Code#:

Spring 2016

INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section Mechanics: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Indicate the seven problems you wish to be graded:**

Section:	§1		§2		§3		§4		§5	
Problems: (Circle your seven choices)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.
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#1: UNDERGRADUATE MECHANICS

`PROBLEM:` In a world devoid of dissipative influences, a golf ball (assume a solid sphere of uniform density) of mass, M, and radius, R, is to be launched across a flat, level surface at a velocity, v, such that it rolls without slipping. Shortly after launch, the ball is to encounter a gentle ramp of angle, θ, and come to a stop at a height, h, just as it encounters a hole, or cup, into which it falls.

(a) (6 points) How much faster or slower must the solid ball be launched relative to the speed, v_0, of an idealized point particle that achieves the same outcome? Express as a ratio, or factor.

(b) (4 points) Now compute the ratio of velocities of a hollow, thin-shelled ball compared to the idealized point particle to achieve the same result.

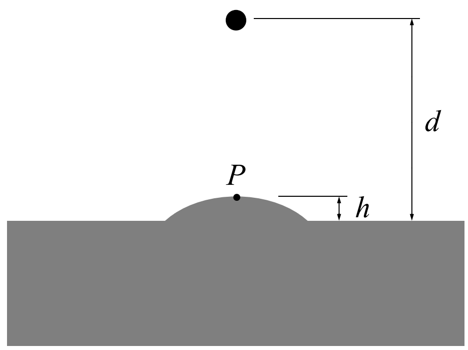
#2: UNDERGRADUATE MECHANICS

`PROBLEM:` In a world riddled with dissipative influences, we cannot ignore air resistance.

(a) (6 points) Approximating the phenomenon (from the reference frame of the object) as robbing the "oncoming" column of air of its kinetic energy, develop an expression for the drag force for an object projecting cross-sectional area, A, to the "wind." Use ρ for the density of air.

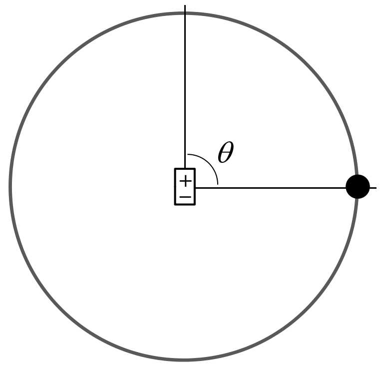
(b) (2 points) What do we find would be the terminal velocity (on Earth) of a human of mass, M, and projected cross-sectional area, A, in symbolic terms?

(c) (2 points) If $\rho \approx 1.25$ kg m^{-3}, $M \approx 70$ kg, and $A \approx 0.5$ m^2, what value do we get for the terminal velocity?



#3: UNDERGRADUATE E&M

`PROBLEM:` A plastic ball carrying a uniform charge Q is suspended by an insulating string on a distance d above the surface of a large container of salted water with a high electrical conductivity. As a result, the surface of the water below the ball raises as shown schematically on the figure. How large is the rise of the water level h at point P below the ball. Ignore surface tension. Consider the case of small deviation of the water surface from a plane surface ($h << d$). The water density is ρ, acceleration of gravity is g.



#4: UNDERGRADUATE E&M

`PROBLEM:` A small electrically charged bead with the mass m and charge Q can slide on a circular insulating string without friction. The radius of the circle is r. A point-like electric dipole is at the center of the circle with the dipole moment P lying in the plane of the circle. Initially the bead is at the angle $\theta = \pi/2 + \delta$, where δ is infinitely small, as shown schematically on the figure.

(a) How does the bead move after it is released? Find the bead velocity as a function of the angle θ.

(b) Find the normal force exerted by the string on the bead.

5: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider two identical gas engines, each containing the same amounts of an ideal gas.

Engine E_1 is a Carnot engine, which goes reversibly in a cycle from states $a \to b \to c \to d \to a$. Here a is a state with pressure p_a, volume V_a, and temperature T_a, and the $a \to b$ process is isothermal expansion to a state with volume $V_b = 5V_a$. The $b \to c$ process is adiabatic expansion to temperature $T_c = \frac{1}{3}T_a$. The $c \to d$ process is isothermal compression. The $d \to a$ process is adiabatic compression.

Engine E_2 goes *irreversibly* in a cycle, $a \to b' \to a$, where a is the same state as in engine E_1. The $a \to b'$ process is isolated, free expansion to volume $V_{b'} = 5V_a$. The $b' \to a$ process is reversible, back to the initial state.

(a) Find the change of entropy for engine E_1 for each of the four processes. Express all of your answers in terms of p_a, V_a, and T_a, and show all your work.

(b) Find the change of entropy for engine E_2 for each of the two processes.

(c) Find the total change of entropy of the universe, $\Delta S_{universe}$ for engines E_1 and E_2, for each step, with some explanation.

(d) Find the total work done by engines E_1 and E_2 in their complete cycles (you don't need to break them down into separate steps, just give the totals).

#6: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a 1d quantum system whose energy levels are $\epsilon_n = an+b$, with a and b constants and $n = 0, 1, 2, \ldots$.

(a) Compute the partition function.

Consider a system of N distinguishable copies of such a system.

(b) Find its average energy, U, as a function of temperature.

(c) Compute the specific heat, C_V, as a function of temperature.

(d) Compute the entropy.

#7: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the following superposition of plane waves (an "eigen-differential"):

$$\psi_{k,\delta k}(x) = \tfrac{1}{2\sqrt{\pi \delta k}} \int_{k-\delta k}^{k+\delta k} dq\, e^{iqx}$$

where the parameter δk is assumed to take values much smaller than the wave number k, i.e., $\delta k \ll k$.

(a) Prove that the wave functions $\psi_{k,\delta k}(x)$ are normalized and orthogonal to each other when, for two different k and k', $|k - k'| > \delta k + \delta k'$.

(b) For a free particle, compute the expectation value of the momentum and the energy in such a state.

#8: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A quantum system has only two energy eigenstates, $|1\rangle$, $|2\rangle$, corresponding to the energy eigenvalues E_1 and E_2. Apart from the energy, the system is also characterized by a physical observable whose operator P acts on the energy eigenstates as follows:

$$P|1\rangle = |2\rangle, \qquad P|2\rangle = |1\rangle.$$

P can be regarded as a “parity” operator.

(a) Find the eigenstates and eigenvalues of P.

(b) At a particular time t a measurement of P is made on the system. What is the probability of finding the system with positive “parity”?

#9: UNDERGRADUATE GENERAL

`PROBLEM:` Below two questions are independent from each other. Each of them is 5 points.

1) Imagine one day the sun ceases to shine. The temperature on the earth will be so cold such that the atmosphere is liquefied, and the earth surface will be covered by an ocean of liquid air. Estimate the average depth of this liquid air ocean. Compared it with the average depth of the usual water oceans which is at the order of 5km.

2) Estimate the largest height of a mountain on the earth. Are the typical heights of mountains on the moon higher or lower than those on the earth?

Assume rocks melt roughly at 1000 Kelvin, and the average mass number of atoms in rock is 30. The rest mass of a proton or neutron is about 1GeV, and 1eV is about 10 thousand Kevin.

(Hint: You only need to use some common sense knowledge and estimate the order of the magnitude.)

#10: UNDERGRADUATE GENERAL

`PROBLEM:` Intensity of a sound wave in a gas decreases as $e^{-\alpha x}$ with the travel distance x. According to the Stokes' law, the attenuation coefficient α is proportional to viscosity η,

$$\alpha \propto \eta \, ,$$

which is measured in poise ($1\,\text{poise} = 1\,\text{g}\,\text{cm}^{-1}\,\text{s}^{-1}$). Make a guess what physical parameters other than η may determine the attenuation and then use dimensional arguments to derive the complete form of the Stokes' law (without numerical coefficients). Express and analyze your result as follows:

a. Write α as a function of the sound wavelength λ or, better yet, wavenumber $k = 2\pi/\lambda$.

b. Using dimensional arguments once again, express η in terms of the mean-free path ℓ of the gas molecules. Combine the two formulas to have α in terms of ℓ, k, and remaining parameters, if any. Estimate numerical value of α for ultrasound of frequency $f = 1\,\text{MHz}$ assuming $\ell = 70\,\text{nm}$ and the speed of sound $v = 332\,\text{m/s}$.

c. Conventional gas dynamics theory of sound is valid on scales longer than ℓ, which means that there is an upper limit on the wavenumber, $k \sim \ell^{-1}$. What is the attenuation length α^{-1} of such an ultimate short-wavelength sound?

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INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

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Section:	§1		§2		§3		§4		§5	
Problems: (Circle your seven choices)	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

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#11: GRADUATE MECHANICS

`PROBLEM:` An infinite 1D diatomic chain consists of masses m_1, m_2 connected by springs of constant k. The masses m_1, m_2 alternate positions.

(a) What are the frequencies of waves which propagate on the spring?

