INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

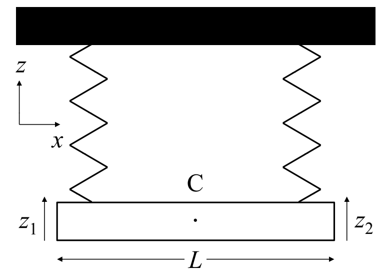
Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.



#1: UNDERGRADUATE MECHANICS

`PROBLEM:`

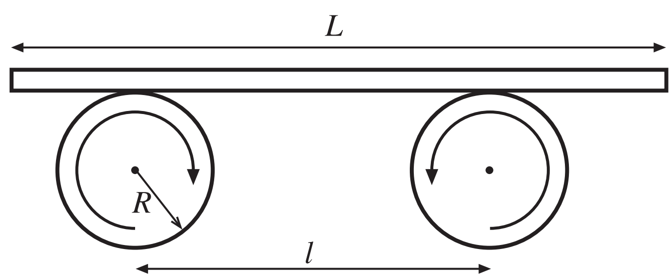
A rigid uniform bar of mass m and length L is supported in equilibrium in a horizontal position by two massless springs with the spring constant k attached one at each end. The motion of the bar is constrained to the xz plane. The bar center of gravity is constrained to move parallel to the vertical z axis. Gravity points down along the z axis. Find the frequency of vibration for symmetric (i.e. symmetric under exchange of the bar ends) mode and for antisymmetric mode of the system.

#2: UNDERGRADUATE MECHANICS

`PROBLEM:`

Two cylinders with equal radii R are rotating with the same angular velocity, Ω, but in opposite directions. Take the axes of the cylinders parallel to the $\widehat{y}$ axis, both at height $z = 0$, and at $x = \pm\ell/2$; see the figure (next page). A board with a mass m and length L is placed on the cylinders slightly off-center: let x_0 be the initial x-location of its center of mass. The coefficient of friction between the cylinders and the board is μ. Describe mathematically the ensuing motion of the board.

14, mechanics problem.pdf



#3: UNDERGRADUATE E&M

`PROBLEM:`

A long coaxial cable consists of two concentric cylindrical conductors. The inner conductor is a **solid** cylinder with a radius a. The outer conductor is a thin cylindrical shell with a radius $b > a$. The cable carries current I, which flows forward along the inner conductor and backward along the outer conductor. The current is evenly distributed over the cross-section of the inner conductor and over the surface of the outer conductor. Find the inductance of the cable per unit length.

#4: UNDERGRADUATE E&M

`PROBLEM:`

Imagine that magnetic monopoles are found to exist. We then need to revise Maxwell's equations to include magnetic charges. We assume that for the static case the magnetic version of Coulomb law is $\vec{B} = \frac{q_m}{r^2}\hat{r}$ (Gauss unit), or $\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$ (SI unit).

1) What is the Gauss's law in the differential form for magnetic monopoles in the static case, i.e. $\nabla\cdot\vec{B}(\vec{r}) =$? Use the $\rho_m(\vec{r})$ to denote the monopole density. In following, we assume this magnetic version of Gauss law is generally valid not just for statics.

2) Magnetic charges must be conserved. Write down the associated differential form of magnetic charge conservation in terms of ρ_m and $\vec{j}_m$, where $\vec{j}_m(\vec{r})$ is the monopole current density.

3) Show that only modifying the $\nabla \cdot \vec{B}$ term of the Maxwell equation would not maintain monopole charge conservation. Find the appropriate modification of Faraday's law, needed to maintain monopole charge conservation.

4) Imagine an infinitely long wire carrying a steady monopole current I_m along the $\hat{z}$-direction. Find the associated electric field $\vec{E}(\rho, \phi, z)$ in cylindrical coordinates.

#5: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` At $t = 0$, an electron, with magnetic moment $\vec{\mu} = \frac{-e\hbar}{2mc}\vec{\sigma}$, is in the spin state

$$\chi(t=0) = \begin{pmatrix} \sqrt{\frac{2}{3}}i \\ \sqrt{\frac{1}{3}} \end{pmatrix}.$$

A magnetic field B is applied in the z direction. **a)** Find the spin state of the particle, as a function of time. **b)** Find the expectation value of S_y as a function of time. **c)** What is the probability, as a function of time, to measure that the electron's spin along the x direction is $\frac{\hbar}{2}$?

#6: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:`

An electron is in a three dimensional harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$. A small electric field, of strength E_z, is applied in the z direction. Calculate the lowest order nonzero correction to the ground state energy. (For the 1D oscillator, $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(A^\dagger A + \frac{1}{2})$ and $[A, A^\dagger] = 1$).

#7: UNDERGRADUATE STAT MECH/THERMO

`PROBLEM:`

Two identical, monatomic, ideal gases with the same pressure P and the same number of particles N but with different temperatures T_1 and T_2 are confined in two vessels of volume V_1 and V_2. Then the vessels are connected. The combined system remains thermally insulated. Find the change in entropy after the system reaches equilibrium. Express your answer in terms of T_1, T_2 and either N and Boltzmann constant or the heat capacity at constant pressure. What would be the change in entropy if $T_1 = T_2$?

#8: UNDERGRADUATE STAT MECH/THERMO

`PROBLEM:`

Consider a system of N **distinguishable** particles, at temperature T, with two available energy levels: the groundstate is nondegenerate, with energy $\epsilon_1 = 0$, and the excited energy level is **doubly** degenerate, with energy $\epsilon_2 \equiv \epsilon$.

(a) Determine the equilibrium values of the occupation numbers N_1 and N_2 (such that $N_1 + N_2 = N$) as a function of temperature.

(b) Determine the energy U of the system, as a function of temperature.

(c) Determine the specific heat (at constant volume) as a function of temperature.

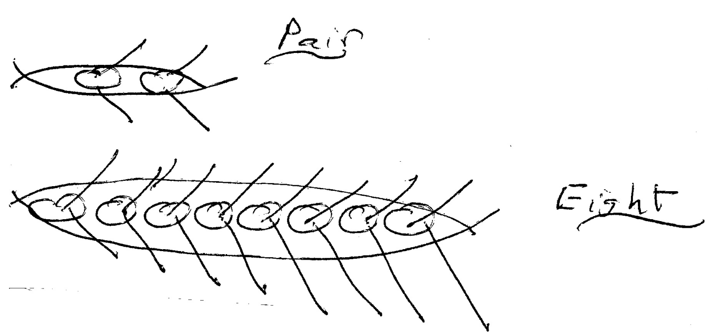
#9: UNDERGRADUATE General

`PROBLEM:`

Consider a scull - a kind of long, thin rowboat. You may assume that the drag force is independent of the mass of the boat and rowers.

Please write how the speed, v, of the scull scales as a function of **only** the following quantities: $v(n, X, \rho, G)$. Here n is the number of rowers; X is the constant power of each rower; ρ is the density of the fluid on which they row; G is the volume per rower.

To improve v, is it better for Admiral Arius to try to scale up n, or X?



You may assume:

- Power/rower $\sim X \sim const.$

- ℓ is length of boat.

- Volume/per rower $G \sim \ell^3/n \sim const.$

#10: UNDERGRADUATE GENERAL

`PROBLEM:`

Describe one experiment (ancient, historical or contemporary) that would allow you to measure the speed of light in a vacuum. You should assume that the length of the meter and duration of the second are both perfectly known. (a) Give with an overview of the method in one or two sentences. (b) Describe the experimental equipment. (c) What measurements are made? (d) What physics ideas (equations, principles, assumptions) are involved? (e) List the primary potential sources of error. (f) Attempt to give a quantitative estimate for the overall error.

INSTRUCTIONS
PART 2 : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11: GRADUATE MECHANICS

`PROBLEM:`

(a) Consider a binary star system, with stars of masses, m_1 and m_2, in a circular orbit of radius r_0, with period τ, thanks to the attraction of classical, Newtonian gravity (you can ignore general relativity effects). Derive Kepler's third law relation between r_0 and τ. [1 point]

(b) Suppose that the masses of the binary star system are $m_1 = M_{sun}$ and $m_2 = 15M_{sun}$ (here M_{sun} is the solar mass unit, the mass of our sun), and that $r_0 = 4AU$ (four times the distance between the earth and the sun). Find the orbit period, τ, in units of years. [4 points]

(c) Some mischievous aliens are playing with their new weapon / toy. By the push of a button, they cause the two stars to suddenly stop moving. Right after they push the button, the stars are separated by the same distance r_0, but have zero velocity. How long do the aliens now have to wait, in years, before they can enjoy watching the two stars collide? (You might, or might not, be interested to know that $\int_0^1 \frac{du}{\sqrt{u^{-1}-1}} = 2\int_0^{\pi/2} \sin^2\theta d\theta = \pi/2$, where $u = \sin^2\theta$ is used in the second step.) [4 points]

(d) Is such an alien toy possible? What physical principle does it possibly violate? Can you suggest a way to make it work, without violating anything? Just a few comments here is sufficient. [1 point].

#12: GRADUATE MECHANICS

`PROBLEM:`

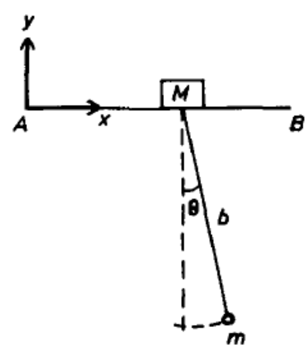
A mass M is constrained to slide without friction on the track AB as shown in the figure. A mass m is connected to M by a massless inextensible string of length b.

(a) Write the Lagrangian for this system.

(b) Write the leading order Lagrangian, assuming small oscillations.

(c) Find the normal coordinates and describe them.

(d) Find expressions for the normal coordinates as a function of time, for completely general initial conditions (identify your constants of integration).



#13: GRADUATE E&M

PROBLEM:

In three spatial dimension (x,y,z) the scalar potential field $\phi(x, y, z, t)$ from a charged particle of charge Q at the origin (x,y,z) = (0,0,0) satisfies

$$\nabla^2 \phi(x, y, z, t) = -4\pi Q \delta(x)\, \delta(y)\, \delta(z). \tag{1}$$

(a) Explain where this equation comes from. It might be a good idea to start with Maxwell's equations.

