**PHYSICS DEPARTMENT EXAM**
**FALL 2012. PART I**

## INSTRUCTIONS

• Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. The questions are grouped in five Sections: Mechanics, E&M, Quantum, StatMech, and General. You must attempt at least one problem from each Section. Credit will be assigned for seven (7) questions only.

• You should not have anything close to you other than your pens, pencils, calculator, and food items. Please deposit your belongings (books, notes, backpacks, etc.) in a corner of the exam room.

• Departmental examination paper is provided. Colored scratch paper is also provided. Please make sure you:

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b. Write only on one side of the paper;

c. Start each problem on the attached examination sheets;

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| Mechanics | | E & M | | Quantum | | Stat Mech | | General | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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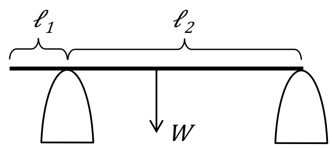
## #1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM:

A uniform rod of weight $W$ rests horizontally on two supports as shown. The right support is suddenly removed. What is the force on the remaining support immediately thereafter?

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SOLUTION:

Since initially there is no angular velocity, the centripetal acceleration is zero, and so the force in question $N$ is directed vertically upward. We can write two equations relating the torques and angular accelerations, one about the remaining support (moment of inertia $I_s$) and the other about the center of mass (moment of inertia $I_c$):

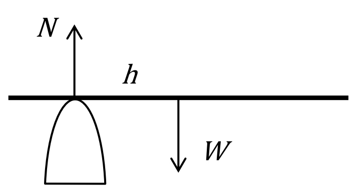
$$I_s\ddot{\theta} = Wh\,, \quad I_c\ddot{\theta} = Nh\,,$$

so that

$$N = \frac{I_c}{I_s}W\,.$$

Calculating the moments of inertia, we get

$$N = \frac{(l_1 + l_2)^2}{4(l_1^2 - l_1 l_2 + l_2^2)}\,W\,.$$



## #2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM:

As it falls vertically under the influence of gravity, the hail stone grows by accretion of microdroplets of water from the atmosphere. The droplets have negligible velocity and their collisions with the hail stone are almost completely inelastic. Suppose that the mass $m(t)$ of the hail stone grows linearly with its velocity $v = v(t)$:

$$\frac{dm}{dt} = kv\,, \quad k = \text{const}\,.$$

Find the function $v(t)$ in the limit of large time $t$ where it becomes independent of the initial conditions, $m(0)$ and $v(0)$.

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SOLUTION:

Use the equations

$$\frac{dm}{dt} = kv\,, \quad \frac{dp}{dt} = \frac{dm}{dt}v + m\frac{dv}{dt} = mg\,,$$

which imply

$$\left(\frac{dm}{dt}\right)^2 + m\frac{d^2m}{dt^2} = \frac{1}{2}\frac{d^2}{dt^2}\,m^2 = kgm\,.$$

Let $X \equiv m^2$, then

$$\frac{d^2X}{dt^2} = 2kgX^{1/2}.$$

This is analogous to the equation of motion of a particle subject to the external force that grows with coordinate as $X^{1/2}$. Such a particle would accelerate without bound so that the initial value of "position" $X$ and "velocity" $dX/dt$ would indeed be unimportant in the long-time limit.

Looking for a solution in the form $X(t) = At^\alpha$, we find $A = (kg/6)^2$, $\alpha = 4$, and so

$$m(t) = \frac{1}{6}kgt^2\,, \quad v(t) = \frac{1}{3}gt\,.$$

Effectively, the free fall acceleration gets reduced three times.

## #3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM:

A plate of cross-section area $A$ and small thickness $h$ is made of metal with electrical resistivity $\rho$. The plate is suddenly placed in an external electric field $E_0$ normal to its surface. Find the total energy lost in the Joule heating.

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SOLUTION:

The heating is expressed as the integral over current density

$$H = \int d^3r dt \rho J^2(\mathbf{r}, t)\,.$$

The current density is the time derivative of the polarization and it also obeys Ohm's law

$$J = \frac{\partial P}{\partial t} = \frac{E}{\rho}\,.$$

In general, to relate $P$ and the electric field $E$ one needs to solve Poisson equation. However, it is well known that the field inside a uniformly polarized plate is also uniform. Using CGS units,

$$E = E_0 - 4\pi P\,.$$

In particular, in the final equilibrium state $P(\mathbf{r}, \infty) = E_0/4\pi$. Since none of the above equations contain spatial coordinates, they can be made consistent with each other. In other words, we can be certain that $P$ is uniform inside the plate at all $t$ as long as

$$\frac{\partial J}{\partial t} = \frac{1}{\rho}\frac{\partial E}{\partial t} = -\frac{4\pi}{\rho}\frac{\partial P}{\partial t} = -\frac{4\pi}{\rho}J\,.$$

The solution of this differential equation is

$$J = \frac{E_0}{\rho}\exp\left(-\frac{4\pi}{\rho}t\right)\,.$$

Substituting it into the first equation for $H$, we find

$$H = \frac{1}{8\pi}E_0^2 Ah\,.$$

We see that the Joule losses are independent of the resistivity $\rho$. A very similar result applies to charging of a capacitor by a battery in the $RC$-circuit: the total Ohmic losses are exactly equal to the charging energy $CV^2/2$ and do not depend on the resistance $R$ of the circuit.

