PHYSICS DEPARTMENT EXAM
SPRING 2013. PART I

INSTRUCTIONS

• You should not have anything close to you other than your pens, pencils, calculator, and food items. Please deposit your belongings (books, notes, backpacks, phone etc.) in a corner of the exam room.

• Departmental examination paper is provided. Colored scratch paper is also provided. Please make sure you:

a. Write the problem number and your ID number on each sheet;

b. Write only on one side of the paper;

c. Start each problem on the attached examination sheets;

d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

• Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem.

• The questions are grouped in 5 Sections: Mechanics, E&M, Quantum, StatMech, and General. You must attempt at least one problem from each Section. You are to do seven (7) of the ten (10) problems. At the conclusion of the examination period, please staple sheets from each problem together. **Circle the 7 problems you wish to be graded:**

Mechanics		E & M		Stat Mech		Quantum		General	
1	2	3	4	5	6	7	8	9	10

• Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 : UNDERGRADUATE MECHANICS PROBLEM

A rock moves with initial velocity v_o on a path that will take it near a spherical planet of mass M and radius r_o.

(a) Find the maximum impact parameter b_0 for striking the planet. Take the planet to be at rest, and ignore all other bodies. The impact parameter is defined as the smallest distance from the the center of the planet if the rock were to continue on its original path in the absence of gravity. See Figure **??**.

(b) Derive an expression for the escape velocity, v_e, the minimum velocity required to escape from the surface of the planet to infinity.

(c) Use v_e to simplify the expression for b_0.

(d) Discuss the result in (c).

(e) What is the minimum velocity of the rock as it just passes over the surface of the the planet?

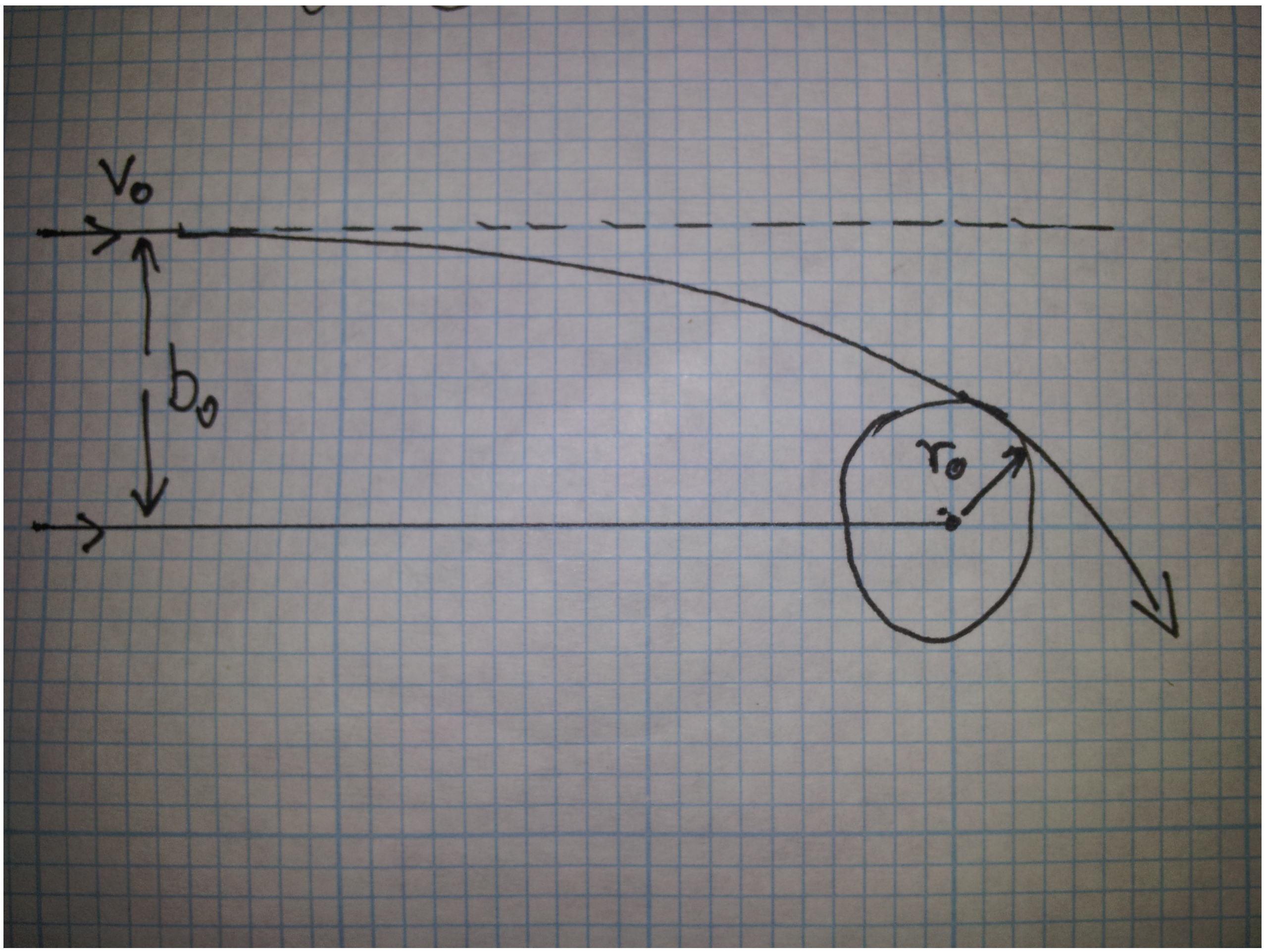

Figure 1: b_o is vertical distance perpendicular to v_o

#1 : UNDERGRADUATE MECHANICS SOLUTION

Consider the case where the rock is moving in from a far away. Define impact parameter b_o to be that where the trajectory bends to bring the rock just parallel to and scraping the surface of the planet.

The angular momentum of the rocket about the center of the planet is conserved since

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = r\hat{r} \times F\hat{r} = 0 \tag{1}$$

where the last F is the central force.

When the rock is far from the planet, the angular momentum is $\ell = mv_o b_o$.

We also know the angular momentum when the rock just touches the surface of planet with some velocity v, $\ell = mvr_o$. Then $mv_o b_o = mvr_o$ or

$$b_o = \frac{vr_o}{v_o}. \tag{2}$$

The orbit also conserves energy. When the rock is far away, approaching with velocity v_o the P.E. (last term below) can be ignored.

$$E_{far} = \frac{1}{2}m{v_o}^2 - \frac{k}{r}, \tag{3}$$

and when it just scrapes the planet

$$E_{scraping} = \frac{1}{2}mv^2 - \frac{GMm}{r_o}. \tag{4}$$

Since $E_{far} = E_{scraping}$

$$\frac{1}{2}mv_o^2 = \frac{1}{2}mv^2 - \frac{GMm}{r_o} \tag{5}$$

$${v_o}^2 = v^2 - \frac{2GM}{r_o} \tag{6}$$

$$v = \sqrt{{v_o}^2 + \frac{2GM}{r_o}} \tag{7}$$

Insert this expression for v into expression for b_o

$$b_o = \frac{r_o}{v_o}\sqrt{{v_o}^2 + \frac{2GM}{r_o}} = r_o\sqrt{1 + \frac{2GM}{r_o {v_o}^2}}. \tag{8}$$

(b) The condition for v_e is that $E_{scraping} = 0$, or $v_e^2 = 2GM/r_o$, then (c)

$$b_o = r_o \sqrt{1 + \left(\frac{v_e}{v_o}\right)^2}. \tag{9}$$

(d) In the limit of initial velocities v_o much larger than v_e, b_o tends to r_o, hence fast moving rocks are least likely to hit the planet. However in the other extreme of $v_0 << v_e$, $b_o \sim r_o(v_e/v_o)$ with a limit of b_o tends to infinity as v_o tends to zero. While initially slow moving, such rocks will pass the planet or hit with a velocity of about v_e. The slower a rock is moving initially, the larger the cross-section for an impact.

(e) This is the condition $E_{far} = E_{scraping} = 0$, so we set $v_o = 0$ in Eqn. **??** to find the minimum velocity is $v = v_e$.

#2 : UNDERGRADUATE MECHANICS PROBLEM

Consider measuring the period of a simple point-mass pendulum to experimentally test Newton's 2nd Law. Assume that the mass of the pendulum string is negligible, friction is negligible, and the pendulum makes small oscillations. Assume that the force due to gravity is given by $F_G = m_G g$, where m_G is the gravitational mass and $g = 9.8 m/s^2$.

(A) What is the predicted period of the pendulum expressed in terms of the pendulum length L, inertial mass m, gravitation mass m_G, and g?

(B) What measurements can you do to show that the inertial mass is equivalent to gravitational mass?

(C) Suppose the length of the pendulum is exactly 1 meter, but the error in your time measurement is ± 1 second. If you measure the pendulum swinging for one minute, with what percent uncertainty can you confirm that Newton's second law is valid?

