INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

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The questions are grouped in five Sections (mechanics, electromagnetism, quantum mechanics, statistical mechanics, general/math). You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded.**

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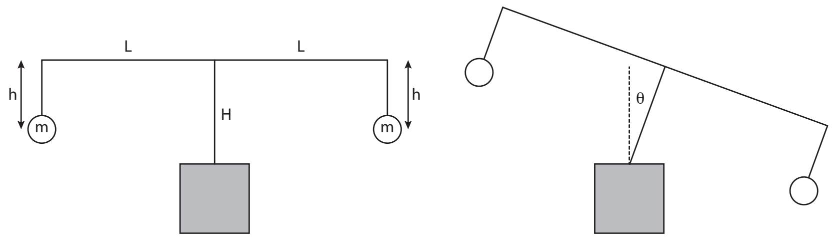
#1: UNDERGRADUATE MECHANICS

`PROBLEM:` The figure below depicts a mass balance, wherein the rigid, symmetric, massless structure (composed of right angles) supports two equal masses and is allowed to pivot in one angular dimension as depicted at right. Polar coordinates about the pivot point are recommended for this problem.



(a) Construct the potential energy for this system as a function of the geometry and angle θ, and evaluate the conditions for stability about the configuration shown at left ($\theta = 0$).

(b) What is the oscillation frequency of this apparatus as a function of the parameters (L, H, h, m), provided parameter values that satisfy stability from part (a)?

#2: UNDERGRADUATE MECHANICS

`PROBLEM:` Orbital precession due to a quadrupole perturbation.

As a result of the Earth's aspherical mass distribution, its gravitational potential has the form

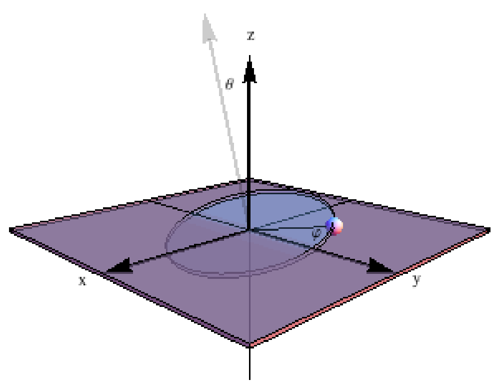
$$U(\vec{r}) = -\alpha \left(\frac{1}{r} + \beta \frac{3z^2 - r^2}{r^5} \right) ,$$

where the z-axis is the Earth's axis of rotation, and α and β are constants. The corresponding gravitational force thus contains a (small) non-central component.

(a) Determine the equations of motion of a (non-relativistic) particle of mass m in this field. (Hint: try using rectangular coordinates.)

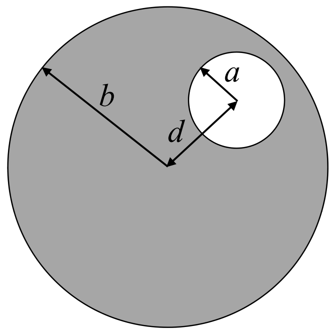
(b) Determine the rate of change of the angular momentum vector $\vec{L}$ of the particle. Write the answer in terms of $\vec{r}, \hat{z}$ and the given constants.

(c) Consider a particle moving initially in an approximately circular orbit, with radius a and angular velocity $\dot{\varphi} = \omega$. Let the normal to the plane of the orbit lie initially in the x-z plane, at an angle θ with the z-axis, as indicated in the figure. Assume $|\frac{d\vec{L}}{dt}|$ to be sufficiently small that over one period the orbit can be considered to be an unperturbed circle. Find the time average $\left\langle \frac{d\vec{L}}{dt} \right\rangle$ over one period of this circular orbit.



#3: UNDERGRADUATE ELECTRODYNAMICS

`PROBLEM:` A cylindrical hole of radius a is drilled in a solid cylinder of radius b. The two cylinder axes are parallel and are at a distance d apart. A constant current I flows in this structure, with uniform current density. Find the magnetic field at the center of the hole.



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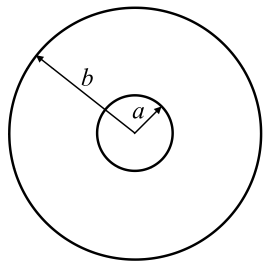
`PROBLEM:` An apparatus is made from two concentric conducting cylinders of radius $a < b$. The inner cylinder is grounded, the outer cylinder is at positive voltage V, and a uniform magnetic field H is directed along the cylinder axis. Electrons leave the inner cylinder with zero velocity and travel to the outer cylinder. What is the threshold value of V below which the current between the inner and outer cylinder is completely suppressed by the magnetic field?



#5: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` The Stark effect (the verbose text is mostly a ton of hints).

(a) Consider the Hydrogen atom, with the electron in its first excited state ($n = 2$). What is the degeneracy (ignoring the small effects that slightly split it)? Give a few details to show that you understand what the states are, including how this number is not totally unrelated to an aspect of the periodic table of the elements.

(b) A constant, uniform electric field is applied, $\vec{E} = E\hat{z}$. Consider, to first order in perturbation theory, its effect on the $n = 2$ states. Write down the perturbing Hamiltonian and write down which of its matrix elements are guaranteed to vanish by rotational symmetry.

(c) Compute the non-zero matrix elements of the first-order, perturbing Hamiltonian for $n = 2$. Don't panic if you don't remember the radial part of the wavefuction, and don't bother to evaluate the radial integral. You will get full credit if you can clearly set up the needed integral, and evaluate what it must be, up to an overall numerical factor (like 3 or something, that you don't need to bother to determine). You should, however, determine the correct parametric dependence on the electric field strength, the electron charge, and the Bohr radius a_0. Also, correctly determine the sign (recalling the sign of the electron's charge).

(d) Use the results of part (c) to determine the splitting of the degeneracy of the $n = 2$ states for $E \neq 0$. Draw the energy levels, indicating what (if any) degeneracy remains. Determine the energy eigenstates for each of these levels.

#6: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a system of three identical particles with the same spin orientation in a one dimensional simple harmonic oscillator potential. The system is spin-polarized, i.e. there is no degeneracy from spin. The system is prepared with a total energy

$$E = \frac{9}{2}\hbar\omega, \tag{1}$$

with ω is the characteristic angular frequency of the oscillator. Assume all possible states of the system with this value of total energy are equally probable, and interactions between particles can be neglected.

If you measure the energy ϵ of a single particle, what would be the most probable result, and what is the probability associated with that value, assuming:

(a) Bose-Einstein statistics?

(b) Fermi statistics?

(c) Maxwell Boltzman statistics?

#7: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a gas of $N \gg 1$ identical atoms in a harmonic trap. They behave like indistinguishable 3-dimensional simple harmonic oscillators obeying Bose-Einstein statistics. The energy of any one of those atoms of mass m, momentum $\vec{p}$ and position $\vec{r}$ is given by the expression

$$\epsilon = \frac{\vec{p}^2}{2m} + \tfrac{1}{2}K\vec{r}^2,$$

so that they all vibrate with a common frequency $\nu = \sqrt{\frac{K}{m}}/2\pi$, with K the spring constant.

(a) Write an integral expression for the expectation value of the total number of atoms in excited states, in thermal equilibrium at temperature T. There is no need to actually do the integral. Rather express the answer in terms of a dimensionless integral.

(b) Show that, at low temperatures, this gas undergoes a Bose-Einstein condensation, with critical temperature

$$T_B \propto (h\nu/k_B)N^{1/3}$$

(c) Determine the manner in which the condensate fraction N_{gs}/N and the specific heat C_V of the gas vary with temperature T for $T < T_B$.

#8: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The speed of sound, w, in any medium is given by the formula

$$w = \frac{1}{\sqrt{\rho \kappa_s}},$$

where $\rho = mN/V$ is the mass density and $\kappa_s = -\left(\frac{\partial V}{\partial P}\right)/V$ is the adiabatic compressibility at constant entropy and number of particles.

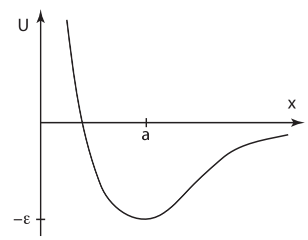
Show that, for an ideal Fermi gas at $T = 0$ K, $w = v_F/\sqrt{3}$, where v_F is the Fermi velocity of the gas (the speed of particles at the Fermi energy).

#9: UNDERGRADUATE GENERAL

`PROBLEM:` A particle of mass 3 (arbitrary units) arrives at a speed of $v = 0.8c$ to smack into a stationary particle of mass 7. Emerging from the collision is a particle of mass 8 traveling at $0.6c$ along the direction of the original incoming particle's velocity vector, and another particle of unknown mass and velocity. Conserving relativistic energy (γmc^2) and relativistic momentum ($\gamma m\vec{v}$), describe the properties of the unknown particle. The speeds given in the problem lead to conveniently rational gamma factors for easy computation.

#10: UNDERGRADUATE GENERAL

`PROBLEM:` The figure below shows a generic potential for atoms in a solid lattice. Assign reasonable numbers for the scale of the equilibrium location, a, and the depth of the potential well, ε for, say, iron. Now imagine a steel fiber 1 m long and 100 μm in diameter suspending a 1 kg mass in normal gravity. How much will the fiber stretch, based on your estimates for a and ε? Hint: first work out a spring constant, k.



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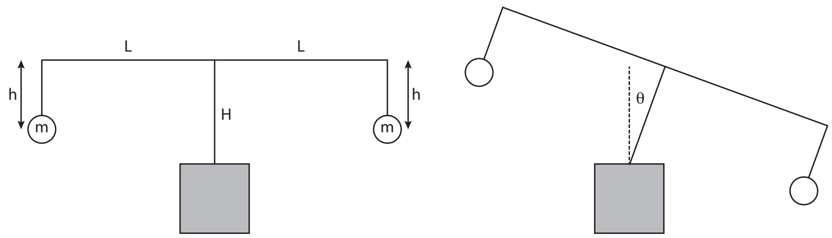

(a) Construct the potential energy for this system as a function of the geometry and angle θ, and evaluate the conditions for stability about the configuration shown at left ($\theta = 0$).

(b) What is the oscillation frequency of this apparatus as a function of the parameters (L, H, h, m), provided parameter values that satisfy stability from part (a)?