## #4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM:

Consider a metallic wire of length $L$ and radius $a$, which runs parallel to a conducting plane a distance $D$ away. Find an approximate expression for the capacitance of this system assuming $L \gg D \gg a$.

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SOLUTION:

According to the method of images, the capacitance $C$ between the wire and the plane is equal to one half of the capacitance $C_2$ between two parallel wires separated by distance $2D$. If the wires have charges $\pm Q$, the electric field each of them creates is approximately equal to that of a single infinite wires of linear charge density $\lambda = \pm Q/L$,

$$E(r) = 2\lambda/r\,, \quad r > a.$$

Integrating this from $r = a$ to $r = 2D$ and adding the (equal) contribution from the other wire, the estimate for the potential difference between the wires is $4\lambda \ln(2D/a)$, so that the capacitance is

$$C = \frac{1}{2}C_2 \simeq \frac{L}{2\ln(2D/a)}\,, \quad L \gg D \gg a\,.$$

The condition $D \gg a$ can be removed by using the method of images in a better way. Consider two fictitious linear charges at distances $\pm\sqrt{D^2 - a^2}$ from the plane. (The first of these is inside the wire.) It is easy to check that these charges make the surface of the wire equipotential. The improved formula for the inverse capacitance is

$$C^{-1} \simeq \frac{2}{L} \ln\left(\frac{D + \sqrt{D^2 - a^2}}{a}\right)\,, \quad L \gg D\,.$$

## #5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM:

Consider the following spin wavefunctions of two particles, $a$ and $b$, each of spin $S = 1$:

$$\begin{aligned}&|0\rangle_a|0\rangle_b,\\&|1\rangle_a|1\rangle_b,\\&\frac{1}{\sqrt{2}}(|0\rangle_a|{-1}\rangle_b - |{-1}\rangle_a|0\rangle_b).\end{aligned}$$

(a) Which of these wavefunctions are simultaneous eigenstates of the total spin $J$ and its $z$-component $J^z$? Label them with the corresponding $|J, J^z\rangle$.

*Hint*: the raising (lowering) operators $S^\pm = S^x \pm iS^y$ act on the basis functions according to

$$S^\pm|S^z\rangle = \sqrt{(S+1\pm S^z)(S\mp S^z)}\;|S^z \pm 1\rangle.$$

(b) A new particle recently discovered at the Large Hadron Collider has the mass of 125 GeV and decays into two energetic photons. What values of $J$ are allowed by symmetry arguments for this particle?

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SOLUTION:

(a) From $\mathbf{J}^2 = \mathbf{S}_a^2 + \mathbf{S}_b^2 + 2\mathbf{S}_a\mathbf{S}_b$ we see that for $S = 1$

$$\mathbf{J}^2 = S_a^+S_b^- + S_a^-S_b^+ + 2S_a^zS_b^z + 4\,.$$

Using this formula and the rule for the raising/lowering operators, one can check which functions of the given set are eigenfunctions of both $\mathbf{J}^2$ and $J^z = S_a^z + S_b^z$. The results are as follows:

$$\begin{aligned}|\mathrm{N/A}, 0\rangle &: |0\rangle_a|0\rangle_b,\\|2,2\rangle &: |1\rangle_a|1\rangle_b,\\|1,-1\rangle &: \frac{1}{\sqrt{2}}(|0\rangle_a|{-1}\rangle_b - |{-1}\rangle_a|0\rangle_b).\end{aligned}$$

(b) Since the photon has unit spin, $J$ must belong to the set $\{0, 1, 2\}$. Out of these, $J = 0$ and 2 are allowed. Spin $J = 1$ is forbidden because its wavefunction is asymmetric under exchange, as exemplified by the last equation in part (a).

Indeed, in the center-of-mass frame the particle appears as a static point object. The emitted photons are $s$-waves, and so the orbital part of the two-photon wavefunction is necessarily symmetric under the exchange. Since the photons are bosons, the two-photon spin wavefunction must be symmetric as well, which rules out $J = 1$. This statement is known as the Landau-Yang theorem.

## #6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM:

A one-dimensional harmonic oscillator has momentum $p$, mass $m$, and angular frequency $\omega$.

(a) Derive the expressions in terms $x$ and $p$ for the raising and lowering operators $a$ and $a^\dagger$ such that that $[a, a^\dagger] = 1$, $H = \hbar\omega(a^\dagger a + 1/2)$.

(b) Calculate the energy shift $\Delta E_n$ of the $n$th state due to the perturbation $U(x) = \lambda x^4$ to the first order in $\lambda$, using your results from part (a).