(b) What is the physics of the low frequency, long wavelength wave? Describe it in terms of how the neighboring masses move.

(c) What is the physics of the high frequency wave? Describe it in terms of how the neighboring masses move.

(d) At what frequencies would you *not* expect to observe a sustained oscillation?

#12: GRADUATE MECHANICS

`PROBLEM:` A long, thin magnetic configuration has the form

$$\underline{B} = B_r \tilde{r} + B_z \tilde{z}.$$

Take $B_z = B_z(z)$, with slow axial variation (i.e. in z direction). $B_z >> B_r$ here. $B_z(z)$ is strongest at ends $z = \pm\ell$.

(a) Determine B_r, exploiting the slow axial variation of B_z.

(b) Derive an *approximate* invariant of charged particle motion in this configuration.

(c) What limits the validity of this invariant?

(d) What class of particles are confined by this configuration? (Hint: Think in terms of region of phase space.)

#13: GRADUATE E&M

`PROBLEM:` A photon moves along the z axis in the laboratory frame where it encounters an object of mass M_1 at rest. Out of this collision comes an object of mass M_2 and a photon moving in the x-z plane at an angle θ with respect to the z-axis.

Find the relation between the wavelength of the incoming photon λ_1, and the wave length of the outgoing photon λ_2, and the two masses. The speed of light c and Planck's constant will enter this relation.

14: GRADUATE E&M

`PROBLEM:` A non-relativistic object with charge Q and mass M moves in an electric and a magnetic field produced by the potentials $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$. Show that the following Lagrangian gives the correct equations of motion for this object:

$$L(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = \tfrac{M\mathbf{v}(t)^2}{2} - Q\phi(\mathbf{x}(t), t) + Q\dot{\mathbf{x}}(t) \cdot \mathbf{A}(\mathbf{x}(t), t).$$

Show that under a gauge transformation on $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ the action

$$S = \int_{t_0}^{t_f} dt\, L(\mathbf{x}(t), \dot{\mathbf{x}}(t), t),$$

is invariant if current is conserved. Writing down $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ for a point particle should help you here.

#15: GRADUATE STAT-MECH

`PROBLEM:` Consider a three dimensional (3D) ferromagnet fully polarized along the z-axis in the ground state. Its low temperature thermodynamic properties are described by the spin-wave excitations, whose quantum version is called "magnons". Magnon excitations cause magnetization deviating from the z-direction. More precisely, each magnon carries a spin-1 moment opposite to the z-axis, i.e., the magnetization of each magnon $m = -g\mu_B$ where g is the Lande factor and μ_B the Bohr magneton.

Similar to photons, magnons can be viewed as bosons with zero chemical potential. The magnon dispersion is $\epsilon(k) = Jk^2a^2$, where k is the magnitude of the 3D wavevector $\vec{k}$, a is the lattice constant, and J is the magnetic exchange energy scale. For the following problems, we impose a cut-off $\Lambda \approx 1/a$ for the wavevector $\vec{k}$.

Consider the low temperature region such that $k_B T \ll J$.
1) Derive the Bloch's law of the temperature dependence of the magnetization $\Delta M(T) = M(T) - M(0)$.

2) Derive the low temperature specific heat $C(T)$.

Both ΔM and C exhibit power-law dependence on T. Please express each of your result in terms a power of temperature multiplied by a dimensionless integral and other constants. You do not need to evaluate the integral.

#16: GRADUATE STAT-MECH

`PROBLEM:` Consider a three dimensional free electron gas with the density n at zero temperature. Neglect interaction effects.

1) What are the values of the Fermi wavevector k_f and the Fermi energy ϵ_f?

2) Apply a magnetic field along the z-direction

$$H_B = -hg\mu_B S_z, \tag{1}$$

where g is the Lande factor and μ_B the Bohr magneton. Consider the small field limit, $h \ll \frac{\epsilon_f}{g\mu_B}$, and neglect the orbital effect of the magnetic field. What is the spin magnetic susceptibility χ at $T = 0$?

3) Consider a free spin-1/2 local moment in the external magnetic field h. Calculate its magnetic susceptibility $\chi(T)$ at temperature T and take the limit of $T \to 0$. Compare $\chi(T)$ at $T \to 0$ of local moments with the zero temperature χ that you obtain in 2). What do you think is the key reason for their difference?

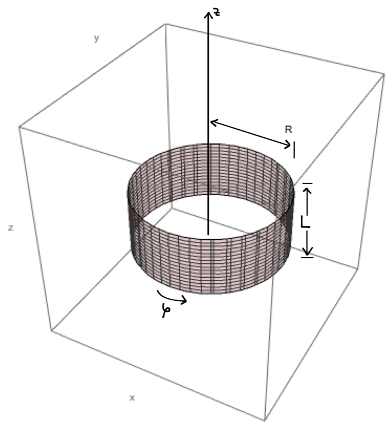
#17: GRADUATE QUANTUM MECHANICS

`PROBLEM:` A non-relativistic particle of mass m moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly-spaced, attractive delta functions:

$$V(x) = -V_0 a \sum_{n=-1}^{1} \delta(x - na), \quad V_0, a > 0.$$

1. Calculate the discontinuity in the first derivative of the wavefunction at $x = -a, 0$, and a.
2. Consider the possible number and locations of nodes in bound state wavefunctions for this system.
   (a) How many nodes are possible in the region $x > a$?
   (b) How many nodes are possible in the region $0 < x < a$?
   (c) Can there be a node at $x = a$?
   (d) Can there be a node at $x = 0$?
3. If V_0 is large enough, the system will have both symmetric and antisymmetric bound states. Sketch *qualitatively* the lowest energy symmetric ($\psi(-x) = \psi(x)$) and antisymmetric ($\psi(-x) = -\psi(x)$) bound states. For V_0 much larger than other energy scales in the problem, how many bound states are there?
4. For the lowest energy antisymmetric bound state, derive a transcendental equation that determines the bound state energy. You do not need to solve the equation.

#18: GRADUATE QUANTUM MECHANICS



`PROBLEM:` An electron of mass M is confined to the surface of a hollow cylinder of radius R and length L (never mind how). In cylindrical coordinates $r = \sqrt{x^2 + y^2}, \varphi = \tan^{-1}(y/x)$, and z, the electron is confined to the *two*-dimensional region with $r = R$ and $0 \leq z \leq L$, with φ unconstrained.

You may find the following formula useful:

$$\nabla^2 \psi|_{r=R} = \left(\partial_z^2 + \frac{1}{R^2} \partial_\varphi^2 \right) \psi(z, \varphi).$$

1. In the absence of any electromagnetic fields, write down the eigenfunctions and eigenvalues of the Hamiltonian. The wavefunction should vanish at the boundaries $z = 0, z = L$. Ignore the spin of the electron and treat the problem non-relativistically.

2. Now a weak uniform magnetic field of magnitude B_0 is applied in the z direction. This corresponds to a vector potential which can be written in cylindrical coordinates as $\vec{A} = \frac{rB_0}{2} \hat{\varphi}$ where $\hat{\varphi}$ is a unit vector in the azimuthal direction. Write down the Schrödinger equation for the electron.

3. Find a gauge transformation in the region $0 < \varphi < 2\pi$ so that the problem in the presence of $\vec{B}$ is equivalent to the $\vec{B} = 0$ problem, but with a different boundary condition for the wavefunction at $\varphi = 2\pi$, $\tilde{\psi}(\varphi = 2\pi, z) = \tilde{\psi}(\varphi = 0, z) e^{\mathbf{i}\alpha}$. Find α.

4. Solve for the energy eigenvalues exactly in the presence of $\vec{B}$.

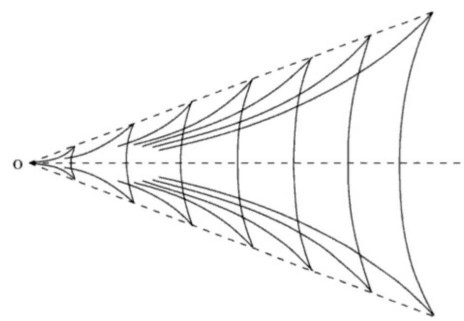
5. Find a condition on B_0 so that the energy spectrum is unchanged by the magnetic field.

#19: GRADUATE GENERAL/MATH

`PROBLEM:` Use quantum uncertainty principle and relativity theory to do a rough numerical estimate of the ultimate computational speed of a computer of mass $m = 1\,\mathrm{kg}$ and volume $V = 10^{-3}\,\mathrm{m}^3$, which is roughly the size of a conventional laptop computer.