#2 : UNDERGRADUATE MECHANICS SOLUTION

(A) The force tangential to the arc of the pendulum's swing is $F_t = m a_t$. The equation of motion is thus

$$-m_G g \theta = mL \frac{d^2\theta}{dt^2} \tag{10}$$

where L=length of the pendulum and θ = angle relative to vertical. The motion is thus simple harmonic motion with period

$$T = 2\pi \sqrt{\left(\frac{m}{m_G}\right) \frac{L}{g}} \tag{11}$$

(B) To show that inertial mass is equivalent to gravitational mass ($m = m_G$, conduct measurements with different masses and using different materials and show that you always measure $T = 2\pi\sqrt{L/g}$.

(C) Newton's second law predicts $T = 2\pi\sqrt{L/g}$ Plugging in L=1 and g=9.8 we get T = 2 seconds. When we measure for 1 minute, we can measure about N=30 periods. Since

$$T_{tot} = NT = 2\pi N\sqrt{\frac{L}{g}} \tag{12}$$

Newton's second law predicts $g = (2\pi N)^2 L T_{tot}^{-2}$ We can thus test Newton's second law by determining g experimentally by measuring T_{tot}. If we find $g = 9.8$ then out measurement is consistent with Newton's second law. If our error in measuring T_{tot} is δT_{tot} then our error in determining g is

$$\delta g = \left|\frac{\partial g}{\partial T_{tot}}\right| \delta T_{tot} = (2\pi N)^2 L(2T_{tot}^{-3}) = 0.33. \tag{13}$$

Therefore our uncertainty in testing Newton's second law is

$$\frac{\delta g}{g} = \frac{0.33}{9.8} = 0.0337, \tag{14}$$

about 3% uncertainty.

#3 : UNDERGRADUATE EM

A homogeneous magnetic field B is perpendicular to a track of width l, which is inclined at an angle α to the horizontal, Figure **??**. A frictionless conducting rod of mass m can move along the two rails of the track as shown in the figure. The resistance of the rod and the rails is negligible. The rod is released from rest.

(a) Find the maximum velocity of the rod if the circuit formed by the rod and the rails is closed by a resistor of resistance R.

(b) Find the acceleration of the rod if the circuit formed by the rod and the rails is closed by a capacitor of capacitance C, instead of the resistor.

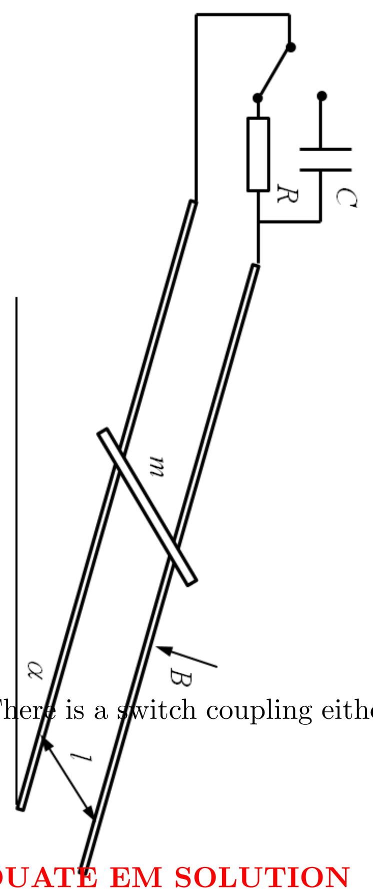

Figure 2: Question 3. There is a switch coupling either the resistor (shown) or capacitor.

#3 : UNDERGRADUATE EM SOLUTION

The rod's equation of motion is

$$ma = mg \sin \alpha - BlI, \tag{15}$$

where a is the acceleration of the rod and I is the current through the rod. The induced voltage in the rod is $V = Blv$, where v is velocity of the rod.

(a) $V = IR$, therefore $I = Blv/R$. The breaking force BlI is proportional to the velocity. The acceleration of the rod decreases and ultimately the rod reaches the maximum velocity. We find the maximum velocity of the rod v_{max} from the equation of motion by setting $a = 0$

$$v_{max} = \frac{mgR \sin \alpha}{B^2 l^2}. \tag{16}$$

(b) $V = Q/C$, where Q is the charge on the capacitor. The current through the rod

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} = CBl\frac{dv}{dt} = CBla. \tag{17}$$

Substituting the current into the equation of motion, we find that the rod moves with constant acceleration

$$a = \frac{mg \sin \alpha}{m + B^2 l^2 C} \tag{18}$$

#4 : UNDERGRADUATE EM

Each element in the chain of resistors shown in the figure is 1 Ω. A current of 1A flows through the final element of this ladder circuit, Figure **??**.

(a) What is the potential difference V across the input of the chain?

(b) What is the resistance of the chain?

(c) If the ladder circuit is extended further and further an infinite chain is obtained. What is the resistance of an infinite chain?

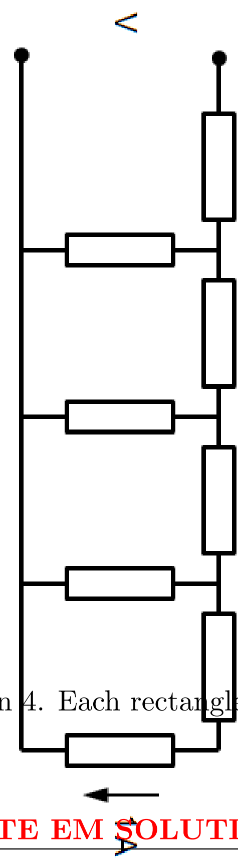

Figure 3: Question 4. Each rectangle is a 1 Ω resistor.

#4 : UNDERGRADUATE EM SOLUTION

(a) Let's number the resistors starting from the last in the chain. The currents through the first and second resistors are equal. The potential drop across the third resistor is equal to the sum of the potential drops across the first and second resistors (Kirchhoff's second law). Therefore the current through the third resistor is the sum of the currents through the first and second resistors, since the resistance of each resistor is 1 Ω. The current through the fourth resistor is the sum of the currents through the third and second resistors (Kirchhoff's first law). Repeating this for each next element of the ladder circuit, we obtain that the current through each resistor is equal to the sum of the currents through the two previous elements. The values of the currents are the terms of the Fibonacci series: 1, 1, 2, 3, 5, 8, 13, 21 A (see Figure ??). The voltage V is equal to the potential drop across the last two resistors $(21 + 13) = 34$ V.

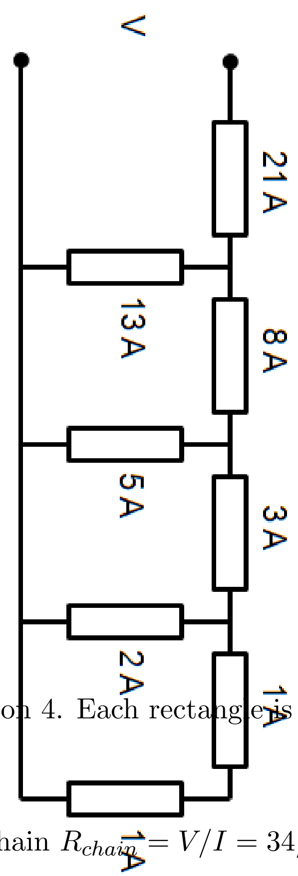

Figure 4: Question 4. Each rectangle is a 1 Ω resistor.

(b) The resistance of the chain $R_{chain} = V/I = 34/21 = 1.619\ \Omega$.

(c) For an infinite chain, adding another pair of 1-Ω resistors should not change the chain resistance R, see figure ??. Therefore

$$R = 1\Omega + \frac{1}{(1/1\Omega) + (1/R)}. \tag{19}$$

This yields the quadratic equation

$$R^2 - R - 1 = 0. \tag{20}$$

The positive root of this equation gives the resistance of an infinite chain

$$R = \frac{1 + \sqrt{5}}{2} \approx 1.618\Omega. \tag{21}$$

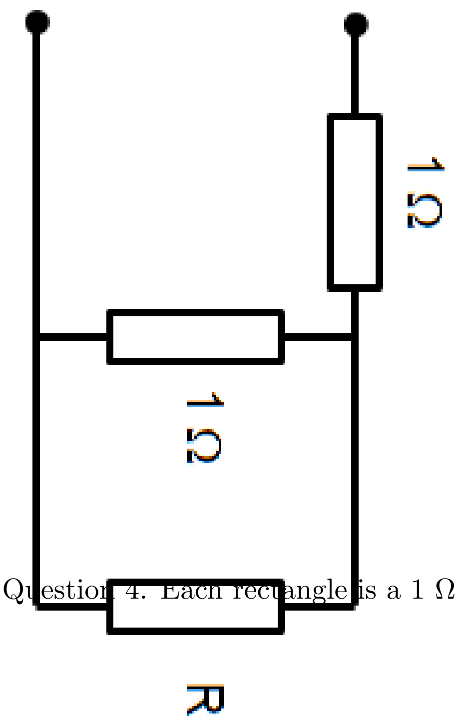

Figure 5: Question 4. Each rectangle is a 1 Ω resistor.