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SOLUTION:

(a) Define $\ell = \sqrt{\hbar/m\omega}$, then

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell} + i\frac{p\ell}{\hbar}\right)\,, \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{\ell} - i\frac{p\ell}{\hbar}\right)\,.$$

(b) Within the first-order perturbation theory

$$\Delta E_n = \lambda\langle n|x^4|n\rangle = \frac{\lambda}{4\ell^2}\langle n|(a^\dagger + a)^4|n\rangle\,.$$

Next,

$$(a^\dagger + a)^4 = [(a^\dagger a^\dagger + aa + (a^\dagger a + aa^\dagger)]^2 = (a^\dagger a^\dagger aa + aaa^\dagger a^\dagger) + (a^\dagger a + aa^\dagger)^2 + \ldots\,,$$

where the dots represent terms with unequal number of $a$ and $a^\dagger$. Using $a|n\rangle = \sqrt{n}\,|n\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n\rangle$ we obtain

$$\Delta E_n = \frac{3\lambda}{4\ell^2}(2n^2 + 2n + 1)\,.$$

## #7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:

In a dovetail (or riffle) shuffle, the deck of 52 distinct playing cards is split and divided into two heaps of 26 cards each. One then chooses at random whether to take a card from either heap, until one runs through all the cards.

Let us apply the methods of statistical mechanics to this problem. Define information entropy as $S = -\sum_n p_n \log_2 p_n$, where $n$ ranges over all possible configurations and $p_n$ is the probability of the state $n$. For example, if a new deck is always prepared in the same order (A♠ 2♠ $\cdots$ K♣), then $S_{\text{new}} = 0$ because $p_n = 1$ for this one configuration and $p_n = 0$ otherwise.

Answer the following questions:

(a) What is the information entropy of completely randomized decks?

(b) What is the increase in information entropy for decks that have been shuffled once (starting from a new deck)?

(c) Assuming each subsequent shuffle results in the same entropy increase, how many shuffles are necessary in order to randomize a deck?

Use Stirling's formula

$$K! \simeq \sqrt{2\pi K}\, K^K e^{-K}\,, \quad K \gg 1$$

to approximate your answers. Note that $\log_2 K = \ln K / \ln 2$.

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SOLUTION:

(a) $S_{\text{random}} = \log_2 52! = 225.581$ because there are 52! possible orders with probability 1/52! each. Stirling's formula gives 225.579.

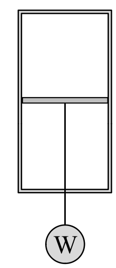
(b) After one shuffle, there are $\Omega = 52!/(26!\,26!)$ possible configurations, so $\Delta S_{\text{riffle}} = \log_2 \Omega = 48.817$. Stirling's formula gives 48.824.

(c) Setting $m\,\Delta S_{\text{riffle}} = S_{\text{random}}$, we have $m = 4.62$. Rounded to the nearest integer, our estimate for the number of shuffles becomes $m = 5$. In comparison, computer experiments show $m = 9$ and a rigorous asymptotic formula for $N \gg 1$ cards is $m \simeq (3/2)\log_2 N$ [D. Bayer and P. Diaconis, The Annals of Applied Probability, Vol.2, No. 2, 294-313 (1992)].

## #8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:

A frictionless piston of negligible mass and heat capacity divides a vertical insulating cylinder of height $2h$ into two halves. Each half contains 1 mol of ideal gas at standard pressure and temperature. The heat capacity ratio $\gamma = C_p/C_v$ of the gas is given. A load of weight $W$ is tied to the piston and suddenly released. After some oscillations the system comes to equilibrium with the piston at rest and equal temperatures of the gases. What is the final displacement of the piston assuming $W$ is very large?




*Hint*: use energy conservation.

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SOLUTION:

Let $y$ be the final displacement. Conservation of energy gives

$$Wy = 2C_v(T - T_0)\,,$$

where $T_0$ and $T$ are the initial and final temperatures. Mechanical equilibrium requires

$$\frac{W}{A} = p_1 - p_2 = \frac{RT}{A}\left(\frac{1}{h-y} - \frac{1}{h+y}\right)\,,$$

where $A$ is the area of the piston and $p_1$ ($p_2$) is the final pressure in the lower (upper) part of the cylinder. Combining these equations, we obtain

$$W = \left(T_0 + \frac{Wy}{2C_v}\right)\frac{2Ry}{h^2 - y^2}\,.$$

If $W$ is very large, $W \gg C_v T_0/h$, we can neglect $T_0$ in this equation. Using $R/C_v = (C_p - C_v)/C_v = \gamma - 1$, we arrive at

$$1 = (\gamma - 1)\frac{y^2}{h^2 - y^2}\,,$$

which implies $y = h/\sqrt{\gamma}$.

## #9 : UNDERGRADUATE GENERAL

`PROBLEM:`

Derive Stirling's formula from Problem #7,

$$K! \simeq \sqrt{2\pi K}\, K^K e^{-K}\,, \quad K \gg 1\,.$$

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`SOLUTION:`

For any natural integer $K$ we have

$$K! = \int\limits_0^\infty dx x^K e^{-x} = K^{K+1} e^{-K} I(K)\,,$$

where

$$I(K) = \int\limits_{-1}^\infty dy e^{Kf(y)}\,, \quad f(y) = \ln(1+y) - y\,.$$

Function $f(y) = -\frac{1}{2}y^2 + \mathcal{O}(y^3)$ has a maximum at $y = 0$, which allows application of the saddle-point method to $I(K)$ with the desired result $I(K) \simeq \sqrt{2\pi/K}$ for $K \gg 1$.