Show the solution to this equation is

$$\phi(x, y, z, t) = \frac{Q}{R}, \qquad R^2 = x^2 + y^2 + z^2. \tag{2}$$

(b) Now using Maxwell's equations again, write the wave equation satisfied by $\phi(x, y, z, t)$ in the Lorentz gauge:

$$\nabla \cdot \mathbf{A}(x, y, z, t) + \frac{1}{c}\frac{\partial \phi(x, y, z, t)}{\partial t} = 0. \tag{3}$$

(c) Solve this equation in an inertial frame where the particle is seen moving along the x-axis at a constant velocity v. Hint: the first part of this problem is quite relevant.

(d) Find the vector potential $\vec{A}$ in the setup of parts (b) and (c).

#14: GRADUATE E&M

`PROBLEM:`

Consider a paramagnetic sphere of radius a, with magnetic permeability μ, and conductivity σ.

(a) The sphere is in the presence of a constant external magnetic field $\vec{B}_{\text{ext}} = B_0\hat{z}$. Find the total magnetic field $\vec{B}$ everywhere, inside and outside the sphere, and the sphere's magnetization density $\vec{M}$.

(b) Now take $\vec{B}_{\text{ext}} = B_0 e^{-i\omega t}\hat{z}$ (the real part). Find $\vec{E}$ inside the sphere, to leading order in $a\omega/c$.

(c) In the same setup and approximation as part (b), find the eddy current distribution in the sphere, and the average power absorbed (heat loss) by the sphere.

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:`

Consider the Hamiltonian of two particles (labeled by the subscripts $1, 2$), each in a 3d harmonic oscillator potential:

$$H = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + \frac{1}{2} m\omega^2 \vec{r}_1^2 + \frac{1}{2} m\omega^2 \vec{r}_2^2 + V_{int},$$

The two particles are **identical and of spin** $\frac{1}{2}$. (There is no spin-orbit coupling.) The interaction

$$V_{int} = -\epsilon \delta^3(\vec{r}_1 - \vec{r}_2)$$

will be treated as a small perturbation.

(a) Working to zero-th order, ϵ^0, (i.e. dropping V_{int}), what are the energy eigenvalues, and what is the form of the energy eigenfunctions? You don't need to derive the precise spatial dependence of the 1d harmonic oscillator eigenfunctions – just write the formal form of the wavefunction in terms of such quantities, taking care to properly label for example how the energy levels appear. Especially take care with regard to the particles being identical and include their spin degrees of freedom, accounting for all possibilities. How many integers etc. are needed to specify the states? Be sure that your energies and eigenfunctions depend on all these labels.

(b) Again, working to order ϵ^0, what is the energy, degeneracy, and total spin quantum number(s) of the ground state? Write the actual ground state wavefunction(s), including the spin degrees of freedom.

(c) What is the shift of the ground state energy to first order in V_{int}, i.e. ϵ^1?

(d) Accounting for V_{int} qualitatively, assuming that it is repulsive ($\epsilon < 0$), what is the spin S and degeneracy of the first excited state?

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM`: **Quantum interference versus measurement of which-way information**

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them $|\uparrow\rangle$ and $|\downarrow\rangle$), representing the possible paths taken by the particles. A particle emerging in state $|\uparrow\rangle$ produces a wavefunction at the screen of the form $\psi_\uparrow(x)$, (where x is a coordinate along the screen) while a particle emerging in state $|\downarrow\rangle$ produces the wavefunction $\psi_\downarrow(x)$. The evolution from the wall with the slits to the screen is linear in the input state.

As the source repeatedly spits out particles, the screen counts how many particles hit at each location x.

Suppose, for simplicity, that $\psi_\uparrow(x) = e^{\mathbf{i}k_\uparrow x}, \psi_\downarrow(x) = e^{\mathbf{i}k_\downarrow x}$, where $k_\uparrow, k_\downarrow$ are some real constants.

1. If the particles are all spat out in the state $|\uparrow\rangle$, what is the x-dependence of the resulting pattern $P_\uparrow(x)$?
2. If the particles are all spat out in the (normalized) state
$$|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle \ ,$$
what is the x-dependence of the resulting pattern, $P_\psi(x)$? Assume μ, λ are real.

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space $\mathcal{H}_E$. Suppose these interactions are described by the hamiltonian

$$\mathbf{H} = \boldsymbol{\sigma}^z \otimes \mathbf{M}$$

acting on $\mathcal{H}_2 \otimes \mathcal{H}_E$, where $\boldsymbol{\sigma}^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ acting on the Hilbert space $\mathcal{H}_2$ of particle paths, and $\mathbf{M}$ is an operator acting on the Hilbert space of the environment.

Suppose the initial state of the whole system is

$$|\Psi_0\rangle \equiv (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_E \ ,$$

and that

$$\mathbf{M} = m\boldsymbol{\sigma}^x = m\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)_E .$$

3. Find $|\Psi(t)\rangle$, the state of the whole system at time t.
4. How does the interference pattern depend on x and t? For simplicity, consider the case where $\mu = \lambda = \frac{1}{\sqrt{2}}$.
5. Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.
6. What would happen if instead the initial state of the environment were an eigenvector of $\mathbf{M}$?

#17: GRADUATE STAT MECH

`PROBLEM`: A black hole as a thermodynamic system

There is a powerful analogy between the physics of black holes and thermodynamics. A static Schwarzchild black hole is a hole in space, characterized by a radius R_H, and a mass M. In equilibrium, these two parameters are related by

$$R_H = 2G_N M/c^2,$$

where G_N is Newton's constant and c is the speed of light. You should regard this as an equation of state.

Here is the dictionary that relates black hole properties to thermodynamic variables: The internal energy of a static Schwarzchild black hole is $E = Mc^2$. The entropy is

$$S_{\text{BH}} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

where A is its surface area, $4\pi R_H^2$. [Note that the resulting thermodynamic system has only one independent thermodynamic variable.]

1. By demanding that the first law of thermodynamics applies to black holes, compute the temperature of a static Schwarzchild black hole of mass M.

2. Compute the specific heat of the Schwarzchild black hole. What is the physical consequence of its sign?

3. An intermediate step which can be useful for the next part, and doesn't involve black holes: What is the entropy of thermal radiation in a box of volume V, $S_{\text{photons}}(T, V)$? You may give your answer in terms of an unspecified constant prefactor.

4. Consider a black hole in a box of volume V with adiabatic walls, in equilibrium with the thermal radiation. We would like to determine whether the black hole evaporates. Proceed as follows:

   Suppose a fraction x of the total energy is in the black hole, and the rest is in the thermal radiation. Derive an equation determining the equilibrium value of x. What is the equilibrium value of x when the volume is large?

#18: GRADUATE STAT MECH

`PROBLEM:`

Electrons in metal can be approximately considered as free electron gas in three dimensions. Let m denote the electron mass. Consider a system with electron density n, and assume that there is no spin polarization.

1. Derive the relation between the Fermi wavevector k_f and the electron density n.

2. Calculate the average kinetic energy E_K per electron at $T = 0K$. Express E_K in terms of the Fermi energy defined as $\epsilon_f = \frac{\hbar^2}{2m} k_f^2$.

3. Qualitatively explain the scaling of specific heat of the metal with temperature T, for $k_B T \ll \epsilon_f$. How is it different from the specific heat of the ideal Boltzmann gas? Can you present an intuitive reason?

4. Now let us further consider the effect of Coulomb interaction among electrons. Estimate the average Coulomb interaction E_c among electrons.

5. Usually when we say interactions are strong or weak, it is not based on the absolute value of E_c, but actually based on the dimensionless ratio defined as $r_s = E_c/E_K$.

   Rewrite r_s as the ratio between two length scales up to a constant at the order of one. What are these two length scales? (Hint: one of them is made of fundamental constants.) Will the interaction effect be weakened or strengthened as the electron density n increases?

   Can you estimate the order of the typical value of r_s in metals? Are they in the weak interaction regime?

#19: GRADUATE MATH

`PROBLEM:`

Recall the Laplace transform, $f(t) \to F(s)$, and its inverse $F(s) \to f(t)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds,$$

where the latter integral is on a contour parallel to the imaginary s axis, and c is a constant.

(a) Find the function $f(t)$ for which

$$F(s) = \frac{b}{(s+a)^2 + b^2}, \qquad a, b = \text{positive constants.}$$

Don't forget to determine $f(t)$ for both positive and negative t, discuss each case explicitly and plot the function $f(t)$.

(b) Consider a series LRC circuit, with inductor, resistor, and capacitor specified by L, R, and C (all constants in t). The circuit consists of these three in a closed circuit, together with a battery that supplies constant voltage V_0. The circuit has a switch, which is open (breaking the current loop) for $t < 0$ and closed for $t \geq 0$. At time $t = 0$, the conductor has a charge $Q_0 = 0$. Write the equations (for $t \geq 0$) which need to be solved in order to find the charge $Q(t)$ on the conductor and the current $I(t)$ in the circuit.

(c) Let $I(s)$ be the Laplace transform of $I(t)$. Using the results of part (b), solve for $I(s)$.

(d) Using the results from the above parts, solve for $I(t)$. Be sure that your solution satisfies the boundary conditions.

#20: GRADUATE GENERAL

`PROBLEM:` When the universe is about 400,000 years old, protons are at 3000 K, their density is $n_p = 2 \times 10^8$ protons per m^3 and only three species are important for electromagnetic interactions: protons, electrons and photons. You may need various constants; $e = 1.6 \times 10^{-19}$ C, $m_e = 9.1 \times 10^{-31}$ kg, $k = 1.38 \times 10^{-23}$ J/K, $\epsilon_0 = 8.85 \times 10^{-12}\text{m}^{-3}\text{kg}^{-1}\text{s}^4\text{A}^2$, $\sigma_T = 6.7 \times 10^{-29}$ m^2 and the baryon to photon (number density) ratio is $n_p/n_\gamma = 6 \times 10^{-10}$.

(a) Why do we not consider neutrinos, dark matter and dark energy when we consider significant electromagnetic interactions?

