INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

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#1 :UNDERGRADUATE MECHANICS

`PROBLEM: Rockets`
A rocket with initial mass m_0 (mass of the rocket plus its fuel) initially has velocity $v_0 = 0$, in empty space. It then starts to expel fuel with (a constant) velocity v_{ex} *relative to the rocket.*

(a) Write, and solve, the equation relating the rocket's velocity v to its mass m at later times. Use non-relativistic expressions. [4 points]

(b) Note that the rocket's momentum initially increases, and then decreases. What is the rocket's maximum momentum (in terms of m_0 and v_{ex})? [3 points]

(c) You might be wondering about the relativistic version of part (a). We won't ask you to work it out here, but will just ask two preliminary questions: (i) if the rocket has velocity $v\widehat{x}$ with respect to some inertial lab frame, and the fuel is ejected with velocity $-v_{ex}\widehat{x}$ relative to the rocket, what is the relativistic expression for the velocity of the fuel relative to the lab frame? (ii) is the total mass of the rocket and the ejected fuel conserved? If yes, why? If no, why, and what is the instead-conserved quantity? [3 points]

#2 :UNDERGRADUATE MECHANICS

`PROBLEM: Lagrange, Euler, and Hamilton`

A bead of mass m slides frictionlessly on a wire, with $y = cx^2$. The force of gravity is $\vec{F}_{grav} = -mg\hat{y}$. The bead is connected with a spring, of spring constant k, to a support at $(x, y) = (0, L)$.

(a) Write the Lagrangian in terms of just one generalized coordinate, x.

(b) What is the conjugate momentum p_x? Is it conserved?

(c) Write the Euler Lagrange equations.

(d) Write the Hamiltonian of the system, in terms of x and p_x. Is it conserved?

#3 :UNDERGRADUATE E&M

`PROBLEM: Spherical Shell of Charge`
A thin uniform spherical shell of charge of radius R and surface charge density σ is centered at the origin of a Cartesian coordinate system. Then a small patch of the charge of area $\delta A << 4\pi R^2$, located where the $+\ x$ axis intersects the shell, is removed. The charge distribution on the rest of the shell remains as it was before the patch was removed.

(a) Find the approximate (i.e., to order $\delta A/R^2$) amount of total work W required to establish this complete (sphere minus patch) charge configuration, assuming all of the charge is brought in from infinity. Briefly explain the procedure you use to calculate W.

(b) If a charge Q is placed inside the shell at the origin after the patch is removed, find the magnitude and direction of the force on Q.

#4 :UNDERGRADUATE E&M

`PROBLEM: Conducting Cylinder`
A long, solid cylindrical conductor has a radius b, length L, and electrical conductivity σ_c. A uniform current flows through it when a voltage difference V_0 is applied between the ends.

(a) Use the Poynting vector to calculate the total power flow into (or out of) the cylindrical region inside radius $s = b/2$. Does the power flow into this region or out of it?

(b) Calculate the Ohmic heating in the region $s \leq b/2$, and compare this to the power flow calculated in part (a).

#5 :UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM: An electric dipole in an electric field.`
Consider a system which has an electric dipole moment $\vec{d}$; this means that in the presence of an electric field $\vec{\mathcal{E}}$ the system is governed by the Hamiltonian

$$\mathbf{H} = -\vec{d} \cdot \vec{\mathcal{E}} + \mathbf{H}_0$$

where $\mathbf{H}_0$ is independent of the electric field.

Quantum mechanically, the dipole moment is an observable. Consider a two-state system with Hilbert space $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$ (these states are orthonormal). Suppose the dipole moment operator is

$$\vec{\mathbf{d}} = d\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\check{x}$$

where d is a constant and $\check{x}$ is a unit vector in 3-space. (This is a simple model for *e.g.* an ammonia molecule.)

Suppose further that the field-independent part of the Hamiltonian is

$$\mathbf{H}_0 \equiv E_0 |1\rangle\langle 1|$$

for some $E_0 > 0$.

1. Show that with this choice of field-independent energetics the ground state (state of lowest energy) in the absence of an electric field is $|0\rangle$.
2. Compute the expectation value of the dipole moment in the ground-state, in zero electric field.
3. Suppose we put the system in a time-independent electric field $\vec{\mathcal{E}} = \mathcal{E}\check{x}$. What is the ground state energy as a function of $\mathcal{E}$?
4. Suppose we put the system in a weak electric field $\frac{d\mathcal{E}}{E_0}$, which varies in space as $\vec{\mathcal{E}} = \mathcal{E}(y)\check{x}$. Calculate the force on the system.

   (Recall that a force results from an energy which depends on position.)

#6 :UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM: Hydrogen Excitation`

A hydrogen atom is in a uniform but weak ($ea_0E_0 << \alpha^2mc^2$) electric field in the z direction which turns on abruptly at $t = 0$ and decays exponentially as a function of time, $E(t) = E_0e^{-t/\tau}$. The atom is initially in its ground state. Find the probability for the atom to have made a transition to the $2P$ state as $t \to \infty$.

Hint: Hydrogen radial wavefundtions: $R_{10} = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
and $R_{21} = \sqrt{\frac{1}{3}}\left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0}e^{-r/2a_0}$.
Spherical Harmonic: $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$.

#7 :UNDERGRADUATE STAT MECH/THERMO

`PROBLEM: Maximum Work`
Consider a thermally isolated system that consists of two bodies of equal and temperature-independent heat capacity C each. The initial temperatures of the bodies are $T_1^{(0)}$ and $T_2^{(0)}$.
(a) Show tht the Energy of the system is bounded from below. *Hint:* For any positive real X and Y, $X + Y \geq 2\sqrt{XY}$.

(b) What is the maximum work that can be extracted from this system? *Hint: The maximum work is obtained if the process is reversible, i.e., if the entropy is conserved.*

#8 :UNDERGRADUATE STAT MECH/THERMO

`PROBLEM: Blackbody Radiation`

1. Derive the Maxwell relation:
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

2. Maxwell found that based on his theory of electromagnetic fields, the pressure p of an isotropic radiation field equals $\frac{1}{3}$ of its energy density $u(T)$, i.e.,
$$p = \frac{1}{3}u(T) = \frac{1}{3}\frac{U(T)}{V},$$
in which V is the cavity volume. Use this result, the Maxwell relation in 1), and the fundamental thermodynamic relation $dU = TdS - pdV$ together to prove that
$$u = \frac{T}{4}\frac{du}{dT}.$$

3. Solve this equation and derive the Stefan's law for the black body radiation.

#9 :UNDERGRADUATE GENERAL PHYSICS

`PROBLEM: Scaling a Helicopter`

On a planet with the acceleration of gravity g and the atmospheric density $\rho_{\rm a}$, a helicopter with linear dimensions $\propto L$ and the average density $\rho_{\rm h}$ can hover when the power of its engine is P. A second helicopter is a copy of the first one with the same average density, but its linear dimensions are two times smaller. What engine power is needed to enable this second helicopter to hover? Assume that friction can be neglected.

Hint: Identify the physical quantities on which the engine power required for hovering depends and assume that the dependence is the product of these quantities, each raised to a (different) power.

#10 :UNDERGRADUATE GENERAL PHYSICS

`PROBLEM: Measuring Planck's Constant`

Describe one experiment (historical or contemporary) that would allow you to measure Planck's constant h. Please answer each of the following. (a) Give with an overview of the method in one or two sentences. (b) Describe the experimental equipment. (c) What measurements are made? (d) What physics ideas (equations, principles, assumptions) are involved? (e) List all the potential sources of error. (f) Give a quantitative estimate for the overall error.

INSTRUCTIONS
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#11 :GRADUATE MECHANICS

`PROBLEM: Pendulum`

A pendulum of length ℓ with bob mass m undergoes an up-and-down oscillation of its support with angular frequency $\omega >> \sqrt{\frac{g}{\ell}}$. The pendulum is rigid. (a) Write the Lagrangian for the pendulum in the angular coordinte ϕ.

(b) Derive the equation of motion.

(c) Consider "fast" and "slow" changes in ϕ. What characteristic of the oscillation is necessary so that the pendulum has a stable equilibrium when inverted?

#12 :GRADUATE MECHANICS

`PROBLEM: Eikonal Equation`
A sound wave propagates in a medium with speed of sound

$$c_s = c_s(x, y, z).$$

a.) Derive the eikonal equation (WKB approximate wave equation) and show the condition for its solution.

b.) How can one obtain Fermat's Principle – i.e. that rays will choose the path of least time from the eikonal equation?

c.) For $c_s = c_s(y)$, derive the equation for the ray path.

#13 :GRADUATE E&M

`PROBLEM: Polarized Waves in a Medium`

Consider circularly polarized EM waves propagating in the direction of a static magnetic field $\vec{B}_0$ in a medium consisting of N electrons per unit volume behaving as bound harmonic oscillators with a single oscillator resonance frequency ω_0 with damping constant γ.

(a) Show that the relationship between the refractive indices for waves of (+) and (-) circular polarization can be written as

$$n_+^2 - n_-^2 = \frac{Ne^2}{\epsilon_0 m}\left[\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega + eB_0\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega - eB_0\omega/m}\right]$$

(b) Neglecting γ, from the above formula, calculate the rotation in radians of the direction of polarization of a *linearly polarized* plane wave of frequency ω, after propagating a length L in the medium when the magnetic field is present.

Hint: Consider a linearly polarized wave as the coherent sum of two oppositely circularly polarized waves.

#14 :GRADUATE E&M

`PROBLEM: Radiation from Circling Charge`

A charge Q is driven around a circle of radius a centered at the origin of a coordinate system. The circle lies in the (x, y) plane with the normal to the circle in the z-direction. The charge moves at constant angular velocity Ω around the circle with $\Omega a << c$. Find the electric and magnetic fields in the "far zone" where $r >> a$. Find the Poynting vector in the "far zone" for this source. Find the power radiated from this configuration.

#15 :GRADUATE QUANTUM MECHANICS

`PROBLEM: Scattering in a Central Potential`
We consider the scattering problem of a particle with mass m in the 3D central potential

$$V(r) = \frac{\alpha}{r^2}, \tag{1}$$

where $\alpha > 0$.

1) Use the 3D partial wave method to solve for the phase shift δ_l with l a non-negative integer. l represents the partial wave channel.