## #10 : UNDERGRADUATE GENERAL

PROBLEM:

Suppose it is necessary, using a thin lens, to couple the output of an incandescent bulb with a filament 1.0 mm in diameter into an optical fiber. The design requires that the total distance from the bulb to the fiber be 110 mm. The fiber has the core diameter 100 $\mu$m. The half-angle $\theta$ of the widest cone of light admitted by the fiber is such that $\sin\theta = 0.25$. What should the focal length and minimum radius of the lens be to collect the most light?

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SOLUTION:

For the image to fit into the fiber, the optical magnification must be $M \leq 100\,\mu\text{m}/1\,\text{mm} = 0.1$. This means that the distance $\ell$ from the bulb to the lens and the distance $\ell'$ from the lens to the fiber must satisfy

$$M = \frac{\ell'}{\ell} = \frac{\ell'}{110\,\text{mm} - \ell'} \leq 0.1\,.$$

This implies that largest allowed $\ell'$ is 10 mm. From the lens equation

$$\frac{1}{f} = \frac{1}{\ell} + \frac{1}{\ell'}\,,$$

we conclude that the largest focal length needed is $f = 100/11\,\text{mm} \approx 9.1\,\text{mm}$. The radius of the lens should be no less than

$$R = \ell' \tan\theta = \ell' \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}\,.$$

To collect the most light from the bulb, we should use the smallest $\ell$, i.e., the largest $\ell'$, for which we get $R \approx 2.6\,\text{mm}$.

**PHYSICS DEPARTMENT EXAM**
**FALL 2012. PART II**

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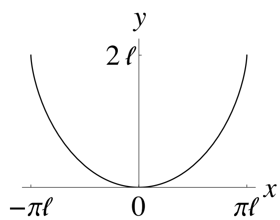
## #11 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:`

A particle of mass $m$ moves without friction along the curve

$$x = \ell(2\phi + \sin 2\phi)\,, \quad y = \ell(1 - \cos 2\phi)\,.$$

in a gravitational field $\mathbf{g} = -\hat{\mathbf{y}}g$. For low enough energy $E$ the motion is periodic in the interval $-\phi_0 \le \phi \le \phi_0$ where $\phi_0 < \pi/2$. Find the oscillation period and show that it does not depend on $E$.

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`SOLUTION:`

$$\begin{aligned}
T &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = 8m\ell^2\cos^2\phi\,\dot{\phi}^2\,,\\
V &= mgy = -2mg\ell\sin^2\phi\,,\\
L &= T - V = 8m\ell^2\cos^2\phi\,\dot{\phi}^2 + 2mg\ell\sin^2\phi\,,
\end{aligned}$$

Choosing the new coordinate $u = \sin\phi$, we obtain the Lagrangian of a harmonic oscillator:

$$L = \frac{1}{2}\mu\dot{u}^2 + \frac{1}{2}\mu\omega^2u^2\,, \quad \mu \equiv 16m\ell^2\,, \quad \omega^2 = \frac{2mg\ell}{8m\ell^2} = \frac{g}{4\ell}\,.$$

Hence, the period $\tau$ is $E$-independent:

$$\tau = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{\ell}{g}}\,.$$

## #12 : GRADUATE CLASSICAL MECHANICS

PROBLEM:

A nonrelativistic particle of charge $e$ and mass $m$ moves in a magnetic field $\mathbf{B}(\mathbf{r}, t) = \text{const.}$

(a) Write down the Lagrangian for a general gauge with a time-independent vector potential $\mathbf{A}(\mathbf{r})$.

(b) Find the three conserved quantities corresponding to translations $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ where $\mathbf{a}$ is an arbitrary constant vector.

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SOLUTION:

(a) For zero electric field and time-independent $\mathbf{A}(\mathbf{r})$ the scalar potential must be zero. Hence, the Lagrangian is (in the CGS units)

$$L = \frac{1}{2} m \dot{\mathbf{r}}^2 + \frac{e}{c} \mathbf{A}(\mathbf{r}) \dot{\mathbf{r}} \,.$$

One possible choice of the vector potential is $\mathbf{A}(\mathbf{r}) = \frac{1}{2}[\mathbf{B} \times \mathbf{r}]$.

(b) If a Lagrangian is invariant with respect to a shift in some coordinate $q$, then $\partial L / \partial q = 0$. The conserved quantity is the canonical momentum $p = \partial L / \partial \dot{q}$ because $dp/dt = \partial L / \partial q = 0$. More generally, the action and hence the equations of motion are invariant under the coordinate shift if there exists $K(q, \dot{q})$ such that

$$\frac{\partial L}{\partial q} = \frac{dK}{dt} \,.$$

The corresponding conserved quantity is

$$\Pi = \frac{\partial L}{\partial \dot{q}} - K \,.$$

This is what we have in the present case. Indeed, we find

$$\frac{\partial L}{\partial r_i} = \frac{e}{c} \frac{\partial A_j}{\partial r_i} \dot{r}_j = \frac{e}{c} \left( \frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j} \right) \dot{r}_j + \frac{e}{c} \frac{\partial A_i}{\partial r_j} \dot{r}_j = \frac{e}{c} \left\{ [\dot{\mathbf{r}} \times (\nabla \times \mathbf{A})] + \frac{d\mathbf{A}}{dt} \right\}_i \,,$$

so that for $\nabla \times \mathbf{A} = \mathbf{B} = \text{const}$,

$$\mathbf{K} = \frac{e}{c} [\mathbf{r} \times \mathbf{B}] + \frac{e}{c} \mathbf{A} \,,$$

$$\mathbf{\Pi} = \frac{\partial L}{\partial \dot{\mathbf{r}}} - \mathbf{K} = m \dot{\mathbf{r}} - \frac{e}{c} [\mathbf{r} \times \mathbf{B}] \,.$$

The three conserved quantities are the three components of $\mathbf{\Pi}$. Up to some constants, two of them are the cyclotron guiding center coordinates in a plane perpendicular to $\mathbf{B}$. The third one is the velocity in the direction along $\mathbf{B}$.