`SOLUTION:` (a) In polar coordinates, the masses are at a distance $r = \sqrt{L^2 + (H-h)^2}$ from the pivot. It helps to define the angular offset of the masses when $\theta = 0$: $\tan\phi = \frac{H-h}{L}$, so that $\sin\phi = \frac{H-h}{r}$ and $\cos\phi = \frac{L}{r}$.

We define angles from the horizontal axis for the left and right masses as $\psi_{\mathrm{R}} = \phi - \theta$ and $\psi_{\mathrm{L}} = \pi - \theta - \phi$. Then we compute the vertical coordinates of each mass as $y_{\mathrm{R}} = r\sin\psi_{\mathrm{R}}$ and $y_{\mathrm{L}} = r\sin\psi_{\mathrm{L}}$ Working through the trigonometry and recognizing that we have expressions for $\sin\phi$ and $\cos\phi$, we arrive at: $y_{\mathrm{R}} = -L\sin\theta + (H-h)\cos\theta$ and $y_{\mathrm{L}} = L\sin\theta + (H-h)\cos\theta$. The potential energy is then $U = mgy_{\mathrm{L}} + mgy_{\mathrm{R}} = 2mg(H-h)\cos\theta$, as if the mass is all located at the center of mass, resonably.

Stability is manifest when the second derivative of the potential is positive (curves up like a harmonic potential). The second derivative of U with respect to θ creates a negative sign from $\cos\theta$, so that we have stability if $h - H > 0$. In other words, $h > H$, which puts the masses below the pivot at the equilibrium position ($\theta = 0$) in a retrospectively obvious manner.

(b) We can form the equation of motion for this system by one of two common approaches. We can write that kinetic energy is $T = \frac{1}{2}mv_{\mathrm{L}}^2 + \frac{1}{2}mv_{\mathrm{R}}^2 = mv^2$ (since both masses must have the same velocity magnitude), so $T = mr^2\dot{\theta}^2$, and then form the Lagrangian $\mathcal{L} = T - U$ and construct the equations accordingly. Or, we can appeal to Newton's Second Law and set the negative gradient of potential equal to mass times acceleration.

The Lagrangian route likely needs no further elaboration, so we will illustrate the latter approach here. $F = -\frac{\partial U}{\partial x}$, where x is some length-dimensioned variable. In this case, the equivalent of x is $r\theta$, so that $F = -\frac{1}{r}\frac{\partial U}{\partial \theta} = -2mg\frac{h-H}{r}\sin\theta$. The acceleration of each mass is $r\ddot{\theta}$, so that the total force on the apparatus must be $2mr\ddot{\theta}$. Equating these two, we find that $\ddot{\theta} = -\frac{g(h-H)}{r^2}\sin\theta$, which is the classic harmonic form with $\omega^2 = \frac{g(h-H)}{r^2} = \frac{g(h-H)}{L^2+(h-H)^2}$.

Note that even though the potential looks like a simple pendulum with mass $2m$ suspended $h-H$ under the pivot, the frequency does *not* look like the usual $\omega^2 = \frac{g}{\ell}$. It slows down as the length, L, increases, but approaches the usual form when $L \ll h-H$. Note that it also gets very slow when $h-H \to 0$, which is easy to explore by balancing a stiff ruler at its midpoint.

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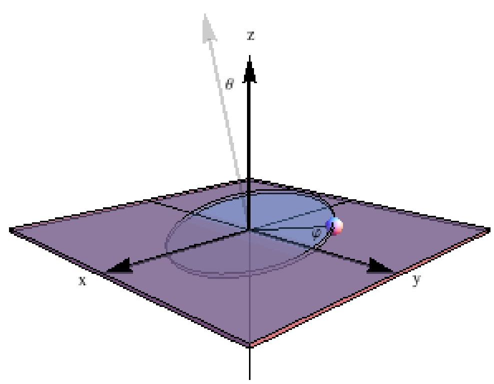


SOLUTION:

(a) Newton's equations are

$$m\partial_t^2\vec{x} = \vec{F} = -m\vec{\nabla}U \ .$$

Or we can use the Lagrangian

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mU$$

and the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}.$$

Using $\frac{\partial r}{\partial x} = \frac{x}{r}$, the result is

$$\ddot{x} = -\frac{\alpha x}{r^3} - \alpha\beta\left(\frac{15xz^2}{r^7} - \frac{3x}{r^5}\right)$$

$$\ddot{y} = -\frac{\alpha y}{r^3} - \alpha\beta\left(\frac{15yz^2}{r^7} - \frac{3y}{r^5}\right)$$

$$\ddot{z} = -\frac{\alpha z}{r^3} - \alpha\beta\left(\frac{15zz^2}{r^7} - \frac{3z}{r^5} - \frac{6z}{r^5}\right)$$

It will be useful for the next part to rewrite the equations of motion as

$$m\partial_t^2\vec{r} = -\alpha\vec{r}\left(\frac{1}{r^3} + 15\beta\frac{z^2}{r^7}\right) + 6\alpha\beta\frac{\hat{z}z}{r^5}.$$

(b) The angular momentum is

$$\vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\partial_t\vec{x}$$

and its time derivative is

$$\partial_t\vec{L} = \vec{r}\times\partial_t\vec{p}.$$

Only the non-central bit matters:

$$\partial_t\vec{L} = \vec{r}\times\hat{z}\frac{6\alpha\beta z}{r^5}.$$

(c) To evaluate the right hand side of the torque, we need to parametrize the orbit. If the plane of the orbit were not rotated away from the z-axis, it would be $\vec{r}_0 = a(\cos\varphi, 0, \sin\varphi)$. We need to rotate this vector by an angle θ about an axis (say $\hat{y}$) perpendicular to the the z-axis. The associated rotation matrix is

$$R(\hat{y},\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

This gives

$$\vec{r} = a(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta\cos\varphi).$$

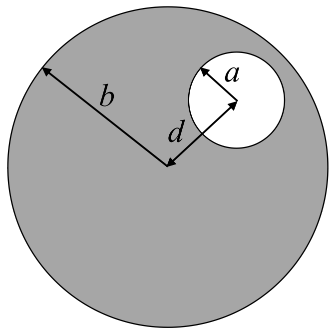
$$\partial_t\vec{L} = \frac{-6\alpha\beta}{r^5}(-yz, xz, 0) = \frac{-6\alpha\beta}{a^3}(\cos\theta\sin\theta\cos\varphi\sin\varphi, -\cos\theta\sin\theta\cos^2\varphi, 0)$$

The average over a period of $\sin\varphi\cos\varphi = \frac{1}{2}\sin 2\varphi$ is zero, while $\langle\cos^2\varphi\rangle = \frac{1}{2}$ so

$$\partial_t\vec{L} = \frac{3\alpha\beta}{a^3}\cos\theta\sin\theta\hat{y}.$$

#3: UNDERGRADUATE ELECTRODYNAMICS

`PROBLEM:` A cylindrical hole of radius a is drilled in a solid cylinder of radius b. The two cylinder axes are parallel and are at a distance d apart. A constant current I flows in this structure, with uniform current density. Find the magnetic field at the center of the hole.



`SOLUTION:`

The linearity of Maxwell's equations allows us to find the magnetic field as a sum of two magnetic fields produced by two currents: A current with density

$$j_b = \tfrac{I}{\pi(b^2-a^2)} \tag{1}$$

carried by the cylinder of radius b and a current with density

$$j_a = -j_b \tag{2}$$

carried by a cylinder of radius a. The sum of these two currents gives the current distribution in the considered structure. From Ampere's circuital law $\oint \mathbf{H}\cdot d\mathbf{l} = \frac{4\pi}{c}\int \mathbf{j}\cdot d\mathbf{S}$ one finds that the current carried by the cylinder of radius b produces a magnetic field at the center of the hole

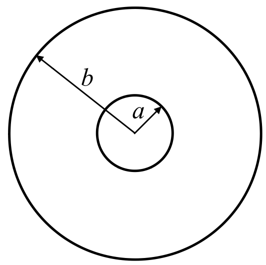
$$H_b = \tfrac{2Id}{c(b^2-a^2)}, \tag{3}$$

while the current carried by the cylinder of radius a produces no magnetic field at the center of the hole, $H_a = 0$. Therefore, the magnetic field at the center of the hole is

$$H = \tfrac{2Id}{c(b^2-a^2)}. \tag{4}$$

#4: UNDERGRADUATE ELECTRODYNAMICS

`PROBLEM:` An apparatus is made from two concentric conducting cylinders of radius $a < b$. The inner cylinder is grounded, the outer cylinder is at positive voltage V, and a uniform magnetic field H is directed along the cylinder axis. Electrons leave the inner cylinder with zero velocity and travel to the outer cylinder. What is the threshold value of V below which the current between the inner and outer cylinder is completely suppressed by the magnetic field?



`SOLUTION:`

In cylindrical coordinates, the angular momentum change due to the Lorentz force gives

$$\frac{d}{dt}(mr^2\dot{\theta}) = \frac{e}{c} r\dot{r}H. \tag{5}$$

The solution of this equation for zero velocity at $r = a$ is

$$mr^2\dot{\theta} = \frac{eH}{2c}(r^2 - a^2). \tag{6}$$

The conservation of energy gives

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - eV(r) = 0. \tag{7}$$

Substituting the expression for $\dot{\theta}$ in the energy equation gives

$$\frac{1}{2}m\left[\dot{r}^2 + \left(\frac{eH(r^2-a^2)}{2mcr}\right)^2\right] = eV(r). \tag{8}$$

The threshold voltage V_t is obtained when $\dot{r} = 0$ at $r = b$. Therefore

$$V_t = \frac{eH^2(b^2-a^2)^2}{8mc^2b^2}. \tag{9}$$

#5: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM`: The Stark effect (the verbose text is mostly a ton of hints).

(a) Consider the Hydrogen atom, with the electron in its first excited state ($n = 2$). What is the degeneracy (ignoring the small effects that slightly split it)? Give a few details to show that you understand what the states are, including how this number is not totally unrelated to an aspect of the periodic table of the elements.

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(d) Use the results of part (c) to determine the splitting of the degeneracy of the $n = 2$ states for $E \neq 0$. Draw the energy levels, indicating what (if any) degeneracy remains. Determine the energy eigenstates for each of these levels.