Hints: $\frac{-\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] R_{nl}(r) + \left(V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right) R_{nl}(r) = E R_{nl}(r)$ *The spherical Bessel function $j_\nu(kr)$ satisfies the equation*

$$\frac{d^2}{dr^2} j_\nu + \frac{2}{r}\frac{d}{dr} j_\nu + (k^2 - \frac{\nu(\nu+1)}{r^2}) j_\nu = 0, \tag{2}$$

and its asymptotic behavior at $r \to \infty$ is

$$j_\nu(kr) \to \frac{1}{kr}\sin(kr - \frac{\nu\pi}{2}), \tag{3}$$

in which ν does not need to be an integer. You need to decide the appropriate value of ν to use.

2) Under the condition that $\frac{m\alpha}{\hbar^2} \ll \frac{1}{8}$, find the approximate formulae for δ_l up to the linear order of $\frac{m\alpha}{\hbar^2}$. Then find an simple analytic form for the scattering amplitude $f(\theta)$ up to the linear order of $\frac{m\alpha}{\hbar^2}$, and the corresponding differential cross section $\sigma(\theta)$.

Hint: You may need to use the formula

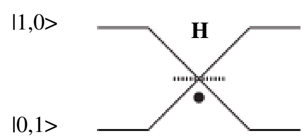
$$\sum_l P_l(\cos\theta) = \frac{1}{\sin\frac{\theta}{2}}, \tag{4}$$

and the relation of the scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_l (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\theta). \tag{5}$$

#16 :GRADUATE QUANTUM MECHANICS

`PROBLEM: Multiple photons on paths of an interferometer`
Consider a qbit ($\equiv$ two-state system) made from the two states of a single photon moving on the upper and lower paths of an interferometer. On this two-state system, a half-silvered mirror $\mathbf{H}$



acts as a beamsplitter:

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}}\left(\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^z\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

But photons are bosons. This means that if

$$\mathbf{a}^\dagger|0,0\rangle \equiv |1,0\rangle \text{ is a state with one photon on the upper path}$$

of the interferometer (and none on the lower path), then

$$\frac{\left(\mathbf{a}^\dagger\right)^n}{\sqrt{n!}}|0,0\rangle \equiv |n,0\rangle \text{ is a state with } n \text{ photons on the upper path.}$$

Similarly, define

$$\frac{\left(\mathbf{b}^\dagger\right)^n}{\sqrt{n!}}|0,0\rangle \equiv |0,n\rangle \text{ to be a state with } n \text{ photons on the lower path}$$

of the interferometer. (Note that $[\mathbf{a}, \mathbf{b}] = 0 = [\mathbf{a}, \mathbf{b}^\dagger]$.)

Questions:

1. How does $\mathbf{H}$ act on $|0,0\rangle$?
2. How does $\mathbf{H}$ act on $|2,0\rangle$ and $|0,2\rangle$?
3. How does $\mathbf{H}$ act on the operators $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$?

4. What is the state which results upon sending a coherent state of photons

$$|\alpha, \beta) \equiv \mathcal{N}_\alpha \mathcal{N}_\beta \; e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger} |0, 0\rangle$$

through a half-silvered mirror? ($\mathcal{N}_\alpha \equiv e^{-|\alpha|^2/2}$ is a normalization constant.)

#17 :GRADUATE STAT MECH/THERMO

`PROBLEM: Container of Classical Gas`

A classical gas of non-interacting atoms is in thermal equilibrium at temperature T in a container of volume V and surface area A. The potential energy of the atoms in the bulk is zero. Atoms adsorbed on the surface have a potential energy $V = -E_a$ and behave as an ideal twodimensional gas.

Find an analytic expression for the surface density $\sigma(n, T) \equiv N_{surface}/A$ in terms of the bulk density $n \equiv N_{bulk}/V$ and the temperature. Be sure to "correct Boltzmann counting".

The following mathematical results may be useful.

$$\ln(N!) \approx N \ln N - N$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 1$$

#18 :GRADUATE STAT MECH/THERMO

`PROBLEM: Ising Model`

A spin-1 Ising model in one dimension is described by the Hamiltonian

$$H_N\{\sigma_i\} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} \qquad (\sigma_i = -1, 0, 1)$$

Write down the transfer matrix (P) (where the partition function $Q_N = TrP^N$) for this interaction and show that the free energy $A_N(T)$ of this model, in the thermodynamic limit, is equal to

$$-NkT\ln\left(\frac{1}{2}\left[(1+2\cosh K) + (8+(2\cosh K-1)^2)^{1/2}\right]\right) \qquad (K \equiv J/KT)$$

Examine the limiting behavior of this quantity as $T \to 0$ and as $T \to \infty$, and discuss the physical interpretation of each limit.

#19 :GRADUATE MATHEMATICAL PHYSICS

`PROBLEM: Fourier Transform`
Calculate the Fourier transform of a hyperbolic tangent, i.e.,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} \tanh x \; dx \, .$$

(If you worry about the formal convergence of the integral, it can be assured by adding a factor like $e^{-\alpha x^2}$ to the integrand with the understanding that we are interested in the $\alpha \to 0$ limit.)

#20 :GRADUATE GENERAL PHYSICS

`PROBLEM: Convection`

When a liquid fills the gap between two parallel horizontal plates and has a positive thermal expansion coefficient $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)$, and the temperature of the bottom plate, T_b, is higher than the temperature of the top plate, T_t , convective motion of the liquid may occur. The onset of the convection corresponds to a certain value of a dimensionless expression called the Rayleigh number (Ra). The expression for Ra involves the distance between the plates, h, the temperature difference, $\Delta T = T_t - T_b$, the thermal expansion coefficient, β, the viscosity and density of the liquid, η and ρ, the gravitational acceleration, g, and the thermal diffusivity, α (measured in m^2/s). Find the expression for Ra from the dimensional analysis of motion of a small volume of the liquid that is moving up.

Hint: As it moves up, a small volume of the liquid enters regions with lower ambient temperature, corresponding to greater local fluid density, that leads to a positive buoyancy force on the liquid volume, F_b. Consider whether T of the rising liquid volume is increasing with time (amplification; convection is maintained) or decreasing with time (convection is suppressed). The diameter of the liquid volume can be assumed to be on the order h and its upward velocity, v, can be assumed to be proportional to $F_b/(h\eta)$. The cooling of the volume is described by the thermal diffusivity equation $\frac{\partial T}{\partial t} = \alpha\nabla^2 T$, where T is the difference between the temperature of the volume and of the liquid around it.

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A rocket with initial mass m_0 (mass of the rocket plus its fuel) initially has velocity $v_0 = 0$, in empty space. It then starts to expel fuel with (a constant) velocity v_{ex} *relative to the rocket.*

(a) Write, and solve, the equation relating the rocket's velocity v to its mass m at later times. Use non-relativistic expressions. [4 points]

(b) Note that the rocket's momentum initially increases, and then decreases. What is the rocket's maximum momentum (in terms of m_0 and v_{ex})? [3 points]

(c) You might be wondering about the relativistic version of part (a). We won't ask you to work it out here, but will just ask two preliminary questions: (i) if the rocket has velocity $v\widehat{x}$ with respect to some inertial lab frame, and the fuel is ejected with velocity $-v_{ex}\widehat{x}$ relative to the rocket, what is the relativistic expression for the velocity of the fuel relative to the lab frame? (ii) is the total mass of the rocket and the ejected fuel conserved? If yes, why? If no, why, and what is the instead-conserved quantity? [3 points]

`SOLUTION:`

(a) The rocket has $dp = (m + dm)(v + dv) + (v - v_{ex})(-dm) - mv = mdv + v_{ex}dm = 0$, so $v = \int_0^v dv = -v_{ex} \int_{m_0}^m \frac{dm}{m} = v_{ex} \ln(m_0/m)$. (The sign of dm is taken to be negative, since $m < m_0$.)

(b) The momentum is $p = mv = mv_{ex} \ln(m_0/m)$. This is maximum when $\frac{dp}{dm} = 0$, so $\ln(m_0/m) - 1 = 0$, i.e. $m = m_0/e$, and there $p_{max} = m_0 v_{ex}/e$.

(c) (i) relativistic addition of velocity: $u_{ex} = (v - v_{ex})/(1 - vv_{ex}/c^2)$; (ii) mass is not conserved in relativity, total energy $E = \sum \gamma mc^2$ is.

#2 :UNDERGRADUATE MECHANICS

`PROBLEM: Lagrange, Euler, and Hamilton`
A bead of mass m slides frictionlessly on a wire, with $y = cx^2$. The force of gravity is $\vec{F}_{grav} = -mg\widehat{y}$. The bead is connected with a spring, of spring constant k, to a support at $(x, y) = (0, L)$.

(a) Write the Lagrangian in terms of just one generalized coordinate, x.

(b) What is the conjugate momentum p_x? Is it conserved?

(c) Write the Euler Lagrange equations.

(d) Write the Hamiltonian of the system, in terms of x and p_x. Is it conserved?

`SOLUTION:`
$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, with $\dot{y} = 2cx\dot{x}$, so $T = \frac{1}{2}m(1 + 4cx^2)\dot{x}^2$. The potential energy is $U = U_{grav} + U_{spring}$, with $U_{grav} = mgy$ and $U_{spring} = \frac{1}{2}k(x^2 + (y - L)^2)$. So

$$\mathcal{L} = \tfrac{1}{2}m(1 + 4c^2x^2)\dot{x}^2 - mgcx^2 - \tfrac{1}{2}k(x^2 + (cx^2 - L)^2).$$

The momentum is

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m(1 + 4c^2x^2)\dot{x}.$$

It is not conserved (no translation symmetry).

Indeed, the EL equation is

$$\frac{d}{dt}p_x = \frac{\partial \mathcal{L}}{\partial x}$$

and the RHS is non-zero. It gives

$$m(1 + 4c^2x^2)\frac{d^2x}{dt^2} + 8mc^2x\dot{x}^2 = -2mgcx - k(x + (cx^2 - L)(2cx)).$$

The Hamiltonian is

$$H = p_x\dot{x} - \mathcal{L} = \tfrac{1}{2}m(1 + 4c^2x^2)\dot{x}^2 + mgcx^2 + \tfrac{1}{2}k(x^2 + (cx^2 - L)^2)$$

which should be expressed with $\dot{x}$ eliminated in favor of p_x, so

$$H = \frac{p_x^2}{2m(1 + 4c^2x^2)} + mgcx^2 + \tfrac{1}{2}k(x^2 + (cx^2 - L)^2)$$

This is guaranteed to be conserved, since there is no explicit t dependence in the Lagrangian or Hamiltonian.