#5 : UNDERGRADUATE STAT MECH/THERMAL PROBLEM

What is the efficiency for a reversible engine operating around the cycle in Fig **??**

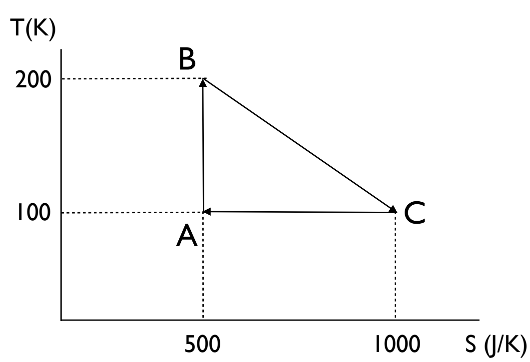

Figure 6: Question 5.

#5 : UNDERGRADUATE STAT MECH/THERMAL SOLUTION

The heat absorbed by the system in the process B $\rightarrow$ C is

$$Q_{\mathrm{BC}} = \int_B^C T ds = 75000 J. \tag{22}$$

The heat emitted during the process C $\rightarrow$ A is -50000J. Since Q_{AB} is zero, we obtain the efficiency of the engine

$$\eta = \mathrm{work}/Q_{\mathrm{BC}} = (Q_{\mathrm{BC}} + Q_{\mathrm{CA}})/Q_{\mathrm{BC}} = 33.3\%. \tag{23}$$

#6 : UNDERGRADUATE STAT MECH/THERMAL PROBLEM

Consider the system in the figure **??** below. A "gravity piston" in a vacuum, pressed by a particle (mass m with heat capacity C_1), is in equilibrium at initial temperature $T_{\rm i}$. A second particle at different temperature T_1 (otherwise the same as the first particle) comes in contact with the piston. The piston slowly reaches a new equilibrium such that the piston expands from its initial length $h_{\rm i}$ to the final equilibrium $h_{\rm f} = 2h_{\rm i}$. Assume that the heat capacity of the piston wall is negligible.

(a) What is the final temperature, $T_{\rm f}$, of the system in terms of initial temperature of the system $T_{\rm i}$?

(b) Calculate the total change in entropy of the particles and the piston chamber. Express the answer in terms of the parameters given and the gas constant R.

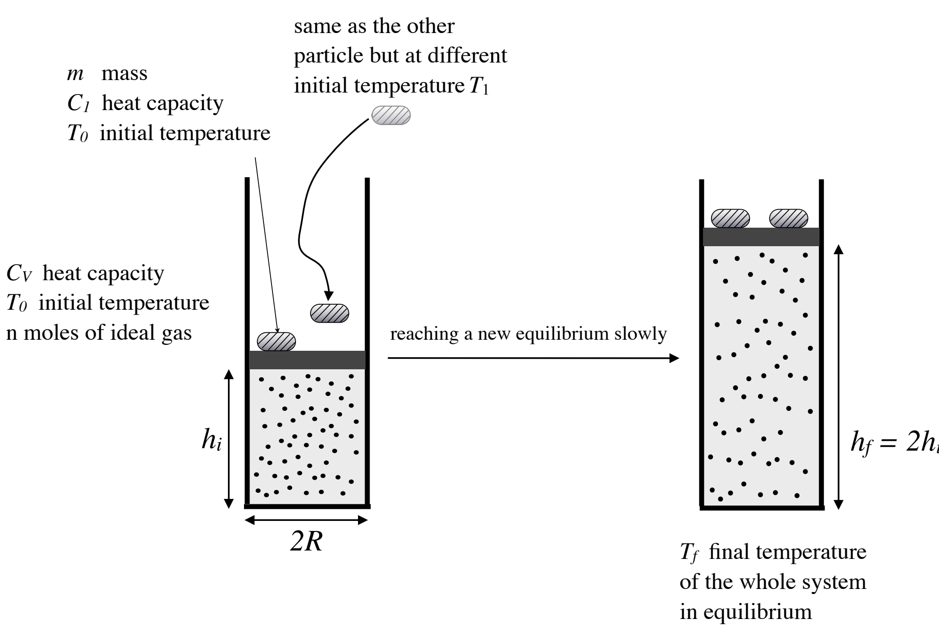

Figure 7: Question 6 Gravity piston.

#6 : UNDERGRADUATE STAT MECH/THERMAL SOLUTION

(a) Starting from $PV = nRT$, we obtain

Initial: $[mg/(\pi R^2)] \cdot (\pi R^2 h) = nRT_\mathrm{i}$
Final: $[2mg/(\pi R^2)] \cdot (\pi R^2 2h) = nRT_\mathrm{f}$
Thus, $T_\mathrm{f} = 4T_\mathrm{i}$

(b) The first particle: $\Delta S = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{dQ}{T} = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{C_1 dT}{T} = C_1 \ln \frac{T_\mathrm{f}}{T_\mathrm{i}} = C_1 \ln 4$.
Similarly, the second particle: $\Delta S = C_1 \ln \frac{T_\mathrm{f}}{T_1} = C_1 \ln \frac{4T_\mathrm{i}}{T_1}$.
For the gas + chamber: $\Delta S = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{dQ}{T} = \int_{T_\mathrm{i}}^{T_\mathrm{f}} \frac{C_\mathrm{v} dT}{T} + \int_{V_\mathrm{i}}^{V_\mathrm{f}} \frac{PdV}{T} = C_1 \ln \frac{T_\mathrm{f}}{T_\mathrm{i}} + nR \int_{V_\mathrm{i}}^{V_\mathrm{f}} \frac{dV}{V} = C_\mathrm{v} \ln 4 + nR \ln 2$.

Thus, the total change in entropy $\Delta S = C_1 \ln 4 + C_1 \ln \frac{4T_\mathrm{i}}{T_1} + C_\mathrm{v} \ln 4 + nR \ln 2$.

#7 : UNDERGRADUATE QM PROBLEM

An oven contains a gas with a molecular weight of $m = 1.8 \cdot 10^{-25}$kg at a temperature T of 600K. A cylindrical hole of diameter a in the side of the oven emits a collimated beam of molecules that make a circular spot of diameter D on a detector a distance L from the oven.

(a) Show that D cannot be made arbitrarily small by reducing a.

(b) Calculate the optimal aperture size a_{opt} that gives the smallest spot size.

(c) Find the minimum spot size D_{min} for $L = 0.2$m.

(Assume, for simplicity, that all atoms have identical momentum in the direction of the beam. $k = 1.38 \cdot 10^{-23}$J/K and $\hbar = 1.05 \cdot 10^{-34}$ Js.)

#7 : UNDERGRADUATE QM SOLUTION

(a) p_x = momentum in beam direction. By equipartition,

$$\frac{p_x^2}{2m} = \frac{1}{2}kt \Rightarrow p_x = \sqrt{mkT}. \tag{24}$$

Quantum uncertainty of momentum in y direction is given by

$$\Delta p_y a = \frac{\hbar}{2}. \tag{25}$$

This leads to a beam spread with angle γ, where

$$tan\gamma = \frac{\frac{1}{2}\Delta p_y}{p_x} = \frac{\hbar}{4a\sqrt{mkT}} \tag{26}$$

The spot diameter is:

$$D = a + 2Ltan\gamma = a + \frac{\hbar L}{2a\sqrt{mkT}}, \tag{27}$$

which cannot be made arbitrarily small by reducing a.

(b)

$$\frac{dD}{da} = 1 - \frac{\hbar L}{2a^2\sqrt{mkT}}. \tag{28}$$

Then $dD/da = 0$ implies

$$a_{opt} = \frac{\sqrt{\hbar L}}{\sqrt[4]{4mkT}} = 0.52\mu m \tag{29}$$

and substituting a_{opt} into the equation for D, $D_{min} = 2a_{opt} = 1.04\mu m$.

#8 : UNDERGRADUATE QM PROBLEM

Consider a hydrogen like bound state between two particles with mass m_1 and m_2 and charges q_1 and q_2.

(a) Write an expression for the ground state energy and wave function.

(b) Provide an expression for the characteristic radius of the system. Show that your result is consistent with the roughly equal sizes of the positronium and hydrogen systems.

Hydrogen atom we have

$$E_n = -\frac{\mu e^4}{2\hbar^2 n^2} \tag{30}$$

$$\psi_{n=0}(r) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \tag{31}$$

$$a_0 = \frac{\hbar^2}{\mu e^2} \tag{32}$$

where e is the charge on the electron and μ the reduced mass.

#8 : UNDERGRADUATE QM SOLUTION

A hydrogen like system, with $\times$ showing the center of mass,

$$\mathbf{m_1, q_1} \bigcirc \cdots \leftarrow \mathbf{r_1} \rightarrow \cdots \times \cdots\cdots \leftarrow \mathbf{r_2} \rightarrow \cdots\cdots \bigcirc \mathbf{m_2, q_2}$$

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{33}$$

"Bohr radius"

$$a \equiv \frac{\hbar^2}{2\mu q_1 q_2} = \frac{\hbar^2 (m_1 + m_2)}{2 q_1 q_2 m_1 m_2} \tag{34}$$

(a) Ground state

$$E_1 = \frac{-\mu q_1^2 q_2^2}{2\hbar^2} = \frac{-m_1 m_2 q_1^2 q_2^2}{2(m_1 + m_2)\hbar^2} \tag{35}$$

$$\Psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a} \tag{36}$$

(b) Characteristic radius r_{char} of system $a \equiv r_1 + r_2$, $r = max(r_1, r_2)$.