(b) Give arguments and equations leading to a quantitative estimate for the time scale (in seconds) for an electron to significantly change its kinetic energy through interactions with protons, when electrons are at T=3000 K. Hint: find the Coulomb interaction cross section by taking the impact parameter for an e-p interaction such that the electromagnetic potential energy equals the electron's kinetic energy. Use this to find the time between interactions.

(c) What is the time scale for photons to exchange energy with the e and p? Hint: find the interaction time between photons and electrons, associated with Thompson scattering. Estimate the associated fractional energy change per scattering. Thereby find the time scale needed to establish thermal equilibrium between electrons and photons.

(**Fact:**) One can likewise compute the time scale for a proton to significantly change its kinetic energy through interactions with electrons, but we will **not** ask you to do this computation here. It turns out that this time scale is intermediate between the times computed in parts (b) and (c). The times are all less than the lifetime of the universe.

(d) What do the results mean? State the very simple prediction that follows from these results.

INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

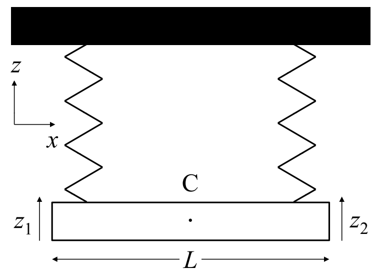
Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.



#1: UNDERGRADUATE MECHANICS

`PROBLEM:`

A rigid uniform bar of mass m and length L is supported in equilibrium in a horizontal position by two massless springs with the spring constant k attached one at each end. The motion of the bar is constrained to the xz plane. The bar center of gravity is constrained to move parallel to the vertical z axis. Gravity points down along the z axis. Find the frequency of vibration for symmetric (i.e. symmetric under exchange of the bar ends) mode and for antisymmetric mode of the system.

`SOLUTION:`

Let the vertical displacements of the bar ends from the equilibrium positions be z_1 and z_2. For the center of mass C, the law of motion gives

$$\frac{m}{2}(\ddot{z}_1 + \ddot{z}_2) = -k(z_1 + z_2) - mg. \tag{1}$$

The torque equation gives for small z_1 and z_2

$$\frac{I_0}{L}(\ddot{z}_2 - \ddot{z}_1) = -\frac{L}{2}k(z_2 - z_1), \tag{2}$$

where $I_0 = mL^2/12$ is the bar moment of inertia about C.

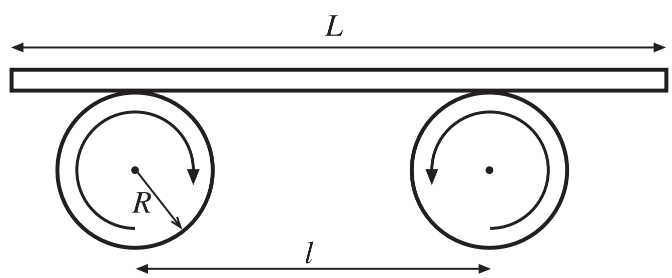
The gravity term merely determines the equilibrium position and does not affect the vibration frequencies. From (1) and (2), the frequency of symmetric mode with $z_1 = z_2$ is $\omega_s = \sqrt{2k/m}$ and the frequency of antisymmetric mode with $z_1 = -z_2$ is $\omega_a = \sqrt{6k/m}$.

#2: UNDERGRADUATE MECHANICS

`PROBLEM:`

Two cylinders with equal radii R are rotating with the same angular velocity, Ω, but in opposite directions. Take the axes of the cylinders parallel to the $\widehat{y}$ axis, both at height $z = 0$, and at $x = \pm\ell/2$; see the figure (next page). A board with a mass m and length L is placed on the cylinders slightly off-center: let x_0 be the initial x-location of its center of mass. The coefficient of friction between the cylinders and the board is μ. Describe mathematically the ensuing motion of the board.

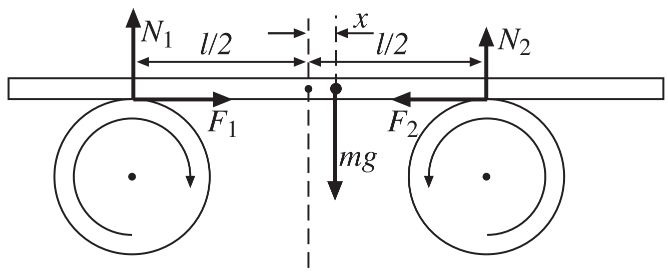
14, mechanics problem.pdf



SOLUTION:

The center of mass of the board is shifted to the right with respect to the plane of symmetry of the two rotating cylinders by a distance x. There are 3 forces acting on the board along the vertical axes: the gravity, mg, and two reaction forces, $\vec{N}_1$ and $\vec{N}_2$. Because the board is not rotating, the torques with respect to a point of the board equidistant from the cylinders should add up to zero: $N_1\ell/2 + mgx - N_2\ell/2 = 0$, resulting in $N_2 - N_1 = 2mgx/\ell$. The net force acting on the board in the horizontal plane is the sum of two friction forces, $F_2 - F_1 = \mu(N_2 - N_1) = 2\mu mgx/\ell$. One can see that this force acts as a restoring force, whose magnitude is proportional to the deviation, x, such that the system is similar to a mass on a spring and $2\mu mg/\ell$ similar to the spring constant. Therefore, $x(t) = x_0 \cos \omega t$, with $\omega = \sqrt{2\mu g/\ell}$.

14, mechanics 1.pdf



#3: UNDERGRADUATE E&M

PROBLEM:

A long coaxial cable consists of two concentric cylindrical conductors. The inner conductor is a **solid** cylinder with a radius a. The outer conductor is a thin cylindrical shell with a radius $b > a$. The cable carries current I, which flows forward along the inner conductor and backward along the outer conductor. The current is evenly distributed over the cross-section of the inner conductor and over the surface of the outer conductor. Find the inductance of the cable per unit length.

SOLUTION:
Solution. The magnetic field at $r < a$, is found from $B \cdot 2\pi r = \mu_0 I_{encl} = \mu_0 I r^2/a^2$ as $B = \mu_0 I r/2\pi a^2$. The magnetic field at $a < r < b$ is $B = \mu_0 I/2\pi r$. The energy of the magnetic field per unit length in the inner conductor is

$$E_1 = \frac{1}{2\mu_0}\int_0^a B^2 2\pi r dr = \frac{\mu_0 I^2}{16\pi}$$

The energy of the magnetic field in the annular gap is

$$E_2 = \frac{1}{2\mu_0}\int_a^b B^2 2\pi r dr = \frac{\mu_0 I^2}{4\pi}\ln(b/a).$$

Hence,

$$E = E_1 + E_2 = \frac{\mu_0}{4\pi}\left(\frac{1}{4} + \ln(b/a)\right) I^2 \equiv \frac{1}{2} L I^2.$$

The inductance per unit length, L, is thus

$$L = \frac{\mu_0}{2\pi}\left(\frac{1}{4} + \ln(b/a)\right).$$

#4: UNDERGRADUATE E&M

PROBLEM:

Imagine that magnetic monopoles are found to exist. We then need to revise Maxwell's equations to include magnetic charges. We assume that for the static case the magnetic version of Coulomb law is $\vec{B} = \frac{q_m}{r^2}\hat{r}$ (Gauss unit), or $\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\hat{r}$ (SI unit).

1) What is the Gauss's law in the differential form for magnetic monopoles in the static case, i.e. $\nabla \cdot \vec{B}(\vec{r}) =?$ Use the $\rho_m(\vec{r})$ to denote the monopole density. In following, we assume this magnetic version of Gauss law is generally valid not just for statics.

2) Magnetic charges must be conserved. Write down the associated differential form of magnetic charge conservation in terms of ρ_m and $\vec{j}_m$, where $\vec{j}_m(\vec{r})$ is the monopole current density.

3) Show that only modifying the $\nabla \cdot \vec{B}$ term of the Maxwell equation would not maintain monopole charge conservation. Find the appropriate modification of Faraday's law, needed to maintain monopole charge conservation.

4) Imagine an infinitely long wire carrying a steady monopole current I_m along the $\hat{z}$-direction. Find the associated electric field $\vec{E}(\rho, \phi, z)$ in cylindrical coordinates.

SOLUTION:

1) $\nabla \cdot \vec{B}(\vec{r}) = 4\pi\rho_m(\vec{r})$, or, $\nabla \cdot \vec{B}(\vec{r}) = \mu_0\rho_m(\vec{r})$.

2)

$$\frac{\partial}{\partial t}\rho_m + \nabla \cdot \vec{j}_m(\vec{r}) = 0. \tag{3}$$

3) From the previous version of $\nabla \times E = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}$, we have $0 = \nabla \cdot (\nabla \times E) = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \vec{B}) = -\frac{4\pi}{c}\frac{\partial}{\partial t}\rho_m(\vec{r})$. This is different from the monopole charge conservation law above.

We should modify the Faraday's law as

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} - \frac{4\pi}{c}\vec{j}_m. \tag{4}$$

Or,

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} - \mu_0 \vec{j}_m. \tag{5}$$

Then

$$\nabla \cdot (\nabla \times \vec{E}) = -\frac{4\pi}{c}\left\{\frac{\partial}{\partial t}\rho_m + \nabla \cdot \vec{j}_m(\vec{r})\right\} = 0 \tag{6}$$

4) For the steady state, we have a monopole current version of the Ampere's law $\nabla \times \vec{E} = -\frac{4\pi}{c}\vec{j}_m$. Then $\int d\vec{l} \cdot \vec{E} = -\frac{4\pi}{c} I_m$, thus

$$2\pi r E = -\frac{4\pi}{c} I_m \tag{7}$$

Or, $E = \frac{2\pi}{cr} I_m$. E is along the tangent direction, and the minus sign means that E's direction follows the left-hand rule.

#5: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` At $t = 0$, an electron, with magnetic moment $\vec{\mu} = \frac{-e\hbar}{2mc}\vec{\sigma}$, is in the spin state

$$\chi(t=0) = \begin{pmatrix} \sqrt{\frac{2}{3}}i \\ \sqrt{\frac{1}{3}} \end{pmatrix}.$$

A magnetic field B is applied in the z direction. **a)** Find the spin state of the particle, as a function of time. **b)** Find the expectation value of S_y as a function of time. **c)** What is the probability, as a function of time, to measure that the electron's spin along the x direction is $\frac{\hbar}{2}$?