#20: GRADUATE GENERAL/MATH

PROBLEM:



A physical explanation for the phenomenon of boat wakes (see Figure), and for the value of the wake angle, was originally given by Lord Kelvin. The key theoretical formula is as follows. Suppose the boat is moving in the x-direction with velocity v, then the vertical disturbance of the water surface at point (x, y) is given by the Fourier integral

$$z(x,y) = \int A(k_x) e^{ik_x x + ik_y y} dk_x,$$

where $A(k_x)$ is a smooth function that depends on the shape of the boat, k_y is the solution of the equation $v^2 k_x^2 = \omega^2 \equiv g\sqrt{k_x^2 + k_y^2}$, and g is the acceleration of gravity. Use the saddle-point (or stationary-phase) method to show that:

a. For given (x, y), the integral is dominated by the wavenumber k_x that satisfies the equation $x + y\dfrac{dk_y}{dk_x} = 0$.

b. The wake pattern is dominated by the two rays $y = \pm x/\sqrt{8}$.
Hint: evaluate the pre-exponential factor in the saddle-point formula.

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(a) (6 points) How much faster or slower must the solid ball be launched relative to the speed, v_0, of an idealized point particle that achieves the same outcome? Express as a ratio, or factor.

(b) (4 points) Now compute the ratio of velocities of a hollow, thin-shelled ball compared to the idealized point particle to achieve the same result.

`SOLUTION:`

For the idealized point particle, we just need energy balance to establish the nominal velocity: $\frac{1}{2}Mv_0^2 = Mgh$. For a rolling solid object, the sum of translational kinetic and rotational kinetic energies combine, so that $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh$. Because the potential energy is the same in both cases, we can ignore it and set $Mv^2 + I\omega^2 = Mv_0^2$, also removing a common factor of $\frac{1}{2}$. Rolling without slipping means $\omega = v/R$, so that the $I\omega^2$ term becomes fMv^2 where $I = fMR^2$.

For both sections, we may use the same development for evaluating the rotational inertia of a (possibly hollow) sphere:

$$I = 2\pi\rho \int_{r_1}^{R} r^2 \mathrm{d}r \int_0^{\pi} \sin\theta \mathrm{d}\theta (r\sin\theta)^2$$

Substituting $u = \mathrm{d}(\cos\theta) = -\sin\theta\mathrm{d}\theta$ turns this into the integral:

$$I = 2\pi\rho \int_{r_1}^{R} r^4 \mathrm{d}r \int_{-1}^{1} (1-u^2)\mathrm{d}u$$

For which we obtain:

$$I = \tfrac{8\pi}{15}\rho(R^5 - r_1^5)$$

Replacing density with mass via $\rho = M/V$, where $V = \frac{4}{3}\pi(R^3 - r_1^3)$ yields:

$$I = \tfrac{2}{5}M\tfrac{R^5 - r_1^5}{R^3 - r_1^3}$$

(a) For a solid sphere, $r_1 = 0$ and we have the familiar relation $I = \frac{2}{5}MR^2$. Referring to the introductory paragraph, $f = \frac{2}{5}$, so we have $Mv^2 + \frac{2}{5}Mv^2 = Mv_0^2$, or $v^2 = \frac{5}{7}v_0^2$, so that $v = v_0\sqrt{5/7} \approx 0.845v_0$.

(b) For a hollow sphere with a thin shell, we say $r_1 = R - \delta$, where δ is small. Then $r_1^n \approx R^n - nR^{n-1}\delta$. The ratio $(R^5 - r_1^5)/(R^3 - r_1^3)$ then becomes approximately $5R^4\delta/3R^2\delta = \frac{5}{3}R^2$, making $I = \frac{2}{3}MR^2$. Following similar steps as before, we find that $v = v_0\sqrt{3/5} \approx 0.775v_0$.

#2: UNDERGRADUATE MECHANICS

`PROBLEM:` In a world riddled with dissipative influences, we cannot ignore air resistance.

(a) (6 points) Approximating the phenomenon (from the reference frame of the object) as robbing the "oncoming" column of air of its kinetic energy, develop an expression for the drag force for an object projecting cross-sectional area, A, to the "wind." Use ρ for the density of air.

(b) (2 points) What do we find would be the terminal velocity (on Earth) of a human of mass, M, and projected cross-sectional area, A, in symbolic terms?

(c) (2 points) If $\rho \approx 1.25$ kg m^{-3}, $M \approx 70$ kg, and $A \approx 0.5$ m^2, what value do we get for the terminal velocity?

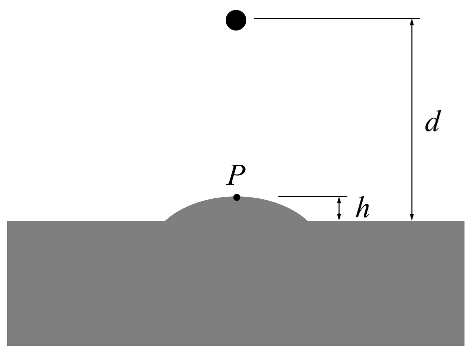
`SOLUTION:`

(a) In the frame of the moving object, a column of air with cross-sectional area, A, approaches. In each time interval, Δt, the length of the column is $v\Delta t$ so that its volume is $Av\Delta t$ and mass is $\rho A v \Delta t$, so that the kinetic energy is $\frac{1}{2}\rho A v^3 \Delta t$. Since this is the kinetic energy robbed from the air in each time interval Δt, the power and associated force is $P = Fv = \frac{1}{2}\rho A v^3$, meaning that the drag force is $F = \frac{1}{2}\rho A v^2$.

(b) Terminal velocity is achieved when drag force and gravitational force balance (are equal). Thus $\frac{1}{2}\rho A v^2 = Mg$, so that the velocity is

$$v = \sqrt{\frac{2Mg}{\rho A}}$$

(c) Inserting $M \approx 70$ kg, $\rho \approx 1.25$ kg m^{-3}, and $A \approx 0.5$ m^2, we compute $v \approx 50$ m/s, which is about right for human freefall.



#3: UNDERGRADUATE E&M

PROBLEM: A plastic ball carrying a uniform charge Q is suspended by an insulating string on a distance d above the surface of a large container of salted water with a high electrical conductivity. As a result, the surface of the water below the ball raises as shown schematically on the figure. How large is the rise of the water level h at point P below the ball. Ignore surface tension. Consider the case of small deviation of the water surface from a plane surface ($h << d$). The water density is ρ, acceleration of gravity is g.

SOLUTION:

Within the method of image charges, the electric field above the surface can be presented as the field produced by charge Q and image charge $-Q$ positioned at the depth d below the surface. The electric field just above the surface at point P is

$$E = E_Q + E_{-Q} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{d^2} + \frac{1}{4\pi\varepsilon_0}\frac{Q}{d^2} = \frac{1}{2\pi\varepsilon_0}\frac{Q}{d^2}$$

According to Gauss's law the surface charge density at P is

$$\sigma = \varepsilon_0 E = \frac{1}{2\pi}\frac{Q}{d^2}.$$

At the water surface the electrostatic force on a unit area is the product of the surface charge density σ and the electric field E_Q due to the ball

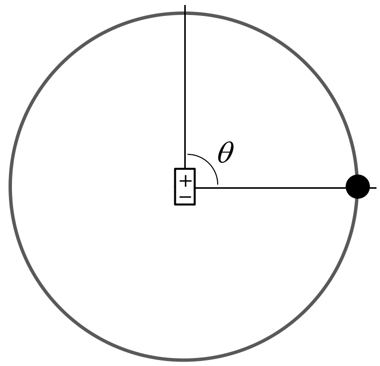
$$\frac{F}{A} = \sigma E_Q.$$

It is balanced by the hydrostatic pressure associated with the rise h in water level

$$\frac{F}{A} = \rho g h.$$

Substituting for E_Q and σ gives the expression for h

$$h = \frac{1}{8\pi^2\varepsilon_0}\frac{1}{\rho g}\frac{Q^2}{d^4}.$$



#4: UNDERGRADUATE E&M

`PROBLEM:` A small electrically charged bead with the mass m and charge Q can slide on a circular insulating string without friction. The radius of the circle is r. A point-like electric dipole is at the center of the circle with the dipole moment P lying in the plane of the circle. Initially the bead is at the angle $\theta = \pi/2 + \delta$, where δ is infinitely small, as shown schematically on the figure.

(a) How does the bead move after it is released? Find the bead velocity as a function of the angle θ.

(b) Find the normal force exerted by the string on the bead.

`SOLUTION:`

(a) Applying the law of conservation of energy for a bead of mass m and charge Q in the field of a dipole with dipole moment P gives

$$\frac{1}{2}mv^2 + QP\frac{\cos\theta}{r^2} = \frac{1}{2}mv_0^2 + QP\frac{\cos(\pi/2)}{r^2} = 0$$

This gives the expression for the velocity of the bead v at angle θ

$$v = \sqrt{\frac{-2QP\cos\theta}{mr^2}}.$$

The bead moves along a circular path until it reaches the point opposite its starting position. The bead stops there and then goes back executing a periodic motion.

(b) The radial component of the force on the charge due to the dipole $F_{r-dipole}$ can be calculated as the derivative of the electric potential energy with respect to r

$$\frac{\partial}{\partial r}\left(QP\frac{\cos\theta}{r^2}\right) = -2QP\frac{\cos\theta}{r^3}.$$

For the circular motion $mv^2/r = F_{r-dipole} + F_{r-string}$. Substituting v and $F_{r-dipole}$ gives the normal force exerted by the string on the bead $F_{r-string} = 0$. If the string were not there, the bead would move along the circular path as with the string.