`SOLUTION`: The good old Stark (Johannes, not Tony) effect.

(a) $n = 2$ can have $\ell = 0, 1$, i.e. $2s$ and $2p$. The $\ell = 1$ state can have $m = 1, 0, -1$. The electron spin can point up or down. All told, the degeneracy is $2(1+3) = 8$. This is not totally unrelated to the number of elements in the second row of the periodic table.

(b) $H_1 = -ezE$, so the matrix elements are $-eE\langle n = 2, \ell', m'|z|n = 2, \ell, m\rangle$. Writing $z = r\cos\theta$, the rotational symmetry (the $\int d\Omega \dots$) implies that the matrix element is zero unless $|\ell' - \ell| = 1$, and $m = m'$. So the only non-zero matrix elements are $\langle 2p, m = 0|H_1|2s, m = 0\rangle$ and its complex conjugate. The perturbation doesn't affect the spin, so we suppress the fact that each ket has a tensor product with the $S_z = \pm\frac{1}{2}$ states.

(c) The non-zero matrix element is $-eE\langle 2p, m' = 0|z|2s, m = 0\rangle =$

$$= -eE \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\cos\theta^2 \int_0^{\infty} dr\ r^2\ r|\psi_{n=2}(r)|^2 = |e|ECa_0$$

where r_0 is the Bohr radius, $e = -|e|$, and C is the positive numerical coefficient that is 3 or something (actually, it is 3).

(d) The energy eigenstates are $|2s, m = 0\rangle \pm |2p, m = 0\rangle$, and $|2p, m = \pm 1\rangle$. The first two are shifted up and down by the perturbation $\Delta E = \pm|e|ECa_0$, and the second two remain unchanged from the $E = 0$ case.

#6: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a system of three identical particles with the same spin orientation in a one dimensional simple harmonic oscillator potential. The system is spin-polarized, i.e. there is no degeneracy from spin. The system is prepared with a total energy

$$E = \frac{9}{2}\hbar\omega, \tag{10}$$

with ω is the characteristic angular frequency of the oscillator. Assume all possible states of the system with this value of total energy are equally probable, and interactions between particles can be neglected.

If you measure the energy ϵ of a single particle, what would be the most probable result, and what is the probability associated with that value, assuming:

(a) Bose-Einstein statistics?

(b) Fermi statistics?

(c) Maxwell Boltzman statistics?

`SOLUTION:`

The single-particle energies are

$$\epsilon = (n + 1/2)\hbar\omega, \tag{11}$$

where $n = 0, 1, 2, 3, ...$ Thus the sum of the quantum numbers n of the three particles is 9/2 - 3/2 = 3. The combinations of quantum numbers n that fulfill this constraint are:
A. (1,1,1)

B. (2,1,0)

C. (3,0,0).

(a) In BE statistics, all three states are possible. The probabilities are:
$n = 3$: $(1/3)(1/3) = 1/9$

$n = 2$: $(1/3)(1/3) = 1/9$

$n = 1$: $(3/3)(1/3)$+$(1/3)(1/3) = 4/9$

$n = 0$: (1/3)(1/3) + (2/3)(1/3) =3/9

Thus the most probable energy corresponds to $n = 1$, with $\epsilon = 3/2\hbar\omega$ and an associated probability of 4/9.

(b) In FD statistics, no two particles of the same spin orientation can be in the same quantum state, thus the only choice is B. Hence, three quantum numbers $n = 0, 1, and 2$ each have a 1/3 probability each, with corresponding energies $\epsilon = 1/2$, 3/2 and 5/2 $\hbar\omega$.

(c) In Maxwell Boltzman statistics, each particle is distinct. Thus the number of distinct states corresponding to the combinations of quantum numbers are: A: 1; B: 3 x 2 = 6, and C: 3; which sums to 10 three particle states. Hence:
$n = 3$: (1/3)(3/10) = 1/10

$n = 2$: (1/3)(6/10) = 2/10

$n = 1$: (3/3)(1/10)+(1/3)(6/10) = 3/10

$n = 0$: (1/3)(6/10) + (2/3)(3/10) =4/10

Thus most probable is $n = 0$ with an associated probability of 40 % and $\epsilon = 1/2$ $\hbar\omega$.

#7: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a gas of $N \gg 1$ identical atoms in a harmonic trap. They behave like indistinguishable 3-dimensional simple harmonic oscillators obeying Bose-Einstein statistics. The energy of any one of those atoms of mass m, momentum $\vec{p}$ and position $\vec{r}$ is given by the expression

$$\epsilon = \frac{\vec{p}^2}{2m} + \tfrac{1}{2}K\vec{r}^2,$$

so that they all vibrate with a common frequency $\nu = \sqrt{\frac{K}{m}}/2\pi$, with K the spring constant.

(a) Write an integral expression for the expectation value of the total number of atoms in excited states, in thermal equilibrium at temperature T. There is no need to actually do the integral. Rather express the answer in terms of a dimensionless integral.

(b) Show that, at low temperatures, this gas undergoes a Bose-Einstein condensation, with critical temperature

$$T_B \propto (h\nu/k_B)N^{1/3}$$

(c) Determine the manner in which the condensate fraction N_{gs}/N and the specific heat C_V of the gas vary with temperature T for $T < T_B$.

`SOLUTION:`

(a) The expectation value of the total number of atoms in the excited states is

$$N_{exc} = \int_0^\infty \tfrac{g(\epsilon)d\epsilon}{e^{\epsilon/k_BT}-1}.$$

By writing the density of states in phase space in spherical coordinates we get

$$N_{exc} = \int_0^\infty \int_0^\infty \tfrac{4\pi p^2 dp\, 4\pi r^2 dr/h^3}{e^{p^2/2mk_BT + Kr^2/2k_BT}-1}.$$

Let us define $p = \sqrt{2mk_BT}\, u$ and $r = \sqrt{2k_BT/K}\, v$. The above integral can then be written as

$$N_{exc} = 128\pi^2 (m/K)^{3/2} (k_BT/h)^3\, I,$$

where I is

$$I = \int_0^\infty \int_0^\infty \tfrac{u^2 v^2 du\, dv}{e^{u^2+v^2}-1},$$

and is simply a numerical factor. This implies that

$$N_{exc} = \text{constant} \times (k_BT/h\nu)^3.$$

(b) The transition temperature is set by the condition $N = N_{exc}$ so that

$$T_B = \text{constant} \times (h\nu/k_B)N^{1/3}.$$

(b) For $T < T_B$

$$N_{exc}/N = (T/T_B)^3$$

and

$$N_{gs}/N = 1 - (T/T_B)^3.$$

The mean thermal energy carried by an atom is of order $k_B T$. The total thermal energy of the gas is then proportional to T^4 so that C_V is proportional to T^3.

#8: UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The speed of sound, w, in any medium is given by the formula

$$w = \frac{1}{\sqrt{\rho \kappa_s}},$$

where $\rho = mN/V$ is the mass density and $\kappa_s = -\left(\frac{\partial V}{\partial P}\right)/V$ is the adiabatic compressibility at constant entropy and number of particles.

Show that, for an ideal Fermi gas at $T = 0$ K, $w = v_F/\sqrt{3}$, where v_F is the Fermi velocity of the gas (the speed of particles at the Fermi energy).

`SOLUTION:` For an ideal Fermi gas at $T = 0$ K, we have

$$U = \tfrac{3}{5} N \epsilon_F = \tfrac{3}{5} N \tfrac{h^2}{2m} \left(\tfrac{3N}{8\pi V}\right)^{2/3},$$

so that

$$P = -\left(\tfrac{\partial U}{\partial V}\right)_{S,N} = \tfrac{2}{3}\tfrac{U}{V} = \tfrac{2}{5}\tfrac{N}{V}\epsilon_F$$

and

$$\left(\tfrac{\partial P}{\partial V}\right)_{S,N} = -\tfrac{5}{3}\tfrac{P}{V}.$$

The compressibility is thus

$$\kappa_s = \tfrac{3}{5P} = \tfrac{3}{2}\tfrac{V}{N}\tfrac{1}{\epsilon_F}.$$

Finally, since $\epsilon_F = m v_F^2/2$

$$w = \tfrac{1}{\sqrt{\rho\kappa_s}} = \sqrt{\tfrac{2\epsilon_F}{3m}} = \tfrac{v_F}{\sqrt{3}}.$$

#9: UNDERGRADUATE GENERAL

`PROBLEM:` A particle of mass 3 (arbitrary units) arrives at a speed of $v = 0.8c$ to smack into a stationary particle of mass 7. Emerging from the collision is a particle of mass 8 traveling at $0.6c$ along the direction of the original incoming particle's velocity vector, and another particle of unknown mass and velocity. Conserving relativistic energy (γmc^2) and relativistic momentum ($\gamma m\vec{v}$), describe the properties of the unknown particle. The speeds given in the problem lead to conveniently rational gamma factors for easy computation.

`SOLUTION:`

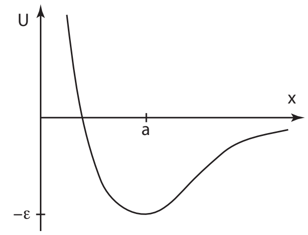
The problem is one-dimensional, so we will take the initial incoming particle to have positive velocity. $\gamma = \frac{5}{3}$ for the incoming particle, so that initial momentum of the incoming particle is $\gamma mv = \frac{5}{3} \cdot 3 \cdot 0.8c = 4c$ (mass units implicit). The stationary particle adds no momentum, so the total system momentum is $P = 4c$. Initial energy is $E = \frac{5}{3} \cdot 3c^2 + 7c^2 = 12c^2$.

After the collision, the mass-8 particle has $\gamma = \frac{5}{4}$, so that its momentum is $\frac{5}{4} \cdot 8 \cdot 0.6 = 6c$ and its energy is $\frac{5}{4} \cdot 8c^2 = 10c^2$.

Conservation of energy and momentum demands that the mystery particle have energy $2c^2$ and momentum $-2c$. Associating the former with γmc^2 and the latter with γmv, we learn that $\gamma m = 2$, so that $v = -c$. The particle travels at the speed of light, so that $\gamma = \infty$ and mass is zero. We might call this a photon. The relation $E = pc$ is upheld.

