INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1: UNDERGRADUTE MECHANICS

`PROBLEM:` Two particles of mass m_1 and m_2, move about each other in circular orbits under the influence of gravitational forces with a period τ. The motion is instantly stopped and the particles are then released and allowed to fall into each other. Find the time after which they collide. Express your answer in terms of τ.

#2: UNDERGRADUTE MECHANICS

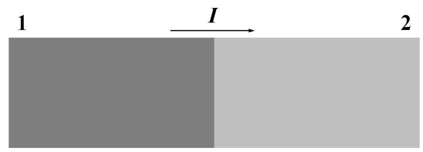
`PROBLEM:` A bead of mass m_1 can slide on a horizontal frictionless rod; call its position X. A (massless) string connects the bead to another mass, m_2, which hangs down because of gravity; call its angle relative to vertical ϕ.

(a) Write the Lagrangian for the above system, in the coordinates given above. Please simplify the expression as much as you can (e.g. use basic trig identities).

(b) Compute the generalized momenta p_X and p_ϕ, and associated generalized forces.

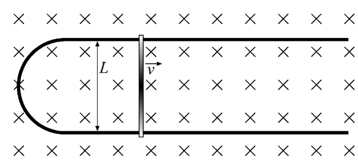
(c) Write the Euler-Lagrange equations of motion.

(d) Find expressions for all conserved quantities.



#3: UNDERGRADUTE E&M

`PROBLEM:` A current I flows through a wire made of a piece of material 1 and a piece of material 2 of identical cross-sections A welded end-to-end as shown in the figure. The resistivity of material 1 is ρ_1, the resistivity of material 2 is ρ_2. How much electric charge accumulates at the boundary between the two materials?



#4: UNDERGRADUTE E&M

`PROBLEM:` A cylindrical rod made of a material with a density ρ, and electrical conductivity σ, has a small diameter and a length L and is moving with a speed v along an infinite U-shaped wire in a uniform magnetic field B, which is applied perpendicularly to the wire and the rod. The distance between the wires is the same as the rod length. The electrical resistance of the wire and the friction between the wide and the rod are negligible, and there is no gravity. Find the distance the wire will move before it stops.

#5: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` Assume particles A,B, and C are bosonic states with angular momentum $J = 0$ and parity $P = -1$. Particle D is a bosonic state with $J = 1$ and $P = -1$, and particle E is boson with $J = 0$ and $P = +1$. Assume that J and P are conserved. Which of the following transitions are allowed? Please state your answer with reasons for each case.

1. $A \rightarrow B + C$
2. $D \rightarrow B + C$
3. $E \rightarrow B + C$
4. $D \rightarrow A + A$

#6: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in a time dependent electric field $\overrightarrow{E} = E(t)\hat{k}$. Consider transitions between the n=1 ground state, and the n=2 excited states given the perturbation $H' = eEz$.

(a) Use symmetry to show that for all n=1 and n=2 states the matrix element $H'_{ii} = 0$.

(b) Use symmetry arguments to show that only one of the n=2 states can be reached from the n=1 ground state.

(c) Calculate the one remaining transition matrix element H'_{ij} between n=1 and n=2 states that is not zero.

#7: UNDERGRADUTE STAT MECH/THERMO

`PROBLEM:`

1.5 mol of Helium considered as an ideal gas is initially at a temperature of 300 K and has a volume of 15 liters. It undergoes an isothermal expansion to a volume of 30 liters and then an adiabatic contraction to the initial volume. By what factor does the number of gas molecules with the x-component of velocity between 200 m/s and 200.1 m/s change? Various constants are (in SI units) Avogadro number: 6.03×10^{23}; Boltzmann constant: 1.38×10^{-23}; mass of proton: 1.67×10^{-27}; molar mass of Helium 0.004; universal gas constant: 8.31; charge of electron: 1.6×10^{-19}. You might find the following integral useful: $\int_{-\infty}^{+\infty} e^{(-\lambda x^2)} = \sqrt{\pi/\lambda}$.

#8: UNDERGRADUTE STAT MECH/THERMO

`PROBLEM:` The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice, at the melting point (absolute temperature T_0). We want to use an ideal, maximally efficient, cyclic (reversible) refrigerator (powered by some external mechanical work) to freeze a mass m amount more of the liquid water in the bucket into ice. The refrigerator rejects all heat to a finite, external reservoir, of constant heat capacity C, that is initially also at temperature T_0.