5: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider two identical gas engines, each containing the same amounts of an ideal gas.

Engine E_1 is a Carnot engine, which goes reversibly in a cycle from states $a \to b \to c \to d \to a$. Here a is a state with pressure p_a, volume V_a, and temperature T_a, and the $a \to b$ process is isothermal expansion to a state with volume $V_b = 5V_a$. The $b \to c$ process is adiabatic expansion to temperature $T_c = \frac{1}{3}T_a$. The $c \to d$ process is isothermal compression. The $d \to a$ process is adiabatic compression.

Engine E_2 goes *irreversibly* in a cycle, $a \to b' \to a$, where a is the same state as in engine E_1. The $a \to b'$ process is isolated, free expansion to volume $V_{b'} = 5V_a$. The $b' \to a$ process is reversible, back to the initial state.

(a) Find the change of entropy for engine E_1 for each of the four processes. Express all of your answers in terms of p_a, V_a, and T_a, and show all your work.

(b) Find the change of entropy for engine E_2 for each of the two processes.

(c) Find the total change of entropy of the universe, $\Delta S_{universe}$ for engines E_1 and E_2, for each step, with some explanation.

(d) Find the total work done by engines E_1 and E_2 in their complete cycles (you don't need to break them down into separate steps, just give the totals).

`SOLUTION:`

(a) For $a \to b$, $dE = 0$, so $dS = pdV/T = NkdV/V$ and $\Delta S_{a\to b} = Nk\ln 5 = (p_aV_a/T_a)\ln 5$. $\Delta S_{b\to c} = \Delta S_{d\to a} = 0$, since they're adiabatic. And $\Delta S_{c\to d} = -\Delta S_{a\to b} = -(p_aV_a/T_a)\ln 5$, since S is a state variable so $\Delta S_{\text{total}} = 0$ for the cyclic process.

(b) For $a \to b'$, $\Delta T = 0$ since $E = \frac{d.o.f}{2}NkT$ is unchanged (no heat transfer and no work). So we can compute ΔS from the reversible isotherm as in the previous part, $\Delta S_{a\to b'} = (p_aV_a/T_a)\ln 5$. Since S is a state variable, $\Delta S_{b'\to a} = -(p_aV_a/T_a)\ln 5$.

(c) For E_1, since reversible, $\Delta S_{universe} = 0$ for each step. For E_2, $\Delta S_{a\to b',universe} =$

$\Delta S_{a\to b'} = (p_a V_a/T_a)\ln 5$ for that reversible step, and $\Delta S_{b'\to a,universe} = 0$ for the reversible step.

(d) For E_1, $\Delta W = \oint p dV = \oint T dS = (T_a - T_c)\Delta S_{a\to b} = \frac{2}{3} p_a V_a \ln 5$. For E_2, no work is done in the first step. The second step is isothermal compression, so $\Delta E = 0$ and $\Delta W = T_a \Delta S = -T_a \Delta S_{a\to b} = -p_a V_a \ln 5$.

#6: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a 1d quantum system whose energy levels are $\epsilon_n = an+b$, with a and b constants and $n = 0, 1, 2, \ldots$.

(a) Compute the partition function.

Consider a system of N distinguishable copies of such a system.

(b) Find its average energy, U, as a function of temperature.

(c) Compute the specific heat, C_V, as a function of temperature.

(d) Compute the entropy.

`SOLUTION:`

(a) $Z = \sum_{n=0}^{\infty} e^{-(an+b)\beta} = \frac{e^{-b\beta}}{1-e^{-a\beta}}$, where $\beta = 1/kT$.

(b) $U = -N\frac{\partial \ln Z}{\partial \beta} = N(b + \frac{a}{e^{a\beta}-1})$.

(c) $C_V = (\frac{\partial U}{\partial T})_N = -(1/kT^2)(\frac{\partial U}{\partial \beta})_N = (N/kT^2)\frac{a^2 e^{a\beta}}{(e^{a\beta}-1)^2}$

(d) $S = -\frac{\partial F}{\partial T} = Nk\frac{\partial}{\partial T}(T\ln Z) = Nk(\ln Z - \beta\frac{\partial}{\partial \beta}\ln Z) = Nk(-b\beta - \ln(1 - e^{-a\beta}) + \beta b + \beta a\frac{1}{e^{a\beta}-1}) = Nk(\frac{\beta a}{e^{a\beta}-1} - \ln(1 - e^{-a\beta}))$.

#7: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the following superposition of plane waves (an "eigendifferential"):

$$\psi_{k,\delta k}(x) = \tfrac{1}{2\sqrt{\pi\delta k}} \int_{k-\delta k}^{k+\delta k} dq\, e^{iqx}$$

where the parameter δk is assumed to take values much smaller than the wave number k, i.e., $\delta k \ll k$.

(a) Prove that the wave functions $\psi_{k,\delta k}(x)$ are normalized and orthogonal to each other when, for two different k and k', $|k-k'| > \delta k + \delta k'$.

(b) For a free particle, compute the expectation value of the momentum and the energy in such a state.

`SOLUTION:`

(a) The proof of normalization goes as follows:

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx\, |\psi_{k,\delta k}(x)|^2 &= \tfrac{1}{4\pi\delta k} \int_{-\infty}^{+\infty} dx \int_{k-\delta k}^{k+\delta k} dq' \int_{k-\delta k}^{k+\delta k} dq'' e^{i(q'-q'')x} \\
&= \tfrac{1}{2\delta k} \int_{k-\delta k}^{k+\delta k} dq' \int_{k-\delta k}^{k+\delta k} dq'' \delta(q'-q'') \\
&= \tfrac{1}{2\delta k} \int_{k-\delta k}^{k+\delta k} dq' = 1
\end{aligned}$$

The proof of orthogonality is similar ($|k-k'| > \delta k + \delta k'$):

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx\, \psi^*_{k,\delta k}(x)\psi_{k',\delta k'}(x) &= \tfrac{1}{2\sqrt{\delta k \delta k'}} \int_{k-\delta k}^{k+\delta k} dq' \int_{k'-\delta k'}^{k'+\delta k'} dq'' \delta(q'-q'') \\
&= \tfrac{1}{2\sqrt{\delta k \delta k'}} \int_{k-\delta k}^{k+\delta k} dq' \Theta(k'+\delta k'-q')\Theta(q'-k'+\delta k') = 0
\end{aligned}$$

(b) The expectation value of the momentum operator on the eigendifferentials is

$$\begin{aligned}\langle p\rangle &= \tfrac{1}{4\pi\delta k}\int_{-\infty}^{+\infty} dx \int_{k-\delta k}^{k+\delta k} dq' \int_{k-\delta k}^{k+\delta k} dq'' e^{-iq'x}(-i\hbar\partial_x)e^{iq''x} \\ &= \tfrac{1}{2\delta k}\int_{k-\delta k}^{k+\delta k} dq' \int_{k-\delta k}^{k+\delta k} dq'' \hbar q'' \delta(q'-q'') \\ &= \tfrac{1}{2\delta k}\int_{k-\delta k}^{k+\delta k} dq' \hbar q' = \tfrac{\hbar}{4\delta k}[(k+\delta k)^2-(k-\delta k)^2] = \hbar k\end{aligned}$$

A similar calculation gives

$$\langle \tfrac{p^2}{2m}\rangle = \tfrac{\hbar^2 k^2}{2m}$$

#8: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A quantum system has only two energy eigenstates, $|1\rangle$, $|2\rangle$, corresponding to the energy eigenvalues E_1 and E_2. Apart from the energy, the system is also characterized by a physical observable whose operator P acts on the energy eigenstates as follows:

$$P|1\rangle = |2\rangle, \qquad P|2\rangle = |1\rangle.$$

P can be regarded as a "parity" operator.

(a) Find the eigenstates and eigenvalues of P.

(b) At a particular time t a measurement of P is made on the system. What is the probability of finding the system with positive "parity"?

`SOLUTION:`

(a) Construct the following linear combinations $|1\rangle \pm |2\rangle$. It is then clear that

$$P(|1\rangle \pm |2\rangle) = \pm(|1\rangle \pm |2\rangle)$$

Therefore, the eigenstates of P are

$$|\pm\rangle = \tfrac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$$

and the inverse relations as

$$|1\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |2\rangle = \tfrac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

The evolved state of the system is then

$$|\psi(t)\rangle = e^{i\alpha t}(\cos(\omega t)\ |+\rangle + i\ \sin(\omega t)\ |-\rangle)$$

with

$$\omega = \tfrac{E_2 - E_1}{2\hbar}, \quad \alpha = -\tfrac{E_2 + E_1}{2\hbar}$$

(b) The probability of finding the system in the state $|+\rangle$ is $\cos^2(\omega t)$.