#10: UNDERGRADUATE GENERAL

`PROBLEM:` The figure below shows a generic potential for atoms in a solid lattice. Assign reasonable numbers for the scale of the equilibrium location, a, and the depth of the potential well, ε for, say, iron. Now imagine a steel fiber 1 m long and 100 μm in diameter suspending a 1 kg mass in normal gravity. How much will the fiber stretch, based on your estimates for a and ε? Hint: first work out a spring constant, k.



`SOLUTION:` A typical scale has atoms on lattice spacings around 2–3 Å, so let's say 2.5×10^{-10} m. Bond energy is a few electron volts; we'll say 2 eV, or about 3×10^{-19} J.

Approximating the trough of the potential as one that has a minimum at $(a,\ -\varepsilon)$, and intersects the x-axis at $a/2$, we get $U \approx -\varepsilon + \frac{4\varepsilon}{a^2}(x-a)^2$. The force is the negative gradient of potential, so $F = -\frac{8\varepsilon}{a^2}\Delta x$, where $\Delta x = (x-a)$ is the displacement from equilibrium. For this single-pair interaction, we associate the spring constant, $k = \frac{8\varepsilon}{a^2}$.

The cross section of the fiber has A/a^2 bonds, where A is the cross-sectional area. Therefore, the force required to separate adjacent lattice planes by Δx is multiplied by this ratio. We aim for a total spring constant, so that $F_{\text{tot}} = -k_{\text{tot}}\Delta X$, ΔX being the total fiber stretch that we seek. We note that each inter-atom displacement is a small fraction of the total fiber displacement by the ratio of inter-atomic spacing to total length: $\Delta x = \frac{a}{L}\Delta X$, where L is the fiber length. We therefore multiply the single-pair spring constant by $\frac{A}{a^2}\frac{a}{L}$ to get the total spring constant, resulting in $k_{\text{tot}} = \frac{8\varepsilon}{a^2}\frac{A}{a^2}\frac{a}{L} = \frac{8\varepsilon}{a^3}\frac{A}{L}$. The first factor is equivalent to an elastic modulus for the material, for which we would compute about 150 GPa (a reasonable value for metal). Multiplying by A/L results in approximately $k_{\text{tot}} \approx 1000$ N/m. Hanging a 1 kg (10 N) mass on this fiber would result in a 0.01 m displacement, which matches crude intuition, and also suggests that the material would probably snap for a strain this large.

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PART II : PHYSICS DEPARTMENT EXAM

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#11: GRADUATE MECHANICS

`PROBLEM:` A sound wave propagates in a medium with acoustic speed $c_s = c_s(x, y, z)$.

(a) Write the equation governing the propagation of a monochromatic acoustic wave of frequency ω.

(b) Derive the eikonal equation for the wave function phase. When does this provide a valid lowest-order solution to the problem? To what mechanics problem is this equivalent?

(c) Suppose that the sound speed has the form

$$\frac{1}{c_s(x,y,z)^2} = \frac{1}{c_0^2}\left(n_1(x)^2 + n_2(y)^2 + n_3(z)^2\right) \ .$$

Solve the eikonal equation for the phase by separation of variables.

#12: GRADUATE MECHANICS

`PROBLEM:` Consider an oscillator of mass m where the spring constant k varies with time, i.e. $k = k_0 f(t)$ where

$$f = 1 + h\cos(\Omega t).$$

(a) For $h < 1$ and $\Omega^2 \ll k_0/m$, *describe* how you would calculate the motion. What approach would you use? Why?

(b) For $h < 1$ and $\Omega \sim 2\sqrt{k_0/m}$ *describe* how you would calculate the motion. What approach would you use? Why?

(c) For $h < 1$ and $\Omega^2 \gg k_0/m$, *describe* how you would calculate the motion. What approach would you use? Why?

(d) Choose *one* of cases (a), (b), (c) and actually implement the calculation of an approximate solution.

#13: GRADUATE ELECTRODYNAMICS

`PROBLEM:` Retarded fields for a moving sheet.

An infinite sheet with uniform surface charge density σ lies in the plane $z = 0$.

(a) Suppose the sheet has been moving for all times with constant speed v in the positive y direction. Calculate the magnetic field that would be produced by such a moving sheet.

Assume now that the sheet has been stationary for $t < 0$. At time $t = 0$ the sheet begins to move in the positive y direction with constant speed v.

Recall the formula for the retarded potential:

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3\vec{x}' \, \frac{\vec{J}\left(\vec{x}', t - \frac{1}{c}\left|\vec{x} - \vec{x}'\right|\right)}{\left|\vec{x} - \vec{x}'\right|}.$$

If you prefer to work with SI units replace the $1/c$ prefactor above by $\mu_0/(4\pi)$.

(b) Write an expression for the current density $\vec{J}(\vec{x}, t)$. Calculate $A_y(z, t)$. Plot its value as a function of t for fixed z (positive and negative) and also as a function of z for fixed time $t > 0$.

(c) Calculate B_x and E_y and plot them as functions of time at a fixed value of z (positive and negative) and as functions of z for a fixed time $t > 0$. Compare with the results obtained in part (a).

#14: GRADUATE ELECTRODYNAMICS

`PROBLEM:` EM Waves in a plasma.

A plane electromagnetic wave of angular frequency ω propagates in a uniform plasma with electron density N_e and overall and local charge neutrality: $\rho = 0$. The EM wave generates periodic currents within the plasma that, as the problem will show, modify the index of refraction of the medium compared to that of the vacuum.

Assume the plasma is collisionless, *i.e.* has no resistivity, and that radiation pressure effects can be neglected. Moreover, assume the ions are much heavier than the electrons so that we can neglect the current due to the ions.

(a) Relate the steady-state current density $\vec{J}(\vec{r}, t)$ in the plasma to the wave's electric field $\vec{E}(\vec{r}, t)$ or derivatives thereof. Assume magnetic forces can be neglected.

(b) Write down the appropriate Maxwell equations and derive the wave equation. Find the dispersion relation $\omega(k)$. Find the lowest frequency EM wave that can propagate in the plasma.

(c) Find the phase and group velocities for EM waves in the plasma. Compare those velocities with the speed c of light in vacuum.

(d) Find the index of refraction n of the plasma as a function of the frequency. If a plane EM wave is incident on a plane interface between vacuum and the plasma, what is the critical angle (measured from the normal to the interface) for total 'external' reflection?

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Helium atom ground state approximation, variational method.

(a) Write down the helium atom Hamiltonian, taking the nucleus to have charge $Z|e|$.

(b) In the approximation where one neglects the Coulomb interaction between the two electrons, what would be the position-space energy eigenfunctions? You don't need to write actual functions, but just a formal expression, indicating how you can get the answer by recycling wavefunctions from another system. Indicate all of the possible quantum numbers, and make sure that it has the correct behavior for electrons being Fermions. What are the possibilities for the total spin?

(c) Write down the **ground state** wavefunction of the Helium atom, again ignoring the electron-electron interaction, this time explicitly writing out the coordinate dependence. Don't forget to write the spin dependence (what is its spin?). Keep the charge Z of the nucleus general.

(d) Use your answer from the previous part to set up a variational method approximation of the Helium ground state energy. Replace $Z \to Z_{eff}$ in the wavefunction; Z_{eff} will be the variational parameter. Keep the Z in the Hamiltonian fixed at $Z = 2$. Write out as much as you can of this variational calculation for maximal partial credit. If you have time to spare, and want to show off, you can do the electron-electron interaction term integral using

$$\tfrac{1}{r_{12}} \equiv \tfrac{1}{|\vec{x}_1 - \vec{x}_2|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma), \qquad \text{and}$$

$$P_\ell(\cos\gamma) = \tfrac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\theta_1,\phi_1) Y_{\ell,m}(\theta_2,\phi_2).$$

If you don't have time or want to show off, just write the integral in terms of the various quantities (like Z, Z_{eff}, a_0, e, etc) up to an undetermined overall coefficient (which evaluates to be 5/4). Evaluate the needed other pieces from H, do the variational calculation to determine Z_{eff}, and thus obtain the ground state energy. (A check: the variational method will determine Z_{eff} to be in the range between 1 and 2, which makes physical sense: each electron sees the nucleus partly screened by the other one.)

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the following game. You are given a sequence of *pairs* of qbits (spin $\frac{1}{2}$s). You are guaranteed that either (A) all pairs are in the state $\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$ *or* (B) all pairs are randomly chosen from $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ with equal probability. You have a Stern-Gerlach apparatus, of course.

(a) In the first version of the game, you are only allowed to perform measurements (as many as you want) on *one* of the two qbits. In this version of the game, is it possible to decide whether the sequence is (A) or (B)? Explain why or why not by deriving the reduced density matrix describing the state of one of the two qbits in each case.

(b) In preparation for the next part, write the density matrices for the *two* qbits which describe sequences A and B respectively.

(c) In the advanced version of the game, you may perform measurements on both qbits. Describe a strategy for distinguishing (A) from (B) and describe its outcome for sequence A and sequence B.

#17: GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a system of N identical but distinguishable particles, each of which has a nondegenerate ground state with energy zero, and a g-fold degenerate excited state with energy $\varepsilon > 0$.

(a) Let the total energy of the system be fixed at $E = M\varepsilon$, where M is the number of particles in an excited state. What is the total number of states $\Omega(E, N)$?

(b) What is the entropy $S(E, N)$? Assume the system is thermodynamically large. You may find it convenient to define $\nu \equiv M/N$, which is the fraction of particles in an excited state.

(c) Find the temperature $T(\nu)$. Invert this relation to find $\nu(T)$.

(d) Show that there is a region where the temperature is negative.

(e) What happens when a system at negative temperature is placed in thermal contact with a heat bath at positive temperature?

#18: GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider an Ising model on a square lattice with Hamiltonian

$$\hat{H} = -J \sum_{i \in \mathrm{A}} \sum_{j \in \mathrm{B}}{}' S_i \, \sigma_j \ ,$$

where the sum is over all nearest-neighbor pairs, such that i is on the A sublattice and j is on the B sublattice (this is the meaning of the prime on the j sum), as depicted in Fig. 1. The A sublattice spins take values $S_i \in \{-1, 0, +1\}$, while the B sublattice spins take values $\sigma_j \in \{-1, +1\}$.