(a) What is the change in the entropies of (i) the contents of the bucket where the water is changed to ice, (ii) the refrigerator, and (iii) the external reservoir? Write your answers as inequalities if appropriate.

(b) What is the change in the Gibbs function of the ice-water mixture in this process?

(c) What is the minimum mechanical work required to run the refrigerator for this process?

#9: UNDERGRADUTE MATH

`PROBLEM:` Evaluate the integral

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$$

around a circle C with equation $|z| = 3$.

#10: UNDERGRADUTE GENERAL

`PROBLEM:` a) Estimate the life time in years of a star that has a total mass 2×10^{30} kg, assuming that it life ends when the nuclear reactions consume 10% of all of the hydrogen, and that the star radiates at a constant rate that gives a flux, $f = 1360$ W/m^2, at $a = 1.5 \times 10^{11}$ m from the star.

b) The center of a star is dense and fully ionized. Estimate the time in years for a photon to escape from the center of the star out to a radius of $r = 10^8$ m, using a mean (plasma) density of $\rho = 10^5$ kg/m^3. Assume only elastic scattering off electrons and that the photon wavelength is much smaller than the mean free path.

The mass of a proton is $m_H = 1.673 \times 10^{-27}$ kg, the mass of an electron is $m_e = 9.11 \times 10^{-31}$ kg and the mass of a helium nucleus is $m_{He} = 6.646 \times 10^{-27}$ kg. The Thomson cross-section is $\sigma_T = 6.65 \times 10^{-29}$ m^2.

#11: GRADUATE MECHANICS

`PROBLEM:` A non-uniform spherical mass M, of radius a and moment of inertia I rolls on the outside of a fixed hemi-spherical surface of radius R. The sphere starts at rest at the top of the hemi-sphere, $\theta_1 = 0$, and then rolls down, without slipping, until it loses contact with the hemisphere. Here θ_1 is the coordinate for the position of the CM of the sphere.

(a) Write the Lagrangian, including the constraints via Lagrange multipliers.

(b) Find the value of θ_1 where the sphere first loses contact with the hemi-sphere, for general moment of inertia I.

(c) Which would lose contact first: a uniform solid ball, or a thin spherical shell?

#12: GRADUATE MECHANICS

`PROBLEM:`

The Hamiltonian for a charged particle moving in a uniform magnetic field $B\hat{z}$ is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2.$$

(a) Find Hamilton's equations of motion for this Hamiltonian.

(b) Find at least three constants of the motion.

(c) Solve Hamilton's equations of motion of part (a) for $x(t)$, $p_x(t)$, $y(t)$ and $p_y(t)$ in terms of initial value data $x(0)$, $p_x(0)$, $y(0)$ and $p_y(0)$.

(d) Find the Lagrangian corresponding to this Hamiltonian. Find the Euler-Lagrange equations of this Lagrangian. Check that your solutions for $x(t)$ and $y(t)$ from part (a) satisfy the Euler-Lagrange equations.

#13: GRADUATE E&M

`PROBLEM:`

The electric potential that results when a metallic sphere of radius R is placed in an external uniform electric field E_0 pointing in the z direction is of the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos\theta) \tag{1}$$

where P_l are Legendre polynomials.

(a) Give the expression $\phi_0(r,\theta)$, for the electric potential due to the external electric field only in spherical coordinates.

(b) By considering the potential $\phi(r,\theta)$ for large values of r, deduce which coefficients A_l, B_l in the expression for ϕ must vanish.

(c) By considering $\phi(r,\theta)$ on the surface of the sphere, deduce the values of all the coefficients A_l, B_l in the formula for ϕ, and hence an expression for $\phi(r,\theta)$.

(d) Find the magnitude of the electric field at $\theta = 0$ right outside the surface of the sphere.

Show all steps in your derivations and justify all steps.