#3 :UNDERGRADUATE E&M

`PROBLEM: Spherical Shell of Charge`

A thin uniform spherical shell of charge of radius R and surface charge density σ is centered at the origin of a Cartesian coordinate system. Then a small patch of the charge of area $\delta A << 4\pi R^2$, located where the $+\ x$ axis intersects the shell, is removed. The charge distribution on the rest of the shell remains as it was before the patch was removed.

(a) Find the approximate (i.e., to order $\delta A/R^2$) amount of total work W required to establish this complete (sphere minus patch) charge configuration, assuming all of the charge is brought in from infinity. Briefly explain the procedure you use to calculate W.

(b) If a charge Q is placed inside the shell at the origin after the patch is removed, find the magnitude and direction of the force on Q.

`SOLUTION:`

(a). The potential outside a uniformly charged spherical shell with total charge q is

$$V(r) = \frac{q}{4\pi\varepsilon_0 r},$$

for $r \geq R$. Thus, bringing in charge a bit at a time and spreading it evenly on the shell, the work to establish the shell will be

$$W = \int_0^{q_0} V(r)dq = \int_0^{q_0} \frac{q\ dq}{4\pi\varepsilon_0 R} = \frac{q_0^2}{8\pi\varepsilon_0 R},$$

where $q_0 = 4\pi R^2\sigma$. Now to establish the hole, we bring in an increment of charge $\delta q = -\sigma\delta A$. This will require an increment of work δW of magnitude

$$\delta W \approx V\delta q = -\frac{\sigma\delta A q_0}{4\pi\varepsilon_0 R}.$$

Thus the total work to establish the charge configuration will be

$$\begin{aligned} W_{tot} = &\ \frac{q_0}{4\pi\varepsilon_0 R}\left(\frac{q_0}{2} - \sigma\delta A\right) \\ = &\ \frac{2\pi\sigma^2 R^3}{\varepsilon_0}\left(1 - \frac{\delta A}{2\pi R^2}\right). \end{aligned}$$

(b) For the uniformly charged shell, $\vec{F_Q} = 0$. In the presence of the missing patch, it will act as a negative charge, $\delta q = -\sigma\delta A$, that will attract Q. Thus

$$\vec{F_Q} = +\ \frac{Q\delta A\sigma}{4\pi\varepsilon_0 R^2}\ \hat{x}$$

#4 :UNDERGRADUATE E&M

`PROBLEM: Conducting Cylinder`

A long, solid cylindrical conductor has a radius b, length L, and electrical conductivity σ_c. A uniform current flows through it when a voltage difference V_0 is applied between the ends.

(a) Use the Poynting vector to calculate the total power flow into (or out of) the cylindrical region inside radius $s = b/2$. Does the power flow into this region or out of it?

(b) Calculate the Ohmic heating in the region $s \leq b/2$, and compare this to the power flow calculated in part (a).

`SOLUTION:`

(a). The electric field will be

$$\vec{E} = \tfrac{V_0}{L}\hat{z}$$

For the magnetic field,

$$2\pi s B(s) = \mu_0 I_{enc} = \mu_0 \pi s^2 J = \mu_0 \pi s^2 \sigma_c E,$$

where s is the radial coordinate, and J is the current density. Thus

$$\vec{B}(s) = \tfrac{\mu_0 s \sigma_c E}{2}\hat{\phi}.$$

Thus

$$\vec{S} = \tfrac{1}{\mu_0}(\vec{E} \times \vec{B}) = -\tfrac{\sigma_c {V_0}^2 s}{2L^2}\hat{s},$$

and

$$P = 2\pi s L S = \tfrac{\pi \sigma_c {V_0}^2 s^2}{L}.$$

and so at $s = b/2$, the power flow will be

$$\vec{P} = -\tfrac{\pi b^2 \sigma_c {V_0}^2}{4L}\hat{s}$$

(i.e., flowing into the volume).

(b) For comparison, the Ohmic power will be

$$P = IV = (\tfrac{\pi b^2}{4}) J V_0 = (\tfrac{\pi b^2}{4}) \sigma_c E V_0 = \tfrac{\pi b^2 \sigma_c {V_0}^2}{4L},$$

and so they are the same.

#5 :UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM: An electric dipole in an electric field.`
Consider a system which has an electric dipole moment $\vec{d}$; this means that in the presence of an electric field $\vec{\mathcal{E}}$ the system is governed by the Hamiltonian

$$\mathbf{H} = -\vec{d} \cdot \vec{\mathcal{E}} + \mathbf{H}_0$$

where $\mathbf{H}_0$ is independent of the electric field.

Quantum mechanically, the dipole moment is an observable. Consider a two-state system with Hilbert space $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$ (these states are orthonormal). Suppose the dipole moment operator is

$$\vec{\mathbf{d}} = d\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\check{x}$$

where d is a constant and $\check{x}$ is a unit vector in 3-space. (This is a simple model for *e.g.* an ammonia molecule.)

Suppose further that the field-independent part of the Hamiltonian is

$$\mathbf{H}_0 \equiv E_0|1\rangle\langle 1|$$

for some $E_0 > 0$.

1. Show that with this choice of field-independent energetics the ground state (state of lowest energy) in the absence of an electric field is $|0\rangle$.
2. Compute the expectation value of the dipole moment in the ground-state, in zero electric field.
3. Suppose we put the system in a time-independent electric field $\vec{\mathcal{E}} = \mathcal{E}\check{x}$. What is the ground state energy as a function of $\mathcal{E}$?
4. Suppose we put the system in a weak electric field $\frac{d\mathcal{E}}{E_0}$, which varies in space as $\vec{\mathcal{E}} = \mathcal{E}(y)\check{x}$. Calculate the force on the system.

   (Recall that a force results from an energy which depends on position.)

`SOLUTION:`

1. The energy eigenstates are $|0\rangle$ and $|1\rangle$:

$$\mathbf{H}_0|0\rangle = 0, \quad \mathbf{H}_0|1\rangle = E_0|1\rangle$$

with energies $0, E_0 > 0$ respectively.

2.

$$\langle\vec{\mathbf{d}}\rangle_{|0\rangle} = d\check{x}\langle 0|\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle = d\check{x}\left(\langle 0|0\rangle\langle 1|0\rangle + \langle 0|1\rangle\langle 0|0\rangle\right) = 0.$$

3. The Hamiltonian in a field is a 2x2 matrix. Any such matrix can be expanded in the familiar Pauli matrices (which I will write in the basis given above, so $|0\rangle \equiv |\uparrow_z\rangle, |1\rangle \equiv |\downarrow_z\rangle$):

$$\mathbf{H} = d\mathcal{E}\boldsymbol{\sigma}^x + \tfrac{1}{2}E_0\left(\mathbb{1} - \boldsymbol{\sigma}^z\right) \ .$$

We know the eigenvalues of such an operator because it is of the form $n_0\mathbb{1} + \vec{n}\cdot\boldsymbol{\sigma}$ (with $n_0 = \frac{1}{2}E_0, \vec{n} = \check{x}d\mathcal{E} + \check{z}(-\frac{1}{2}E_0)$) whose eigenvalues are $n_0 \pm \sqrt{\vec{n}\cdot\vec{n}}$. So the energies are

$$E_\pm = +\tfrac{1}{2}E_0 \pm \sqrt{d^2\mathcal{E}^2 + \frac{E_0^2}{4}} \ ,$$

$E_- < E_+$ is the ground state energy.

4. The ground state is

$$|\chi_-\rangle = |\uparrow_{\check{n}}\rangle = \cos\frac{\theta}{2}e^{\frac{\mathrm{i}\varphi}{2}}|0\rangle + \sin\frac{\theta}{2}e^{\frac{-\mathrm{i}\varphi}{2}}|1\rangle$$

where $\vec{n} = \check{x}d\mathcal{E} + \check{z}(-\frac{1}{2}E_0)$ means that

$$\varphi = \pi, \quad \tan\theta = \frac{d\mathcal{E}}{\frac{1}{2}E_0} \ .$$

In the stated limit, $\theta \ll 1$ means

$$\theta \sim \tan\theta = 2\frac{d\mathcal{E}}{E_0}$$

so the groundstate is

$$|\chi_-\rangle \simeq |0\rangle - \frac{\theta}{2}|1\rangle \simeq |0\rangle - \frac{d\mathcal{E}}{E_0}|1\rangle.$$

In the weak-field limit, the groundstate energy is

$$E_- \simeq 0 - \tfrac{1}{2}\frac{d^2\mathcal{E}^2}{E_0}$$

and the force is

$$\vec{F} = -\vec{\nabla}E_- = -\frac{d^2}{2E_0}\vec{\nabla}\mathcal{E}^2$$

which points in the $\pm\hat{y}$ direction in the case indicated.

#6 :UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM: Hydrogen Excitation`
A hydrogen atom is in a uniform but weak ($ea_0E_0 << \alpha^2mc^2$) electric field in the z direction which turns on abruptly at $t = 0$ and decays exponentially as a function of time, $E(t) = E_0e^{-t/\tau}$. The atom is initially in its ground state. Find the probability for the atom to have made a transition to the $2P$ state as $t \to \infty$.
Hint: Hydrogen radial wavefundtions: $R_{10} = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$
and $R_{21} = \sqrt{\frac{1}{3}}\left(\frac{1}{2a_0}\right)^{3/2}\frac{r}{a_0}e^{-r/2a_0}$.
Spherical Harmonic: $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$.