## #13 : GRADUATE ELECTROMAGNETISM

PROBLEM:

Consider a segment of length $\ell$ of an infinite nonmagnetic solid cylinder of radius $R$. For both parts (a) and (b), you are asked to compute the time derivative of the electromagnetic field energy inside the cylinder and the flux of energy through the cylinder's surface. If they do not agree, give a physical reason and account for the discrepancy quantitatively.

(a) The cylinder is wrapped with $N$ turns of a wire, carrying current $I(t)$ that slowly increases with time.

(b) The cylinder is not wrapped with wire but rather is made from a material with conductivity $\sigma$ and there is a constant (in both space and time) current density along the axis of the cylinder.

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SOLUTION:

(a) Using the Ampere and Faraday laws we find the fields inside the cylinder to be

$$\vec{B} = \frac{4\pi}{c}\frac{N}{\ell}I\hat{z}\,,$$

$$\vec{E} = -\frac{2\pi r}{c^2}\frac{N}{\ell}\dot{I}\hat{\theta}\,.$$

Since $\dot{I}$ is small, $E^2 \ll B^2$, we drop it in computing the field energy density:

$$U = \int \frac{B^2}{8\pi} d^3r = \frac{2\pi^2}{\ell c^2} N^2 I^2 R^2.$$

Find the energy flux $\Phi = \int \vec{S}\cdot d\vec{a}$ through the surface of the cylinder from

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{2\pi\dot{I}I}{\ell^2}\left(\frac{N}{\ell}\right)^2 \vec{r}, \quad \Phi = \int \vec{S}\cdot d\vec{a} = -\frac{4\pi^2}{\ell c^2}\dot{I}IN^2R^2\,.$$

The energy conservation is satisfied:

$$\int \vec{S}\cdot d\vec{a} + \frac{d}{dt}U = 0.$$

The negative sign of $\Phi$ accounts for the fact that energy is flowing *into* the cylinder, accounting for the increase of $U$.

(b) Use $\vec{J} = \widehat{z} I/(\pi R^2) = \sigma \vec{E}$, so inside the cylinder

$$\vec{E} = \frac{I}{\pi \sigma R^2}\,\widehat{z}.$$

Find $\vec{B}$ inside the cylinder from $\oint \vec{B}\cdot d\vec{\ell} = 2\pi r B_\theta = (4\pi/c)\pi r^2 J$, so

$$\vec{B} = \frac{2Ir}{R^2 c}\,\widehat{\theta}\,.$$

We can compute $U = \frac{1}{8\pi}\int (E^2 + B^2) d^3\vec{r}$ and find $dU/dt = 0$, since $\dot{I} = 0$. On the other hand, the energy flux $\Phi$ at $r = R$ is found from

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{c}{4\pi}\frac{2I^2}{\pi R^3 \sigma}\,\widehat{r}\,, \quad \Phi = -\frac{I^2 \ell}{\pi R^2 \sigma}\,.$$

The negative sign means that field energy is flowing *into* the cylinder. Energy conservation works because this energy is dissipated as Joule heat:

$$\frac{dQ}{dt} = IV = IE\ell = \frac{I^2\ell}{\pi R^2 \sigma}\,,$$

so that the energy conservation is obeyed:

$$\frac{dQ}{dt} + \frac{dU}{dt} + \Phi = 0\,.$$

## #14 : GRADUATE ELECTROMAGNETISM

PROBLEM:

The half-space $z < 0$ is occupied by a material with dielectric constant $\epsilon_1$. The half-space $z > 0$ is vacuum, $\epsilon_0 = 1$. An electric dipole $\mathbf{p} = (p_x, 0, p_z)$ is placed at the point $\mathbf{r}_d = (0, 0, d)$. Use the method of images to answer the following questions:

(a) What are the location and magnitude (in components) of the image dipole?

(b) What is the electric potential distribution in the half-space $z > 0$?

(c) What is the magnitude and direction of the electrostatic force acting on the dipole?

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SOLUTION:

(a) The image dipole

$$\mathbf{p}_i = (-\beta p_x, 0, \beta p_z)\,, \quad \beta = \frac{\epsilon_1 - \epsilon_0}{\epsilon_1 + \epsilon_0}$$

is at the mirror reflection point $\mathbf{r}_i = (0, 0, -d)$.