`SOLUTION:` One Bohr magneton is $\mu_B = \frac{e\hbar}{2mc}$. Let $\omega = \frac{eB}{2mc}$

a) $\chi(t) = \begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}$

b) $\langle\chi(t)|S_y|\chi(t)\rangle = \frac{\hbar}{2}\langle\chi(t)|\sigma_y|\chi(t)\rangle = \frac{\hbar}{2}\begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}$

$\langle\chi(t)|S_y|\chi(t)\rangle = \frac{\hbar}{2}\begin{pmatrix} -i\sqrt{\frac{2}{3}}e^{i\omega t} & \sqrt{\frac{1}{3}}e^{-i\omega t} \end{pmatrix}\begin{pmatrix} -i\sqrt{\frac{1}{3}}e^{i\omega t} \\ -\sqrt{\frac{2}{3}}e^{-i\omega t} \end{pmatrix}$

$\langle\chi(t)|S_y|\chi(t)\rangle = -\frac{\sqrt{2}}{3}\frac{\hbar}{2}\left(e^{2i\omega t} + e^{-2i\omega t}\right) = -\frac{\sqrt{2}}{3}\frac{\hbar}{2}2\cos(2\omega t) = -\frac{\sqrt{2}}{3}\hbar\cos(2\omega t)$

c) $P(t) = \left|\begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} i\sqrt{\frac{2}{3}}e^{-i\omega t} \\ \sqrt{\frac{1}{3}}e^{i\omega t} \end{pmatrix}\right|^2 = \left|\frac{i}{\sqrt{3}}e^{-i\omega t} + \frac{1}{\sqrt{6}}e^{i\omega t}\right|^2$

$P(t) = \frac{1}{3}+\frac{1}{6}+\frac{i}{\sqrt{18}}e^{-2i\omega t}-\frac{i}{\sqrt{18}}e^{2i\omega t} = \frac{1}{2}+\frac{i}{\sqrt{18}}(-2i\sin(2\omega t)) = \frac{1}{2}+\frac{\sqrt{2}}{3}\sin(2\omega t)$

#6: UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:`

An electron is in a three dimensional harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$. A small electric field, of strength E_z, is applied in the z direction. Calculate the lowest order nonzero correction to the ground state energy. (For the 1D oscillator, $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega(A^\dagger A + \frac{1}{2})$ and $[A, A^\dagger] = 1$).

`SOLUTION:` Working in Cartesian coordinates the ground state had all three of the oscillators in the n=0 state. The perturbation is $V = eEz = eE\sqrt{\frac{\hbar}{2m\omega}}(A_z + A_z^\dagger)$. First order perturbation theory gives zero energy shift since

$$\langle 000|A_z + A_z^\dagger|000\rangle = 0.$$

In second order, we have

$$E_{000}^{(2)} = \sum_{n_x n_y n_z \neq 000} \frac{|\langle n_x n_y n_z|V|000\rangle|^2}{E_{000}^{(0)} - E_{n_x n_y n_z}^{(0)}} = e^2 E^2 \frac{\hbar}{2m\omega} \frac{|\langle 001|A_z^\dagger|000\rangle|^2}{\frac{3}{2}\hbar\omega - \frac{5}{2}\hbar\omega} = \frac{-e^2 E^2}{2m\omega^2}$$

#7: UNDERGRADUATE STAT MECH/THERMO

`PROBLEM:`

Two identical, monatomic, ideal gases with the same pressure P and the same number of particles N but with different temperatures T_1 and T_2 are confined in two vessels of volume V_1 and V_2. Then the vessels are connected. The combined system remains thermally insulated. Find the change in entropy after the system reaches equilibrium. Express your answer in terms of T_1, T_2 and either N and Boltzmann constant or the heat capacity at constant pressure. What would be the change in entropy if $T_1 = T_2$?

`SOLUTION:`

The final entropy does not depend on how the final state is reached and can be calculated as if the final state was reached isobarically because the final pressure $P_f = P$. The final temperature $T_f = (T_1 + T_2)/2$. For each part separately,

$$TdS = C_P dT, \tag{8}$$

where $C_P = \frac{5}{2}Nk$ is the heat capacity at constant pressure. This gives

$$\Delta S_1 = C_P \ln \frac{T_f}{T_1} \quad \text{and} \quad \Delta S_2 = C_P \ln \frac{T_f}{T_2} \tag{9}$$

Therefore,

$$\Delta S = \Delta S_1 + \Delta S_2 = C_P \ln \frac{T_f^2}{T_1 T_2} = \frac{5}{2} Nk \ln \frac{T_f^2}{T_1 T_2}. \tag{10}$$

ΔS vanishes if $T_1 = T_2$ as expected.

#8: UNDERGRADUATE STAT MECH/THERMO

`PROBLEM:`

Consider a system of N **distinguishable** particles, at temperature T, with two available energy levels: the groundstate is nondegenerate, with energy $\epsilon_1 = 0$, and the excited energy level is **doubly** degenerate, with energy $\epsilon_2 \equiv \epsilon$.

(a) Determine the equilibrium values of the occupation numbers N_1 and N_2 (such that $N_1 + N_2 = N$) as a function of temperature.

(b) Determine the energy U of the system, as a function of temperature.

(c) Determine the specific heat (at constant volume) as a function of temperature.

`SOLUTION:`

(a)Use $N_1^* = e^{\mu/kT}$, and $N_2^* = 2e^{(\mu-\epsilon)/kT}$. Fixing $N_1^* + N_2^*$ determines μ: $e^{\mu/kT}(1 + 2e^{-\epsilon/kT}) = e^{\mu/kT}Z = N$. So

$$N_1^* = \frac{N}{1 + 2e^{-\epsilon/kT}}, \qquad N_2^* = \frac{2Ne^{-\epsilon/kT}}{1 + 2e^{-\epsilon/kT}}.$$

(b)

$$U = N_2^*\epsilon = \frac{2N\epsilon e^{-\epsilon/kT}}{1 + 2e^{-\epsilon/kT}} = \frac{2N\epsilon}{2 + e^{\epsilon/kT}}.$$

(c)

$$C_V = \frac{dU}{dT}\Big|_\epsilon = \frac{2N\epsilon^2}{kT^2}\frac{e^{\epsilon/kT}}{(2 + e^{\epsilon/kT})^2}.$$

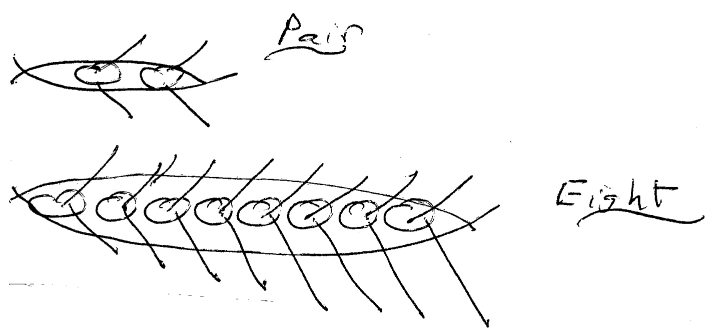
#9: UNDERGRADUATE General

`PROBLEM:`

Consider a scull - a kind of long, thin rowboat. You may assume that the drag force is independent of the mass of the boat and rowers.

Please write how the speed, v, of the scull scales as a function of **only** the following quantities: $v(n, X, \rho, G)$. Here n is the number of rowers; X is the constant power of each rower; ρ is the density of the fluid on which they row; G is the volume per rower.

To improve v, is it better for Admiral Arius to try to scale up n, or X?



You may assume:

- Power/rower $\sim X \sim const.$

- ℓ is length of boat.

- Volume/per rower $G \sim \ell^3/n \sim const.$

`SOLUTION: -` <u>`Dimensional Analysis`</u>

Drag Force: $F_d \sim \rho v^2 \ell^2$

so

Power required: $P \sim vF_d \sim \rho v^3 \ell^2 \sim nX$

Using $\ell \sim (nG)^{1/3}$

we solve for v:

$\boxed{v \sim n^{1/9} X^{1/3} / \rho^{1/3} G^{2/9}}$

- speed scales as $n^{1/9}$

- better to increase rower power X rather than increase number of rowers!

Tell Admiral Arius....

#10: UNDERGRADUATE GENERAL

PROBLEM:

Describe one experiment (ancient, historical or contemporary) that would allow you to measure the speed of light in a vacuum. You should assume that the length of the meter and duration of the second are both perfectly known. (a) Give with an overview of the method in one or two sentences. (b) Describe the experimental equipment. (c) What measurements are made? (d) What physics ideas (equations, principles, assumptions) are involved? (e) List the primary potential sources of error. (f) Attempt to give a quantitative estimate for the overall error.

SOLUTION:

Details of the answer will depend on the method chosen. Some common methods are as follows. 1) Romer calculated c from the change in apparent period of Jupiter's moon Io as a function of the changing distance of the Earth from Jupiter (and Io). This method was the topic of a question in a recent qual. 2) Bradley used the change in apparent position of stars due to the motion of the Earth around the sun (aberration) to estimate the ratio of c to the Earth's orbital velocity which itself is know from the year and the Earth - sun distance. 3) The time for radio signals to travel from Earth to spacecraft at positions known from the size of the solar system and the gravity of planets they are orbiting gives the most accurate astronomical measurement with an error of 2×10^{-11}. 4) Fizeau and Foucault measured the time for light to travel a known distance on Earth. Fizeau measured time by requiring the the beam of light pass between different teeth in a cog wheel rotating at a known speed. Foucault reflected both the outgoing and return beams of light on a mirror rotating at a known speed and then measured time from the angle between the two beams. A 1% error can be obtained today in the class room using a fast oscilloscope to measure time delay of a pulse from a laser or led. 6) From Maxwell, $c^2 = 1/(\epsilon_0\mu_0)$. With μ_0 defined as $4\pi \times 10^{-7}$ H/m, we measure ϵ_0 from the dimensions and capacitance of a capacitor. The error in 1907 was $< 10^{-4}$. 7) We can find $c = f\lambda$ by measuring the wavelength λ of a wave of known frequency f. Wavelength can be measured as twice the distance between nodes of standing waves in a cavity, a microwave cavity (error 10^{-5} in 1950) or, the arms of an interferometer using a laser whose frequency is tied to a more accurately known lower frequency signal (error 3×10^{-9} by 1972).