#14: GRADUATE E&M

`PROBLEM:`

The radiation fields $\vec{E}, \vec{B}$ of an oscillating electric dipole $\vec{P}(t)$ are expressed in the SI unit as

$$\begin{aligned} \vec{B}(\vec{r},t) &= -\tfrac{\mu_0}{4\pi rc}\hat{e}_r \times \tfrac{\partial^2}{\partial_t^2}\vec{P}(t-\tfrac{r}{c}) \\ \vec{E}(\vec{r},t) &= -c\hat{e}_r \times \vec{B}(\vec{r},t), \end{aligned} \tag{2}$$

where the dipole is put at the origin, μ_0 is the vacuum permeability; c is the light velocity; $\hat{e}_r$ is the unit vector along the radial direction. Using these formulae, solve the following problems.

a) Put a point charge q at the origin. It is driven by a planar polarized electromagnetic plane wave $\vec{E} = \hat{x}E_0 e^{i(kz-\omega t)}$ where $k = \frac{\omega}{c}$. What is the radiation field $\vec{E}$ and $\vec{B}$ of this point charge.

b) Use the spherical coordinate, derive the angular distribution of the radiation power intensity (Poynting vector) $\vec{S}(\theta, \phi)$.

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:`

A particle with charge q and mass M is confined in the 2D xy-plane and moves in the presence of an external magnetic field $\vec{B} = B\hat{z}$. Its Hamiltonian is written as

$$H = \frac{(-i\hbar\vec{\nabla} - q\vec{A})^2}{2M}, \tag{3}$$

Use the symmetric gauge $\vec{A} = \frac{B}{2}\hat{z} \times \vec{r}$.

(a) Solve for the semiclassical motion. According to the electron's classic equation of motion and Bohr-Sommerfeld quantization condition, what is the radius l_B of the smallest cyclotron orbit? What is the angular frequency ω associated with this smallest cyclotron orbit?

(b) The "mechanical momentum" is defined as $\vec{P}_m = -i\hbar\vec{\nabla} - q\vec{A}$. Define operators a and $a^\dagger$ as $a = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} + iP_{m,y})$ and $a^\dagger = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} - iP_{m,y})$. Work out their commutation relations $[a, a^+]$.

(c) Show that the Hamiltonian can be written in terms of a and a^+ and find all its eigenvalues.

(d) Define the "guiding center coordinates" $\vec{R}_g = \vec{r} - \hat{z} \times \frac{\vec{P}_m}{M\omega}$ and the operators $b = \frac{1}{\sqrt{2}l_B}(R_{g,x} - iR_{g,y})$ and $b^\dagger = \frac{1}{\sqrt{2}l_B}(R_{g,x} + iR_{g,y})$. Work out the commutation relation $[b, b^\dagger]$. Prove that $[a, b] = [a^\dagger, b^\dagger] = 0$, and $[a, b^\dagger] = [a^\dagger, b] = 0$.

(e) Express the canonical angular momentum $L_z = (\vec{r} \times -i\hbar\vec{\nabla}) \cdot \hat{z}$ in terms of a, $a^\dagger$, b and $b^\dagger$ and find all its eigenvalues.

(f) For each energy level of Eq. 3, figure out its allowed eigenvalues of L_z.

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in an external perturbing potential

$$V = f(r)\left(x^2 + y^2\right)$$

You are given the matrix elements

$$\begin{aligned}
\langle 2s, m=0|f(r)x^2|2s, m=0\rangle &= v_x \\
\langle 2s, m=0|f(r)y^2|2s, m=0\rangle &= v_y \\
\langle 2s, m=0|f(r)z^2|2s, m=0\rangle &= v_z \\
\langle 2p, m=0|f(r)x^2|2p, m=0\rangle &= w_x \\
\langle 2p, m=0|f(r)y^2|2p, m=0\rangle &= w_y \\
\langle 2p, m=0|f(r)z^2|2p, m=0\rangle &= w_z
\end{aligned}$$

(a) There are two relations between v_x, v_y and v_z. Find these to write v_x and v_y in terms of v_z.

(b) There is one relation between w_x, w_y and w_z. Find this to eliminate w_y.

(c) Find the first order shift in the energy levels of the $n = 2$ states in terms of v_z and $w_{x,z}$.