For $r_2 > r_1$,

$$r_2 = a - r_{cm} = \frac{\hbar^2(m_1 + m_2)}{2q_1q_2m_1m_2} - \frac{m_1r_1 + m_2r_2}{m_1 + m_2} \tag{37}$$

$$r_2 = \frac{\hbar^2(m_1 + m_2)}{2q_1q_2m_1m_2} - \frac{0 + m_2a}{m_1 + m_2} \tag{38}$$

$$r_2 = \frac{\hbar^2}{2q_1q_2}\frac{(m_1 + m_2)}{m_1m_2} - \frac{\hbar^2(m_1 + m_2)m_2}{2q_1q_2m_1m_2(m_1 + m_2)} = \frac{\hbar^2}{2q_1q_2m_2} \tag{39}$$

For $r_1 > r_2$, $r_1 = r_{cm}$ and

$$r_1 = a - r_{cm} = \frac{\hbar^2}{2q_1q_2m_1} \tag{40}$$

$$\Rightarrow r_{char} = \frac{\hbar^2}{2q_1q_2} Max\left(\frac{1}{m_1}, \frac{1}{m_2}\right) \tag{41}$$

where the last term selects the smaller mass particle, which is the electron for both H and positronium. Hence for

Hydrogen:

$$r_{char} = \frac{\hbar^2}{2\mu e^2} \simeq \frac{\hbar^2}{m_e e^2} \qquad \text{for}(m_p >> m_e) \tag{42}$$

Positronium:

$$r_{char} = \frac{\hbar^2}{m_e e^2}. \tag{43}$$

You can also go directly from second term in (34) to answer in (42), (43) skipping (37)-(41).

#9 : UNDERGRADUATE GENERAL PROBLEM This is a math problem. Laplace's method from 1774 give an estimate for an integral of the form

$$I(\alpha) = \int dz\, e^{-\alpha f(z)}, \tag{44}$$

as α becomes large.

In this limit, one approximates the integral by seeking a saddle point z_0 where $f'(z_0) = 0$; $f''(z_0) > 0$, and expanding the integral in the vicinity of z_0. Near z_0 $f(z) \approx f(z_0) + f^{(2)}(z_0)\frac{(z-z_0)^2}{2} + ...$, and the integral may be approximated as

$$I(\alpha) \approx e^{-\alpha f(z_0)} \int_{-\infty}^{+\infty} dz\, e^{-\alpha[f^{(2)}(z_0)\frac{(z-z_0)^2}{2}]}, \tag{45}$$

or

$$I(\alpha) \approx e^{-\alpha f(z_0)} \sqrt{\frac{2\pi}{\alpha f^{(2)}(z_0)}} \left\{ \int_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}}\, e^{-u^2} \right\}, \tag{46}$$

after changing variables to $u = \sqrt{\frac{\alpha f^{(2)}(z_0)}{2}}(z - z_0)$.

Expanding in $\frac{1}{\alpha}$ leads to

$$I(\alpha) \approx e^{-\alpha f(z_0)} \sqrt{\frac{2\pi}{\alpha f^{(2)}(z_0)}} \left\{ 1 + O(\frac{1}{\alpha}) \right\}. \tag{47}$$

Use this method on the gamma function

$$\Gamma(a) = \int_0^{\infty} dx\, x^{a-1} e^{-x} \tag{48}$$

to get the leading behavior for large a. For a equal an integer N, $\Gamma(N) = (N-1)!$. You may wish to change variables in Equation (**??**) from x to u where x = (a-1)u.

#9 : UNDERGRADUATE GENERAL SOLUTION

(1) The part 1 solution method is sketched in the problem. When one keeps the quadratic term in the exponential and expands the u^3, u^4 terms, we get zero for the term linear in u^3 as it is odd and the integral is over even limits.

The linear term in u^4 and the term in u^6 give contributions of order α^{-1}. One needs to do three integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} du\, e^{-u^2} &= \sqrt{\pi} \\
\int_{-\infty}^{\infty} du\, u^4 e^{-u^2} &= \frac{3}{4}\sqrt{\pi} \\
\int_{-\infty}^{\infty} du\, u^6 e^{-u^2} &= \frac{15}{8}\sqrt{\pi}.
\end{aligned} \tag{49}$$

(2) For part 2,

$$\Gamma(a) = \int_0^{\infty} dx\, x^{a-1} e^{-x}, \tag{50}$$

write $x = (a-1)u$ and find

$$\Gamma(a) = (a-1)^a \int_0^{\infty} du e^{-(a-1)(u-\log u)}. \tag{51}$$

$f(u) = u - \log u$ has a first derivative of $1 - \frac{1}{u}$ and second derivative $\frac{1}{u^2}$. Using Laplace's result we find for the leading term in $\Gamma(a)$ for large a:

$$\Gamma(a) \approx a^a e^{-(a-1)} \sqrt{\frac{2\pi}{a}}. \tag{52}$$

#10 : UNDERGRADUATE GENERAL PROBLEM

We will discuss an atmosphere of constant composition with mean molecular mass m. Let z be increasing distance going vertically up from the surface, $\rho(z)$ is the density, $T(z)$ the temperature and $P(z)$ the pressure.

a) Find a general expression for the change in pressure with hight z in an atmosphere in hydrostatic equilibrium.

b) Find an expression for $P(z)$ in terms of $T(z)$.

c) Find an expression for the distance H(z) over which the pressure decreases by e i.e. the distance such that $1/H(z) = -d \ln P/dz$.

#10 : UNDERGRADUATE GENERAL SOLUTION

a) An stable atmosphere is in hydrostatic equilibrium with the pressure balancing the gravity. Consider a slab of thickness $\Delta(z)$ and density $\rho(z)$. This slab exerts a force due to its weight on the slab below. Per unit area this force is a pressure. The increase in pressure across the slab ΔP is the weight of a column of height Δz and density $\rho(z)$,

$$\Delta P = -g(z)\rho(z)\Delta z, \quad \text{or} \quad \frac{dP}{dz} = -g(z)\rho(z) \tag{53}$$

where $g(z)$ is the acceleration due to gravity, and the negative sign (important!) shows that the pressure drops with increasing height.

b) We use the ideal gas law

$$P = \frac{\rho k T}{m}, \tag{54}$$

where m is the molecular mass. We use this to substitute for $\rho = Pm/(kT)$, giving

$$dP = -\frac{g(z)P(z)m}{kT(z)} dz \tag{55}$$

This integrates to give

$$P(z) = P(0) exp \left(- \int_0^z \frac{g(z)m}{kT(z)} dr \right) \tag{56}$$

where $r = 0$ at some z where the pressure is $P(0)$, e.g. the bottom of the atmosphere.

c) The expression for the distance H(z) over which the pressure decreases by e, is given by the above, where

$$H(z) = \frac{kT(z)}{g(z)m}. \tag{57}$$

PHYSICS DEPARTMENT EXAM
SPRING 2013. PART II GRADUATE PROBLEMS

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Mechanics		E & M		Stat Mech		Quantum		General	
11	12	13	14	15	16	17	18	19	20

• Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 : GRADUATE MECHANICS

A point object of mass m moves in the (x, y) plane. It is constrained to remain a distance l from the origin $(0, 0)$. If this mass experiences a uniform gravitational acceleration in the -y direction, show the motion of the point is that of a pendulum. You might wish to utilize Lagrange multipliers.

Show that small perturbations to the pendulum hanging straight down (toward the negative y direction) lead to stable motion. Also show that small perturbations to the pendulum hanging straight up (toward +y) lead to unstable motion.

#12 : GRADUATE MECHANICS

This Graduate Classic Mechanics problem involves Canonical transformations. Consider a Hamiltonian system $H(p,q)$ whose equation of motion is determined by the canonical equation, where p and q are canonical momentum and coordinate, respectively. Now we define a new set of variables P and Q which are functions of p and q as $P(p,q)$ and $Q = Q(p,q)$. In order to maintain the canonical form of the equation of motions in terms of variable P and Q, i.e., $\dot{P} = -\frac{\partial H}{\partial Q}$ and $\dot{Q} = \frac{\partial H}{\partial P}$, prove that the following condition needs to be satisfied

$$M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1}$$

where M is the Jacobian matrix

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix}. \tag{2}$$

Begin by expressing $\dot{Q}$ and $\dot{P}$ in terms of $\dot{q}$ and $\dot{p}$.

#13 : GRADUATE E & M

A particle has mass m and charge q, and is in a constant electric field $\vec{E} = E\widehat{x}$. Take $qE > 0$.

(a) Write the fully relativistic equations of motion, for position x^μ as a function of proper time τ.

(b) The initial conditions are: at $\tau = 0$, $x^\mu(\tau = 0) = 0$, and its initial spatial momentum is $\vec{p}(\tau = 0) = p_0\widehat{y}$. Find $\frac{dt}{d\tau}$ at $\tau = 0$.