INSTRUCTIONS
PART 2 : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11: GRADUATE MECHANICS

`PROBLEM:`

(a) Consider a binary star system, with stars of masses, m_1 and m_2, in a circular orbit of radius r_0, with period τ, thanks to the attraction of classical, Newtonian gravity (you can ignore general relativity effects). Derive Kepler's third law relation between r_0 and τ. [1 point]

(b) Suppose that the masses of the binary star system are $m_1 = M_{sun}$ and $m_2 = 15M_{sun}$ (here M_{sun} is the solar mass unit, the mass of our sun), and that $r_0 = 4AU$ (four times the distance between the earth and the sun). Find the orbit period, τ, in units of years. [4 points]

(c) Some mischievous aliens are playing with their new weapon / toy. By the push of a button, they cause the two stars to suddenly stop moving. Right after they push the button, the stars are separated by the same distance r_0, but have zero velocity. How long do the aliens now have to wait, in years, before they can enjoy watching the two stars collide? (You might, or might not, be interested to know that $\int_0^1 \frac{du}{\sqrt{u^{-1}-1}} = 2\int_0^{\pi/2} \sin^2\theta d\theta = \pi/2$, where $u = \sin^2\theta$ is used in the second step.) [4 points]

(d) Is such an alien toy possible? What physical principle does it possibly violate? Can you suggest a way to make it work, without violating anything? Just a few comments here is sufficient. [1 point].

`SOLUTION:`

(a) Recall $\mu = m_1 m_2/(m_1 + m_2)$ and that central circular motion has

$$F = \frac{Gm_1m_2}{r_0^2} = \mu\omega^2 r_0 \tag{1}$$

which gives

$$r_0^3 = \frac{G(m_1 + m_2)\tau^2}{4\pi^2} \tag{2}$$

(b) For the earth-sun system, $m_1 + m_2 \approx M_{sun}$ in the above formula. By comparison, here r_0 is 4 times bigger, and $m_1 + m_2$ is 16 times bigger, so τ is 2 years.

(c) This is a special case of central, non-circular motion. Use the energy conservation equation

$$\frac{1}{2\mu}\dot{r}^2 - \frac{Gm_1m_2}{r} = E = -\frac{Gm_1m_2}{r_0} \tag{3}$$

solve for dr/dt and integrate to get Δt. So the time is (with $u \equiv r/r_0$)

$$t_{collide} = \frac{r_0^{3/2}}{\sqrt{2G(m_1+m_2)}} \int_0^1 \frac{du}{\sqrt{u^{-1}-1}} = \frac{\tau}{2\pi\sqrt{2}}(\pi/2) = \frac{\tau}{4\sqrt{2}} = \frac{1}{2\sqrt{2}} years \tag{4}$$

(d) It removed energy and angular momentum, which should be conserved. Perhaps they are radiated off by lots of photons.

#12: GRADUATE MECHANICS

PROBLEM:

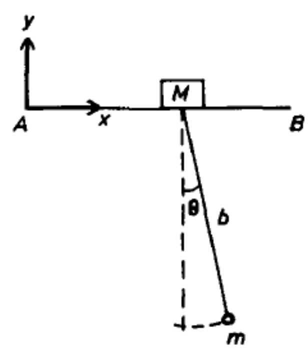
A mass M is constrained to slide without friction on the track AB as shown in the figure. A mass m is connected to M by a massless inextensible string of length b.

(a) Write the Lagrangian for this system.

(b) Write the leading order Lagrangian, assuming small oscillations.

(c) Find the normal coordinates and describe them.

(d) Find expressions for the normal coordinates as a function of time, for completely general initial conditions (identify your constants of integration).



SOLUTION:

(a) Use the coordinates as shown in the figure. M and m have coordinates

$$(x, 0), (x + b\sin\theta, -b\cos\theta) \tag{5}$$

respectively. The Lagrangian of the system is then

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta) + mgb\cos\theta \tag{6}$$

(b) For small oscillations, θ and $\dot{\theta}$ are small quantities and we have the approximate Lagrangian

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta) + mgb\left(1 - \frac{\theta^2}{2}\right) \tag{7}$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{8}$$

then give $(m+M)\dot{x} + mb\dot{\theta} = C$, a constant, and $\ddot{x} + b\ddot{\theta} + g\theta = 0$.

(c) In the above, the first equation can be written as

$$(m+M)\dot{\eta} = C \tag{9}$$

by setting

$$\eta = x + \frac{mb\theta}{m+M}. \tag{10}$$

As $(m+M)\ddot{x} + mb\ddot{\theta} = 0$, the second equation can be written as

$$\frac{mb\ddot{\theta}}{m+M} + g\theta = 0. \tag{11}$$

The two new equations of motion are now independent of each other. Hence η and θ are the new normal coordinates of the system. The center of mass of the system occurs at a distance $\frac{mb}{m+M}$ from M along the string. Hence *eta* is the x-coordinate of the center of mass. Equation (5) shows that the motion of the center of mass is uniform. The other normal coordinate, θ, is the angle the string makes with the vertical.

(d) Equation (5) has the solution

$$\eta = \frac{Ct}{m+M} + D, \tag{12}$$

and Equation (7) has solution

$$\theta = A\cos(\omega t), \tag{13}$$

where

$$\omega = \sqrt{\frac{(m+M)g}{Mb}} \tag{14}$$

is the angular frequency of small oscillations of the string, and A, B, C, D are constants.

#13: GRADUATE E&M

`PROBLEM:`

In three spatial dimension (x,y,z) the scalar potential field $\phi(x, y, z, t)$ from a charged particle of charge Q at the origin (x,y,z) = (0,0,0) satisfies

$$\nabla^2 \phi(x, y, z, t) = -4\pi Q \delta(x)\,\delta(y)\,\delta(z). \tag{15}$$

(a) Explain where this equation comes from. It might be a good idea to start with Maxwell's equations.

Show the solution to this equation is

$$\phi(x, y, z, t) = \frac{Q}{R}, \qquad R^2 = x^2 + y^2 + z^2. \tag{16}$$

(b) Now using Maxwell's equations again, write the wave equation satisfied by $\phi(x, y, z, t)$ in the Lorentz gauge:

$$\nabla \cdot \mathbf{A}(x, y, z, t) + \frac{1}{c}\frac{\partial \phi(x, y, z, t)}{\partial t} = 0. \tag{17}$$

(c) Solve this equation in an inertial frame where the particle is seen moving along the x-axis at a constant velocity v. Hint: the first part of this problem is quite relevant.

(d) Find the vector potential $\vec{A}$ in the setup of parts (b) and (c).

`SOLUTION:`

(a) It comes from $\vec{\nabla} \cdot \vec{E} = 4\pi\vec{\rho}$, and $\vec{E} = -\nabla\phi$ (static), with $\rho = Q\delta^3(\vec{x})$ for a point charge at the origin. Then $\phi = Q/r$ gives $\vec{E} = Q\widehat{r}/r^2$, and $\nabla^2\phi(\vec{r})$ vanishes for $\vec{r} \neq 0$. The delta function at the origin can be seen via $\oint \vec{E} \cdot d\vec{a} = 4\pi Q$.

(b)

The scalar potential satisfies the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(x, y, z, t) = -4\pi Q \delta(y)\delta(z)\delta(x - vt), \tag{18}$$

which follows from Maxwell's equations in the Lorentz gauge.

(c) Change variables to $X = \gamma(x - vt), Y = y, Z = z$, and the equation becomes

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)\phi(X, Y, Z) = -4\pi Q\gamma\delta^3(X), \tag{19}$$

with $\gamma = \frac{1}{\sqrt{1-v^2/c^2)}}$. The solution of this, as above, is

$$\phi(x, y, z, t) = \frac{Q\gamma}{\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2)}}. \tag{20}$$

(d) The vector potential $\mathbf{A}(x, y, z, t)$ satisfies, in the Lorentz gauge,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(x, y, z, t) = -4\pi Q\frac{\mathbf{v}}{c}\delta(y)\delta(z)\delta(x - vt), \tag{21}$$

leading to

$$\mathbf{A}(x, y, z, t) = \frac{Q\frac{\mathbf{v}}{c}\gamma}{\sqrt{\gamma^2(x - vt)^2 + y^2 + z^2)}}. \tag{22}$$

Simple differentiation then shows that the Lorentz gauge condition is satisfied.

#14: GRADUATE E&M

PROBLEM:

Consider a paramagnetic sphere of radius a, with magnetic permeability μ, and conductivity σ.

(a) The sphere is in the presence of a constant external magnetic field $\vec{B}_{\rm ext} = B_0\hat{z}$. Find the total magnetic field $\vec{B}$ everywhere, inside and outside the sphere, and the sphere's magnetization density $\vec{M}$.

(b) Now take $\vec{B}_{\rm ext} = B_0 e^{-i\omega t}\hat{z}$ (the real part). Find $\vec{E}$ inside the sphere, to leading order in $a\omega/c$.

(c) In the same setup and approximation as part (b), find the eddy current distribution in the sphere, and the average power absorbed (heat loss) by the sphere.