`SOLUTION:`
This is just a time dependent pertubation problem. It is not a harmonic perturbing potential. We use the most basic time dependent perturbation forumla.

$$c_n(t) = \frac{1}{i\hbar}\int_0^t dt'\; e^{i(E_n-E_i)t'/\hbar}\langle\phi_n|V(t')|\phi_i\rangle$$

The perturbing potential is $V(t) = eE_0e^{-t/\tau}z = eE_0e^{-t/\tau}r\cos\theta = eE_0e^{-t/\tau}r\sqrt{\frac{4\pi}{3}}Y_{10}$. The initial state is the 1S state and the final state is the 2P state.

$$\begin{aligned}
c_{2p}(\infty) &= \frac{eE_0}{i\hbar}\sqrt{\frac{4\pi}{3}}\int_0^\infty dt'\; e^{i(E_n-E_i)t'/\hbar}e^{-t'/\tau}\langle\psi_{21m}|rY_{10}|\psi_{100}\rangle \\
c_{2p} &= \frac{eE_0}{i\hbar}\sqrt{\frac{4\pi}{3}}\sqrt{\frac{1}{4\pi}}\int_0^\infty dt'\; e^{[i\omega_{21}-1/\tau]t'}\int r^2drd\Omega R^*_{21}Y^*_{1m}rY_{10}R_{10} \\
c_{2p} &= \frac{eE_0}{i\hbar}\sqrt{\frac{1}{3}}\frac{1}{[i\omega_{21}-1/\tau]}\left[e^{[i\omega_{21}-1/\tau]t'}\right]_0^\infty\sum_m\int drR^*_{21}r^3R_{10}\delta_{m0} \\
c_{2p} &= \frac{eE_0}{i\hbar}\sqrt{\frac{1}{3}}\frac{[0-1]}{[i\omega_{21}-1/\tau]}\int dr\sqrt{\frac{1}{3}}\left(\frac{1}{2a_0}\right)^{3/2}e^{-r/2a_0}r^3 2\left(\frac{1}{a_0}\right)^{3/2}\frac{r}{a_0}e^{-r/a_0} \\
c_{2p} &= \frac{eE_0}{3\sqrt{2}i\hbar}\frac{-1}{[i\omega_{21}-1/\tau]}\frac{1}{a_0^4}\int drr^4e^{-3r/2a_0} \\
c_{2p} &= \frac{eE_0}{3\sqrt{2}i\hbar}\frac{-1}{[i\omega_{21}-1/\tau]}\frac{1}{a_0^4}\frac{4!}{\left(\frac{3}{2a_0}\right)^5} \\
c_{2p} &= \frac{ieE_0}{3\sqrt{2}}\frac{1}{[i\hbar\omega_{21}-\hbar/\tau]}a_0\frac{24\cdot 2^5}{3^5} \\
P_{2p} &= \frac{2^{15}e^2E_0^2a_0^2}{3^{10}[(\frac{3}{8}\alpha^2mc^2)^2+\frac{\hbar^2}{\tau^2}]}
\end{aligned}$$

#7 :UNDERGRADUATE STAT MECH/THERMO

`PROBLEM: Maximum Work`

Consider a thermally isolated system that consists of two bodies of equal and temperature-independent heat capacity C each. The initial temperatures of the bodies are $T_1^{(0)}$ and $T_2^{(0)}$.

(a) Show tht the Energy of the system is bounded from below. *Hint:* For any positive real X and Y, $X + Y \geq 2\sqrt{XY}$.

(b) What is the maximum work that can be extracted from this system? *Hint: The maximum work is obtained if the process is reversible, i.e., if the entropy is conserved.*

`SOLUTION:`

Up to an irrelevant constant, the energy of the first body is equal to

$$E_1 = CT_1 \,.$$

The entropy of this body is

$$S_1 = \int \tfrac{dQ_1}{T_1} = \int C \tfrac{dT_1}{T_1} = C \ln T_1 \,,$$

where the constant term is again neglected. If the total entropy of our two-body system is to remain constant, we must have

$$S = S_1 + S_2 = C \ln T_1 + C \ln T_2 = C \ln T_1 T_2 = \text{const} \,,$$

which implies

$$T_1 T_2 = T_1^{(0)} T_2^{(0)} \,.$$

The energy of the system is bounded from below by

$$E = C(T_1 + T_2) \geq 2C\sqrt{T_1 T_2} = 2C\sqrt{T_1^{(0)} T_2^{(0)}} \,.$$

The bound is attained when the temperatures equilibrate, in agreement with the physical intuition. The work W extracted from the system is given by the decrease of its energy:

$$W = E^{(0)} - E \leq C(T_1^{(0)} + T_2^{(0)}) - 2C\sqrt{T_1^{(0)} T_2^{(0)}} = C\left(\sqrt{T_1^{(0)}} - \sqrt{T_2^{(0)}}\right)^2 \,.$$

#8 :UNDERGRADUATE STAT MECH/THERMO

`PROBLEM: Blackbody Radiation`

1. Derive the Maxwell relation:
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$
2. Maxwell found that based on his theory of electromagnetic fields, the pressure p of an isotropic radiation field equals $\frac{1}{3}$ of its energy density $u(T)$, i.e.,
$$p = \frac{1}{3}u(T) = \frac{1}{3}\frac{U(T)}{V},$$
in which V is the cavity volume. Use this result, the Maxwell relation in 1), and the fundamental thermodynamic relation $dU = TdS - pdV$ together to prove that
$$u = \frac{T}{4}\frac{du}{dT}.$$
3. Solve this equation and derive the Stefan's law for the black body radiation.

`SOLUTION:`

1)

$$\begin{aligned} dF &= -SdT - pdV \\ \left(\frac{\partial F}{\partial T}\right)_V &= -S, \\ \left(\frac{\partial F}{\partial V}\right)_T &= -p \end{aligned}$$

Using $\frac{\partial^2 F}{\partial V \partial T} = \frac{\partial^2 F}{\partial T \partial V}$, we have

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

2)

$$\begin{aligned} U(T,V) &= u(T)V \\ \left(\frac{\partial U}{\partial V}\right)_T &= T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p \\ u &= \frac{T}{3}\frac{du}{dT} - \frac{1}{3}u \\ T\frac{du}{dT} &= 4u \end{aligned}$$

3) We solve $u = aT^4$ in which a is the integral constant, which is the Stefan's law.

#9 :UNDERGRADUATE GENERAL PHYSICS

`PROBLEM: Scaling a Helicopter`
On a planet with the acceleration of gravity g and the atmospheric density ρ_{a}, a helicopter with linear dimensions $\propto L$ and the average density ρ_{h} can hover when the power of its engine is P. A second helicopter is a copy of the first one with the same average density, but its linear dimensions are two times smaller. What engine power is needed to enable this second helicopter to hover? Assume that friction can be neglected.

Hint: Identify the physical quantities on which the engine power required for hovering depends and assume that the dependence is the product of these quantities, each raised to a (different) power.

`SOLUTION:`

The required power for the hovering helicopter depends on the acceleration of gravity g, the linear size of the helicopter L, the average density of the helicopter ρ_{h}, and the atmospheric density ρ_{a}. The required power depends on these quantities:

$$P \propto g^{\alpha} \times L^{\beta} \times \rho_{\mathrm{h}}^{\gamma} \times \rho_{\mathrm{a}}^{\delta}.$$

The dimensions on the left- and right-hand sides must be equal:

$$\frac{\mathrm{kgm}^2}{\mathrm{s}^3} = \left(\frac{\mathrm{m}}{\mathrm{s}^2}\right)^{\alpha} \times m^{\beta} \times \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{\gamma} \times \left(\frac{\mathrm{kg}}{\mathrm{m}^3}\right)^{\delta},$$

which yields

$$\begin{aligned} \gamma + \delta &= 1 \\ \alpha + \beta - 3(\gamma + \delta) &= 2 \\ -2\alpha &= -3. \end{aligned}$$

The solution of this system of linear equations is $\beta = \frac{7}{2}$, $\alpha = \frac{3}{2}$, $\gamma + \delta = 1$. Therefore, $P \propto L^{\frac{7}{2}}$ and the second helicopter should have an engine producing power $0.5^{\frac{7}{2}} P \approx 0.09P$.

#10 :UNDERGRADUATE GENERAL PHYSICS

`PROBLEM: Measuring Planck's Constant`
Describe one experiment (historical or contemporary) that would allow you to measure Planck's constant h. Please answer each of the following. (a) Give with an overview of the method in one or two sentences. (b) Describe the experimental equipment. (c) What measurements are made? (d) What physics ideas (equations, principles, assumptions) are involved? (e) List all the potential sources of error. (f) Give a quantitative estimate for the overall error.

`SOLUTION:`

Details of the answer will depend on the method chosen. Some methods are as follows.

1) Planck (1901) used his new idea of quantizing energy together with measurement of the black body spectrum to find a value of h within 2% of the modern value. Today, a diffraction grating can be used to disperse light from a incandescent light (black body) of known temperature, and a photodiode is used to measure the intensity I at one wavelength as a function of the temperature. A value of h can be obtained comparing the I at two or more temperatures. The temperature of a filament varies approximately linearly with its resistance for T near 2500 K. This apparatus has order of magnitude errors from the uncertainty in the temperature and the response of the photodiode. Using color filters instead of the diffraction grating, and a pyrometer to measure temperature, the error on h drops to 2%.

2) The dependence of the kinetic energy (KE) of photoelectrons on the frequency of light causing the photoemission was used by Millikan in 1916 to measure h. We need a vacuum tube containing a metal surface a wire grid and an electrometer (detector). We must clean the metal surface while it is in vacuum. We use light source(s) with various near ultraviolet wavelengths (eg Mercury emission spectrum). We can measure the KE of the photoelectrons by determining the negative voltage on the grid that stops them moving to a detector behind the grid. We plot how the current drops with the retarding potential and extrapolate to find the maximum kinetic energy. $KE = h\nu - E_o$, where ν is the frequency of the light and E_o is the energy required to escape from the metal, the work function. A plot of the KE values against ν has a slope h. We should repeat after re-cleaning the surface

since contamination is an error source, and with different alkali metals.

3) We use X-ray crystallography to measure the atomic spacing in a crystal of pure Silicon. When compared to the volume of 1 mole of the substance we find Avogadro's number, N_A and this is related to h, since the Hydrogen spectrum gives accurate values for the molar Planck constant $N_A h$. Silicon is suitable since very high purity samples are made for use in semiconductors. The volume of 1 mole can be obtained from the density and the atomic weight of the Silicon used.