(c) Find the fully relativistic solution of the equations of motion, as a function of its proper time τ, i.e. find $t(\tau)$, $x(\tau)$, $y(\tau)$, $z(\tau)$, using the boundary conditions of part (b).

(d) Find $\frac{dt}{d\tau}$ as a function of τ, and verify that it approaches c for τ large.

#14 : GRADUATE E & M

A long rod of *paramagnetic material* with linear magnetic permeability μ, and cross-sectional area a, is inserted a distance x into a long solenoid of length d, cross-sectional area A, carrying current I with total number of turns N. The current is held fixed by an external battery that can do work. Find the force (magnitude and direction) on the rod.

#15 : GRADUATE STAT MECH

a) Consider non-interacting non-relativistic Bose gas in 2D free space. The boson mass is M and the boson density is n_{2d}. Write an expression for n_{2d} at temperature T and chemical potential μ. Your expression should involve integrals, which should be in dimensionless form, but you do not need to evaluate them.

b) Repeat part (a) for 1D bose gas with density n_{1d}.

c) Prove that in 1D and 2D there are no Bose-Einstein condensations at any non-zero temperature T, unlike in the case in 3D.

#16 : GRADUATE STAT MECH

Derive the formula for the low-temperature specific heat $C_V(T)$ of an ideal Fermi gas with the density of states g at the Fermi level. Keep only the leading term.

Hint: Use the Sommerfeld expansion

$$\int\limits_0^\infty \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int\limits_0^\mu \phi(\varepsilon)d\varepsilon + \frac{\pi^2}{6}\frac{1}{\beta^2}\phi'(\mu) + \mathcal{O}\left(\frac{1}{\beta^3}\right) ,$$

which is valid for $\mu \gg \beta^{-1}$ and any well-behaved function $\phi(\varepsilon)$.

#17 : GRADUATE QUANTUM

Calculate the differential scattering cross section $d\sigma/d\Omega$ for a particle of mass m and momentum p to scatter off the spherical shell δ-function potential

$$V = \lambda\delta(r - r_0) \tag{3}$$

Assume the potential is weak enough that the Born approximation is valid. Write the result in terms of the scattering angle θ and the incident momentum. Make a crude plot of the scattering cross section as a function of p for 90° scattering.

#18 : GRADUATE QUANTUM

A free particle of mass M is confined to a cylindrical cavity (Figure **??**) of height L, inner radius a and outer radius b, with boundary conditions that the wavefunction ψ vanishes on the boundary. A magnetic field with total flux Φ_B passes through the central core of the cylinder, but vanishes in the region where the particle is trapped.

a) Write the Hamiltonian.

b) Find the allowed wavefunctions for the system. You can leave the quantization condition in terms of relations on Bessel functions.

c) Find the condition on Φ_B so that the energies are the same as the system with no magnetic field.

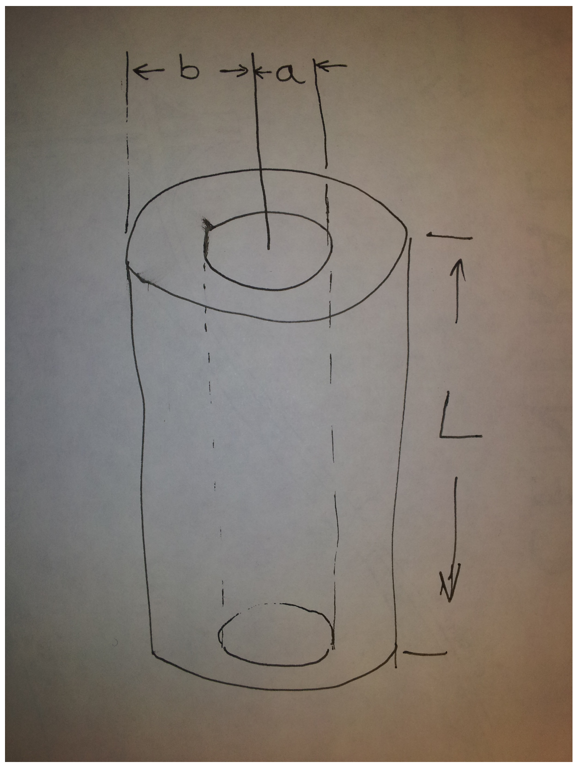


Figure 1: Question 18 Particle in cylinder.

#19 : GRADUATE GENERAL

To derive the Sommerfeld expansion in Problem 16,

$$\int_0^\infty \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \phi(\varepsilon)d\varepsilon + \frac{\pi^2}{6}\frac{1}{\beta^2}\phi'(\mu) + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

solve the following auxiliary problems:

(a) Show that

$$I_1(a) = \int_{-\infty}^\infty \frac{e^{ax}dx}{e^x+1} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1\,.$$

Hint: Consider an integral along the contour made of the real axis plus the line $\mathrm{Im}\, x = 2\pi$.

(b) Evaluate

$$I_2 = \int_0^\infty \frac{x dx}{e^x + 1}\,.$$

Hint: Show that I_2 is related to I_1:

$$I_2 = \frac{1}{2}\lim_{a\to 0^+}\frac{d}{da}\left[I_1(a) - \frac{1}{a}\right] = \frac{\pi^2}{12}\,.$$

(c) Finally, use the result of part (b) to derive the Sommerfeld formula assuming that $\mu \gg \beta^{-1}$ and that function $\phi(\varepsilon)$ is well-behaved.

#20 : GRADUATE GENERAL

The times below show when a satellite orbiting a planet is at a particular point (measured relative to distant stars) in its circular orbit. All four times are measured on Earth in seconds from some time on March 7th when the Earth is nearest to the planet, assuming 365.25 days (31,557,600 s) per year. The satellite makes four complete orbits between the times given on March 7 and March 14, and again between the times given on June 7 and June 14. The distance from the Sun to the Earth is $1AU = 1.5 \times 10^{11}$m, from the sun to the planet 7.8×10^{11}m, and from the planet to the satellite 422,000 km. $G = 6.67 \times 10^{-11}\mathrm{Nm^2kg^{-2}}$.

a) Compare the period in March and June (1 point).

b) Qualitatively explain your answer for (a) (3 points).

c) Using only the numbers and constant given in the question, calculate a famous physical constant (3 points).

d) Quantitatively explain the interval between the March 7 and June 7 times. Assume the planet is stationary (3 points).

Day	time (s)
March 7	28,705
March 14	640,350
June 7	7,980,587
June 14	8,592,293

PHYSICS DEPARTMENT EXAM
SPRING 2013. PART II GRADUATE PROBLEMS

INSTRUCTIONS

• You should not have anything close to you other than your pens, pencils, calculator, and food items. Please deposit your belongings (books, notes, backpacks, phone etc.) in a corner of the exam room.

• Departmental examination paper is provided. Colored scratch paper is also provided. Please make sure you:

a. Write the problem number and your ID number on each sheet;

b. Write only on one side of the paper;

c. Start each problem on the attached examination sheets;

d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

• Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem.

• The questions are grouped in 5 Sections: Mechanics, E&M, Quantum, StatMech, and General. You must attempt at least one problem from each Section. You are to do seven (7) of the ten (10) problems. At the conclusion of the examination period, please staple sheets from each problem together.
Circle the 7 problems you wish to be graded:

Mechanics		E & M		Stat Mech		Quantum		General	
11	12	13	14	15	16	17	18	19	20

• Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 : GRADUATE MECHANICS

A point object of mass m moves in the (x, y) plane. It is constrained to remain a distance l from the origin $(0, 0)$. If this mass experiences a uniform gravitational acceleration in the -y direction, show the motion of the point is that of a pendulum. You might wish to utilize Lagrange multipliers.

Show that small perturbations to the pendulum hanging straight down (toward the negative y direction) lead to stable motion. Also show that small perturbations to the pendulum hanging straight up (toward +y) lead to unstable motion.

#11 : GRADUATE MECHANICS SOLUTION

The Lagrangian for this situation is

$$L(x, y, \dot{x}, \dot{y}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy + \lambda(x^2 + y^2 - l^2). \tag{1}$$

From the Euler Lagrange equations, we have

$$m\ddot{x} = 2\lambda x, \tag{2}$$

and

$$m\ddot{y} = 2\lambda y + mg, \tag{3}$$

and

$$l^2 = x^2 + y^2. \tag{4}$$

Write $y = l \sin\theta$ and $x = l \sin\theta$, then eliminating the Lagrange multiplier λ we arrive at the pendulum equation:

$$\ddot{\theta} + \omega^2 \sin\theta = 0; \; \omega^2 = \frac{g}{l} \tag{5}$$

There are two fixed points for this equation $\theta = 0$ and $\theta = \pi$. Writing $\theta = 0 + \delta\theta(t)$, we find to linear order in $\delta\theta$ around this fixed point

$$\delta\ddot{\theta}(t) + \omega^2 \delta\theta(t) = 0, \tag{6}$$

which executes stable periodic motions in the neighborhood of $\theta = 0$.