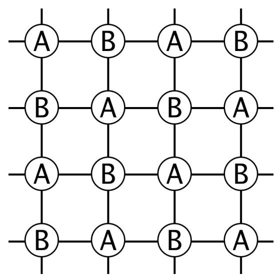


Figure 1: The square lattice and its A and B sublattices.

(a) Make the mean field assumptions $\langle S_i \rangle = m_{\mathrm{A}}$ for $i \in$ A and $\langle \sigma_j \rangle = m_{\mathrm{B}}$ for $j \in$ B. Find the mean field free energy $F(T, N, m_{\mathrm{A}}, m_{\mathrm{B}})$. Adimensionalize as usual, writing $\theta \equiv k_{\mathrm{B}} T / zJ$ (with $z = 4$ for the square lattice) and $f = F/zJN$. Then write $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$.

(b) Write down the two mean field equations (one for m_{A} and one for m_{B}).

(c) Expand the free energy $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$ up to fourth order in the order parameters m_{A} and m_{B}.

(d) The part of $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$ which is quadratic in m_{A} and m_{B} may be written as a quadratic form, *i.e.*

$$f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}}) = f_0 + \tfrac{1}{2} \begin{pmatrix} m_{\mathrm{A}} & m_{\mathrm{B}} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} m_{\mathrm{A}} \\ m_{\mathrm{B}} \end{pmatrix} + \mathcal{O}\big(m_{\mathrm{A}}^4, m_{\mathrm{B}}^4\big) \ ,$$

where the matrix M is symmetric, with components $M_{aa'}$ which depend on θ. The critical temperature θ_{c} is identified as the largest value of θ for which $\mathsf{det}\, M(\theta) = 0$. Find θ_{c} and explain why this is the correct protocol to determine it.

#19: GRADUATE MATH

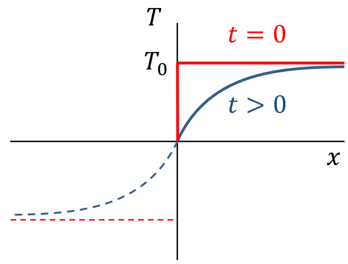
`PROBLEM:` The standard definitions of the direct and inverse Fourier transforms (FT) in D dimensions are

$$\tilde{v}\left(\vec{k}\right) = \int d^D r e^{-i\vec{k}\cdot\vec{r}} v\left(\vec{r}\right) , \qquad v\left(\vec{r}\right) = \int \tfrac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{r}} \tilde{v}\left(\vec{k}\right) .$$

A well-known result is that the $D = 3$ FT of function $v\left(\vec{r}\right) = 1/r = 1/\sqrt{x^2 + y^2 + z^2}$ is equal to $\tilde{v}\left(\vec{k}\right) = 4\pi/|\vec{k}|^2$. Calculate the 2D version of this result, that is, compute the $D = 2$ FT $\tilde{v}\left(\vec{k}\right)$ of $v\left(\vec{r}\right) = 1/r = 1/\sqrt{x^2 + y^2}$.

#20: GRADUATE GENERAL

`PROBLEM:` Let us assume that the Earth has started as a ball of molten rock at uniform initial temperature $T_0 = 2000°\mathrm{C}$ and has since gradually cooled down by radiation from the surface, which remained at $0°\mathrm{C}$ at all times $t > 0$. The Earth's temperature as a function of depth x is sketched by the solid lines in the Figure (the surface is at $x = 0$, and x increases toward the interior). The temperature gradient at the surface $\gamma = \partial T/\partial x|_{x=+0}$ has decreased over time, from $\gamma = \infty$ at $t = 0$ to the present-day value of $\gamma = 0.037°\mathrm{C/m}$. Assume that the temperature



distribution is described by the heat diffusion equation

$$\frac{\partial T}{\partial t} - D\frac{\partial^2 T}{\partial x^2} = Q\,, \quad 0 < x < \infty\,, \quad D = 1.2 \times 10^{-6}\mathrm{m}^2\mathrm{s}^{-1}.$$

Not knowing about heat production by radioactive decay inside the Earth, Lord Kelvin (who was the first to work on this problem) set Q to zero. (This was his famous mistake.) Rederive Kelvin's estimate for the age of the Earth in years, using $1\,\text{year} = 3.15 \times 10^7\,\mathrm{s}$.

Hints: 1) Use the method of images to reformulate the problem on the full line by formally extending function $T(x, t)$ to negative x as shown by the dashed lines in the Figure. 2) The retarded Green's function of the diffusion equation is $G(x, t) = \theta(t)\exp(-x^2/\,4Dt)/\sqrt{4\pi Dt}\,$.

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`PROBLEM:` A sound wave propagates in a medium with acoustic speed $c_s = c_s(x, y, z)$.

(a) Write the equation governing the propagation of a monochromatic acoustic wave of frequency ω.

(b) Derive the eikonal equation for the wave function phase. When does this provide a valid lowest-order solution to the problem? To what mechanics problem is this equivalent?

(c) Suppose that the sound speed has the form

$$\frac{1}{c_s(x,y,z)^2} = \frac{1}{c_0^2}\left(n_1(x)^2 + n_2(y)^2 + n_3(z)^2\right) \ .$$

Solve the eikonal equation for the phase by separation of variables.

`SOLUTION:`

(a) The wave equation is

$$\left(c_s(x)^2\nabla^2 - \partial_t^2\right)\psi(x,t) = 0.$$

For $\psi(x,t) = e^{-\mathbf{i}\omega t}\psi(x)$, we find the Helmholtz equation

$$\left(c_s(x)^2\nabla^2 + \omega^2\right)\psi(x) = 0.$$

(b) Letting $\psi(x) = e^{\mathbf{i}\phi(x)/h}$, the eikonal equation implies

$$(\nabla\phi)^2 + \frac{\omega^2}{c_s(x)^2} = \text{terms proportional to } h$$

The eikonal equation arises by setting the right hand side to zero. This is a good approximation if the phase is rapidly varying compared to c_s

$$\left|\frac{\nabla c_s}{c_s}\right| \ll k \equiv \left|\frac{\nabla\psi}{\psi}\right|$$

in which case we may treat h as small. This is the Hamilton-Jacobi equation for a particle in a potential $V = \frac{\omega^2}{c_s^2(x)}$.

(c) Letting $\phi = \phi_1(x) + \phi_2(y) + \phi_3(z)$, we have

$$(\partial_x\phi_1)^2 - k_0^2 n_1(x)^2 = 0$$

with $k_0^2 \equiv \frac{\omega^2}{c_0^2}$, so

$$\phi_1(x) = k_0\int^x dx' n_1(x')$$

and similarly for $\phi_{2,3}$. The density factorizes:

$$\psi(x,y,z) = e^{-\mathbf{i}\omega t}e^{\mathbf{i}\phi_1(x)}e^{\mathbf{i}\phi_2(y)}e^{\mathbf{i}\phi_3(z)}.$$

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`PROBLEM:` Consider an oscillator of mass m where the spring constant k varies with time, i.e. $k = k_0 f(t)$ where

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(a) For $h < 1$ and $\Omega^2 \ll k_0/m$, *describe* how you would calculate the motion. What approach would you use? Why?

(b) For $h < 1$ and $\Omega \sim 2\sqrt{k_0/m}$ *describe* how you would calculate the motion. What approach would you use? Why?

(c) For $h < 1$ and $\Omega^2 \gg k_0/m$, *describe* how you would calculate the motion. What approach would you use? Why?

(d) Choose *one* of cases (a), (b), (c) and actually implement the calculation of an approximate solution.

`SOLUTION:`

(a) For $\Omega^2 \ll w_0^2 = k_0/m$

we have

$w^2(t) = w_0^2(1 + h\cos(\Omega t))$

with $\dot{w}/w \ll 1$ ($h < 1$), so the perturbation is *adiabatic*.

Approach via *adiabatic invariant*

i.e. $N = E/w(t) = const$

E $\rightarrow$ energy
and calculate orbit.

(b) For $\Omega \sim 2w_0$, *parametric resonance* is possible between $w^2(t)$ variation at $2w_0$ and fundamental frequency w_0. This feeds back onto fundamental, and can lead to instability.

Approach by multiple time scale perturbation theory, i.e.

$\Omega = 2w_0 + \varepsilon$
$x = a(t)\cos[(w_0 + \frac{\varepsilon}{2})t] + b(t)\sin[(w_0 + \frac{\varepsilon}{2})t]$

use $\dot{a}/a$, $\dot{b}/b \ll w_0$, and obtain and solve coupled equations for $\dot{a}$, $\dot{b}$. Instability can arise depending on ε, h.

(c) For $\Omega \gg w$, have a situation where a fast varying parameter enters the oscillator evolution. This can generate a *ponderomotive potential* and *pondermotive force.*

Approach: Method of averaging

i.e. $\ddot{x} + w_0^2 x + \alpha(\Omega t)x = 0$

$\Omega \gg w_0$
$\alpha(\Omega t)$ is oscillatory.

$x = x_f + \bar{x}$
$\bar{x}$ is slow envelope
x_f is fast variation.

$\bar{x} + w_0^2\bar{x} + \langle\alpha(\Omega t)x_f\rangle = 0$

$\langle \quad \rangle = 1/(2\pi/\Omega)\int_0^{2\pi/\Omega} dt$

$\ddot{x}_f + w_0^2 x_f = -\alpha(\Omega t)\bar{x}$

$x_f = +\frac{\alpha(\Omega t)\bar{x}}{\Omega^2}$

$\Rightarrow \ddot{\bar{x}} + w_0^2\bar{x} + \frac{\langle\alpha(\Omega t)^2\rangle\bar{x}}{\Omega^2} = 0$

etc.

(d) Simplest, by far, is (a).

$$\begin{aligned} E &= 2(\tfrac{1}{2}kx^2) \\ &= m w_0^2 x^2 \end{aligned}$$

$w(t) = w_0(1 + h\cos(\Omega t))^{1/2}$

$$\begin{aligned} N = E/w &= m w_0 x(t)^2/(1 + h\cos\Omega t)^{1/2} \\ &= const. = N_o \end{aligned}$$

$x(t) = \frac{N_0^{1/2}(1+h\cos\Omega t)^{1/4}}{(m w_0)^{1/2}}$

#13: GRADUATE ELECTRODYNAMICS

`PROBLEM:` Retarded fields for a moving sheet.