4) We use a Watt balance to equate the force of gravity and the force of a current carrying coil inside a radial magnetic field B. The mechanical power and electrical power are both measured in Watts. We need a knife edge balance, a coil of wire, a magnet and a laser to measure the tilt of the balance and a way to measure the velocity of the coil attached to the balance arm. Apparatus made from Lego and costing $< \$500$ can give h to 1% (http://arxiv.org/abs/1412.1699). The apparatus is used first in velocity mode, where Faraday's Law gives the induced voltage $V = BLv$, where L is the length of the wire in the coil and v the velocity of the coil through the magnetic field. The apparatus is then used in force mode, where the gravitational force on mass m is balanced by an upwards electromagnetic force F in the coil that now carries a current I, where $F = BLI = mg$. Mechanical and electrical power are then related by $P = VI = mgv$. The Josephson effect and the quantum Hall effect allow us to express the electrical power in terms of h, $P = Ch$, where the C is a known constant.

5) There are other ways to use the Josephson effect to measure h. $K_J = \nu/U = 2e/h$, where U is the potential difference generated by the Josephson effect at a large number of junctions wired in series, with microwave radiation of frequency ν. We can measure K_J from a measurement of the U generated by junctions.

The best modern methods, the Watt balance and X-ray crystallography, give h with an uncertainty of a few 10^{-8} but there are unexplained differences between these two methods. The definition of mass is changing in 2018 so that the kg will depend on a fixed defined value for h and not on the lump of metal stored in Paris.

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:

   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 :GRADUATE MECHANICS

`PROBLEM: Pendulum`

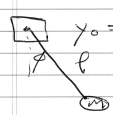
A pendulum of length ℓ with bob mass m undergoes an up-and-down oscillation of its support with angular frequency $\omega >> \sqrt{\frac{g}{\ell}}$. The pendulum is rigid. (a) Write the Lagrangian for the pendulum in the angular coordinte ϕ.

(b) Derive the equation of motion.

(c) Consider "fast" and "slow" changes in ϕ. What characteristic of the oscillation is necessary so that the pendulum has a stable equilibrium when inverted?

`SOLUTION:`

→ (Classic) Example – Inverted Pendulum
(recall movie)



$y_0 = y_0 \cos\omega t$ $\omega >> g/\ell$

$y = y_0 \cos\omega t + \ell\cos\phi$
$x = \ell\sin\phi$

Consider "the usual" pendulum with vertically oscillating point of support.

11.

$$L = T - U$$

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - mg\ell(1-\cos\phi)$$

$$L = \frac{1}{2} m \left(\ell^2\cos^2\phi\,\dot{\phi}^2 + (\omega y_0 \sin\omega t + \ell\sin\phi\,\dot{\phi})^2\right) - mg\ell(1-\cos\phi)$$

$$= \frac{1}{2} m \left((\ell^2\cos^2\phi + \ell^2\sin^2\phi)\dot{\phi}^2 + 2\omega y_0 \ell\dot{\phi}\sin\omega t\sin\phi + \omega^2 y_0^2 \sin^2\omega t \right)$$

h.o. on y_0
⇒ weak
(ℓ small)

$\therefore$

$$L = \frac{1}{2} m \left(\ell^2\dot{\phi}^2 + 2\omega y_0 \ell\dot{\phi}\sin\phi\sin\omega t\right) + mg\ell\cos\phi$$

oscillating term.

12.

1. EOM

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi}$$

$\Rightarrow$

$$\frac{d}{dt}\left(m\ell^2\dot\phi + m\omega y_0 \ell \sin\omega t \sin\phi\right)$$

$$= -mg\ell\sin\phi + m\omega y_0 \ell\dot\phi \sin\omega t\cos\phi$$

$\Rightarrow$

$$m\ell^2\ddot\phi + m\omega y_0\ell\left(\omega\cos\omega t\sin\phi + \sin\omega t\cos\phi\dot\phi\right)$$

$$= -mg\ell\sin\phi + m\omega y_0\ell\dot\phi\sin\omega t\cos\phi$$

$$\boxed{m\ell^2\ddot\phi = -mg\ell\sin\phi - m\omega^2 y_0\ell\sin\phi\cos\omega t}$$

$$\boxed{\ddot\phi = -\frac{g}{\ell}\sin\phi - \omega^2\frac{y_0}{\ell}\sin\phi\cos\omega t}$$

13.

in form:

$$\ddot\phi = -\frac{dU(\phi)}{d\phi} + f_1(\phi)\cos\omega t$$

$\Leftrightarrow$

$$\ddot\phi = -\frac{g}{\ell}\sin\phi - \omega^2\frac{y_0}{\ell}\sin\phi\cos\omega t$$

$\Rightarrow$

$$U(\phi) = \frac{g}{\ell}\cos\phi \qquad \omega \gg \sqrt{g/\ell}$$

$$f_1(\phi) = -\frac{\omega^2 y_0}{\ell}\sin\phi$$

To solve:

$$\phi = \langle\phi\rangle + \tilde\phi$$

$$= \bar\phi + \tilde\phi$$

notational brevity

$$\ddot{\bar\phi} + \ddot{\tilde\phi} = -\frac{g}{\ell}\left(\sin[\bar\phi + \tilde\phi]\right) - \frac{\omega^2 y_0}{\ell}\sin(\bar\phi + \tilde\phi)\cos\omega t$$

14.

expanding $\Rightarrow$

$$\ddot{\bar\phi} + \ddot{\tilde\phi} = \overset{①}{-\frac{g}{\ell}\sin\bar\phi} \overset{②}{- \tilde\phi\frac{g}{\ell}\cos\bar\phi}$$

$$\overset{③}{-\omega^2\frac{y_0}{\ell}\sin\bar\phi\cos\omega t} \overset{④}{-\omega^2\frac{y_0}{\ell}\cos\bar\phi\,\tilde\phi\cos\omega t}$$

then average:

$$\langle②\rangle = \langle③\rangle = 0$$

$$\ddot{\bar\phi} = -\frac{g}{\ell}\sin\bar\phi - \frac{\omega^2 y_0}{\ell}\cos\bar\phi\langle\tilde\phi\cos\omega t\rangle$$

To compute $\langle\ \rangle$, need $\tilde\phi$

$$\ddot{\tilde\phi} = \cancel{-\tilde\phi\frac{g}{\ell}\cos\bar\phi} - \omega^2\frac{y_0}{\ell}\sin\bar\phi\cos\omega t$$

$$g/\ell \ll \omega^2$$

$$\tilde\phi = a(\bar\phi)\cos\omega t$$

15.

$$-\omega^2 a(\bar\phi)\cos\omega t = -\omega^2\frac{y_0}{\ell}\sin\bar\phi\cos\omega t$$

$$a(\bar\phi) = \frac{y_0}{\ell}\sin\bar\phi$$

$$\langle\tilde\phi\cos\omega t\rangle = \left\langle\frac{y_0}{\ell}\sin\bar\phi\cos^2\omega t\right\rangle$$

$$= \frac{y_0}{2\ell}\sin\bar\phi$$

$\Rightarrow$

$$\langle④\rangle = -\frac{\omega^2 y_0^2}{2\ell^2}\sin\bar\phi\cos\bar\phi$$

$$= -\frac{\omega^2 y_0^2}{4\ell^2}\sin 2\bar\phi$$

$\Rightarrow$

$$\boxed{\ddot{\bar\phi} = -\frac{g}{\ell}\sin\bar\phi - \frac{\omega^2 y_0^2}{4\ell^2}\sin 2\bar\phi}$$

16.

$$-\omega^2 a(\bar{\phi}) \cos\omega t = -\omega^2 \frac{y_0}{\ell} \sin\bar{\phi} \cos\omega t$$

$$a(\bar{\phi}) = \frac{y_0}{\ell} \sin\bar{\phi}$$

$$\langle \tilde{\phi} \cos\omega t \rangle = \frac{y_0}{\ell} \langle \sin\bar{\phi} \cos^2\omega t \rangle$$

$$= \frac{y_0}{2\ell} \sin\bar{\phi}$$

so

$$\langle \Phi \rangle = -\frac{\omega^2 y_0^2}{2\ell^2} \sin\bar{\phi} \cos\bar{\phi}$$

$$= -\frac{\omega^2 y_0^2}{4\ell^2} \frac{d}{d\bar{\phi}} \sin^2\bar{\phi}$$

so

$$\ddot{\bar{\phi}} = -\frac{d}{d\bar{\phi}} \left(-\frac{g}{\ell} \cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2} \sin^2\bar{\phi} \right)$$

17.

$$\boxed{\begin{aligned} U_{eff} &= U_0 + U_{pond} \\ U_0 &= -\frac{g}{\ell} \cos\bar{\phi} \\ U_{pond} &= \frac{\omega^2 y_0^2}{4\ell^2} \sin^2\bar{\phi} \end{aligned}}$$

effective potential.

Now, with inverted pendulum in mind, look for minima (1D)!

$$U(\bar{\phi}) = -\frac{g}{\ell} \cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2} \sin^2\bar{\phi}$$

$$= -\frac{g}{\ell} \cos\bar{\phi} + \frac{\omega^2 y_0^2}{4\ell^2} \left(\frac{1}{2} - \frac{1}{2} \cos 2\bar{\phi} \right)$$

extrema:

$$\frac{dU}{d\bar{\phi}} = \frac{g}{\ell} \sin\bar{\phi} + \frac{\omega^2 y_0^2 \sin 2\bar{\phi}}{4\ell^2} = 0$$

$\bar{\phi} = 0$ is minimum

$\bar{\phi} = \pi$ is extremum.

18.

Now,

$$\frac{d^2 U}{d\bar{\phi}^2} = \frac{g}{\ell} \cos\bar{\phi} + \frac{\omega^2 y_0^2}{2\ell^2} \cos 2\bar{\phi}$$

for $\bar{\phi} = \pi$

$$= -\frac{g}{\ell} + \frac{\omega^2 y_0^2}{2\ell^2}$$

$\Rightarrow$ extremum can be minimum for $y_0^2\omega^2/2\ell^2 > g/\ell$.

so for $\frac{\omega^2 y_0^2}{2\ell^2} > \frac{g}{\ell}$, $\bar{\phi} = \pi$ is a (stable) minimum.

$\Rightarrow$

- pendulum effective potential has second minimum at inversion ($\bar{\phi} = \pi$) for $\boxed{\frac{\omega^2 y_0^2}{2\ell^2} > \frac{g}{\ell}}$
- established critical ponderomotive potential strength.