#9: UNDERGRADUATE GENERAL

`PROBLEM:` Below two questions are independent from each other. Each of them is 5 points.

1) Imagine one day the sun ceases to shine. The temperature on the earth will be so cold such that the atmosphere is liquefied, and the earth surface will be covered by an ocean of liquid air. Estimate the average depth of this liquid air ocean. Compared it with the average depth of the usual water oceans which is at the order of 5km.

2) Estimate the largest height of a mountain on the earth. Are the typical heights of mountains on the moon higher or lower than those on the earth?

Assume rocks melt roughly at 1000 Kelvin, and the average mass number of atoms in rock is 30. The rest mass of a proton or neutron is about 1GeV, and 1eV is about 10 thousand Kevin.

(Hint: You only need to use some common sense knowledge and estimate the order of the magnitude.)

`SOLUTION:` 1) The atmosphere pressure is 10 meters high water. The air is mostly oxygen and nitrogen. The liquid air density should be at the same order of water. So the ocean of liquid air should be at the order of 10 meters

Another method: Due to gravity, the atmosphere density decays exponentially as height increases. At the highest mountain   8 km, the atmosphere pressure is about one half at the sea level. So we can estimate the character tic height of air as 10km. When air liquefies, its volume shrinks about 1000 times, so the height also reduces 1000 times, which gives rise to the same order of 10m.

The water ocean average depth is about 5000m. So we have much more water on the earth than air.

2) The foot of the mountain needs to be strong enough to hold the weight of the entire mountain. Suppose the entire mountain shift downward for one layer of atoms. This is equivalent to move the top layer of atoms to the bottom, and thus each atom gains the gravity energy mgh. If this energy is equivalent to the thermal energy $k_B T_M$ where T_M is the melting point of

rocks, then the foot of the mountain will melt. So we estimate

$$h = k_B T_M/(mg). \tag{1}$$

Considering $mc^2 \approx 30 \times 10^9 eV$, and $k_B T_B \approx 0.1eV$. We have

$$h = \frac{0.1}{3\times10^{10}} c^2/g = 1/3 \times 10^{-13} \times 9 \times 10^{16} m \approx 30km. \tag{2}$$

It is higher than the actual value of 8.8km but at the same order. On the moon, the gravity is 1/6 of the earth, and thus the typical height of mountains on the moon is much higher than on the earth.

#10: UNDERGRADUATE GENERAL

`PROBLEM:` Intensity of a sound wave in a gas decreases as $e^{-\alpha x}$ with the travel distance x. According to the Stokes' law, the attenuation coefficient α is proportional to viscosity η,

$$\alpha \propto \eta\,,$$

which is measured in poise ($1\,\text{poise} = 1\,\text{g}\,\text{cm}^{-1}\,\text{s}^{-1}$). Make a guess what physical parameters other than η may determine the attenuation and then use dimensional arguments to derive the complete form of the Stokes' law (without numerical coefficients). Express and analyze your result as follows:

a. Write α as a function of the sound wavelength λ or, better yet, wavenumber $k = 2\pi/\lambda$.

b. Using dimensional arguments once again, express η in terms of the mean-free path ℓ of the gas molecules. Combine the two formulas to have α in terms of ℓ, k, and remaining parameters, if any. Estimate numerical value of α for ultrasound of frequency $f = 1\,\text{MHz}$ assuming $\ell = 70\,\text{nm}$ and the speed of sound $v = 332\,\text{m/s}$.

c. Conventional gas dynamics theory of sound is valid on scales longer than ℓ, which means that there is an upper limit on the wavenumber, $k \sim \ell^{-1}$. What is the attenuation length α^{-1} of such an ultimate short-wavelength sound?

`SOLUTION:`

a. Physically reasonable set of parameters consists of the wavenumber k, the mass density ρ, the speed of sound v, and of course, viscosity. In principle, the thermal speed of molecules is one more parameter but it is equal to v up to a numerical factor. The combination that is linear in η and gives the correct unites for α is unique: $\alpha = \eta k^2/\rho v$.

b. $\eta = \rho v \ell$, which leads to $\alpha = k^2 \ell$. In terms of f, we get the estimate $\alpha = (2\pi f/v)^2 \ell = 25\,\text{m}^{-1}$.

c. At $k = \ell^{-1}$ we have $\alpha^{-1} \sim \ell$. This makes sense because at such k the sound crosses over from being a collective mode to being a single-particle excitation.

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section Mechanics: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Indicate the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have indicated the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11: GRADUATE MECHANICS

`PROBLEM:` An infinite 1D diatomic chain consists of masses m_1, m_2 connected by springs of constant k. The masses m_1, m_2 alternate positions.

(a) What are the frequencies of waves which propagate on the spring?

(b) What is the physics of the low frequency, long wavelength wave? Describe it in terms of how the neighboring masses move.

(c) What is the physics of the high frequency wave? Describe it in terms of how the neighboring masses move.

(d) At what frequencies would you *not* expect to observe a sustained oscillation?

`SOLUTION:` (a) Equations:

$$m_1\ddot{x}_{2n} = -k(2x_{2n} - x_{2n-1} - x_{2n+1})$$
$$m_2\ddot{x}_{2n+1} = -k(2x_{2n+1} - x_{2n} - x_{2n+2})$$

Solutions:

$$x_{2n} = Ae^{2in\ell\alpha}e^{-iwt}$$
$$x_{2n+1} = Be^{i(2n+1)\ell\alpha}e^{-iwt}$$

so

$$(w^2 - \tfrac{2k}{m_1})(w^2 - \tfrac{2k}{m_2}) - \tfrac{4k^2cos^2\ell\alpha}{m_1m_2} = 0$$

and

$$(\tfrac{1}{\mu} = \tfrac{1}{m_1} + \tfrac{1}{m_2})$$

$w^2 = \frac{k}{\mu} \pm \frac{k}{\mu}\{1 - \frac{4\mu^2 sin^2(\alpha\ell)}{m_1m_2}^{1/2}\}$ is dispersion relation.

(b)

Low frequency $\rightarrow$ acoustic mode (ala' sound)
$\rightarrow$ analogous to mode of monatomic chain

$w \sim \alpha(\frac{k\ell^2}{m_1+m_2})^{1/2}$

Neighboring masses vibrate *in* phase, $\alpha\ell \to 0$

(c)

$$\begin{array}{lll} \text{High frequency} & \to & \text{optical mode} \\ & \to & \text{resembles EM wave in plasma} \end{array}$$

$w \sim (\frac{2k}{\mu})^{1/2}$

Neighboring mass vibrate *out of* phase, as $\alpha\ell \to 0$.

(d) No propagation in *gap*:

$(\frac{2k}{m_2})^{1/2} < w < (\frac{2k}{m_1})^{1/2}$.

#12: GRADUATE MECHANICS

`PROBLEM:` A long, thin magnetic configuration has the form

$$\underline{B} = B_r \tilde{r} + B_z \tilde{z}.$$

Take $B_z = B_z(z)$, with slow axial variation (i.e. in z direction). $B_z >> B_r$ here. $B_z(z)$ is strongest at ends $z = \pm\ell$.

(a) Determine B_r, exploiting the slow axial variation of B_z.

(b) Derive an *approximate* invariant of charged particle motion in this configuration.

(c) What limits the validity of this invariant?

(d) What class of particles are confined by this configuration? (Hint: Think in terms of region of phase space.)

`SOLUTION:` Magnetic Mirror

(a)

$\underline{\nabla} \cdot \underline{B} = 0$

$\partial_z B_z \neq 0 \Rightarrow \partial_r B_r \neq 0$

$B_r = \frac{-1}{r} \int_0^r dr' \; r' \; \frac{\partial B_z}{\partial z} \cong \frac{-r}{2} \frac{\partial B_z}{\partial z}$

(b) For adiabatic invariant

$$I = \oint_{cycl} \underline{p}_\perp \cdot dq_\perp \qquad \underline{p} \to \underline{p} - \tfrac{e}{0}\underline{A}$$

$$= \int_{cycl} m\underline{v}_\perp \cdot d\underline{q}_\perp - \tfrac{e}{c}\int_c \underline{A} \cdot d\underline{q}_\perp$$

$$= mv_\perp(2\pi\rho_L) - \tfrac{e}{c}\pi\rho_2^2 B$$

$$= \tfrac{mv_\perp^2}{2B}\underbrace{(\tfrac{2\pi M_c}{|e|})}_{\to \text{ irrelevant constant}} \qquad (\rho_L = \tfrac{v_\perp}{\Omega})$$

so

$I = \frac{mv_\perp^2}{2B}$, magnetic moment

(c) The invariant is *adiabatic*, valid on $t > \Omega^{-1}$, with Ω the cyclotron frequency.

(d) For confinement:

$$(v_\|^2 + v_\perp^2)\Big|_\ell = (v_{\|0}^2 + v_{\perp 0}^2)$$

$$\frac{v_\perp^2|(\ell)|}{B(\ell)} = \frac{v_{\perp(0)}^2}{B(0)}$$

ℓ is mirror length.