An infinite sheet with uniform surface charge density σ lies in the plane $z = 0$.

(a) Suppose the sheet has been moving for all times with constant speed v in the positive y direction. Calculate the magnetic field that would be produced by such a moving sheet.

Assume now that the sheet has been stationary for $t < 0$. At time $t = 0$ the sheet begins to move in the positive y direction with constant speed v.

Recall the formula for the retarded potential:

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3\vec{x}' \frac{\vec{J}\left(\vec{x}', t - \frac{1}{c}|\vec{x} - \vec{x}'|\right)}{|\vec{x} - \vec{x}'|}.$$

If you prefer to work with SI units replace the $1/c$ prefactor above by $\mu_0/(4\pi)$.

(b) Write an expression for the current density $\vec{J}(\vec{x}, t)$. Calculate $A_y(z, t)$. Plot its value as a function of t for fixed z (positive and negative) and also as a function of z for fixed time $t > 0$.

(c) Calculate B_x and E_y and plot them as functions of time at a fixed value of z (positive and negative) and as functions of z for a fixed time $t > 0$. Compare with the results obtained in part (a).

`SOLUTION:`

(a) The sheet produces a surface current σv in the y direction. The resulting field is independent of x and y. The integrated form of Ampere's law, $\oint_R \vec{B} \cdot d\vec{\ell} = \frac{4\pi}{c} \int_R \vec{J} \cdot d\vec{a}$, integrated over a square loop ∂R with normal in the y direction and which penetrates the sheet gives

$$B_x(z) = \begin{cases} \frac{2\pi}{c}\sigma v, & z > 0 \\ -\frac{2\pi}{c}\sigma v, & z < 0 \end{cases}.$$

(b) The current density is

$$\vec{J}(\vec{x}, t) = \sigma v \delta(z)\theta(t)\hat{y}$$

where $\theta(t)$ is the step function which vanishes for $t < 0$ and is 1 for $t > 0$. Using the above formula, only the y component of the vector potential is nonzero: $\vec{A}(\vec{x}, t) = \hat{y} A_y(\vec{x}, t)$, and

$$A_y(\vec{x}, t) = \frac{\sigma v}{c} \int d^3x' \frac{\delta(z')}{|\vec{x} - \vec{x}'|} \theta(t - |\vec{x} - \vec{x}'|/c).$$

This depends only on z, t, since the x, y-dependence can be removed by a change of integration variable. Evaluate it at $x = y = 0$:

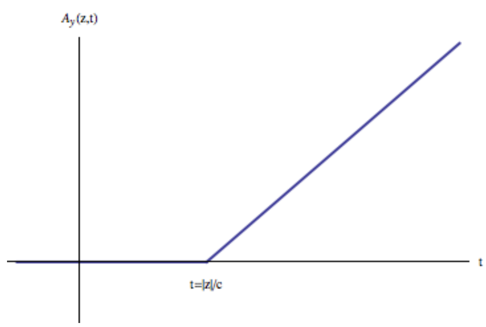
$$A_y(z,t) = \frac{\sigma v}{c} \int dx' dy' \frac{1}{\sqrt{x'^2 + y'^2 + z^2}} \theta(t - \sqrt{x'^2 + y'^2 + z^2}/c).$$

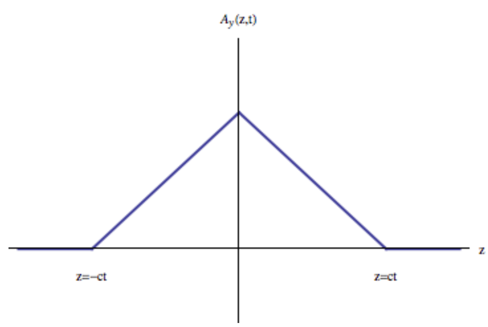
In polar coordinates: $x'^2 + y'^2 = R^2, dx'dy' = d\theta R dR$

$$A_y(z,t) = \frac{\sigma v}{c} 2\pi \int_0^\infty \frac{RdR}{\sqrt{R^2 + z^2}} \theta(t - \sqrt{R^2 + z^2}/c) = \begin{cases} 0, & t < |z|/c \\ \frac{\sigma v}{c} 2\pi \int_0^{R_t} \frac{RdR}{\sqrt{R^2+z^2}}, & t > |z|/c \end{cases}$$

where the upper limit is $R_t \equiv \sqrt{c^2t^2 - z^2}$. Substituting $u = R^2$, the answer when it is nonzero is

$$A_y(z,t) = \frac{\sigma v}{c} 2\pi \left(\sqrt{R_t^2 + z^2} - \sqrt{z^2} \right) = \frac{2\pi\sigma v}{c} \left(ct - |z| \right), \quad ct > |z|.$$





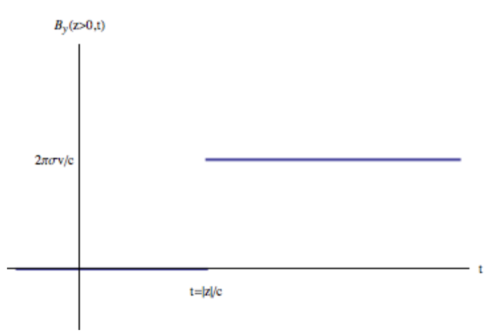
(c) Taking the curl $\vec{B} = \vec{\nabla} \times \vec{A} = B_x \hat{x}$, with

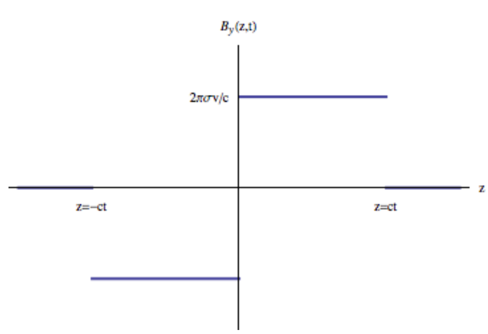
$$B_x = -\partial_z A_y = \mathrm{sign}(z) \frac{2\pi\sigma v}{c}, \quad ct > |z|$$

The only component of E is

$$E_y = -\frac{1}{c} \partial_t A_y(z,t) = -2\pi\sigma v, \quad ct > |z|.$$

The magnetic field is the same as the answer in the case of a static current wherever it is nonzero. The electric field is a new feature of the time-dependent case, and is a sign that work must be done to move the sheet, since $\vec{E} \times \vec{B} \neq 0$.






E_y (z,t)
t
t=|z|/c
−2πσv
E_y (z,t)
z
z=−ct
z=ct
−2πσv

#14: GRADUATE ELECTRODYNAMICS

`PROBLEM:` EM Waves in a plasma.

A plane electromagnetic wave of angular frequency ω propagates in a uniform plasma with electron density N_e and overall and local charge neutrality: $\rho = 0$. The EM wave generates periodic currents within the plasma that, as the problem will show, modify the index of refraction of the medium compared to that of the vacuum.

Assume the plasma is collisionless, *i.e.* has no resistivity, and that radiation pressure effects can be neglected. Moreover, assume the ions are much heavier than the electrons so that we can neglect the current due to the ions.

(a) Relate the steady-state current density $\vec{J}(\vec{r}, t)$ in the plasma to the wave's electric field $\vec{E}(\vec{r}, t)$ or derivatives thereof. Assume magnetic forces can be neglected.

(b) Write down the appropriate Maxwell equations and derive the wave equation. Find the dispersion relation $\omega(k)$. Find the lowest frequency EM wave that can propagate in the plasma.

(c) Find the phase and group velocities for EM waves in the plasma. Compare those velocities with the speed c of light in vacuum.

(d) Find the index of refraction n of the plasma as a function of the frequency. If a plane EM wave is incident on a plane interface between vacuum and the plasma, what is the critical angle (measured from the normal to the interface) for total 'external' reflection?

`SOLUTION:`

(a) The motion of an electron at location $\vec{r}$ is governed by

$$m\ddot{\vec{s}} = -e\vec{E}(\vec{r}, t)$$

where $\vec{s}$ is the (small) displacement vector of the electron, and hence $\dot{\vec{s}} = \vec{v}(\vec{r}, t)$ is the electron velocity field at $\vec{r}$. Ignore transient solutions; the steady-state solution for the displacement is harmonic with the applied frequency: $\ddot{\vec{s}} = -\omega^2 \vec{s}$. The current density is therefore

$$\vec{J}(\vec{r}, t) = -eN_e\dot{\vec{s}}$$

and in steady state

$$\dot{\vec{J}}(\vec{r}) = -eN_e\ddot{\vec{s}} = N_e\frac{e^2}{m}\vec{E}$$

or

$$\vec{J}(\vec{r}) = -\frac{e^2 N_e}{m\omega^2}\partial_t \vec{E}(\vec{r}).$$

(b) In Gaussian units, Maxwell's equations are

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}, \quad \vec{\nabla} \times \vec{B} = +\frac{1}{c}\partial_t \vec{E} + \frac{4\pi}{c}\vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{E} = 4\pi\rho = 0$$

where the last equation comes from the local neutrality condition. Using our expression for the current density from the previous part, Ampere's law becomes

$$\vec{\nabla} \times \vec{B} = +\frac{1}{c}\left(1 - \frac{4\pi e^2 N_e}{m\omega^2}\right)\partial_t \vec{E}$$

Taking the curl and using Faraday and $\vec{\nabla} \cdot \vec{E} = 0$, we get

$$\nabla^2 \vec{E} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\frac{1}{c^2}\partial_t^2 \vec{E}$$

where the plasma frequency is

$$\omega_p \equiv \sqrt{\frac{4\pi e^2 N_e}{m}}.$$

Therefore the dispersion relation is

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right), \quad \text{or} \quad \omega^2 = c^2 k^2 + \omega_p^2$$

which has no real solutions for $\omega < \omega_p$.

(c) The phase and group velocities are

$$v_p = \frac{\omega(k)}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \geq c$$

$$\text{and} \quad v_g = \frac{d\omega}{dk} = \frac{kc^2}{\omega} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \leq c.$$

The index of refraction is $n = c/v_p = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1$. Snell's law says $n_{\text{vac}} \sin\theta_{\text{inc}} = n \sin\theta$ where θ_{inc} is the angle of incidence with respect to the outer normal to the interface, and θ is the angle between the propagation direction in the plasma with respect to the inner normal to the interface. Total external reflection means $\theta > \pi/2$, which happens at the critical angle

$$\sin\theta_{\text{inc}}^\star = n$$

which is possible for $n < 1$.