Comments:

$\rightarrow$ have encountered ponderomotive force

#12 :GRADUATE MECHANICS

`PROBLEM: Eikonal Equation`

A sound wave propagates in a medium with speed of sound

$$c_s = c_s(x, y, z).$$

a.) Derive the eikonal equation (WKB approximate wave equation) and show the condition for its solution.

b.) How can one obtain Fermat's Principle – i.e. that rays will choose the path of least time from the eikonal equation?

c.) For $c_s = c_s(y)$, derive the equation for the ray path.

---

a.) $\nabla^2\psi + \frac{\omega^2}{c_s^2}\psi = 0$

$1/c_s^2 = \left(1/c_o^2\right)\left(n(x,y,z)\right)^2$

$\psi = e^{i\phi}$

$(\nabla\phi)^2 = n(x,y,z)^2\frac{\omega^2}{c_o^2}$ is eikonal equation

if $n^2 = n_1(x)^2 + n_2(y)^2 + n_3(z)^2$

eikonal equation is separable, i.e.

$\phi = \phi_1(x) + \phi_2(y) + \phi_3(z)$

$$\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_2}{\partial y}\right)^2 + \left(\frac{\partial\phi_3}{\partial z}\right)^2 = \frac{\omega^2}{c_o^2}\left(n_1(x)^2 + n_2(y)^2 + n_3(z)^2\right)$$

etc.

b.) Analogous to abbreviated action $\delta_0 = \int p\,dq$, we have

$\Phi = \int \underline{k}\cdot d\underline{x} = \int \underline{\nabla}\phi \cdot d\underline{x}$

$\delta\Phi = 0$ gives ray path

so

$\delta\Phi = \delta\int \underline{\nabla}\phi\cdot d\underline{x} = \delta\int|\nabla\phi|\,ds$

$= \delta\int \frac{\omega}{c_o} n(\underline{x})\,ds$

$\delta\frac{\omega}{c_o}\int n(\underline{x})\,ds = 0$ recovers Fermat's Principle.

c.) $n = n(y)$

$ds = \left(dx^2 + dy^2\right)^{1/2} = dx\left(1 + (dy/dx)^2\right)$

SOLUTION:

$$\delta \int n(y)\left(1+\dot{y}^2\right)^{1/2} dy = 0$$

$\Rightarrow$

$$\frac{d}{dx}\left(\frac{n(y)}{\left(1+\dot{y}^2\right)^{1/2}}\dot{y}\right) = \left(1+\dot{y}^2\right)^{1/2}\frac{\partial n}{\partial y}$$

gives ray path.

#13 :GRADUATE E&M

`PROBLEM: Polarized Waves in a Medium`

Consider circularly polarized EM waves propagating in the direction of a static magnetic field $\vec{B}_0$ in a medium consisting of N electrons per unit volume behaving as bound harmonic oscillators with a single oscillator resonance frequency ω_0 with damping constant γ.

(a) Show that the relationship between the refractive indices for waves of (+) and (-) circular polarization can be written as

$$n_+^2 - n_-^2 = \frac{Ne^2}{\epsilon_0 m}\left[\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega + eB_0\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega - eB_0\omega/m}\right]$$

(b) Neglecting γ, from the above formula, calculate the rotation in radians of the direction of polarization of a *linearly polarized* plane wave of frequency ω, after propagating a length L in the medium when the magnetic field is present.

Hint: Consider a linearly polarized wave as the coherent sum of two oppositely circularly polarized waves.

`SOLUTION:`

Part (a):

Eqn. of motion for electron is

$$m(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}) - e(\vec{B}_0 \times \dot{\vec{x}}) = -e\vec{E}$$ (B_0 along z-axis)

Write electric field as a circularly polarized wave:

$$\vec{E} = E_0(\vec{\varepsilon}_1 \pm \vec{\varepsilon}_2)e^{-i\omega t}$$ ($\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$ along x,y)

Choose solutions of the form

$$\vec{x} = x_0(\vec{\varepsilon}_1 \pm i\vec{\varepsilon}_2)e^{-i\omega t}$$

Substituting in Eq. of motion, we get

$$[-m\omega^2 - im\omega\gamma + m\omega_0^2]x_0(\vec{\varepsilon}_1 \pm \vec{\varepsilon}_2) + i\omega eB_0x_0(\mp i\vec{\varepsilon}_1 + \vec{\varepsilon}_2) = -eE_0(\vec{\varepsilon}_1 \pm \vec{\varepsilon}_2)$$

Equating coefficients of $\boldsymbol{\varepsilon}_1$ we get

$$m(\omega_0^2 - \omega^2 - i\gamma\omega)x_0 \pm eB_0\omega x_0 = -eE_0$$

or

$$x_0 = -\frac{e}{m}\frac{E_0}{(\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega / m)}$$

Equating coefficients of $\boldsymbol{\varepsilon}_2$ gives same solution. So,

$$\vec{P}_{\pm} = -e\vec{x}_{\pm} = \frac{e^2}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega/m}$$

So

$$[\frac{\varepsilon(\omega)}{\varepsilon_0}]_{\pm} = n_{\pm}^2 = 1 + \frac{Ne^2}{\varepsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega \pm eB_0\omega/m}$$

which yields

$$n_+^2 - n_-^2 = \frac{Ne^2}{\varepsilon_0 m}[\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega + eB_0\omega/m} - \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega - eB_0\omega/m}]$$

Part (b):

Consider a plane wave which propagates as the sum of 2 oppositely circularly polarized waves with a relative phase difference of ϕ.

Complex electric field is written as

$$\tilde{\vec{E}} = E_0[\frac{1}{2}(\vec{\varepsilon}_1 + i\vec{\varepsilon}_2) + \frac{1}{2}(\vec{\varepsilon}_1 - i\vec{\varepsilon}_2)e^{i\phi}]e^{i(kz-\omega t)} \qquad (1)$$

$\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$ along x- and y-axes respectively. At z=0, t=0, actual field is real part of Eq. (1):

If ϕ =0, $\vec{E} = E_0\vec{\varepsilon}_1$ i.e. wave is plane polarized along x-axis.

If $\phi \neq 0$,

$$\vec{E} = E_0 \,\mathrm{Re}\{[\frac{1+\cos\phi}{2}\vec{\varepsilon}_1 + \frac{1}{2}\sin\phi\vec{\varepsilon}_2] + i[\frac{1-\cos\phi}{2}\vec{\varepsilon}_2 + \frac{1}{2}\sin\phi\vec{\varepsilon}_1]\}$$

$$= E_0\{\vec{\varepsilon}_1\cos^2\frac{\phi}{2} + \vec{\varepsilon}_2\sin\frac{\phi}{2}\cos\frac{\phi}{2}\} = E_0\cos\frac{\phi}{2}[\vec{\varepsilon}_1\cos\frac{\phi}{2} + \vec{\varepsilon}_2\sin\frac{\phi}{2}]$$

#14 :GRADUATE E&M

`PROBLEM: Radiation from Circling Charge`
A charge Q is driven around a circle of radius a centered at the origin of a coordinate system. The circle lies in the (x, y) plane with the normal to the circle in the z-direction. The charge moves at constant angular velocity Ω around the circle with $\Omega a << c$. Find the electric and magnetic fields in the "far zone" where $r >> a$. Find the Poynting vector in the "far zone" for this source. Find the power radiated from this configuration.

`SOLUTION:` The simplest solution is to use Lieneard-Wiechert potentials and fields and keep the $\frac{1}{R}$ term in the field, dropping the $\frac{1}{R^2}$ term.

$$\vec{E}(\vec{x}, t) \to \frac{Q}{c}\left[\frac{\hat{n}}{\kappa^3 R} \times \left[(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right]\right]_{retarded}$$

The "retarded" means that we pick the charge location where the field moves at the speed of light to get to the point at which we evaluate, but this is not a big problem: $t' = t - \frac{R}{c}$.

$$\vec{B} = \hat{n} \times \vec{E}$$

$$\kappa = 1 - \hat{n} \cdot \vec{\beta} \to 1$$

$$\vec{R} = \vec{x} - \hat{n} \cdot \vec{r}(t')$$

$$\vec{r}(t') = a(\cos(\Omega t')\hat{x} + \sin(\Omega t')\hat{y})$$

$$\vec{\beta}(t') = \frac{a\Omega}{c}(-\sin(\Omega t')\hat{x} + \cos(\Omega t')\hat{y})$$

$$\dot{\vec{\beta}}(t') = -\frac{a\omega^2}{c}(\cos(\Omega t')\hat{x} + \sin(\Omega t')\hat{y}) = -\frac{\Omega^2}{c}\vec{r}(t')$$

We can keep $\hat{n}$ in the $x-z$ plane without loss of generality, so using a normal spherical coordinate system where θ is the angle from the z axis, we choose $\phi = 0$ for simplicity.

$$\vec{E}(\vec{x}, t) \to -\frac{Q\Omega^2}{c^2}\left[\frac{1}{R}\hat{n} \times [\hat{n} \times \vec{r}]\right]_{retarded}$$

$$\hat{n} \times [\hat{n} \times \vec{r}] = a \sin\theta \cos(\Omega t')\hat{n} - \vec{r} = a\left[\sin\theta\cos\theta\hat{z} - \cos^2\theta\hat{x}\right]\cos(\Omega t')$$

$$\vec{E}(\vec{x}, t) \to -\frac{aQ\Omega^2}{c^2}\left[\frac{1}{R}\left[\sin\theta\cos\theta\hat{z} - \cos^2\theta\hat{x}\right]\cos(\Omega t')\right]_{retarded}$$

$$\vec{B} = \vec{n} \times \vec{E}$$
$$\vec{S} = \frac{c}{4\pi} |E|^2 \, \hat{n}$$
$$\frac{d\vec{P}}{d\Omega} = R^2 S = \frac{c}{4\pi} \frac{a^2 Q^2 \Omega^4}{c^4} \left[\sin^2\theta \cos^2\theta + \cos^4\theta\right] \cos^2(\Omega t')$$

Computing the time averaged power as a function of θ,

$$\left\langle \frac{d\vec{P}}{d\Omega} \right\rangle = \frac{a^2 Q^2 \Omega^4}{8\pi c^3} \left[\cos^2\theta - \cos^4\theta + \cos^4\theta\right] = \frac{a^2 Q^2 \Omega^4}{8\pi c^3} \cos^2\theta$$

$$P = \frac{a^2 Q^2 \Omega^4}{4c^3} \frac{2}{3} = \frac{a^2 Q^2 \Omega^4}{6c^3}$$

#15 :GRADUATE QUANTUM MECHANICS

`PROBLEM: Scattering in a Central Potential`

We consider the scattering problem of a particle with mass m in the 3D central potential

$$V(r) = \frac{\alpha}{r^2}, \tag{1}$$

where $\alpha > 0$.