$\Rightarrow \frac{v_{\|(0)}^2}{v_{\perp(0)}^2} + 1 < \frac{B(\ell)}{B(0)}$ are confined.

i.e. $\frac{v^2_{\|(0)}}{v^2_{\perp(0)}} < \frac{B(\ell)}{B(0)} - 1$

#13: GRADUATE E&M

`PROBLEM:` A photon moves along the z axis in the laboratory frame where it encounters an object of mass M_1 at rest. Out of this collision comes an object of mass M_2 and a photon moving in the x-z plane at an angle θ with respect to the z-axis.

Find the relation between the wavelength of the incoming photon λ_1, and the wave length of the outgoing photon λ_2, and the two masses. The speed of light c and Planck's constant will enter this relation.

`SOLUTION:` The four vectors of the various objects, in the rest frame of the target of mass M_1 are these:

incoming photon $k_1 = (\frac{hf_1}{c}, 0, 0, \frac{hf_1}{c})$

target mass $p_1 = (M_1 c, 0, 0, 0)$

outgoing photon $k_2 = (\frac{hf_2}{c}\sin\theta, 0, \frac{hf_2}{c}\cos\theta)$

outgoing particle of mass M_2 p_2 do not need to know the components. Note $f = \frac{c}{\lambda}$

Conserve energy and momentum

$$k_1 + p_1 = k_2 + p_2$$

So

$$(k_1 + p_1 - k_2)^2 = p_2^2 = M_2^2 c^2.$$

Using the four vectors written above in components find

$$\lambda_2 - \lambda_1 = \lambda_c(1 - \cos\theta) + \frac{\lambda_1\lambda_2}{2\lambda_c}(\frac{M_2^2}{M_1^2} - 1).$$

where $\lambda_c = \frac{h}{M_1 c}$.

14: GRADUATE E&M

`PROBLEM:` A non-relativistic object with charge Q and mass M moves in an electric and a magnetic field produced by the potentials $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$. Show that the following Lagrangian gives the correct equations of motion for this object:

$$L(\mathbf{x}(t),\dot{\mathbf{x}}(t),t) = \tfrac{M\mathbf{v}(t)^2}{2} - Q\phi(\mathbf{x}(t),t) + Q\dot{\mathbf{x}}(t)\cdot\mathbf{A}(\mathbf{x}(t),t).$$

Show that under a gauge transformation on $\phi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ the action

$$S = \int_{t_0}^{t_f} dt\, L(\mathbf{x}(t),\dot{\mathbf{x}}(t),t),$$

is invariant if current is conserved. Writing down $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ for a point particle should help you here.

`SOLUTION:` The Euler-Lagrange equations are

$$\tfrac{d}{dt}\tfrac{\partial L(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{x_a}(t)} = \tfrac{\partial L(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial x_a(t)}$$

$$\tfrac{\partial L(\mathbf{x}(t),\dot{\mathbf{x}}(t),t)}{\partial \dot{x_a}(t)} = M\dot{x_a}(t) + QA_a(\mathbf{x}(t),t)$$

So

$$\begin{aligned} M\tfrac{d^2x_a(t)}{dt^2} &= Q\Bigg(-\nabla_{\mathbf{x}}\phi(\mathbf{x},t) - \tfrac{\partial A_a(\mathbf{x}(t),t)}{\partial t}\Bigg) + \\ Q\dot{x_b}(t)&\Bigg(\tfrac{\partial A_b(\mathbf{x},t)}{\partial x_a} - \tfrac{\partial A_a(\mathbf{x},t)}{\partial x_b}\Bigg) \\ M\tfrac{d^2x_a(t)}{dt^2} &= Q\Bigg(\mathbf{E}(\mathbf{x}(t),t) + \mathbf{v}(t)\times\mathbf{B}(\mathbf{x}(t),t))\Bigg). \end{aligned}$$

Write the action as

$$S = \int_{t_0}^{t_f} dt\,\tfrac{M\mathbf{v}(t)^2}{2} + \int_{t_0}^{t_f} dt \int d^3x\,\Bigg[\rho(\mathbf{x},t)\phi(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t)\cdot\mathbf{A}(\mathbf{x},t)\Bigg].$$

Under a gauge transformation we have

$$\begin{aligned}\phi(\mathbf{x},t) &\to \phi(\mathbf{x},t) - \tfrac{\partial\chi(\mathbf{x},t)}{\partial t}\\ \mathbf{A}(\mathbf{x},t) &\to \mathbf{A}(\mathbf{x},t) + \nabla\chi(\mathbf{x},t),\end{aligned}$$

for an arbitrary $\chi(\mathbf{x},t)$.

Make this transformation in the action and find that the change in the action is

$$-Q\int_{t_0}^{t_f} dt \int d^3x\,\chi(\mathbf{x},t)\left(\tfrac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{x},t)\right),$$

and this msut be zero if the action is gauge invariant. This means, as $\chi(\mathbf{x},t)$ is arbitary

$$\tfrac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0.$$

#15: GRADUATE STAT-MECH

`PROBLEM:` Consider a three dimensional (3D) ferromagnet fully polarized along the z-axis in the ground state. Its low temperature thermodynamic properties are described by the spin-wave excitations, whose quantum version is called "magnons". Magnon excitations cause magnetization deviating from the z-direction. More precisely, each magnon carries a spin-1 moment opposite to the z-axis, i.e., the magnetization of each magnon $m = -g\mu_B$ where g is the Lande factor and μ_B the Bohr magneton.

Similar to photons, magnons can be viewed as bosons with zero chemical potential. The magnon dispersion is $\epsilon(k) = Jk^2a^2$, where k is the magnitude of the 3D wavevector $\vec{k}$, a is the lattice constant, and J is the magnetic exchange energy scale. For the following problems, we impose a cut-off $\Lambda \approx 1/a$ for the wavevector $\vec{k}$.

Consider the low temperature region such that $k_BT \ll J$.
1) Derive the Bloch's law of the temperature dependence of the magnetization $\Delta M(T) = M(T) - M(0)$.

2) Derive the low temperature specific heat $C(T)$.

Both ΔM and C exhibit power-law dependence on T. Please express each of your result in terms a power of temperature multiplied by a dimensionless integral and other constants. You do not need to evaluate the integral.

`SOLUTION:` 1) The number of magnons at a low temperature T is

$$n = \int_0^\Lambda \frac{d\vec{k}^3}{(2\pi)^3} \frac{1}{e^{\epsilon(k)/(k_BT)}-1} = \frac{1}{2\pi^2} \int_0^\Lambda k^2 dk \frac{1}{e^{\epsilon(k)/(k_BT)}-1}$$

Set $x = Ja^2k^2/(k_BT)$,

$$n = \frac{1}{4\pi^2} \left(\frac{k_BT}{Ja^2}\right)^{\frac{3}{2}} \int_0^{Ja^2\Lambda^2/(k_BT)} \frac{x^{1/2}dx}{e^x-1}.$$

At low temperatures, we can set the upper limit of the integration to infinity, we arrive

$$\Delta M(T) = ng\mu_B \approx \frac{1}{4\pi^2} g\mu_B \left(\frac{k_BT}{Ja^2}\right)^{\frac{3}{2}} \int_0^{+\infty} \frac{x^{1/2}dx}{e^x-1}.$$

2) The low temperature internal energy

$$U = \int_0^\Lambda \frac{d^3\vec{k}}{(2\pi)^3} \frac{\epsilon(k)}{e^{\epsilon(k)/(k_B T)} - 1} = \frac{1}{2\pi^2} \int_0^\Lambda k^2 dk \frac{\epsilon(k)}{e^{\epsilon(k)/(k_B T)} - 1}$$

Again, after setting $x = Ja^2k^2/(k_B T)$, and taking the upper limit of the integral to infinity, we arrive at

$$U \approx \frac{1}{4\pi^2} \frac{1}{(Ja^2)^{3/2}} (k_B T)^{\frac{5}{2}} \int_0^{+\infty} \frac{x^{3/2} dx}{e^x - 1},$$

$$C \approx \frac{5}{8\pi^2} \frac{1}{(Ja^2)^{3/2}} k_B^{\frac{5}{2}} T^{\frac{3}{2}} \int_0^{+\infty} \frac{x^{3/2} dx}{e^x - 1}.$$

#16: GRADUATE STAT-MECH

`PROBLEM:` Consider a three dimensional free electron gas with the density n at zero temperature. Neglect interaction effects.

1) What are the values of the Fermi wavevector k_f and the Fermi energy ϵ_f?

2) Apply a magnetic field along the z-direction

$$H_B = -hg\mu_B S_z, \tag{1}$$

where g is the Lande factor and μ_B the Bohr magneton. Consider the small field limit, $h \ll \frac{\epsilon_f}{g\mu_B}$, and neglect the orbital effect of the magnetic field. What is the spin magnetic susceptibility χ at $T = 0$?