Linearizing about the fixed point at $\theta = \pi$, we find

$$\delta\ddot{\theta}(t) - \omega^2 \delta\theta(t) = 0, \tag{7}$$

which has unstable solutions or exponential growth in time.

#12 : GRADUATE MECHANICS

This Graduate Classic Mechanics problem involves Canonical transformations. Consider a Hamiltonian system $H(p,q)$ whose equation of motion is determined by the canonical equation, where p and q are canonical momentum and coordinate, respectively. Now we define a new set of variables P and Q which are functions of p and q as $P(p,q)$ and $Q = Q(p,q)$. In order to maintain the canonical form of the equation of motions in terms of variable P and Q, i.e., $\dot{P} = -\frac{\partial H}{\partial Q}$ and $\dot{Q} = \frac{\partial H}{\partial P}$, prove that the following condition needs to be satisfied

$$M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8}$$

where M is the Jacobian matrix

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix}. \tag{9}$$

Begin by expressing $\dot{Q}$ and $\dot{P}$ in terms of $\dot{q}$ and $\dot{p}$.

#12 : GRADUATE MECHANICS SOLUTION

First,

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}. \tag{10}$$

Then

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P} & \frac{\partial p}{\partial P}. \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}. \tag{11}$$

we have

$$\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P} & \frac{\partial p}{\partial P}. \end{pmatrix} \tag{12}$$

And since

$$\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial p} & \frac{\partial P}{\partial p}. \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P} & \frac{\partial p}{\partial P}. \end{pmatrix}^T = 1, \tag{13}$$

we have

$$M \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{14}$$

#13 : GRADUATE E & M

A particle has mass m and charge q, and is in a constant electric field $\vec{E} = E\widehat{x}$. Take $qE > 0$.

(a) Write the fully relativistic equations of motion, for position x^μ as a function of proper time τ.

(b) The initial conditions are: at $\tau = 0$, $x^\mu(\tau = 0) = 0$, and its initial spatial momentum is $\vec{p}(\tau = 0) = p_0\widehat{y}$. Find $\frac{dt}{d\tau}$ at $\tau = 0$.

(c) Find the fully relativistic solution of the equations of motion, as a function of its proper time τ, i.e. find $t(\tau)$, $x(\tau)$, $y(\tau)$, $z(\tau)$, using the boundary conditions of part (b).

(d) Find $\frac{dt}{d\tau}$ as a function of τ, and verify that it approaches c for τ large.

#13 : GRADUATE E & M SOLUTION

(a) $m\frac{d^2x^\mu}{d\tau^2} = \frac{q}{c}F^{\mu\nu}\frac{dx_\nu}{d\tau}$, gives

$$\frac{d^2ct}{d\tau^2} = \frac{qE}{mc}\frac{dx}{d\tau}, \qquad \frac{d^2x}{d\tau^2} = \frac{qE}{mc}\frac{dct}{d\tau}, \qquad \frac{d^2y}{d\tau^2} = \frac{d^2z}{d\tau^2} = 0.$$

(b) $\frac{dx^\mu}{d\tau} = \frac{1}{m}p^\mu$. At $\tau = 0$, we have $p^\mu = (E_0/c, 0, p_0, 0)$, with $E_0 = \sqrt{(mc^2)^2 + c^2p_0^2}$. So $c\frac{dt}{d\tau} = E_0/mc$.

(c,d) $y = p_0\tau/m$, and $z = 0$. Combining the EOM, we get $\frac{d^2}{d\tau^2}\frac{dx}{d\tau} = (qE/mc)^2\frac{dx}{d\tau}$. Using the above initial conditions, integrating the EOM once gives

$$\frac{dx}{d\tau} = \frac{E_0}{mc}\sinh(qE\tau/mc), \qquad \frac{dct}{d\tau} = \frac{E_0}{mc}\cosh(qE\tau/mc)$$

The ratio of these gives the answer to part (d): $dx/dt = c\tanh(qE\tau/mc)$, which indeed approaches c. Integrating them again gives

$$x = \frac{E_0}{qE}\left(\cosh(qE\tau/mc) - 1\right), \qquad ct = \frac{E_0}{qE}\sinh(qE\tau/mc).$$

#14 : GRADUATE E & M

A long rod of *paramagnetic material* with linear magnetic permeability μ, and cross-sectional area a, is inserted a distance x into a long solenoid of length d, cross-sectional area A, carrying current I with total number of turns N. The current is held fixed by an external battery that can do work. Find the force (magnitude and direction) on the rod.

#14 : GRADUATE E & M SOLUTION

The change in magnetic energy W_m due to insertion of rod $= \frac{1}{2}\Delta L\, I^2$ where ΔL is the change in the inductance of the system due to the rod. For a closed system this would be the work done on the rod. However, since current is fixed, the external battery also does work on the system as the rod is moved. The work done by the battery is $-I \int dt\, EMF$ where $EMF = I\partial L/\partial t$, so the work by the battery on the rod is $-\Delta L\, I^2$.

Thus, the total work done on the rod is $\Delta E = -\frac{1}{2}\Delta L\, I^2$. The force F on the rod is $F = -\partial \Delta E/\partial x$. It remains only to find an expression for ΔL. This can be found from the general expression

$$W_m = \frac{1}{2} L\, I^2 = \int dV\, \mu H^2/8\pi.$$

Maxwell's equation implies $H = 4\pi IN/(dc)$, everywhere in the solenoid and the rod (neglecting end effects since the rod is long). This implies

$$\begin{aligned} \Delta L &= \frac{H^2}{4\pi I^2}(\mu a x + (A-a)x + A(d-x) - Ad) \\ &= \frac{H^2}{4\pi I^2} a(\mu - 1)x. \end{aligned}$$

Therefore $\Delta E = -\frac{H^2}{8\pi} a(\mu-1)x$, so $F = \frac{H^2}{8\pi} a(\mu - 1)$. The force is in the direction of increasing x (into the solenoid), since for a paramagnetic material $\mu > 1$.

#15 : GRADUATE STAT MECH

a) Consider non-interacting non-relativistic Bose gas in 2D free space. The boson mass is M and the boson density is n_{2d}. Write an expression for n_{2d} at temperature T and chemical potential μ. Your expression should involve integrals, which should be in dimensionless form, but you do not need to evaluate them.

b) Repeat part (a) for 1D bose gas with density n_{1d}.

c) Prove that in 1D and 2D there are no Bose-Einstein condensations at any non-zero temperature T, unlike in the case in 3D.

#15 : GRADUATE STAT MECH SOLUTION

The 2D case.

$$N_{2D}(\mu, T) = \frac{A}{(2\pi)^2} \int d^2\vec{k} \frac{1}{e^{\frac{\hbar^2 k^2}{2mk_BT} - \frac{\mu}{k_BT}} - 1}. \tag{15}$$

Set $x = \frac{\hbar^2 k^2}{2mk_BT}$, we have

$$\begin{aligned} n_{2d}(\mu, T) &= \frac{2\pi}{(2\pi)^2} \frac{2mk_BT}{\hbar^2} \int_0^\infty \frac{1}{2} \frac{xdx}{ze^x - 1} \\ &= \frac{mk_BT}{(2\pi)\hbar^2} \int_0^\infty \frac{dx}{ze^x - 1}, \end{aligned} \tag{16}$$

where $z = e^{-\frac{\mu}{k_BT}}$.

Similarly, in 1D,

$$N_{1D}(\mu, T) = \frac{L}{2\pi} \int dk \frac{1}{e^{\frac{\hbar^2 k^2}{2mk_BT} - \frac{\mu}{k_BT}} - 1}, \tag{17}$$

we have

$$\begin{aligned} n_{1d}(\mu, T) &= \frac{1}{2\pi} \sqrt{\frac{2mk_BT}{\hbar^2}} \frac{1}{2} \int_0^\infty \frac{x^{\frac{1}{2}} dx}{ze^x - 1} \\ &= \frac{\sqrt{mk_BT}}{2\sqrt{2}\pi\hbar} \int_0^\infty \frac{x^{-\frac{1}{2}} dx}{ze^x - 1}. \end{aligned} \tag{18}$$

If BEC can occur, which means that $\mu = 0$ and thus $z = 1$. In this case, both integrals in 2D and 1D diverges. Let us introduce a infrared cutoff x_0.

In the 2D case,

$$I_{2D} = \int_{x_0}^{\infty} \frac{dx}{e^x - 1} \tag{19}$$

diverges logarithmically as $\ln \frac{1}{x_0}$. In the 1D case,

$$I_{1D} = \int_{x_0}^{\infty} \frac{x^{-\frac{1}{2}} dx}{e^x - 1} \tag{20}$$

diverges as $x_0^{-\frac{1}{2}}$. Thus no matter how large the densities are, at any finite temperature T, there are always negative values of μ's, such that $0 < z < 1$, to satisfy the equations of $N_{2D,1D}(\mu, T)$.

#16 : GRADUATE STAT MECH

Derive the formula for the low-temperature specific heat $C_V(T)$ of an ideal Fermi gas with the density of states g at the Fermi level. Keep only the leading term.