(b) In the CGS units, the potential is

$$V(\mathbf{r}) = \frac{\mathbf{p}(\mathbf{r} - \mathbf{r}_d)}{|\mathbf{r} - \mathbf{r}_d|^3} + \frac{\mathbf{p}_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}\,, \quad z > 0\,.$$

(c) The dipole does not exert a force on itself, so the electrostatic force is due to the image dipole only:

$$F = -\mathbf{p}\nabla V_i(\mathbf{r}) = 3\frac{(\mathbf{r}_d - \mathbf{r}_i)\mathbf{p}_d \cdot (\mathbf{r}_d - \mathbf{r}_i)\mathbf{p}_i}{|\mathbf{r}_d - \mathbf{r}_i|^5} - \frac{\mathbf{p}_d\mathbf{p}_i}{|\mathbf{r} - \mathbf{r}_i|^3} = \frac{\beta}{8d^3}(p_x^2 + 2p_z^2)\,.$$

## #15 : GRADUATE QUANTUM MECHANICS

`PROBLEM:`

A hydrogen atom in its ground state is subject to an external electric field $\mathbf{E}\cos\omega t$, where $\omega > \omega_0 = \hbar/(2ma^2)$ and $a = \hbar^2/me^2$. Calculate the ionization rate per unit time using Fermi's Golden Rule and the interaction Hamiltonian in the form

$$H_{int} = \frac{e}{mc}\,\mathbf{p}\cdot\mathbf{A}\,, \quad \mathbf{A} = \frac{c}{\omega}\,\mathbf{E}\sin\omega t\,.$$

The ground-state wavefunction is

$$\psi_{1s}(r) = \frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}.$$

Approximate the wavefunctions of ionized states by plane waves.

---

`SOLUTION:`

The transition rate due to a harmonic perturbation $V(t) = \hat{V}(\omega)e^{-i\omega t}$ is

$$\Gamma = \frac{2\pi}{\hbar}\sum_f |\langle f|\hat{V}(\omega)|i\rangle|^2\,\delta\left(E_f + \frac{\hbar^2}{2ma^2} - \hbar\omega\right)\,.$$

In our case $\hat{V}(\pm\omega) = \pm e\,(\mathbf{E}\mathbf{p})/(2im\omega)$ but only the $+\omega$-term gives nonzero contribution. The corresponding matrix element is

$$M = \frac{eE_j}{2im\omega}\,\langle\mathbf{k}|p_j|1s\rangle = \pm\frac{e\hbar\,\mathbf{E}\mathbf{k}}{2im\omega\sqrt{\pi a^3}}I\,,$$
$$I = \int d^3r e^{-r/a-i\mathbf{k}\mathbf{r}} = \frac{8\pi a^3}{(1+k^2a^2)^2}\,.$$

Hence,

$$|M|^2 = \frac{16\pi a^3}{(1+k^2a^2)^4}\left(\frac{e\hbar}{m\omega}\right)^2(\mathbf{E}\mathbf{k})^2\,.$$

Averaging over directions of $\mathbf{k}$ yields $(\mathbf{Ek})^2 \to E^2k^2/3$; therefore,

$$\Gamma = \frac{2\pi}{\hbar}\left(\frac{e\hbar}{m\omega}\right)^2 \int \frac{d^3k}{(2\pi)^3}\frac{E^2k^2}{3}\frac{16\pi a^3}{(1+k^2a^2)^4}\delta\left(\frac{\hbar^2}{2m}(k^2+a^{-2})-\hbar\omega\right)$$
$$= \frac{16a^3}{3\hbar}\left(\frac{e\hbar}{m\omega}\right)^2 \frac{E^2}{(1+k_\omega^2a^2)^4}\frac{mk_\omega^3}{\hbar^2}, \quad k_\omega = \sqrt{\frac{2m\omega}{\hbar}-\frac{1}{a^2}}\,.$$
$$\Gamma = \frac{64}{3\hbar}E^2a^3\left(\frac{\omega_0}{\omega}\right)^2\left(\frac{\omega}{\omega_0}-1\right)^{3/2}, \quad \omega > \omega_0\,.$$

Note that in a nonperturbative treatment of the problem $\Gamma(\omega)$ is nonzero even at $\omega < \omega_0$ albeit exponentially small, e.g., $\Gamma(0) \sim \exp(-2e\,/\,3Ea^2)$ because of tunneling effect (Oppenheimer, 1928). Also, the approximation of ionized states by plane waves is accurate only for $\omega \gg \omega_0$, in which case

$$\Gamma \simeq \frac{64}{3\hbar}E^2a^3\sqrt{\frac{\omega_0}{\omega}}, \quad \omega \gg \omega_0\,.$$

## #16 : GRADUATE QUANTUM MECHANICS

PROBLEM:

Electron states in graphene are described by the two-component Schrödinger equation

$$\begin{bmatrix} -\varepsilon & v(\pi_x - i\pi_y) \\ v(\pi_x + i\pi_y) & -\varepsilon \end{bmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \quad v \approx \frac{c}{300} .$$

When a uniform magnetic field $-\hat{\mathbf{z}}B$ is present, the generalized momentum operators are

$$\pi_x = -i\hbar\partial_x + \frac{e}{c}A_x , \quad \pi_y = -i\hbar\partial_y + \frac{e}{c}A_y ,$$

and the vector potential obeys $\partial_x A_y - \partial_y A_x = -B$, as usual.

