0 \exp\left(-\frac{h - |x|}{\delta}\right), \qquad\qquad \delta \ll h\,, \\
&\simeq H_0 \left(1 - \frac{h^4 - x^4}{3\delta^4}\right), \qquad\qquad \delta \gg h\,.
\end{aligned}$$

#3 : CLASSICAL ELECTRODYNAMICS

PROBLEM:

A "leaky" capacitor consists of two thin concentric spherical metal shells, with the inner shell at radius b and the outer shell at radius c. The volume $b < r < c$ in between is filled with a weakly conducting material with electric permittivity ϵ_0 and electrical conductivity, $\sigma = \sigma_0 b/r$, where σ_0 is a constant and r is the distance to the center of the spheres. The inner shell, at $r = b$ is at voltage V_0 and the outer shell, at radius $r = c$, is at voltage $V = 0$. Current flows from the inter shell to the outer one. (Ignore perturbing effects from whatever wires and battery would be needed to actually maintain the current and V_0.)

(a) Find the resistance R of the capacitor.

(b) Find the free charge $Q(r)$ inside any radius r (i.e., the charge which produces the electric field to drive the current) in terms of V_0, r, and the constants given.

(c) Find the capacitance C of this device.

(d) If the battery supplying the voltage V_0 is removed, what is the characteristic decay time of the charge on the inner shell? (Be sure to justify briefly your answer.)

SOLUTION:

(a) $I = 4\pi r^2 J = 4\pi r^2 \sigma(r) E(r) = 4\pi\sigma_0 r b E$ must be r independent (charge conservation), so $E = I/4\pi\sigma_0 r b$ which integrates to give $V_0 = \int_b^c \vec{E} \cdot d\vec{r} = \frac{I}{4\pi\sigma_0 b} \ln(c/b)$, so $R = V_0/I = (4\pi\sigma_0 b)^{-1} \ln(c/b)$.

(b) Gauss' law gives $\int \vec{E} \cdot d\vec{a} = Q/\epsilon_0 = Ir/\sigma_0 b$, so $Q(r) = 4\pi V_0 r \epsilon_0 / \ln(c/b)$.

(c) $C = Q(c)/V_0 = 4\pi\epsilon_0 c/\ln(c/b)$. (Credit also if the question is interpreted in terms of $Q(b)$ rather than $Q(c)$.)

(d) $\dot{Q} = -I = -V/R = C\dot{V}$ gives $\frac{d}{dt}(\ln V) = -1/RC$ so $V \sim e^{-t/\tau}$ with $\tau = RC = (\epsilon_0/\sigma_0)(c/b)$. (Credit also if interpreted in terms of $Q(b)$.)