Hint: Use the Sommerfeld expansion

$$\int_0^\infty \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \phi(\varepsilon)d\varepsilon + \frac{\pi^2}{6}\frac{1}{\beta^2}\phi'(\mu) + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

which is valid for $\mu \gg \beta^{-1}$ and any well-behaved function $\phi(\varepsilon)$.

#16 : GRADUATE STAT MECH SOLUTION

$$C_V(T) = \frac{\partial E}{\partial T} = \frac{\partial}{\partial T}\int_0^\infty \varepsilon g(\varepsilon) n_F\left(\frac{\varepsilon-\mu}{T}\right)d\varepsilon\,, \quad n_F(z) = \frac{1}{e^z+1}\,.$$

For simplicity, let us assume the energy-independent density of states, $g(\varepsilon) =$ const, then (for fixed particle density) μ is approximately constant with T. The above expression for C_V becomes

$$C_V(T) \simeq -g\int_0^\infty \varepsilon\frac{\varepsilon-\mu}{T}\frac{\partial n_F}{\partial\varepsilon}d\varepsilon\,.$$

The Sommerfeld formula implies

$$\int_0^\infty d\varepsilon\phi(\varepsilon)\frac{\partial n_F}{\partial\varepsilon} \simeq -\phi(\mu) - \frac{\pi^2}{6}\frac{1}{\beta^2}\phi''(\mu)\,,$$

which can be shown using integration by parts. In our case $\phi(\varepsilon) = (\varepsilon-\mu)\varepsilon$ and we arrive at

$$C_V(T) \simeq \frac{\pi^2}{3}gT\,.$$

#17 : GRADUATE QUANTUM

Calculate the differential scattering cross section $d\sigma/d\Omega$ for a particle of mass m and momentum p to scatter off the spherical shell δ-function potential

$$V = \lambda\delta(r - r_0) \tag{21}$$

Assume the potential is weak enough that the Born approximation is valid. Write the result in terms of the scattering angle θ and the incident momentum. Make a crude plot of the scattering cross section as a function of p for $90°$ scattering.

#17 : GRADUATE QUANTUM SOLUTION

The scattering cross section in the Born approximation is

$$\frac{d\sigma}{d\Omega} = \left|\frac{1}{4\pi}\widetilde{U}(\mathbf{q})\right|^2 \tag{22}$$

where

$$\begin{aligned}
\widetilde{U}(\mathbf{q}) &= \frac{2m}{\hbar^2}\int d^3\mathbf{r}\; e^{-i\mathbf{q}\cdot\mathbf{r}}\; V(\mathbf{r}) \\
&= \frac{2m\lambda}{\hbar^2}\int d^3\mathbf{r}\; e^{-i\mathbf{q}\cdot\mathbf{r}}\; \delta(r - r_0) \\
&= \frac{2m\lambda}{\hbar^2}\int r^2 dr\, d\cos\theta\, d\phi\; e^{-iqr\cos\theta}\; \delta(r - r_0) \\
&= \frac{4\pi m\lambda r_0^2}{\hbar^2}\int d\cos\theta\; e^{-iqr_0\cos\theta} \\
&= \frac{4\pi m\lambda r_0^2}{\hbar^2}\frac{2\sin qr_0}{qr_0}
\end{aligned} \tag{23}$$

$$\frac{d\sigma}{d\Omega} = \frac{4m^2\lambda^2 r_0^2}{\hbar^4}\frac{\sin^2 qr_0}{q^2} \tag{24}$$

q is the momentum transfer,

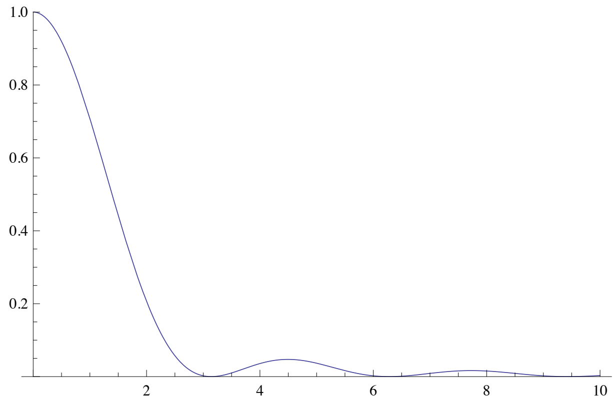
$$\begin{aligned}
\hbar\mathbf{q} &= \mathbf{p}' - \mathbf{p} \\
\hbar^2 q^2 &= 2p^2(1 - \cos\theta) = 4p^2\sin^2\theta/2
\end{aligned} \tag{25}$$

since $|\mathbf{p}'| = |\mathbf{p}|$

$$\frac{d\sigma}{d\Omega} = \frac{m^2\lambda^2 r_0^2}{\hbar^2 p^2} \frac{\sin^2\left[2pr_0/\hbar \sin\theta/2\right]}{\sin^2\theta/2} \tag{26}$$

For $\theta = \pi/4$,

$$\frac{d\sigma}{d\Omega} = \frac{2m^2\lambda^2 r_0^2}{\hbar^2 p^2} \sin^2\left[\sqrt{2}pr_0/\hbar\right] \tag{27}$$



#18 : GRADUATE QUANTUM

A free particle of mass M is confined to a cylindrical cavity (Figure 1) of height L, inner radius a and outer radius b, with boundary conditions that the wavefunction ψ vanishes on the boundary. A magnetic field with total flux Φ_B passes through the central core of the cylinder, but vanishes in the region where the particle is trapped.

a) Write the Hamiltonian.

b) Find the allowed wavefunctions for the system. You can leave the quantization condition in terms of relations on Bessel functions.

c) Find the condition on Φ_B so that the energies are the same as the system with no magnetic field.

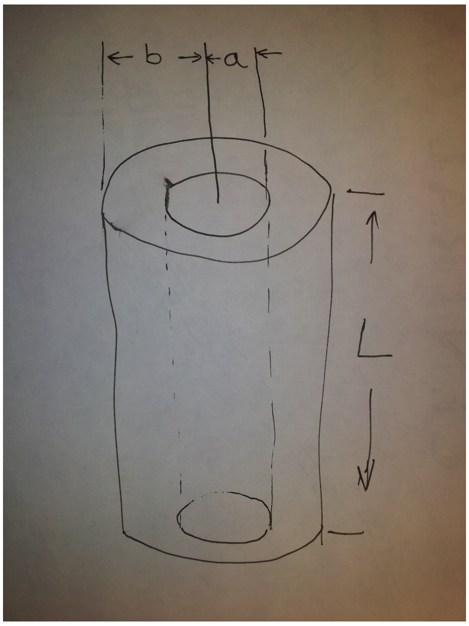


Figure 1: Question 18 Particle in cylinder.

#18 : GRADUATE QUANTUM SOLUTION

The Hamiltonian is

$$H = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 \tag{28}$$

The vector potential in the cavity is (cylindrical coordinates ρ, θ, z)

$$\mathbf{A} = \frac{\Phi_B}{2\pi\rho}\hat{\boldsymbol{\theta}} \tag{29}$$

$$\begin{aligned} H &= \frac{1}{2M}\left(p^2 - \frac{e}{c}(\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}) + \frac{e^2}{c^2}A^2\right) \\ &= \frac{1}{2M}\left(p^2 - \frac{e\Phi_B}{\pi c\rho}\mathbf{p}\cdot\hat{\boldsymbol{\theta}} + \frac{e^2}{c^2}\frac{\Phi_B^2}{4\pi^2\rho^2}\right) \\ &= \frac{1}{2M}\left(p^2 - \frac{e\Phi_B}{\pi c\rho^2}L_z + \frac{e^2}{c^2}\frac{\Phi_B^2}{4\pi^2\rho^2}\right) \end{aligned} \tag{30}$$

since

$$\mathbf{p}\cdot\hat{\boldsymbol{\theta}} = \frac{L_z}{\rho} \tag{31}$$

The wavefunction has the form

$$\psi = \psi(\rho)e^{im\theta}\sin(kz) \qquad kL = n\pi \tag{32}$$

$$H\psi = \frac{\hbar^2}{2M}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{n^2\pi^2}{L^2}\psi + \frac{m^2}{\rho^2}\psi - \frac{e\Phi_B m}{\hbar\pi c\rho^2}\psi + \frac{e^2\Phi_B^2}{4\hbar^2\pi^2c^2\rho^2}\psi\right] = E\psi \tag{33}$$

$$\begin{aligned} \rho^2\frac{\partial^2\psi}{\partial\rho^2} + \rho\frac{\partial\psi}{\partial\rho} - m^2\psi + \frac{e\Phi_B m}{\hbar\pi c}\psi - \frac{e^2\Phi_B^2}{4\hbar^2\pi^2c^2}\psi &= -2M\mathcal{E}\rho^2\psi \\ \mathcal{E} &= \frac{E}{\hbar^2} - \frac{n^2\pi^2}{L^2} \end{aligned} \tag{34}$$

so that

$$\rho^2\frac{\partial^2\psi}{\partial\rho^2} + \rho\frac{\partial\psi}{\partial\rho} + 2M\mathcal{E}\rho^2\psi - \left(m - \frac{e\Phi_B}{2\hbar\pi c}\right)^2\psi = 0 \tag{35}$$