SOLUTION:

(a) The sphere gets a magnetization density and acts like a magnetic moment for $r > a$, and has a constant magnetic field for $r < a$. One can use the magnetic potential, and Legendre polynomials, finding that only $\ell = 1$ contributes. Gauss' law requires that $\hat{r}\cdot\vec{B}$ is continuous at $r = a$. Absence of surface current density sources requires that $\hat{r}\times\vec{H}$ is continuous at $r = a$. One thus finds $\vec{B}_{in} = \mu\vec{H}_{in}$, $\vec{H}_{in} = \vec{B}_0 - \frac{4\pi}{3}\vec{M}$, and

$$\vec{B}_{in} = \vec{B}_0 + \frac{8\pi}{3}\vec{M} = \frac{3\mu}{\mu+2}\vec{B}_0, \qquad \vec{M} = \frac{3}{4\pi}\frac{\mu-1}{\mu+2}\vec{B}_0, \tag{23}$$

$$\vec{B}_{out} = \vec{B}_0 + \frac{3(\hat{r}\cdot\vec{m})\hat{r}-\vec{m}}{r^3}, \qquad \vec{m} = \frac{4}{3}\pi a^3\vec{M}. \tag{24}$$

(b) In this approximation, $\vec{B}_{in}$ is as found above, simply replacing $\vec{B}_0 \to \vec{B}_0 e^{-i\omega t}$, and $\vec{E}_{in}$ is found from $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial\vec{B}/\partial t$. If we don't recall how to compute the curl in spherical coordinates, we can use the integrated form: $\oint_{\partial S}\vec{E}\cdot d\vec{\ell} = i\omega\frac{1}{c}\int_S \vec{B}_{in}\cdot d\vec{a}$. Clearly $\vec{E}$ points in the $\hat{\phi}$ direction, so we should take S to be a disk with boundary along the $\hat{\phi}$ direction, i.e. a disk of radius $r\sin\theta$. The electric field is thus (the real part of)

$$\vec{E}_{in} \approx i\frac{\omega}{2c}\frac{3\mu B_0}{\mu+2} r\sin\theta\hat{\phi}e^{-i\omega t}. \tag{25}$$

(c) The eddy current density is $\vec{J}_{in} = \sigma \vec{E}_{in}$, with $\vec{E}_{in}$ as above. The time averaged power loss is given by integrating $\frac{1}{2}\vec{J}_{in} \cdot \vec{E}_{in} = \frac{1}{2}\sigma \vec{E}_{in}^2$ over the sphere:

$$P = \frac{1}{2}\sigma\omega^2 \frac{9\mu^2 B_0^2}{4c^2(\mu+2)^2}(2\pi)\int_0^a dr r^4 \int_{-1}^{1} d(\cos\theta)(1-\cos^2\theta) \tag{26}$$

$$= \frac{3\pi a^5 \sigma \omega^2 \mu^2 B_0^2}{5c^2(\mu+2)^2}. \tag{27}$$

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:`

Consider the Hamiltonian of two particles (labeled by the subscripts $1, 2$), each in a 3d harmonic oscillator potential:

$$H = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + \frac{1}{2}m\omega^2\vec{r}_1^2 + \frac{1}{2}m\omega^2\vec{r}_2^2 + V_{int},$$

The two particles are **identical and of spin** $\frac{1}{2}$. (There is no spin-orbit coupling.) The interaction

$$V_{int} = -\epsilon\delta^3(\vec{r}_1 - \vec{r}_2)$$

will be treated as a small perturbation.

(a) Working to zero-th order, ϵ^0, (i.e. dropping V_{int}), what are the energy eigenvalues, and what is the form of the energy eigenfunctions? You don't need to derive the precise spatial dependence of the 1d harmonic oscillator eigenfunctions – just write the formal form of the wavefunction in terms of such quantities, taking care to properly label for example how the energy levels appear. Especially take care with regard to the particles being identical and include their spin degrees of freedom, accounting for all possibilities. How many integers etc. are needed to specify the states? Be sure that your energies and eigenfunctions depend on all these labels.

(b) Again, working to order ϵ^0, what is the energy, degeneracy, and total spin quantum number(s) of the ground state? Write the actual ground state wavefunction(s), including the spin degrees of freedom.

(c) What is the shift of the ground state energy to first order in V_{int}, i.e. ϵ^1?

(d) Accounting for V_{int} qualitatively, assuming that it is repulsive ($\epsilon < 0$), what is the spin S and degeneracy of the first excited state?

`SOLUTION:`

`(a)` Writing the 1d SHO energy eigenstates as $\psi_n(x)$, and the 3d eigenstates as $\psi_{\vec{n}}(\vec{x}) = \psi_{n_1}(x)\psi_{n_2}(y)\psi_{n_3}(z)$, the eigenstates we're after are antisymmetric upon exchanging the two identical fermions:

$$\text{Spin 0:} \quad (\psi_{\vec{n}}(\vec{r}_1)\psi_{\vec{m}}(\vec{r}_2) + \psi_{\vec{m}}(\vec{r}_1)\psi_{\vec{n}}(\vec{r}_2)) \otimes |S = 0\rangle,$$

$$\text{Spin 1:} \quad (\psi_{\vec{n}}(\vec{r}_1)\psi_{\vec{m}}(\vec{r}_2) - \psi_{\vec{m}}(\vec{r}_1)\psi_{\vec{n}}(\vec{r}_2)) \otimes |S = 1, S_z = 1, 0, -1\rangle,$$

where $|S = 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ is antisymmetric, and $|S = 1, S_z = 1, 0, -1\rangle$ is the symmetric triplet. The energy is

$$E = (n_1 + n_2 + n_3 + m_1 + m_2 + m_3 + 3)\hbar\omega.$$

The state is labeled by the six integers in $\vec{n}$ and $\vec{m}$. The spin 1 states are also labeled by the three choices $S_z = 1, 0, -1$.

(b) The ground state has $n_i = m_i = 0$, and $E = 3\hbar\omega$. The spatial part is symmetric, so the spin $S = 0$. The state is non-degenerate. Its energy eigenfunction is

$$\psi_{\vec{0}}(\vec{r}_1)\psi_{\vec{0}}(\vec{r}_2) \otimes |S = 0\rangle,$$

where

$$\psi_{\vec{0}}(\vec{r}) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} \exp(-\hbar\vec{r} \cdot \vec{r}/2m\omega).$$

(c) The perturbation shifts the energy to leading order by

$$\Delta E = \int d^3\vec{r_1} d^3\vec{r_2} \psi(\vec{r_1}, \vec{r_2})^* V(\vec{r_1}, \vec{r_2})\psi(\vec{r_1}, \vec{r_2}) = -\epsilon \int d^3\vec{r} |\psi(\vec{r}, \vec{r})|^2,$$

which is non-zero for the spatially symmetric wavefunction, and zero for the spatially antisymmetric. In particular, for the groundstate, the shift is by

$$\Delta E = -\epsilon \left(\frac{m\omega}{\pi\hbar}\right)^3 \int d^3\vec{r} \exp(-2\hbar\vec{r}\cdot\vec{r}/m\omega) = -\epsilon \left(\frac{m\omega}{\pi\hbar}\right)^3 \left(\frac{\pi 2\hbar}{m\omega}\right)^{3/2} = -\epsilon \left(\frac{2m\omega}{\pi\hbar}\right)^{3/2}$$

Note that ϵ has units of EL^3, where L is a length, and $m\omega/\hbar$ has units of $1/L^2$, so the units are copacetic. The groundstate energy to $\mathcal{O}(\epsilon)$ is

$$E_0 = 3\hbar\omega - \epsilon \left(\frac{2m\omega}{\pi\hbar}\right)^{3/2}.$$

(d) The first excited state has say $\vec{n} = (2, 1, 1)$ (and permutations), and $\vec{m} = 0$. Since the interaction is repulsive, the spatially antisymmetric state would have lower energy. So the spin is symmetric. The degeneracy is thus the product of the spatial and spin degeneracies: $3 \cdot 3 = 9$.

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM`: **Quantum interference versus measurement of which-way information**

Consider a double-slit interference experiment, described by a quantum system with two orthonormal states (call them $|\uparrow\rangle$ and $|\downarrow\rangle$), representing the possible paths taken by the particles. A particle emerging in state $|\uparrow\rangle$ produces a wavefunction at the screen of the form $\psi_\uparrow(x)$, (where x is a coordinate along the screen) while a particle emerging in state $|\downarrow\rangle$ produces the wavefunction $\psi_\downarrow(x)$. The evolution from the wall with the slits to the screen is linear in the input state.

As the source repeatedly spits out particles, the screen counts how many particles hit at each location x.

Suppose, for simplicity, that $\psi_\uparrow(x) = e^{\mathbf{i}k_\uparrow x}, \psi_\downarrow(x) = e^{\mathbf{i}k_\downarrow x}$, where $k_\uparrow, k_\downarrow$ are some real constants.

1. If the particles are all spat out in the state $|\uparrow\rangle$, what is the x-dependence of the resulting pattern $P_\uparrow(x)$?
2. If the particles are all spat out in the (normalized) state

   $$|\psi\rangle = \mu|\uparrow\rangle + \lambda|\downarrow\rangle \ ,$$

   what is the x-dependence of the resulting pattern, $P_\psi(x)$? Assume μ, λ are real.

Now we wish to take into account interactions with the environment, which we will model by another two-state system, with Hilbert space $\mathcal{H}_E$. Suppose these interactions are described by the hamiltonian

$$\mathbf{H} = \boldsymbol{\sigma}^z \otimes \mathbf{M}$$

acting on $\mathcal{H}_2 \otimes \mathcal{H}_E$, where $\boldsymbol{\sigma}^z = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$ acting on the Hilbert space $\mathcal{H}_2$ of particle paths, and $\mathbf{M}$ is an operator acting on the Hilbert space of the environment.

Suppose the initial state of the whole system is

$$|\Psi_0\rangle \equiv (\mu|\uparrow\rangle + \lambda|\downarrow\rangle) \otimes |\uparrow\rangle_E \ ,$$

and that

$$\mathbf{M} = m\boldsymbol{\sigma}^x = m\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)_E.$$

3. Find $|\Psi(t)\rangle$, the state of the whole system at time t.
4. How does the interference pattern depend on x and t? For simplicity, consider the case where $\mu = \lambda = \frac{1}{\sqrt{2}}$.
5. Interpret the previous result in terms of the time-dependence of the entanglement between the two qbits.
6. What would happen if instead the initial state of the environment were an eigenvector of $\mathbf{M}$?

`SOLUTION:`

1.
$$P_\uparrow(x) = |\psi_\uparrow(x)|^2 = |e^{\mathbf{i}k_\uparrow x}|^2 = 1.$$
It is featureless.