(a) Verify that that $[\pi_x, \pi_y] = i\hbar^2/\ell^2$, where $\ell = \sqrt{\hbar c/eB}$.

(b) Construct an operator $a$ and its Hermitian conjugate $a^\dagger$ from $\pi_x$ and $i\pi_y$ such that $[a, a^\dagger] = 1$.

(c) Rewrite the Schrödinger equation above in terms of $a$ and $a^\dagger$ and convert it into two independent equations for $\psi_A$ and $\psi_B$.

(d) Using an analogy to the harmonic oscillator problem, find the quantized energy levels $\varepsilon_n$ (known as Landau levels). Can $\varepsilon_n$ be negative?

---

SOLUTION:

(a) Easily checked by substitution.

(b) We find

$$a = \frac{\pi_x + i\pi_y}{\sqrt{2}\,\hbar}\,\ell , \quad a^\dagger = \frac{\pi_x - i\pi_y}{\sqrt{2}\,\hbar}\,\ell .$$

(c) Let $\varepsilon_1 = \sqrt{2}\hbar v/\ell$, then the Schrödinger equation becomes

$$\frac{\varepsilon}{\varepsilon_1}\,\psi_A = a^\dagger\psi_B , \quad \frac{\varepsilon}{\varepsilon_1}\,\psi_B = a\psi_A , \quad \text{or} \quad \frac{\varepsilon^2}{\varepsilon_1^2}\,\psi_A = a^\dagger a\psi_A , \quad \frac{\varepsilon^2}{\varepsilon_1^2}\,\psi_B = aa^\dagger\psi_B .$$

(d) By the indicated analogy $\varepsilon_n^2/\varepsilon_1^2$ must be a nonnegative integer. We can write the set of energy levels (which includes $\varepsilon_n < 0$) as

$$\varepsilon_n = \text{sign}(n)\frac{\hbar v}{\ell}\,|2n|^{1/2} , \quad n \in \mathbb{Z} .$$

## #17 : GRADUATE STATISTICAL MECHANICS

PROBLEM:

Consider a single component noninteracting Bose gas with particle number $N$ and volume $V$. The mass of each boson is $m$ and the system temperature is $T$.

(a) Find the single-particle density of states $D(\epsilon)$ in three dimensions as a function of energy $\epsilon$.

(b) Find the expression for temperature $T_c$ of the Bose-Einstein condensation. Express the result in terms of a dimensionless integral.

(c) Find the expression for the internal energy $U(T,V)$ at $T < T_c$. Express the result in terms of dimensionless integrals.

---

SOLUTION:

(a) The density of states per unit volume is

$$D(\epsilon) = \frac{d}{d\epsilon}\left[\frac{1}{(2\pi\hbar)^3}\frac{4\pi}{3}p^3(\epsilon)\right], \quad p(\epsilon) = \sqrt{2m\epsilon}\,,$$
$$D(\epsilon) = \frac{1}{\sqrt{2}\,\pi^2}\frac{m^{3/2}}{\hbar^3}\epsilon^{1/2}.$$

(b) The condensation occurs at $\mu = 0$.

$$\frac{N}{V} = \int_0^\infty \frac{D(\epsilon)d\epsilon}{e^{\epsilon/T_c} - 1} = \frac{I_{1/2}}{\sqrt{2}\,\pi^2}\frac{m^{3/2}}{\hbar^3}T_c^{3/2}\,,$$
$$I_\nu \equiv \int_0^\infty \frac{x^\nu dx}{e^x - 1}\,, \quad I_{1/2} = 2.315\,.$$

Hence,

$$T_c = \left(\frac{\sqrt{2}\,\pi^2}{I_{1/2}}\frac{N}{V}\right)^{2/3}\frac{\hbar^2}{m}\,.$$

(c) The internal energy at $T < T_c$ is

$$U = V \int_0^\infty \frac{D(\epsilon)\epsilon d\epsilon}{e^{\epsilon/T} - 1} = \frac{I_{3/2}}{\sqrt{2}\,\pi^2} \frac{m^{3/2}}{\hbar^3} V T^{5/2} = \frac{I_{3/2}}{I_{1/2}} \frac{N T^{5/2}}{T_c^{3/2}}\,, \quad I_{3/2} = 1.783\,.$$

## #18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:`

Consider a spin-1 Ising model with Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\, S_i S_j \ , \quad S_i \in \{-1, 0, 1\} \, .$$

The system is on a simple cubic lattice, with nearest-neighbor coupling $J_1 = 40\,\mathrm{K}$. (We use $k_B \equiv 1$.)

(a) Find the mean-field free energy $F(T, N, m)$ where $m = \langle S_i \rangle$.

(b) Find the mean-field equation for $m$ by setting $\partial F/\partial m = 0$.

(c) Find the mean-field transition temperature $T_c$.

---

`SOLUTION:`

(a) The mean field Hamiltonian is

$$H_{\mathrm{MF}} = \frac{1}{2} N \hat{J}\, m^2 - m \sum_i S_i \ ,$$

where

$$\hat{J} = \sum_j J_{ij} = z_1 J_1 = 240\,\mathrm{K} \ ,$$

since there are $z_1 = 6$ nearest neighbors on the simple cubic lattice.