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Helium atom ground state approximation, variational method.

(a) Write down the helium atom Hamiltonian, taking the nucleus to have charge $Z|e|$.

(b) In the approximation where one neglects the Coulomb interaction between the two electrons, what would be the position-space energy eigenfunctions? You don't need to write actual functions, but just a formal expression, indicating how you can get the answer by recycling wavefunctions from another system. Indicate all of the possible quantum numbers, and make sure that it has the correct behavior for electrons being Fermions. What are the possibilities for the total spin?

(c) Write down the **ground state** wavefunction of the Helium atom, again ignoring the electron-electron interaction, this time explicitly writing out the coordinate dependence. Don't forget to write the spin dependence (what is its spin?). Keep the charge Z of the nucleus general.

(d) Use your answer from the previous part to set up a variational method approximation of the Helium ground state energy. Replace $Z \to Z_{eff}$ in the wavefunction; Z_{eff} will be the variational parameter. Keep the Z in the Hamiltonian fixed at $Z = 2$. Write out as much as you can of this variational calculation for maximal partial credit. If you have time to spare, and want to show off, you can do the electron-electron interaction term integral using

$$\tfrac{1}{r_{12}} \equiv \tfrac{1}{|\vec{x}_1 - \vec{x}_2|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma), \qquad \text{and}$$

$$P_\ell(\cos\gamma) = \tfrac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell,m}(\theta_1, \phi_1) Y_{\ell,m}(\theta_2, \phi_2).$$

If you don't have time or want to show off, just write the integral in terms of the various quantities (like Z, Z_{eff}, a_0, e, etc) up to an undetermined overall coefficient (which evaluates to be 5/4). Evaluate the needed other pieces from H, do the variational calculation to determine Z_{eff}, and thus obtain the ground state energy. (A check: the variational method will determine Z_{eff} to be in the range between 1 and 2, which makes physical sense: each electron sees the nucleus partly screened by the other one.)

`SOLUTION:`

(a)

$$H = \tfrac{1}{2m}(\vec{p}_1^{\,2} + \vec{p}_2^{\,2}) - \tfrac{Ze^2}{r_1} - \tfrac{Ze^2}{r_2} + \tfrac{e^2}{r_{12}}$$

(b) In this approximation, the wavefunctions are tensor products of hydrogen atom wavefunctions, together with the spins of the two electrons, such that the overall wavefunction is antisymmetric under exchanging the two electrons (identical Fermions). The total spin can be zero or one (corresponding to χ_- and χ_+ below)

$$\psi(\vec{x}_1, \vec{z}_2)^{(0)} = \tfrac{1}{\sqrt{2}} \left[\psi_{n,\ell,m}(\vec{x}_1)\psi_{n',\ell',m'}(\vec{x}_2) \pm \psi_{n,\ell,m}(\vec{x}_2)\psi_{n',\ell',m'}(\vec{x}_1)\right] \otimes \chi_{\mp}$$

$$\chi_- \equiv \tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle), \qquad \chi_+ = |++\rangle,\ \tfrac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle),\ |--\rangle.$$

(c)

$$\psi_{1,0,0}(\vec{x}_1)\psi_{1,0,0}(\vec{x}_2)\chi_- = \tfrac{Z^3}{\pi a_0^3} e^{-Z(r_1+r_2)/a_0}\chi_-$$

(d)

$$\tilde{E}(Z_{eff}) = \langle\tilde{0}|\tfrac{1}{2m}(\vec{p}_1^{\,2} + \vec{p}_2^{\,2}) - \tfrac{Ze^2}{r_1} - \tfrac{Ze^2}{r_2} + \tfrac{e^2}{r_{12}}|\tilde{0}\rangle$$

$$\langle\vec{x}_1, \vec{x}_2|\tilde{0}\rangle = \tfrac{Z_{eff}^3}{\pi a_0^3} e^{-Z_{eff}(r_1+r_2)/a_0}\chi_-$$

Replace $\vec{p}_1^{\,2} \to -\hbar^2\nabla_1^2 \to -\hbar^2 \frac{1}{r}\partial_{r_1}^2(r_1 \cdot ..)$ to get

$$\tilde{E}(Z_{eff}) = (Z_{eff}^2 - 2ZZ_{eff} + \tfrac{5}{4}Z_{eff})(e^2/a_0),$$

so the minimum value is at $Z_{eff,min} = 2 - \frac{5}{16}$ and then $E_{groundstate} \approx \tilde{E}(Z_{eff,min})$

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the following game. You are given a sequence of *pairs* of qbits (spin $\frac{1}{2}$s). You are guaranteed that either (A) all pairs are in the state $\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)$ *or* (B) all pairs are randomly chosen from $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ with equal probability. You have a Stern-Gerlach apparatus, of course.

(a) In the first version of the game, you are only allowed to perform measurements (as many as you want) on *one* of the two qbits. In this version of the game, is it possible to decide whether the sequence is (A) or (B)? Explain why or why not by deriving the reduced density matrix describing the state of one of the two qbits in each case.

(b) In preparation for the next part, write the density matrices for the *two* qbits which describe sequences A and B respectively.

(c) In the advanced version of the game, you may perform measurements on both qbits. Describe a strategy for distinguishing (A) from (B) and describe its outcome for sequence A and sequence B.

`SOLUTION:`

(a) In both cases, the reduced density matrix for one of the two qbits (they are identical) is proportional to the identity. You can learn nothing by measuring one of them.

(b) The density matrix for both qbits in case (A) is

$$\boldsymbol{\rho}_A = \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\uparrow| + \langle\downarrow\downarrow|\right)$$

(a pure state). For case (B) we have classical uncertainty about the ket :

$$\boldsymbol{\rho}_B = \tfrac{1}{2}\left(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|\right).$$

(c) Measure $\boldsymbol{\sigma}^x \otimes \boldsymbol{\sigma}^x$. This has

$$\langle\boldsymbol{\sigma}^x \otimes \boldsymbol{\sigma}^x\rangle_A = \mathrm{tr}\boldsymbol{\rho}_A\boldsymbol{\sigma}^x \otimes \boldsymbol{\sigma}^x = \mathrm{tr}\tfrac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} = 1$$

(I've written the matrices in the $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$ basis) but

$$\langle\boldsymbol{\sigma}^x \otimes \boldsymbol{\sigma}^x\rangle_B = \mathrm{tr}\boldsymbol{\rho}_B\boldsymbol{\sigma}^x \otimes \boldsymbol{\sigma}^x = \mathrm{tr}\tfrac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} = 0.$$

#17: GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a system of N identical but distinguishable particles, each of which has a nondegenerate ground state with energy zero, and a g-fold degenerate excited state with energy $\varepsilon > 0$.

(a) Let the total energy of the system be fixed at $E = M\varepsilon$, where M is the number of particles in an excited state. What is the total number of states $\Omega(E,N)$?

(b) What is the entropy $S(E,N)$? Assume the system is thermodynamically large. You may find it convenient to define $\nu \equiv M/N$, which is the fraction of particles in an excited state.

(c) Find the temperature $T(\nu)$. Invert this relation to find $\nu(T)$.

(d) Show that there is a region where the temperature is negative.

(e) What happens when a system at negative temperature is placed in thermal contact with a heat bath at positive temperature?

`SOLUTION:`

(a) Since each excited particle can be in any of g degenerate energy states, we have

$$\Omega(E,N) = \binom{N}{M} g^M = \frac{N!\, g^M}{M!\,(N-M)!} \ .$$

(b) Using Stirling's approximation, we have

$$S(E,N) = k_{\mathrm{B}} \ln \Omega(E,N) = -N k_{\mathrm{B}} \Big\{ \nu \ln \nu + (1-\nu)\ln(1-\nu) - \nu \ln g \Big\} \ ,$$

where $\nu = M/N = E/N\varepsilon$.

(c) The inverse temperature is

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{N\varepsilon}\left(\frac{\partial S}{\partial \nu}\right)_N = \frac{k_{\mathrm{B}}}{\varepsilon} \cdot \left\{ \ln\!\left(\frac{1-\nu}{\nu}\right) + \ln g \right\} \ ,$$

hence

$$k_{\mathrm{B}} T = \frac{\varepsilon}{\ln\big(\frac{1-\nu}{\nu}\big) + \ln g} \ .$$

Inverting,

$$\nu(T) = \frac{g\, e^{-\varepsilon/k_{\mathrm{B}}T}}{1 + g\, e^{-\varepsilon/k_{\mathrm{B}}T}} \ .$$

(d) The temperature diverges when the denominator in the above expression for $T(\nu)$ vanishes. This occurs at $\nu = \nu^* \equiv g/(g+1)$. For $\nu \in (\nu^*, 1)$, the temperature is negative! This is technically correct, and a consequence of the fact that the energy is bounded for this system: $E \in [0, N\varepsilon]$. The entropy as a function of ν therefore has a maximum at $\nu = \nu^*$. The model is unphysical though in that it neglects various excitations such as kinetic energy (*e.g.* lattice vibrations) for which the energy can be arbitrarily large.

(e) When a system at negative temperature is placed in contact with a heat bath at positive temperature, heat flows from the system to the bath. The energy of the system therefore decreases, and since $\frac{\partial S}{\partial E} < 0$, this results in a net entropy increase, which is what is demanded by the Second Law of Thermodynamics.

#18: GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider an Ising model on a square lattice with Hamiltonian

$$\hat{H} = -J \sum_{i \in \mathrm{A}} {\sum_{j \in \mathrm{B}}}' S_i \, \sigma_j \ ,$$

where the sum is over all nearest-neighbor pairs, such that i is on the A sublattice and j is on the B sublattice (this is the meaning of the prime on the j sum), as depicted in Fig. 1. The A sublattice spins take values $S_i \in \{-1, 0, +1\}$, while the B sublattice spins take values $\sigma_j \in \{-1, +1\}$.

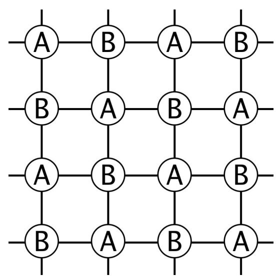


Figure 1: The square lattice and its A and B sublattices.