2.
$$P_\psi(x) = |\mu\psi_\uparrow(x) + \lambda\psi_\downarrow|^2 = |\mu|^2 P_\uparrow + |\lambda|^2 P_\downarrow + (\mu\lambda^\star e^{\mathbf{i}k_\uparrow x} e^{-\mathbf{i}k_\downarrow x} + cc)$$
$$= |\mu|^2 + |\lambda|^2 + 2\mathrm{Re}\left(\mu\lambda^\star e^{\mathbf{i}x(k_\uparrow - k_\downarrow)}\right) = 1 + 2\mu\lambda\cos\left(x(k_\uparrow - k_\downarrow)\right).$$
(In the last step we used the fact that $|\psi\rangle$ is normalized.)

3.
$$|\Psi(t)\rangle = e^{-\mathbf{i}\mathbf{H}t}|\Psi_0\rangle = e^{-\mathbf{i}(\boldsymbol{\sigma}^z)\otimes\mathbf{M}t}\left(\mu|\uparrow\rangle + \lambda|\downarrow\rangle\right)\otimes|m\rangle_E$$
$$= \mu|\uparrow\rangle \otimes e^{-\mathbf{i}\mathbf{M}t}|\psi_0\rangle_E + \lambda|\downarrow\rangle\otimes e^{+\mathbf{i}\mathbf{M}t}|\psi_0\rangle_E$$

4. The observable we are measuring acts as the identity on the environment, so the resulting pattern is
$$P(x) = ||\mu\psi_\uparrow(x)e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}|\psi_0 = \uparrow\rangle_E + \lambda\psi_\downarrow e^{+\mathbf{i}mt\boldsymbol{\sigma}^x}|\psi_0 = \uparrow\rangle_E||^2$$
$$= \left(|\mu|^2 P_\uparrow + |\lambda|^2 P_\downarrow\right)\langle\uparrow|e^{\mathbf{i}mt\boldsymbol{\sigma}^x}e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E + 2\mathrm{Re}\left(\mu\lambda^\star\psi_\uparrow(x)\psi_\downarrow^\star(x)\langle\uparrow|e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E\right)$$
$$= \left(|\mu|^2 + |\lambda|^2\right) + 2\mathrm{Re}\left(\mu\lambda^\star\psi_\uparrow(x)\psi_\downarrow^\star(x)\langle\uparrow|e^{-2\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E\right)$$
$$= 1 + 2\mathrm{Re}\left(\mu\lambda^\star\psi_\uparrow(x)\psi_\downarrow^\star(x)\cos 2mt\right) = 1 + 2\mu\lambda\cos\left(x(k_\uparrow - k_\downarrow)\right)\cos 2mt$$

5. The strength of the interference pattern oscillates with frequency $2m$. When $t = \frac{\pi}{m}$, the interference pattern disappears. This is when the two qbits are maximally entangled – at this time, the environment qbit has 'measured' which slit the particle traversed.

6. Given the assumption that

$$\mathbf{M}|\psi_0\rangle_E = m|\psi_0\rangle_E$$

the state at time t is

$$|\Psi(t)\rangle = \mu|\uparrow\rangle \otimes e^{-\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E + \lambda|\downarrow\rangle \otimes e^{+\mathbf{i}mt\boldsymbol{\sigma}^x}|\uparrow\rangle_E.$$

and the pattern is

$$P(x) = ||\mu\psi_\uparrow(x)e^{-\mathbf{i}mt}|\psi_0\rangle_E + \lambda\psi_\downarrow e^{+\mathbf{i}mt}|\psi_0\rangle_E||^2$$
$$= 1 + 2\mu\lambda\cos(x(k_\uparrow - k_\downarrow) - 2mt)$$

– in this case the amplitude of the pattern is time-independent (though the pattern will 'shimmer' a bit). In this case the environment never measures the which-way information.

#17: GRADUATE STAT MECH

PROBLEM: A black hole as a thermodynamic system

There is a powerful analogy between the physics of black holes and thermodynamics. A static Schwarzchild black hole is a hole in space, characterized by a radius R_H, and a mass M. In equilibrium, these two parameters are related by

$$R_H = 2G_N M/c^2,$$

where G_N is Newton's constant and c is the speed of light. You should regard this as an equation of state.

Here is the dictionary that relates black hole properties to thermodynamic variables: The internal energy of a static Schwarzchild black hole is $E = Mc^2$. The entropy is

$$S_{\mathrm{BH}} = \frac{k_B c^3}{\hbar} \frac{A}{4G_N}$$

where A is its surface area, $4\pi R_H^2$. [Note that the resulting thermodynamic system has only one independent thermodynamic variable.]

1. By demanding that the first law of thermodynamics applies to black holes, compute the temperature of a static Schwarzchild black hole of mass M.

2. Compute the specific heat of the Schwarzchild black hole. What is the physical consequence of its sign?

3. An intermediate step which can be useful for the next part, and doesn't involve black holes: What is the entropy of thermal radiation in a box of volume V, $S_{\mathrm{photons}}(T, V)$? You may give your answer in terms of an unspecified constant prefactor.

4. Consider a black hole in a box of volume V with adiabatic walls, in equilibrium with the thermal radiation. We would like to determine whether the black hole evaporates. Proceed as follows:

   Suppose a fraction x of the total energy is in the black hole, and the rest is in the thermal radiation. Derive an equation determining the equilibrium value of x. What is the equilibrium value of x when the volume is large?

`SOLUTION:`

1. $$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{1}{c^2}\frac{\partial S_{\rm BH}}{\partial M} = \frac{8\pi G_N}{\hbar c^3}M.$$

2. From above, we have
$$E(T) = M(T)c^2 = \frac{\hbar c}{8\pi G_N}\frac{1}{T}$$
so
$$C_V = \frac{\partial E}{\partial T} = -\frac{\hbar c}{8\pi G_N}\frac{1}{T^2} < 0\ .$$
Negative specific heat means instability. In fact, a black hole evaporates, because it radiates like a blackbody at temperature T into empty space. Alternatively, a material made from an ensemble of black holes is unstable, since the black holes will merge into a larger black hole.

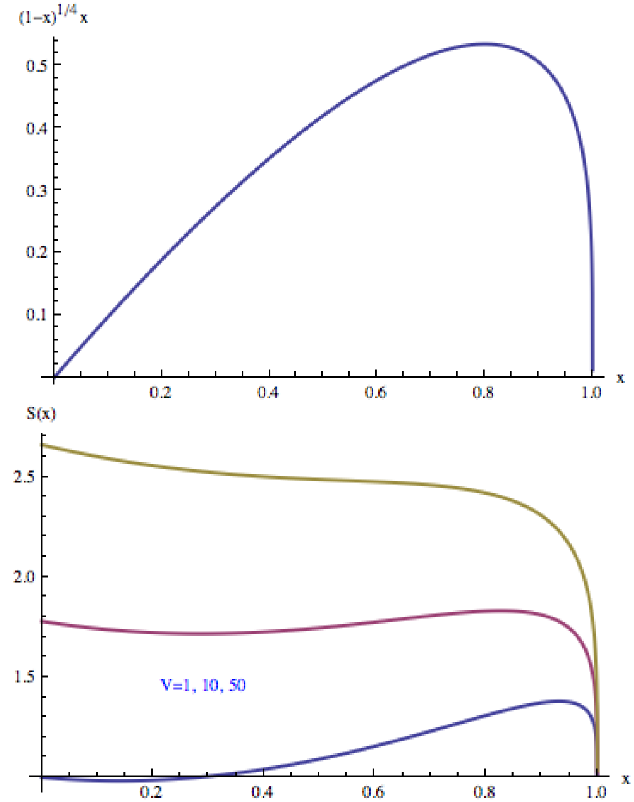
3. $S(T,V) \propto VT^3$. This follows from dimensional analysis: the only scales in the problem are the temperature and the volume; the entropy is extensive, so it must be proportional to V; to make something dimensionless we must multiply by T^3. Alternatively, recall that $E = bVT^4$. Then $dE|_V = TdS$ is a differential equation for S,

4. The equilibrium configuration will maximize the total entropy
$$S(x) = S_{\rm BH}(E_{\rm BH} = xE = M(x)c^2) + S_{\rm photons}(E_{\rm photons} = (1-x)E),$$
keeping the total energy $E = E_{\rm BH} + E_{\rm photons}$ fixed. We can get the same answer by setting the temperatures equal. The energy in the blackbody radiation is
$$(1-x)E = bVT^4 \quad ,$$
where the constant b is related to the Stefan-Boltzmann constant by $b = 4\sigma/c$. The energy in the BH is $E_{\rm BH} = xE = M(x)c^2$, but from above the temperature is
$$T_{\rm BH} = T_{\rm photons} = T = \frac{\hbar c}{8\pi G_N}\frac{1}{M(x)}$$
which gives
$$xE = M(x)c^2 = \frac{\hbar c}{8\pi G_N}\frac{1}{T} \quad .$$

We now have two conditions on x, and we find:

$$x(1-x)^{1/4} = \frac{\alpha c^2 (bV)^{1/4}}{E^{5/4}}.$$

The LHS is bounded above by some fraction of 1. So if the RHS is small, there's an equilibrium solution for x. But if we make the box big enough (large V), we'll eventually exceed that fraction. In that case $x \to 0$ is where the entropy is biggest: no black hole.



#18: GRADUATE STAT MECH

`PROBLEM:`

Electrons in metal can be approximately considered as free electron gas in three dimensions. Let m denote the electron mass. Consider a system with electron density n, and assume that there is no spin polarization.

1. Derive the relation between the Fermi wavevector k_f and the electron density n.
2. Calculate the average kinetic energy E_K per electron at $T = 0K$. Express E_K in terms of the Fermi energy defined as $\epsilon_f = \frac{\hbar^2}{2m}k_f^2$.
3. Qualitatively explain the scaling of specific heat of the metal with temperature T, for $k_B T \ll \epsilon_f$. How is it different from the specific heat of the ideal Boltzmann gas? Can you present an intuitive reason?
4. Now let us further consider the effect of Coulomb interaction among electrons. Estimate the average Coulomb interaction E_c among electrons.
5. Usually when we say interactions are strong or weak, it is not based on the absolute value of E_c, but actually based on the dimensionless ratio defined as $r_s = E_c/E_K$.