The solution is

$$\psi = \alpha J_\nu\left(\sqrt{2M\mathcal{E}}\,\rho\right) + \beta N_\nu\left(\sqrt{2M\mathcal{E}}\,\rho\right)$$
$$\nu = m - \frac{e\Phi_B}{2\hbar\pi c} \tag{36}$$

α and β have to be chosen so that ψ vanishes at $\rho = a, b$.

$$\psi = A\left[N_\nu\left(\sqrt{2M\mathcal{E}}\,a\right) J_\nu\left(\sqrt{2M\mathcal{E}}\,\rho\right) - J_\nu\left(\sqrt{2M\mathcal{E}}\,a\right) N_\nu\left(\sqrt{2M\mathcal{E}}\,\rho\right)\right] \tag{37}$$

vanishes at $\rho = a$. The vanishing at b requires

$$N_\nu\left(\sqrt{2M\mathcal{E}}\,a\right) J_\nu\left(\sqrt{2M\mathcal{E}}\,b\right) = J_\nu\left(\sqrt{2M\mathcal{E}}\,a\right) N_\nu\left(\sqrt{2M\mathcal{E}}\,b\right) \tag{38}$$

For the energies to be the same as $\Phi_B = 0$ requires that ν be an integer, so that

$$\frac{e\Phi_B}{2\hbar\pi c} = \text{integer} \tag{39}$$

#19 : GRADUATE GENERAL

To derive the Sommerfeld expansion in Problem 16,

$$\int_0^\infty \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \phi(\varepsilon)d\varepsilon + \frac{\pi^2}{6}\frac{1}{\beta^2}\phi'(\mu) + \mathcal{O}\left(\frac{1}{\beta^3}\right),$$

solve the following auxiliary problems:

(a) Show that

$$I_1(a) = \int_{-\infty}^{\infty} \frac{e^{ax}dx}{e^x+1} = \frac{\pi}{\sin \pi a}, \quad 0 < a < 1\,.$$

Hint: Consider an integral along the contour made of the real axis plus the line $\operatorname{Im} x = 2\pi$.

(b) Evaluate

$$I_2 = \int_0^\infty \frac{x dx}{e^x+1}\,.$$

Hint: Show that I_2 is related to I_1:

$$I_2 = \frac{1}{2}\lim_{a\to 0^+}\frac{d}{da}\left[I_1(a) - \frac{1}{a}\right] = \frac{\pi^2}{12}\,.$$

(c) Finally, use the result of part (b) to derive the Sommerfeld formula assuming that $\mu \gg \beta^{-1}$ and that function $\phi(\varepsilon)$ is well-behaved.

#19 : GRADUATE GENERAL SOLUTION

(a) The indicated contour integral, traversed in the counterclockwise direction, is equal to $I_1(1 - e^{2\pi i a})$. On the other hand, it is also equal to $2\pi i$ times the residue of the simple pole at $x = i\pi$. Calculating the residue, one obtains the quoted result.

(b) Once the equation given as the hint is verified, elementary algebra leads to $I_2 = \pi^2/12$.

(c) The left-hand side can be split into two integrals, one from 0 to μ and the other from μ to infinity. The first integral can then be transformed as follows:

$$\int_0^\mu \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \phi(\varepsilon)d\varepsilon - \int_0^\mu \frac{\phi(\varepsilon)d\varepsilon}{e^{\beta(\mu-\varepsilon)}+1} \, .$$

Consider the second term in this expression. Due to exponentially fast convergence, the lower integration limit can be extended to negative infinity. For the same reason, function $\phi(x)$ can be expanded in Taylor series around $\varepsilon = \mu$. Using the result of part (b), this term evaluates to $-\phi(\mu)\ln 2 + (\pi^2/12)\phi'(\mu)$. The integral from μ to infinity can be done in the same fashion, giving $+\phi(\mu)\ln 2 + (\pi^2/12)\phi'(\mu)$. Adding all three contributions, we arrive at the Sommerfeld formula.

#20 : GRADUATE GENERAL

The times below show when a satellite orbiting a planet is at a particular point (measured relative to distant stars) in its circular orbit. All four times are measured on Earth in seconds from some time on March 7th when the Earth is nearest to the planet, assuming 365.25 days (31,557,600 s) per year. The satellite makes four complete orbits between the times given on March 7 and March 14, and again between the times given on June 7 and June 14. The distance from the Sun to the Earth is $1AU = 1.5 \times 10^{11}$m, from the sun to the planet 7.8×10^{11}m, and from the planet to the satellite 422,000 km. $G = 6.67 \times 10^{-11}\text{Nm}^2\text{kg}^{-2}$.

a) Compare the period in March and June (1 point).

b) Qualitatively explain your answer for (a) (3 points).

c) Using only the numbers and constant given in the question, calculate a famous physical constant (3 points).

d) Quantitatively explain the interval between the March 7 and June 7 times. Assume the planet is stationary (3 points).

Day	time (s)
March 7	28,705
March 14	640,350
June 7	7,980,587
June 14	8,592,293

#20 : GRADUATE GENERAL SOLUTION

a) The first time is arbitrary since the satellite is not synchronized to the orbital motion of the Earth or the planet. In the table below, the interval is the number of seconds between the two paired observations. This interval is also the time for four orbits. The period is 1/4 of the interval. The period in June is $\Delta P = 15.3$ s longer.

Day	time (s)	interval (s)	period (s)
March 7	28,705		
March 14	640,350	611,645	152,911.2
June 7	7,980,587		
June 14	8,592,293	611,706	152,926.5

b) The period in June is longer because of the Doppler effect. In March the Earth is closest to the planet, and hence the orbital motions of both are perpendicular to the path between the two (Figure 2 below). Orbital motions can be ignored. June 7 is 92 days later than March 7, which is 1/4 of an Earth year later, when the Earth's motion about the sun is directly away from the planet and this gives the Doppler shift and the longer period.

c) We can find (dimensional analysis) the speed of light, since we know distances and times. In June the Earth's velocity $v = 2\pi 1.5 \times 10^{11} m/3.15 \times 10^7 s = 30,000$ m/s is directed away from the planet and its satellite. The proper period of the satellite (as measured from the planet) is $P = 152,911.2$ s. In one proper period the Earth moves $d = vP = 4.58 \times 10^9$m. The speed of light is then $c = d/t = vP/\Delta P = 4.58 \times 10^9/15.3 = 3 \times 10^8$m/s, where we are using a form of the Doppler formula, $\Delta P/P = v/c$.

d) The interval between the March 7 and June 7 times $t_b = 7,951,382$ s is 52 proper periods plus 500 s. The 500 seconds is the time for light to travel the extra distance from the planet to the Earth, which is approximately 1 AU (more accurately 1.095 AU from Figure 2 when we ignore the motion of the planet). We can again get the speed of light from the 1 AU the 500 s. The interval t_b is exactly 52 apparent (Doppler shifted) periods. The apparent periods are larger by the component of the Earth's velocity along the line to the planet.

The (fake) data refer to the orbit of the volcanic satellite Io around Jupiter. Observations by Ole Rømer around 1670 lead to the first accurate measurement of the speed of light. Given data spread over a year you can see the Doppler effect (not described till 1842!) change sign when the Earth approaches the planet, and the pattern repeats each time the Earth is nearest to the planet.

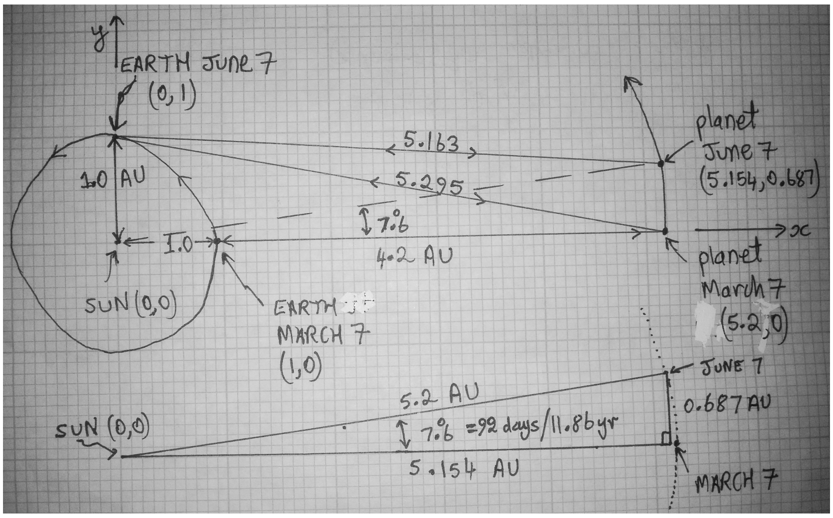


Figure 2: Question 20 with (x-y) coordinates centered on Sun in AU. The lower right angled triangle (not part of the question) can be used to find the position of the planet on June 7 using the known 5.2 AU and 7.6 degree motions. The error we make assuming the planet is stationary is the difference between 5.163 and 5.295 AU.