3) Consider a free spin-1/2 local moment in the external magnetic field h. Calculate its magnetic susceptibility $\chi(T)$ at temperature T and take the limit of $T \to 0$. Compare $\chi(T)$ at $T \to 0$ of local moments with the zero temperature χ that you obtain in 2). What do you think is the key reason for their difference?

`SOLUTION:` 1) The electron density n and Fermi wavevector k_f are related

$$n = 2\int_0^{k_f} \frac{d^3\vec{k}}{(2\pi)^3} = \frac{k_f^3}{3\pi^2},$$

where 2 counts form the spin degeneracy. We have $k_f = (3\pi^2 n)^{\frac{1}{3}}$. The Fermi energy $\epsilon_f = \frac{\hbar^2}{2m}(3\pi^2 n)^{\frac{2}{3}}$.

2) Under external field h, assume that Fermi wavevectors for spin up and down electrons are $k_\uparrow = k_f(1+x)$, and $k_\downarrow = k_f(1-x')$, respectively. Both x and x' should be linear to h. To maintain particle number conservation, and correct to h linear order, we have $x = x'$. Then we have

$$\frac{\hbar^2 k_f^2}{2m}((1+x)^2 - (1-x)^2) = 2\epsilon_f x = hg\mu_B,$$

thus $x = \frac{hg\mu_B}{2\epsilon_f}$. The magnetization M is

$$M = g\mu_B(4\pi k_f^2(k_{f\uparrow} - k_{f\downarrow})) = 8\pi k_f^3 x g\mu_B = \frac{8\pi m}{\hbar^2} g^2\mu_B^2 k_f h,$$

and thus $\chi = \frac{8\pi m}{\hbar^2} g^2 \mu_B^2 k_f$. It is a constant proportional to density of states at the Fermi surface, which is called Pauli paramagnetism.

3) For a free local moment under a magnetic field, its magnetization

$$M = g\mu_B \tanh \frac{hg\mu_B}{2k_B T} \approx (g\mu_B)^2 \frac{h}{2k_B T},$$

$\chi(T) = \frac{(g\mu_B)^2}{2k_B T}$. For the local moment, $\chi(T)$ diverge as $1/T$ but for itinerant electrons it saturates to a constant.

The reason for its difference is the existence of Fermi surface in itinerant systems. Polarizing electrons cost kinetic energy, even at zero temperature only within a small energy window of $\epsilon_f \pm \frac{1}{2} hg\mu_B$ contributes to magnetization. But for local moment, there are no kinetic energy cost, at zero temperature, a infinitesimal field can completely polarize it, thus χ diverges.

#17: GRADUATE QUANTUM MECHANICS

`PROBLEM:` A non-relativistic particle of mass m moves in one dimension, subject to a potential energy function $V(x)$ which is the sum of three evenly-spaced, attractive delta functions:

$$V(x) = -V_0 a \sum_{n=-1}^{1} \delta(x - na), \quad V_0, a > 0.$$

1. Calculate the discontinuity in the first derivative of the wavefunction at $x = -a, 0$, and a.
2. Consider the possible number and locations of nodes in bound state wavefunctions for this system.
   (a) How many nodes are possible in the region $x > a$?
   (b) How many nodes are possible in the region $0 < x < a$?
   (c) Can there be a node at $x = a$?
   (d) Can there be a node at $x = 0$?
3. If V_0 is large enough, the system will have both symmetric and antisymmetric bound states. Sketch *qualitatively* the lowest energy symmetric ($\psi(-x) = \psi(x)$) and antisymmetric ($\psi(-x) = -\psi(x)$) bound states. For V_0 much larger than other energy scales in the problem, how many bound states are there?
4. For the lowest energy antisymmetric bound state, derive a transcendental equation that determines the bound state energy. You do not need to solve the equation.

`SOLUTION:`

1.
$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\psi(x) = E\psi(x).$$

Integrating across any of the delta functions gives

$$-\frac{\hbar^2}{2m}\left(\partial_x\psi|_{x=\xi-\epsilon} - \partial_x\psi|_{x=\xi+\epsilon}\right) - V_0 a\psi(\xi) = 0$$

for $\xi = -a, 0, a$.

2. Away from the delta functions, the eigenfunctions satisfy

$$-\frac{\hbar^2}{2m}\partial_x^2\psi = E\psi.$$

For a bound state in a potential which goes to zero at $x = \infty$ we need $E < 0$, so the general solution is

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}, \quad \kappa \equiv \frac{1}{\hbar}\sqrt{-2mE}, \text{ real.}$$

The only way this function can vanish is if A and B have opposite signs, in which case there is a unique zero. The solution takes this form in each of the regions in between the delta functions, with coefficients A, B to be determined in each region.

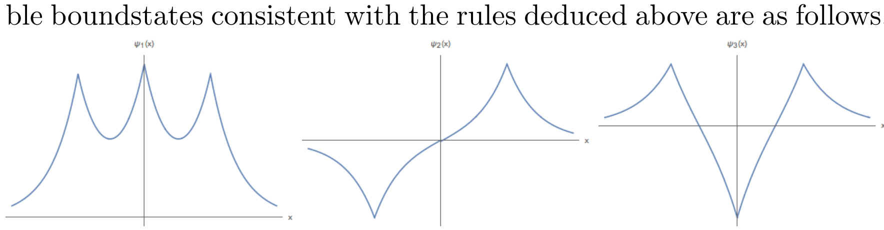
(a) In the region $x > a$, we must have $A = 0$ so the solution is normalizable. Therefore there are no zeros, since $Be^{-\kappa x}$ has no zeros for finite x unless $B = 0$. But if $B = 0$ then the solution vanishes everywhere, since the discontinuity in the derivative is proportional to the value of the function.

(b) There can be zero or one nodes in the region $0 < x < a$, as described above.

(c) Can there be a node at $x = a$? No, because if there were the function would have to vanish for all $x > a$.

(d) Can there be a node at $x = 0$? Sure, why not?

3. Three is the largest possible number of boundstates. The three possible boundstates consistent with the rules deduced above are as follows.



In addition to the no-node symmetric solution and the one-node antisymmetric solution, there can be a two-node symmetric solution. There is no solution with a node in the region $-a < x < 0$ but none in $0 < x < a$. One way to see this is that it would have a parity image which was orthogonal and degenerate. But there can be no degeneracy for discrete states in 1d.

4. The bound state solution with $\psi(x) = \psi(-x)$ is

$$\psi(x) = \begin{cases} Ae^{-\kappa x}, & a \le x \\ B\left(e^{\kappa x} - e^{-\kappa x}\right), & 0 \le x \le a \end{cases}$$

The two boundary conditions at $x = a$ are continuity

$$Ae^{-\kappa a} = B\left(e^{\kappa a} - e^{-\kappa a}\right)$$

and the jump condition on the derivative:

$$\kappa\left(B\left(e^{\kappa a} + e^{-\kappa a}\right) + Ae^{-\kappa a}\right) = -\frac{2m}{\hbar^2} a V_0 A e^{-\kappa a}.$$

There is no jump at $x = 0$, and the jump at $x = -a$ works automatically. Eliminating A by the continuity condition, we arrive at

$$\frac{\kappa a}{1 - e^{-2\kappa a}} = \frac{m V_0 a^2}{\hbar^2}$$

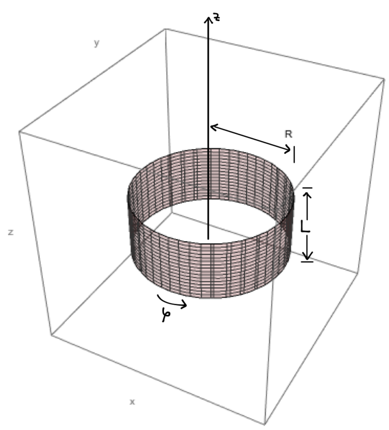
The threshold value of V_0 occurs when the RHS equals the minimum over κ of the LHS. That happens at $\kappa = 0$, and equals

$$\lim_{x \to 0} \frac{x}{1 - e^{-2x}} = \lim_{x \to 0} \frac{x}{1 - 1 + 2x + \mathcal{O}(x^2)} = \frac{1}{2}.$$

Therefore

$$V_0^\star = \frac{\hbar^2}{2ma^2}.$$

#18: GRADUATE QUANTUM MECHANICS



`PROBLEM:` An electron of mass M is confined to the surface of a hollow cylinder of radius R and length L (never mind how). In cylindrical coordinates $r = \sqrt{x^2+y^2}, \varphi = \tan^{-1}(y/x)$, and z, the electron is confined to the *two*-dimensional region with $r = R$ and $0 \le z \le L$, with φ unconstrained.
You may find the following formula useful:

$$\nabla^2\psi|_{r=R} = \left(\partial_z^2 + \frac{1}{R^2}\partial_\varphi^2\right)\psi(z,\varphi).$$

1. In the absence of any electromagnetic fields, write down the eigenfunctions and eigenvalues of the Hamiltonian. The wavefunction should vanish at the boundaries $z = 0, z = L$. Ignore the spin of the electron and treat the problem non-relativistically.