1) Use the 3D partial wave method to solve for the phase shift δ_l with l a non-negative integer. l represents the partial wave channel.

Hints: $\frac{-\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] R_{nl}(r) + \left(V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right) R_{nl}(r) = E R_{nl}(r)$ *The spherical Bessel function $j_\nu(kr)$ satisfies the equation*

$$\frac{d^2}{dr^2} j_\nu + \frac{2}{r}\frac{d}{dr} j_\nu + \left(k^2 - \frac{\nu(\nu+1)}{r^2}\right) j_\nu = 0, \tag{2}$$

and its asymptotic behavior at $r \to \infty$ is

$$j_\nu(kr) \to \frac{1}{kr}\sin\left(kr - \frac{\nu\pi}{2}\right), \tag{3}$$

in which ν does not need to be an integer. You need to decide the appropriate value of ν to use.

2) Under the condition that $\frac{m\alpha}{\hbar^2} \ll \frac{1}{8}$, find the approximate formulae for δ_l up to the linear order of $\frac{m\alpha}{\hbar^2}$. Then find an simple analytic form for the scattering amplitude $f(\theta)$ up to the linear order of $\frac{m\alpha}{\hbar^2}$, and the corresponding differential cross section $\sigma(\theta)$.

Hint: You may need to use the formula

$$\sum_l P_l(\cos\theta) = \frac{1}{\sin\frac{\theta}{2}}, \tag{4}$$

and the relation of the scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_l (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\theta). \tag{5}$$

`SOLUTION:`

1) The radial Schrödinger equation for the radial wavefunction R_l in the l-th partial wave channel is

$$\frac{d^2}{dr^2}R_l + \frac{2}{r}\frac{d}{dr}R_l + \left(k^2 - \frac{l(l+1)}{r^2} - \frac{2m\alpha}{\hbar^2 r^2}\right)R_l = 0 \tag{6}$$

Define μ satisfying

$$\nu(\nu+1) = l(l+1) + \frac{2m\alpha}{\hbar^2}, \tag{7}$$

or

$$\nu = \left[(l+\tfrac{1}{2})^2 + \tfrac{2m\alpha}{\hbar^2}\right]^{\frac{1}{2}} - \frac{1}{2}, \tag{8}$$

we have

$$\frac{d^2}{dr^2}R_l + \frac{2}{r}\frac{d}{dr}R_l + \left(k^2 - \frac{\nu(\nu+1)}{r^2}\right)R_l = 0. \tag{9}$$

It's solution is

$$R_l(r) = \sqrt{\frac{\pi}{2kr}}j_{\nu+\frac{1}{2}}(kr) \to \frac{1}{kr}\sin(kr - \frac{\nu\pi}{2}) = \frac{1}{kr}\sin(kr - \frac{l\pi}{2} + \delta_l). \tag{10}$$

Then we extract the value of δ_l as

$$\delta_l = -\frac{\pi}{2}(\nu - l) = -\frac{\pi}{2}\left[\sqrt{(l+\tfrac{1}{2})^2 + \tfrac{2m\alpha}{\hbar^2}} - (l+\tfrac{1}{2})\right], \tag{11}$$

2)

$$\begin{aligned} \nu + \tfrac{1}{2} &\approx (l+\tfrac{1}{2}) + \frac{m\alpha}{(l+\frac{1}{2})\hbar^2} \\ \delta_l &\approx -\frac{\pi m\alpha}{(2l+1)\hbar^2}. \end{aligned} \tag{12}$$

Since $\delta_l \ll 1$, we have

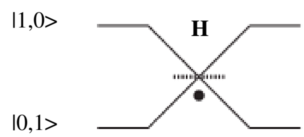
$$f(\theta) \approx \frac{1}{k}\sum_l (2l+1)\delta_l P_l(\cos\theta) = -\frac{\pi\mu\alpha}{\hbar^2 k}\sum_{l=0}^{\infty} P_l(\cos\theta) = -\frac{\pi\mu\alpha}{2\hbar^2 k\sin\frac{\theta}{2}}. \tag{13}$$

The differential cross section

$$\sigma(\theta) = |f(\theta)|^2 = \frac{\pi^2\mu^2\alpha^2}{4\hbar^4 k^2 \sin^2\frac{\theta}{2}}. \tag{14}$$

#16 :GRADUATE QUANTUM MECHANICS

`PROBLEM: Multiple photons on paths of an interferometer`
Consider a qbit ($\equiv$ two-state system) made from the two states of a single photon moving on the upper and lower paths of an interferometer. On this two-state system, a half-silvered mirror $\mathbf{H}$



acts as a beamsplitter:

$$\mathbf{H} \equiv \frac{1}{\sqrt{2}} \left(\boldsymbol{\sigma}^x + \boldsymbol{\sigma}^z\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

But photons are bosons. This means that if

$\mathbf{a}^\dagger|0,0\rangle \equiv |1,0\rangle$ is a state with one photon on the upper path

of the interferometer (and none on the lower path), then

$\frac{(\mathbf{a}^\dagger)^n}{\sqrt{n!}}|0,0\rangle \equiv |n,0\rangle$ is a state with n photons on the upper path.

Similarly, define

$\frac{(\mathbf{b}^\dagger)^n}{\sqrt{n!}}|0,0\rangle \equiv |0,n\rangle$ to be a state with n photons on the lower path

of the interferometer. (Note that $[\mathbf{a},\mathbf{b}] = 0 = [\mathbf{a},\mathbf{b}^\dagger]$.)

Questions:

1. How does $\mathbf{H}$ act on $|0,0\rangle$?
2. How does $\mathbf{H}$ act on $|2,0\rangle$ and $|0,2\rangle$?
3. How does $\mathbf{H}$ act on the operators $\mathbf{a}^\dagger$ and $\mathbf{b}^\dagger$?

4. What is the state which results upon sending a coherent state of photons
$$|\alpha, \beta) \equiv \mathcal{N}_\alpha \mathcal{N}_\beta \; e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger} |0, 0\rangle$$
through a half-silvered mirror? ($\mathcal{N}_\alpha \equiv e^{-|\alpha|^2/2}$ is a normalization constant.)

SOLUTION:

The hilbert space under discussion here is that of two harmonic oscillators, and above we have defined $|n, m\rangle$ to be the state where the respective number operators $\mathbf{a}^\dagger \mathbf{a}$ and $\mathbf{b}^\dagger \mathbf{b}$ have eigenvalues n, m respectively. From the definition of the photon-path-as-qbit, we have:

$$\mathbf{H}|1, 0\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle + |0, 1\rangle) = \frac{1}{\sqrt{2}} \left(\mathbf{a}^\dagger + \mathbf{b}^\dagger \right) |0, 0\rangle,$$
$$\mathbf{H}|0, 1\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle - |0, 1\rangle) = \frac{1}{\sqrt{2}} \left(\mathbf{a}^\dagger - \mathbf{b}^\dagger \right) |0, 0\rangle.$$

Now $|0, 0\rangle$ is a state with $n = 0$ photons on the upper path and $n = 0$ photons on the lower path. No photons at all. So we have $\mathbf{H}|0, 0\rangle = |0, 0\rangle$ since a mirror does nothing to no photons! (It just sits there.) This means further that $\mathbf{H}$ acts on the creation operators by

$$\mathbf{H}\mathbf{a}^\dagger \mathbf{H} = \frac{1}{\sqrt{2}} \left(\mathbf{a}^\dagger + \mathbf{b}^\dagger \right), \mathbf{H}\mathbf{b}^\dagger \mathbf{H} = \frac{1}{\sqrt{2}} \left(\mathbf{a}^\dagger - \mathbf{b}^\dagger \right),$$

in order to be consistent with the action on the one-photon states. So we can conclude that

$$\mathbf{H}|2, 0\rangle = \mathbf{H} \frac{\left(\mathbf{a}^\dagger\right)^2}{\sqrt{2!}} |0, 0\rangle = \left(\mathbf{H}\mathbf{a}^\dagger \mathbf{H} \right)^2 \frac{1}{\sqrt{2!}} |0, 0\rangle = \frac{1}{2}\frac{1}{2} \left(\mathbf{a}^\dagger + \mathbf{b}^\dagger \right)^2 |0, 0\rangle = \frac{1}{2} (|2, 0\rangle + 2|1, 1\rangle + |0, 2\rangle)$$

$$\mathbf{H}|0, 2\rangle = \mathbf{H} \frac{\left(\mathbf{b}^\dagger\right)^2}{\sqrt{2!}} |0, 0\rangle = \left(\mathbf{H}\mathbf{b}^\dagger \mathbf{H} \right)^2 \frac{1}{\sqrt{2!}} |0, 0\rangle = \frac{1}{2}\frac{1}{2} \left(\mathbf{a}^\dagger - \mathbf{b}^\dagger \right)^2 |0, 0\rangle = \frac{1}{2} (|2, 0\rangle - 2|1, 1\rangle + |0, 2\rangle).$$

And finally,

$$\mathbf{H} e^{\alpha \mathbf{a}^\dagger + \beta \mathbf{b}^\dagger} |0, 0\rangle = e^{\alpha \mathbf{H}\mathbf{a}^\dagger \mathbf{H} + \beta \mathbf{H}\mathbf{b}^\dagger \mathbf{H}} |0, 0\rangle = e^{\frac{1}{\sqrt{2}} \left(\alpha \left(\mathbf{a}^\dagger + \mathbf{b}^\dagger \right) + \beta \left(\mathbf{a}^\dagger - \mathbf{b}^\dagger \right) \right)} |0, 0\rangle = e^{\frac{\alpha+\beta}{\sqrt{2}} \mathbf{a}^\dagger + \frac{\alpha-\beta}{\sqrt{2}} \mathbf{b}^\dagger} |0, 0\rangle$$

It acts on the coherent state labels just like it does on the quantum amplitudes. These coherent state labels are the data that label the lightwave in *e.g.* a laser.