Computing the partition function and taking the logarithm, we find the mean field free energy

$$F = \frac{1}{2} N \hat{J}\, m^2 - NT\, \ln\!\left[1 + 2\cosh\left(\frac{\hat{J} m}{T}\right)\right] .$$

(b) The mean-field equation for $m$ is

$$m = \frac{2\sinh\left(\frac{m}{\theta}\right)}{1 + 2\cosh\left(\frac{m}{\theta}\right)} \ , \quad \theta = \frac{T}{\hat{J}} \, .$$

(c) The equation for $T_c$ is obtained by setting the derivatives with respect to $m$ of the LHS and RHS of the above equation to be equal at $m = 0$. Thus, $\theta_c = 2/3$ and $T_c = 2\hat{J}/3 = 160\,\mathrm{K}$.

## #19 : GRADUATE GENERAL

`PROBLEM:`

Using a suitable contour, evaluate

$$I(a,b) = \int_{-\infty}^{\infty} \frac{\cos bx}{x^2 + a^2} dx\,,$$

where both $a$ and $b$ are real and positive.

---

`SOLUTION:`

It is easy to see that the cosine can be replaced by the exponential, after which the contour can be closed by a large arc in the upper half-plane of complex $x$. Using the residue theorem, we get

$$I(a,b) = \int_{-\infty}^{\infty} \frac{e^{ibx} dx}{x^2 + a^2} = 2\pi i \operatorname*{res}_{ia} \frac{e^{ibx}}{x^2 + a^2} = \frac{\pi}{a}\, e^{-ab}\,.$$

**#20 : GRADUATE GENERAL**

`PROBLEM:`

The circular membrane of a kettledrum has the radius $a = 0.40\,\text{m}$, the density $\rho = 0.40\,\text{kg/m}^2$, and the tension $\sigma = 1.00 \times 10^4\,\text{N/m}$.

(a) What is the speed $v$ of transverse waves on the membrane? How does it compare with the speed of sound $c = 331\,\text{m/s}$?

(b) What is the fundamental frequency $f = \omega/(2\pi)$ of the membrane *without* the kettle?

(c) If $v \ll c$, the pressure resulting from any compression or expansion of the enclosed air is nearly uniform within the entire volume. Show that in this case the displacement $\Psi(r)e^{-i\omega t}$ of the membrane obeys the equation

$$\nabla^2\Psi + \frac{\omega^2}{v^2}\Psi = A\int \Psi(\mathbf{r})d^2r\,.$$

(d) Find the coefficient $A$ for air at normal conditions assuming the drum cavity is hemispherical and airtight.

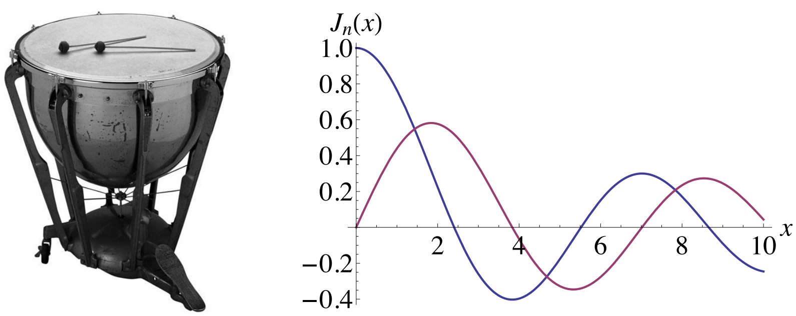


Figure: (Left) Kettledrum. (Right) Bessel functions $J_0(x)$ and $J_1(x)$.

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`SOLUTION:`

(a) $v = \sqrt{\sigma/\rho} = 158\,\text{m/s} \approx 0.48c$.

(b) $\Psi(r) = J_0(kr)$ such that $ka \approx 2.40$ (the first root of $J_0$ from the figure).

$$f = \frac{\omega}{2\pi} = \frac{kv}{2\pi} \approx \frac{2.40}{2\pi}\frac{v}{a} = 151\,\text{Hz}\,.$$

(c), (d) The right-hand side of the wave equation is equal to the perturbation of the air pressure divided by $\sigma$. Noticing that the integral is the change in volume, we get

$$\begin{aligned} A &= -\frac{1}{\sigma}\frac{dP}{dV} = \frac{1}{\sigma}\frac{\gamma P}{V} = \frac{3\gamma}{2\pi}\frac{P}{\sigma a^3} \\ &= \frac{3\times 1.4}{2\pi}\frac{10^5\,\mathrm{Pa}}{10^4\,\mathrm{N/m}\times(0.40\,\mathrm{m})^3} = 104\,\mathrm{m}^{-4}\,. \end{aligned}$$

Note that the fundamental solution satisfying the boundary condition gets modified to

$$\Psi = J_0(kr) - J_0(ka)\,,$$

where $ka$ is no longer equal to the root of $J_0$. Instead, it has to be found substituting $\Psi$ back into equation, which yields

$$-k^2 J_0(ka) = A\int_0^a 2\pi r\Psi(r)dr = \pi A a^2 J_2(ka)\,.$$

For the chosen parameters the numerical solution is $ka = 3.35$, so that $f = 211\,\mathrm{Hz}$, which is a significant change in frequency compared to part (b).