2. Now a weak uniform magnetic field of magnitude B_0 is applied in the z direction. This corresponds to a vector potential which can be written in cylindrical coordinates as $\vec{A} = \frac{rB_0}{2}\hat{\varphi}$ where $\hat{\varphi}$ is a unit vector in the azimuthal direction. Write down the Schrödinger equation for the electron.

3. Find a gauge transformation in the region $0 < \varphi < 2\pi$ so that the problem in the presence of $\vec{B}$ is equivalent to the $\vec{B} = 0$ problem, but with a different boundary condition for the wavefunction at $\varphi = 2\pi$, $\tilde{\psi}(\varphi = 2\pi, z) = \tilde{\psi}(\varphi = 0, z)e^{\mathbf{i}\alpha}$. Find α.

4. Solve for the energy eigenvalues exactly in the presence of $\vec{B}$.

5. Find a condition on B_0 so that the energy spectrum is unchanged by the magnetic field.

`SOLUTION:`

1. $H = \frac{\vec{p}^2}{2m}$ which acting on wavefunctions restricted to $r = R$ is

$$H = -\frac{\hbar^2}{2m}\left(\partial_z^2 + R^{-2}\partial_\varphi^2\right).$$

Since the Hamiltonian is the sum of two commuting terms, the eigenfunctions factorize into eigenfunctions of ∂_z and ∂_φ :

$$\psi_{m,k}(\varphi, z) = f_m(\varphi) g_k(z)$$

with

$$f_m(\varphi) = e^{\mathbf{i}m\varphi}$$

and

$$g_k(z) = e^{\mathbf{i}kz}.$$

Single-valuedness under $\varphi \to \varphi + 2\pi$ requires , $m \in \mathbb{Z}$. The boundary condition at $z = 0$ requires the linear combination $\sin kz$ and the vanishing at $z = L$ says kL is an integer multiple of π, so k must be of the form

$$k_n = n\frac{\pi}{L}, n \in \mathbb{Z}.$$

So altogether the eigenfunctions are

$$\psi_{m,n}(\varphi, z) \propto \sin(k_n z) e^{\mathbf{i}m\varphi}$$

with eigenvalues

$$E_{m,n} = \frac{\hbar^2}{2m}\left(k_n^2 + \frac{m^2}{R^2}\right).$$

2. The hamiltonian is

$$H = \frac{1}{2M}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2.$$

So the Schrödinger equation is $H\psi = E\psi$ with

$$H = -\frac{\hbar^2}{2m}\left(\partial_z^2 + R^{-2}\left(\partial_\varphi - \mathbf{i}\frac{eR^2 B_0}{2\hbar c}\right)^2\right).$$

3. Redefine $\tilde{\psi} \equiv e^{\chi\varphi\mathbf{i}}\psi$ for some constant χ so that

$$\left(\partial_\varphi - \mathbf{i}\frac{eR^2 B_0}{2\hbar c}\right)\psi = e^{-\chi\varphi\mathbf{i}}\left(\partial_\varphi - \mathbf{i}\chi - \mathbf{i}\frac{eR^2 B_0}{2\hbar c}\right)\tilde{\psi}$$

Now choose $\chi = -\frac{eR^2B_0}{2\hbar c}$ to cancel the term with B_0, and $\tilde{\psi}$ satisfies the no-field equation. Since ψ is periodic,

$$\tilde{\psi}(2\pi, z) = e^{\mathbf{i}\chi 2\pi}\tilde{\psi}(0, z)$$

so the phase in the problem statement is

$$\alpha = 2\pi\chi = -\frac{eR^2B_0}{2\hbar c}2\pi = -2\pi\frac{\Phi_B}{\Phi_0}$$

where $\Phi_B = \pi R^2 B_0$ is the magnetic flux through the cylinder and $\Phi_0 \equiv \frac{hc}{e}$ is the flux quantum.

4. The azimuthal wavefunctions now must be

$$f_{\tilde{m}}(\varphi) = e^{\mathbf{i}\varphi\tilde{m}} \stackrel{!}{=} e^{\mathbf{i}(\varphi+2\pi)\tilde{m}}$$

which says

$$\tilde{m} = m - \frac{\Phi_B}{\Phi_0}, \quad m \in \mathbb{Z}$$

and the energies are

$$E_{m,n} = \frac{\hbar^2}{2M}\left(k_n^2 + R^{-2}\left(m - \frac{\Phi_B}{\Phi_0}\right)^2\right).$$

5. The spectrum maps to itself if Φ_B is an integer multiple of the flux quantum.

#19: GRADUATE GENERAL/MATH

`PROBLEM:` Use quantum uncertainty principle and relativity theory to do a rough numerical estimate of the ultimate computational speed of a computer of mass $m = 1\,\mathrm{kg}$ and volume $V = 10^{-3}\,\mathrm{m}^3$, which is roughly the size of a conventional laptop computer.

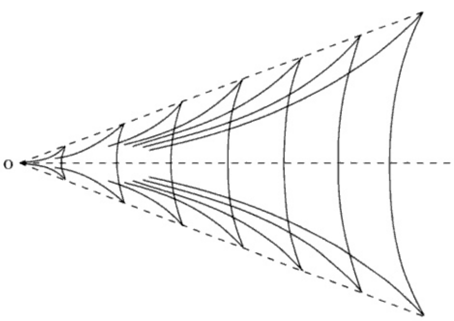
`SOLUTION:`

The computational speed (operations per second) is $f \sim 1/\Delta t$, where Δt is time to do a single operation. The uncertainty principle imposes the bound $\Delta t > \hbar/E$, where E is the energy. The relativity theory says that $E = mc^2$. Hence,

$$f \sim mc^2/\hbar \sim 10^{51}\,\mathrm{s}^{-1}\,.$$

#20: GRADUATE GENERAL/MATH

PROBLEM:



A physical explanation for the phenomenon of boat wakes (see Figure), and for the value of the wake angle, was originally given by Lord Kelvin. The key theoretical formula is as follows. Suppose the boat is moving in the x-direction with velocity v, then the vertical disturbance of the water surface at point (x, y) is given by the Fourier integral

$$z(x,y) = \int A(k_x) e^{ik_x x + ik_y y} dk_x,$$

where $A(k_x)$ is a smooth function that depends on the shape of the boat, k_y is the solution of the equation $v^2 k_x^2 = \omega^2 \equiv g\sqrt{k_x^2 + k_y^2}$, and g is the acceleration of gravity. Use the saddle-point (or stationary-phase) method to show that:

a. For given (x, y), the integral is dominated by the wavenumber k_x that satisfies the equation $x + y\dfrac{dk_y}{dk_x} = 0$.

b. The wake pattern is dominated by the two rays $y = \pm x/\sqrt{8}$.
Hint: evaluate the pre-exponential factor in the saddle-point formula.

SOLUTION:

Denote $f(k_x) = k_x x + k_y y$. According to the stationary-phase method, the integral can be approximated by

$$z(x,y) \sim \sqrt{\tfrac{2\pi i}{f''(k_x^*)}}\, A(k_x^*) e^{if(k_x^*)}, \qquad (*)$$

where k_x^* is the solution of $f'(k_x^*) = 0$, i.e.,

$$x + y\,\frac{dk_y}{dk_x}\bigg|_{k_x^*} = 0\,,$$

in agreement with part a. To do part b, we first solve for k_y:

$$k_y = \pm k_x\sqrt{k_x^2 - \kappa^2}\,, \quad \kappa \equiv \frac{g}{v^2}\,,$$

and then evaluate the necessary derivatives:

$$\frac{dk_y}{dk_x} = \pm\frac{2\left(\frac{k_x}{\kappa}\right)^2 - 1}{\sqrt{\left(\frac{k_x}{\kappa}\right)^2 - 1}}\,, \qquad \frac{d^2k_y}{dk_x^2} = \pm\frac{2k_x}{\kappa^2}\,\frac{\left(\frac{k_x}{\kappa}\right)^2 - \frac{3}{2}}{\left[\left(\frac{k_x}{\kappa}\right)^2 - 1\right]^{3/2}}\,.$$

We see that the quantity

$$f''(k_x) = y\,\frac{d^2k_y}{dk_x^2}$$

vanishes at $\frac{k_x}{\kappa} = \sqrt{\frac{3}{2}}$, so that the pre-exponential factor in formula (*) diverges. While this means that formula is not sufficient for evaluating $z(x, y)$, it also implies that the wake disturbance is likely to reach a maximum. The set of (x, y) for which k_x^* is equal to the above critical number are the rays with the slope

$$\frac{y}{x} = -\left(\frac{dk_y}{dk_x}\right)^{-1} = \pm\frac{1}{\sqrt{8}}\,,$$

The corresponding wake angle is $\arctan\frac{1}{\sqrt{8}} = 19.5°$, as first found by Kelvin.