#17 :GRADUATE STAT MECH/THERMO

`PROBLEM: Container of Classical Gas`

A classical gas of non-interacting atoms is in thermal equilibrium at temperature T in a container of volume V and surface area A. The potential energy of the atoms in the bulk is zero. Atoms adsorbed on the surface have a potential energy $V = -E_a$ and behave as an ideal twodimensional gas.

Find an analytic expression for the surface density $\sigma(n, T) \equiv N_{surface}/A$ in terms of the bulk density $n \equiv N_{bulk}/V$ and the temperature. Be sure to "correct Boltzmann counting".

The following mathematical results may be useful.

$$\ln(N!) \approx N \ln N - N$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = 1$$

`SOLUTION:`

First do the bulk gas. Use $N_{bulk} = N$ for brevity, and let the energy of a particle be $\epsilon = (p_x^2 + p_y^2 + p_z^2)/2m$.

$$Z_{bulk}(N, T, V) = \frac{V^N}{N! h^{3N}} \left[\int_{-\infty}^{\infty} e^{-p_x^2/(2mkT)}\, dp_x \right]^{3N} = \frac{V^N (2\pi mkT)^{3N/2}}{N! h^{3N}}$$

Using $\ln(N!) \approx N \ln N - N$, we get

$$F_{bulk} = -kT \ln Z_{bulk} \approx -NkT \ln \left[\frac{(2\pi mkT)^{3/2}}{h^3} \frac{V}{N} \right] - NkT$$

$$\mu_{bulk} = \left. \frac{\partial F_{bulk}}{\partial N} \right|_{T,V} = -kT \ln \left[\frac{(2\pi mkT)^{3/2}}{h^3} \frac{V}{N} \right]$$

Next do the surface

Use $N_{surface} = N'$ for brevity, and let the energy of a particle be

$$\epsilon = -E_0 + (p_x^2 + p_y^2)/2m.$$

$$Z_{surface}(N', T, A) = \frac{A^{N'}}{N'!h^{2N'}} \left\{ e^{E_0/kT} \left[\int_{-\infty}^{\infty} e^{-p_x^2/(2mkT)}\ dp_x \right]^2 \right\}^{N'} = \frac{1}{N'!} \left[\frac{e^{E_0/kT}(2\pi mkT)A}{h^2} \right]^{N'}$$

$$F_{surface} = -kT \ln Z_{surface} = -N'kT \ln \left[\frac{(2\pi mkT)}{h^2} \frac{A}{N'} e^{E_0/kT} \right] - N'kT$$

$$\mu_{surface} = \left. \frac{\partial F_{surface}}{\partial N'} \right|_{T,V} = -kT \ln \left[\frac{(2\pi mkT)}{h^2} \frac{A}{N'} e^{E_0/kT} \right]$$

Now put the bulk and surface pieces in equilibrium. Set $\mu_{bulk} = \mu_{surface}$ at equilibrium. Let $n \equiv N_{bulk}/V$ and $\sigma \equiv N_{surface}/A$ be the bulk and surface density, respectively.

$$\ln \left[\frac{(2\pi mkT)^{3/2}}{h^3} \frac{V}{N_{bulk}} \right] = \ln \left[\frac{2\pi mkT)}{h^2} \frac{A}{N_{surface}} e^{E_0/kT} \right]$$

$$\frac{\sqrt{2\pi mkT}}{h} n = \frac{1}{\sigma} e^{E_0/kT}$$

$$\sigma(n, T) = n e^{E_0/kT} \frac{h}{\sqrt{2\pi mkT}}$$

#18 :GRADUATE STAT MECH/THERMO

`PROBLEM: Ising Model`

A spin-1 Ising model in one dimension is described by the Hamiltonian

$$H_N\{\sigma_i\} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} \qquad (\sigma_i = -1, 0, 1)$$

Write down the transfer matrix (P) (where the partition function $Q_N = TrP^N$) for this interaction and show that the free energy $A_N(T)$ of this model, in the thermodynamic limit, is equal to

$$-NkT\ln\left(\frac{1}{2}\left[(1+2\cosh K)+(8+(2\cosh K-1)^2)^{1/2}\right]\right) \qquad (K \equiv J/KT)$$

Examine the limiting behavior of this quantity as $T \to 0$ and as $T \to \infty$, and discuss the physical interpretation of each limit.

`SOLUTION:`

Assuming a closed endless structure, the partition function is

$$\begin{aligned} Q_N(T) &= \sum_{\{\sigma_i\}} exp\left[\beta\sum_{i=1}^{N}J\sigma_i\sigma_{i+1}\right] \qquad (\sigma_{N+1}=\sigma_1) \\ &= \sum_{\{\sigma_i\}} exp(\beta J\sigma_1\sigma_2)exp(\beta J\sigma_2\sigma_3)...exp(\beta J\sigma_N\sigma_1) \\ &= \sum_{\{\sigma_i\}} \langle\sigma_1|P|\sigma_2\rangle\langle\sigma_2|P|\sigma_3\rangle...\langle\sigma_N|P|\sigma_1\rangle \end{aligned}$$

where $\langle\sigma_i|P|\sigma_{i+1}\rangle$ are the matrix elements of the transfer matrix $(P) = (e^{\beta J\sigma_i\sigma_{i+1}})$. Writing out this 3×3 matrix (with 1s in the middle row and column and $e^{\pm\beta J}$ in the corners), the eigenvalues are found to be

$$\lambda_{1,2} = \frac{1}{2}\left[(1+2\cosh K)\pm(8+(2\cosh K-1)^2)^{1/2}\right] \qquad \lambda_3 = 2\sinh K$$

It follows that $Q_N(T) = Tr(P^N) = \lambda_1^N + \lambda_2^N + \lambda_3^N$.

In the thermodynamic limit, only the largest eigenvalue, viz. λ_1, matters – with the result that $A_N(T) = -kT\ln Q_N(T) \approx -NkT\ln\lambda_1$, which gives the stated result.

In the limit $T \to 0$, i.e. $K \to \infty$, the function $\cosh K \approx \frac{1}{2}e^K$ and hence $A \approx -NJ$; this corresponds to a state of perfect order, with $U = -NJ$ and $S = 0$.

In the limit $T \to \infty$, i.e. $K \to 0$, the function $\cosh K \to 1$ and hence $A \to NkT \ln 3$; this corresponds to a state of complete randomness, with 3^N equally likely microstates, which entails $U = 0$ and $S = Nk \ln 3$.

#19 :GRADUATE MATHEMATICAL PHYSICS

`PROBLEM: Fourier Transform`
Calculate the Fourier transform of a hyperbolic tangent, i.e.,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} \tanh x \, dx \,.$$

(If you worry about the formal convergence of the integral, it can be assured by adding a factor like $e^{-\alpha x^2}$ to the integrand with the understanding that we are interested in the $\alpha \to 0$ limit.)

`SOLUTION:`

Per remark in the formulation of the problem, the desired Fourier transform is the $\alpha \to 0$ limit of the function $\tilde{f}_\alpha(k)$ defined by

$$\tilde{f}_\alpha(k) = \int_{-\infty}^{\infty} e^{-ikx-\alpha x^2} \tanh x \, dx \,.$$

Consider an integral I of the same function over a box-like contour in the complex plane of x: $-R \to R \to R + i\pi \to -R + i\pi \to -R$. In the limit $R \to +\infty$, the contributions of the horizontal 'sides' of the contour vanish. The contribution of the bottom becomes equal to $\tilde{f}_\alpha(k)$. The contribution of the top is proportional to that of the bottom, up to subleading corrections that vanish as $\alpha \to 0$. Thus, we have

$$I = \tilde{f}_\alpha(k) \left(1 - e^{\pi k}\right) + o(1) \,.$$

The integrand of I is analytic everywhere inside the contour except at $x = i\pi/2$ where it has a simple pole; therefore,

$$I = 2\pi i \operatorname*{res}_{i\pi/2} e^{-ikx-\alpha x^2} = 2\pi i e^{-\pi k/2} + o(1) \,.$$

Combining the two results, we obtain

$$\tilde{f}(k) = -\tfrac{i\pi}{\sinh(\pi k/2)} \,, \quad k \neq 0 \,.$$

Note: With further analysis, it can be shown that the singularity at $k = 0$ should be understood in the sense of the Principal Value:

$$\tilde{f}(k) = -P \, \tfrac{i\pi}{\sinh(\pi k/2)} \,.$$

#20 :GRADUATE GENERAL PHYSICS

`PROBLEM: Convection`

When a liquid fills the gap between two parallel horizontal plates and has a positive thermal expansion coefficient $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)$, and the temperature of the bottom plate, T_b, is higher than the temperature of the top plate, T_t , convective motion of the liquid may occur. The onset of the convection corresponds to a certain value of a dimensionless expression called the Rayleigh number (Ra). The expression for Ra involves the distance between the plates, h, the temperature difference, $\Delta T = T_t - T_b$, the thermal expansion coefficient, β, the viscosity and density of the liquid, η and ρ, the gravitational acceleration, g, and the thermal diffusivity, α (measured in m^2/s). Find the expression for Ra from the dimensional analysis of motion of a small volume of the liquid that is moving up.

Hint: As it moves up, a small volume of the liquid enters regions with lower ambient temperature, corresponding to greater local fluid density, that leads to a positive buoyancy force on the liquid volume, F_b. Consider whether T of the rising liquid volume is increasing with time (amplification; convection is maintained) or decreasing with time (convection is suppressed). The diameter of the liquid volume can be assumed to be on the order h and its upward velocity, v, can be assumed to be proportional to $F_b/(h\eta)$. The cooling of the volume is described by the thermal diffusivity equation $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$, where T is the difference between the temperature of the volume and of the liquid around it.

`SOLUTION:`

The equation for T is (dimensional analysis)

$$\frac{dT}{dt} = \nu \frac{\Delta T}{h} - \alpha \frac{T}{h^2} = \frac{\rho g h^3 \beta T}{h\eta}\frac{\Delta T}{h} - \alpha \frac{T}{h^2}$$

So the condition of $\frac{dT}{dt} > 0$ translates into

$$\frac{\rho g h \beta \Delta T}{\eta} - \alpha \frac{1}{h^2} > 0$$

or

$$\frac{\rho g h^3 \beta \Delta T}{\alpha \eta} > 1$$

So the expressiion for Ra is

$$Ra = \frac{\rho g h^3 \beta \Delta T}{\alpha \eta}$$