(a) Make the mean field assumptions $\langle S_i \rangle = m_{\mathrm{A}}$ for $i \in$ A and $\langle \sigma_j \rangle = m_{\mathrm{B}}$ for $j \in$ B. Find the mean field free energy $F(T, N, m_{\mathrm{A}}, m_{\mathrm{B}})$. Adimensionalize as usual, writing $\theta \equiv k_{\mathrm{B}} T / zJ$ (with $z = 4$ for the square lattice) and $f = F/zJN$. Then write $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$.

(b) Write down the two mean field equations (one for m_{A} and one for m_{B}).

(c) Expand the free energy $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$ up to fourth order in the order parameters m_{A} and m_{B}.

(d) The part of $f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}})$ which is quadratic in m_{A} and m_{B} may be written as a quadratic form, *i.e.*

$$f(\theta, m_{\mathrm{A}}, m_{\mathrm{B}}) = f_0 + \tfrac{1}{2} \begin{pmatrix} m_{\mathrm{A}} & m_{\mathrm{B}} \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} m_{\mathrm{A}} \\ m_{\mathrm{B}} \end{pmatrix} + \mathcal{O}\big(m_{\mathrm{A}}^4, m_{\mathrm{B}}^4\big) \ ,$$

where the matrix M is symmetric, with components $M_{aa'}$ which depend on θ. The critical temperature θ_{c} is identified as the largest value of θ for which $\mathsf{det}\, M(\theta) = 0$. Find θ_{c} and explain why this is the correct protocol to determine it.

`SOLUTION:` (a) Writing $S_i = m_{\mathrm{A}} + \delta S_i$ and $\sigma_j = m_{\mathrm{B}} + \delta \sigma_j$ and dropping the terms proportional to $\delta S_i \, \delta \sigma_j$, which are quadratic in fluctuations, one obtains the mean field Hamiltonian

$$\hat{H}_{\mathrm{MF}} = \tfrac{1}{2} N z J m_{\mathrm{A}} m_{\mathrm{B}} - z J m_{\mathrm{B}} \sum_{i \in A} S_i - z J m_{\mathrm{A}} \sum_{j \in B} \sigma_j \ ,$$

with $z = 4$ for the square lattice. Thus, the internal field on each A site is $H_{\text{int,A}} = zJm_{\text{B}}$, and the internal field on each B site is $H_{\text{int,B}} = zJm_{\text{A}}$. The mean field free energy, $F_{\text{MF}} = -k_{\text{B}}T \ln Z_{\text{MF}}$, is then

$$F_{\text{MF}} = \tfrac{1}{2}NzJm_{\text{A}}m_{\text{B}} - \tfrac{1}{2}Nk_{\text{B}}T \ln\Big[1 + 2\cosh(zJm_{\text{B}}/k_{\text{B}}T)\Big] - \tfrac{1}{2}Nk_{\text{B}}T \ln\Big[2\cosh(zJm_{\text{A}}/k_{\text{B}}T)\Big] \ .$$

Adimensionalizing,

$$f(\theta, m_{\text{A}}, m_{\text{B}}) = \tfrac{1}{2}m_{\text{A}}m_{\text{B}} - \tfrac{1}{2}\theta \ln\Big[1 + 2\cosh(m_{\text{B}}/\theta)\Big] - \tfrac{1}{2}\theta \ln\Big[2\cosh(m_{\text{A}}/\theta)\Big] \ .$$

(b) The mean field equations are obtained from $\partial f/\partial m_{\text{A}} = 0$ and $\partial f/\partial m_{\text{B}} = 0$. Thus,

$$m_{\text{A}} = \frac{2\sinh(m_{\text{B}}/\theta)}{1 + 2\cosh(m_{\text{B}}/\theta)}$$

$$m_{\text{B}} = \tanh(m_{\text{A}}/\theta) \ .$$

(c) Using

$$\ln\big(2\cosh x\big) = \ln 2 + \frac{x^2}{2} - \frac{x^4}{12} + \mathcal{O}(x^6) \quad , \quad \ln\big(1 + 2\cosh x\big) = \ln 3 + \frac{x^2}{3} - \frac{x^4}{36} + \mathcal{O}(x^6) \ ,$$

we have

$$f(\theta, m_{\text{A}}, m_{\text{B}}) = f_0 + \tfrac{1}{2}m_{\text{A}}m_{\text{B}} - \frac{m_{\text{A}}^2}{4\,\theta} - \frac{m_{\text{B}}^2}{6\,\theta} + \frac{m_{\text{A}}^4}{24\,\theta^3} + \frac{m_{\text{B}}^4}{72\,\theta^3} + \ldots \ ,$$

with $f_0 = -\frac{1}{2}\theta \ln 6$.

(d) From the answer to part (c), we read off

$$M(\theta) = \begin{pmatrix} -\frac{1}{2\theta} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{3\theta} \end{pmatrix} \ ,$$

from which we obtain $\mathsf{det}\ M = \frac{1}{6}\theta^{-2} - \frac{1}{4}$. Setting $\mathsf{det}\ M = 0$ we obtain $\theta_{\text{c}} = \sqrt{\frac{2}{3}}$.

#19: GRADUATE MATH

`PROBLEM:` The standard definitions of the direct and inverse Fourier transforms (FT) in D dimensions are

$$\tilde{v}\left(\vec{k}\right) = \int d^D r e^{-i\vec{k}\cdot\vec{r}} v\left(\vec{r}\right) , \qquad v\left(\vec{r}\right) = \int \tfrac{d^D k}{(2\pi)^D} e^{i\vec{k}\cdot\vec{r}} \tilde{v}\left(\vec{k}\right) .$$

A well-known result is that the $D = 3$ FT of function $v\left(\vec{r}\right) = 1/r = 1/\sqrt{x^2+y^2+z^2}$ is equal to $\tilde{v}\left(\vec{k}\right) = 4\pi/|\vec{k}|^2$. Calculate the 2D version of this result, that is, compute the $D = 2$ FT $\tilde{v}\left(\vec{k}\right)$ of $v\left(\vec{r}\right) = 1/r = 1/\sqrt{x^2+y^2}$.

`SOLUTION:` The 2D version of function $1/r$ is equal to the 3D one with the third coordinate z set to zero. We can represent this by means of the inverse 3D FT in the momentum space $(\vec{q}, q_z)$, where $\vec{q} = (q_x, q_y)$:

$$\tfrac{1}{r} = \int \tfrac{d^2 q}{(2\pi)^2} \tfrac{dq_z}{2\pi}\, e^{i\vec{q}\cdot\vec{r} + iq_z\cdot 0} \tfrac{4\pi}{|\vec{q}|^2 + q_z^2} .$$

Substituting this into the integral for the 2D FT, we get

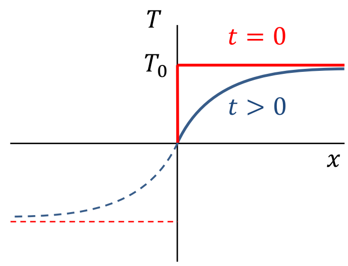
$$\tilde{v}\left(\vec{k}\right) = \int d^2 r e^{-i\vec{k}\vec{r}} \int \tfrac{d^2 q}{(2\pi)^2} \tfrac{dq_z}{2\pi}\, e^{i\vec{q}\vec{r}} \tfrac{4\pi}{|\vec{q}|^2 + q_z^2} .$$

Doing the integration over r first, we obtain a factor $\delta\left(\vec{k} - \vec{q}\right)$, which removes the $\vec{q}$-integral, leaving us with

$$\tilde{v}\left(\vec{k}\right) = \int \tfrac{dq_z}{2\pi} \tfrac{4\pi}{|\vec{k}|^2 + q_z^2} = \tfrac{2\pi}{|\vec{k}|} .$$

#20: GRADUATE GENERAL

`PROBLEM:` Let us assume that the Earth has started as a ball of molten rock at uniform initial temperature $T_0 = 2000°\text{C}$ and has since gradually cooled down by radiation from the surface, which remained at $0°\text{C}$ at all times $t > 0$. The Earth's temperature as a function of depth x is sketched by the solid lines in the Figure (the surface is at $x = 0$, and x increases toward the interior). The temperature gradient at the surface $\gamma = \partial T/\partial x|_{x=+0}$ has decreased over time, from $\gamma = \infty$ at $t = 0$ to the present-day value of $\gamma = 0.037°\text{C/m}$. Assume that the temperature



distribution is described by the heat diffusion equation

$$\frac{\partial T}{\partial t} - D\frac{\partial^2 T}{\partial x^2} = Q\,, \quad 0 < x < \infty\,, \quad D = 1.2 \times 10^{-6}\text{m}^2\text{s}^{-1}.$$

Not knowing about heat production by radioactive decay inside the Earth, Lord Kelvin (who was the first to work on this problem) set Q to zero. (This was his famous mistake.) Rederive Kelvin's estimate for the age of the Earth in years, using $1\,\text{year} = 3.15 \times 10^7\,\text{s}$.

Hints: 1) Use the method of images to reformulate the problem on the full line by formally extending function $T(x,t)$ to negative x as shown by the dashed lines in the Figure. 2) The retarded Green's function of the diffusion equation is $G(x,t) = \theta(t)\exp(-x^2/4Dt)/\sqrt{4\pi Dt}$.

`SOLUTION:` First, we extend the domain of x as suggested in the *Hint*. Second, we observe that the gradient $g(x,t) \equiv \partial T/\partial x$ satisfies the heat diffusion equation

$$\frac{\partial g}{\partial t} - D\frac{\partial^2 g}{\partial x^2} = 0$$

as well, with the initial value $g(x,0) = 2T_0\,\delta(x)$. Therefore, $g(x,t)$ is equal to $2T_0$ times the Green's function of the heat equation:

$$g(x,t) = \frac{2T_0}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right).$$

For the surface gradient $\gamma = g(0,t)$ this implies

$$\gamma(t) = \frac{T_0}{\sqrt{\pi Dt}}\,,$$

and so

$$t = \frac{1}{\pi D}\left(\frac{T_0}{\gamma}\right)^2 = 2.5 \times 10^7\,\text{years}.$$

The accepted estimate is two orders of magnitude longer, 4.5×10^9 years.