   Rewrite r_s as the ratio between two length scales up to a constant at the order of one. What are these two length scales? (Hint: one of them is made of fundamental constants.) Will the interaction effect be weakened or strengthened as the electron density n increases?

   Can you estimate the order of the typical value of r_s in metals? Are they in the weak interaction regime?

`SOLUTION:`

1)

$$n = 2\int \frac{d^3k}{(2\pi)^3} \tag{28}$$

$$n = \frac{2}{8\pi^3}\frac{4\pi}{3}k_f^3 \tag{29}$$

$$k_f = (3\pi^2 n)^{1/3} \tag{30}$$

2)

$$E_K = \frac{\int_0^{k_f} k^2 dk \frac{\hbar k^2}{2m}}{\int_0^{k_f} k^2 dk} \tag{31}$$

$$= \frac{\hbar^2}{2m} \frac{k_f^5}{5} / \frac{k_f^3}{3} = \frac{3}{5}\epsilon_f \tag{32}$$

3) The Fermi distribution function $n(k)$ at $T = 0K$ is a step function $\theta(k < k_f)$. At finite temperature it is smeared around the $k \sim k_f$ within an energy window of $k_B T$ around ϵ_f.

Due to this reason, most fermions deeply inside the Fermi surface are frozen, and thus do not contribute to specific heat. Fermions that contribute to specific heat are those within a shell around the Fermi surface with a thickness of $k_B T$, and thus the specific heat of Fermi gas scales linearly with T as $T \ll T_f$.

4) $E_c = e^2/d$ with $d = n^{-1/3}$.

5)

$r_s = E_c/E_K \approx \frac{e^2/d}{\hbar^2/(md^2)} \approx \frac{d}{\hbar^2/(me^2)} = d/a_B$, where a_B is the Bohr radius.

The more dense the electron gas is, the weaker the interaction effect is.

The typical value of lattice constants in metal is a few Å, and thus r_s is typically at the order of $5 \sim 10$, and thus actually electrons in metal are not in the weak interaction regime.

#19: GRADUATE MATH

`PROBLEM:`

Recall the Laplace transform, $f(t) \to F(s)$, and its inverse $F(s) \to f(t)$:

$$F(s) = \int_0^\infty f(t)e^{-st}dt, \qquad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds,$$

where the latter integral is on a contour parallel to the imaginary s axis, and c is a constant.

(a) Find the function $f(t)$ for which

$$F(s) = \frac{b}{(s+a)^2 + b^2}, \qquad a, b = \text{positive constants.}$$

Don't forget to determine $f(t)$ for both positive and negative t, discuss each case explicitly and plot the function $f(t)$.

(b) Consider a series LRC circuit, with inductor, resistor, and capacitor specified by L, R, and C (all constants in t). The circuit consists of these three in a closed circuit, together with a battery that supplies constant voltage V_0. The circuit has a switch, which is open (breaking the current loop) for $t < 0$ and closed for $t \geq 0$. At time $t = 0$, the conductor has a charge $Q_0 = 0$. Write the equations (for $t \geq 0$) which need to be solved in order to find the charge $Q(t)$ on the conductor and the current $I(t)$ in the circuit.

(c) Let $I(s)$ be the Laplace transform of $I(t)$. Using the results of part (b), solve for $I(s)$.

(d) Using the results from the above parts, solve for $I(t)$. Be sure that your solution satisfies the boundary conditions.

`SOLUTION:`

(a)

$$f(t) = \frac{1}{2\pi i}\int e^{st}\frac{b}{(s+a)^2 + b^2}$$

For $t < 0$ we close the contour at infinity at positive real s, and no poles are enclosed, so $f(t < 0) = 0$. For $t > 0$ we close the contour at infinity at negative real s, and get poles at $s = -a \pm ib$. Evaluating the residues there, this gives

$$f(t) = \Theta(t) e^{-at} \sin bt,$$

where $\Theta(t)$ is the step function: $\Theta(t < 0) = 0$, $\Theta(t > 0) = 1$.

(b)

$$IR + L\frac{dI}{dt} + \frac{Q}{C} = V(t) = V_0 \Theta(t), \qquad I(t) = \frac{dQ}{dt}$$

taking a derivative gives

$$R\frac{dI}{dt} + L\frac{d^2 I}{dt^2} + \frac{I}{C} = V_0 \delta(t).$$

(c) The Laplace transform converts $\frac{d}{dt} \to -s$, so we get

$$(-Rs + Ls^2 + \frac{1}{C}) I(s) = V_0.$$

We can write this as

$$I(s) = \frac{V_0}{L} \frac{1}{(s+a)^2 + b^2}, \qquad a \equiv R/2L, \qquad b \equiv \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

(d) Using the result from parts (a) and (c), we have

$$I(t) = \frac{V_0}{L} \frac{1}{b} \Theta(t) e^{-at} \sin(bt).$$

#20: GRADUATE GENERAL

`PROBLEM:` When the universe is about 400,000 years old, protons are at 3000 K, their density is $n_p = 2 \times 10^8$ protons per m^3 and only three species are important for electromagnetic interactions: protons, electrons and photons. You may need various constants; $e = 1.6 \times 10^{-19}$ C, $m_e = 9.1 \times 10^{-31}$ kg, $k = 1.38 \times 10^{-23}$ J/K, $\epsilon_0 = 8.85 \times 10^{-12}$m^{-3}kg^{-1}s^{4}A^2, $\sigma_T = 6.7 \times 10^{-29}$ m^2 and the baryon to photon (number density) ratio is $n_p/n_\gamma = 6 \times 10^{-10}$.

(a) Why do we not consider neutrinos, dark matter and dark energy when we consider significant electromagnetic interactions?

(b) Give arguments and equations leading to a quantitative estimate for the time scale (in seconds) for an electron to significantly change its kinetic energy through interactions with protons, when electrons are at T=3000 K. Hint: find the Coulomb interaction cross section by taking the impact parameter for an e-p interaction such that the electromagnetic potential energy equals the electron's kinetic energy. Use this to find the time between interactions.

(c) What is the time scale for photons to exchange energy with the e and p? Hint: find the interaction time between photons and electrons, associated with Thompson scattering. Estimate the associated fractional energy change per scattering. Thereby find the time scale needed to establish thermal equilibrium between electrons and photons.

(**Fact:**) One can likewise compute the time scale for a proton to significantly change its kinetic energy through interactions with electrons, but we will **not** ask you to do this computation here. It turns out that this time scale is intermediate between the times computed in parts (b) and (c). The times are all less than the lifetime of the universe.

(d) What do the results mean? State the very simple prediction that follows from these results.

`SOLUTION:`

(a) These are unimportant because all are observed to be electromagnetically neutral. Neutrinos are as common as photons but do not participate in electromagnetic interactions. Dark matter is not known to be a particle. If it is, its mass density exceeds that of protons by about a factor of 7, but astronomical observations show that it has minimal if any electromagnetic interactions. Dark energy has no known electromagnetic iteration and observations indicate it likely has negligible energy density when the universe is 3000 K.

(b) There is a significant change in the KE of an electron if its direction of motion changes by some large angle, say 90 degrees. This happens when the impact parameter for a e-p interaction, r_c gives electromagnetic potential energy equal to the electron KE, or $e^2/(4\pi\epsilon_0 r_c) \approx \frac{1}{2}mv^2 \approx \frac{3}{2}kT$. Then the Coulomb interaction cross-section

$$\sigma_c = \pi r_c^2 \approx \pi \left(\frac{e^2}{4\pi\epsilon_0 \frac{3}{2}kT}\right)^2 \approx 1.4 \times 10^{-17} \left(\frac{T}{3000K}\right)^{-2} \mathrm{m}^2 \qquad (33)$$

The time between interactions is $t_c = 1/(n_e \sigma_c v)$ where $n_e = n_p$ by charge neutrality (we ignore the extra proton and electron with the Helium nuclei), and the mean velocity of electrons is $v = \sqrt{3kT/m_e} = 400,000$ m/s, or $t_c = 180$ s.

(Fact aside (not graded):) Typical interactions between protons and electrons will not significantly change the proton KE because the proton mass is larger. Since momentum is conserved, the interaction considered in (b) will change the proton velocity by an amount smaller by a factor of (m_e/m_p). If we assume the e and p have the same temperature (to be justified by the results) then p and e have the same mean KE, and the proton velocities are smaller by $v_p = v_e\sqrt{m_e/m_p}$. The fractional change in the KE of the proton per interaction is

$$\frac{\Delta \mathrm{KE_p}}{\mathrm{KE_p}} \approx \frac{(\Delta \mathrm{v_p})^2}{\mathrm{v_p^2}} \approx \left(\frac{\mathrm{m_e}}{\mathrm{m_p}}\right) \frac{\Delta \mathrm{KE_e}}{\mathrm{KE_e}}, \qquad (34)$$

where the last factor is of order 1 in 180 s from (b). We need (m_p/m_e) steps taking 300,000 s to significantly change the proton KE. This is the time scale to establish thermal equilibrium between p and e. Since this is much shorter than the age of the universe, and the time scale on which the T changes, this equilibrium will be exact.

(c) The time between interactions of photons with elections is given by the Thomson cross-section, $t_T = 1/(n_\gamma \sigma_T c) = 170$ s when the number density of photons is $n_\gamma = 2 \times 10^8/6 \times 10^{-10} = 3 \times 10^{17}$ m^{-3}. The energy change per scattering is a fraction

$$\frac{\Delta E_e}{E_e} \approx \frac{h\nu}{m_e c^2} \approx \frac{3kT}{m_e c^2} \approx 1.5 \times 10^{-6} \qquad (35)$$

of the total electron energy at 3000 K. The time scale to establish thermal equilibrium between electrons and photons is then 170 s/$1.5 \times 10^{-6} = 7 \times 10^{7}$ s, again much less than the age of the universe and the time scale for significant T change. This establishes that the photons and electrons are in exact thermal equilibrium.

(d) We have shown that p, e and photons are all in thermal equilibrium, and thence that all relevant matter and radiation is in thermal equilibrium. This means that the photons from that time (today they are microwaves in the cosmic microwave background) will have a precise black body spectrum, as was shown by the COBE satellite in 1992.