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

	J	J	…
	M	M	…
m_1 m_2			
m_1 m_2	Coefficients		
⋮ ⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

1/2×1/2

m_1 m_2	$J=1$, $M=+1$
+1/2 +1/2	1

m_1 m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=1$, $M=-1$
−1/2 −1/2	1

1×1/2

m_1 m_2	$J=3/2$, $M=+3/2$
+1 +1/2	1

m_1 m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	$J=3/2$, $M=-3/2$
−1 −1/2	1

2×1/2

m_1 m_2	$J=5/2$, $M=+5/2$
+2 +1/2	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	$J=5/2$, $M=-5/2$
−2 −1/2	1

3/2×1/2

m_1 m_2	$J=2$, $M=+2$
+3/2 +1/2	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	$J=2$, $M=-2$
−3/2 −1/2	1

2×1

m_1 m_2	$J=3$, $M=+3$
+2 +1	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	$J=3$, $M=-3$
−2 −1	1

1×1

m_1 m_2	$J=2$, $M=+2$
+1 +1	1

m_1 m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	$J=2$, $M=-2$
−1 −1	1

3/2×1

m_1 m_2	$J=5/2$, $M=+5/2$
+3/2 +1	1

m_1 m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	$J=5/2$, $M=-5/2$
−3/2 −1	1

3/2×3/2

m_1 m_2	$J=3$, $M=+3$
+3/2 +3/2	1

m_1 m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	$J=3$, $M=-3$
−3/2 −3/2	1

2×3/2

m_1 m_2	$J=7/2$, $M=+7/2$
+2 +3/2	1

m_1 m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	$J=7/2$, $M=-7/2$
−2 −3/2	1

2×2

m_1 m_2	$J=4$, $M=+4$
+2 +2	1

m_1 m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	$J=4$, $M=-4$
−2 −2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

#17: GRADUATE STAT MECH

`PROBLEM:`

In the Ising model of ferromagnetism, the spin at each lattice site σ_i can take the values ± 1. The energy for each configuration of spins is

$$E = -J \sum_{<ij>} \sigma_i \sigma_j \tag{4}$$

where $J > 0$ and the sum $< ij >$ is over nearest neighbor sites. Assume each site has z nearest neighbors.

Call $m =< \sigma_j >$ the thermal average of the spin σ_j at temperature T. In the mean field approximation, the interaction of the spin σ_i with a neighbor σ_j is approximated by replacing σ_j by $< \sigma_j >= m$.

(a) Set up a self-consistency condition for the value of m using the fact that all lattice sites are equivalent, that will determine the value of m as function of J, z, and the temperature T. Hint: Use the canonical ensemble.

(b) Show that the self-consistency condition has a solution with $m \neq 0$ only for T lower than a critical value T_c, and find an expression for T_c in terms of z and J.

(c) By expanding the self-consistency condition for small m, show that m for T close to T_c can be written as

$m = C(T_c - T)^\beta$

with β and C constants, and find the values of β and C.

Show all steps in your derivations and justify all steps.

#18: GRADUATE STAT MECH

`PROBLEM:` Consider a gas of non-interacting, non-relativistic spin-1 bosons in an external magnetic field $B\hat{z}$. The one-particle Hamiltonian is

$$\mathcal{H}(\vec{p}, s_z) = \frac{\vec{p}^{\,2}}{2m} - \mu_0 s_z B,$$

where $\mu_0 \equiv \frac{e\hbar}{mc}$ and s_z, the spin quantum number in the $\hat{z}$ direction, can take three possible values $-1, 0, 1$.

(a) In the grand canonical ensemble, what are the average occupation numbers $\langle n_+(\vec{k})\rangle$, $\langle n_0(\vec{k})\rangle$ and $\langle n_-(\vec{k})\rangle$ of one-particle states with wavenumber $\vec{k} = \frac{\vec{p}}{\hbar}$ and with $s_z = -1$, 0, $+1$?

Use these average occupation numbers to find the average total numbers N_+, N_0, N_- of bosons with $s_z = -1$, 0, $+1$ in terms of the functions

$$f_m^+(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty dx \; \frac{x^{m-1}}{z^{-1}e^x - 1},$$

where $z \equiv e^{\beta\mu}$.

(b) The total number density is equal to

$$n = \frac{N_+ + N_0 + N_-}{V}.$$

Find the temperature $T_c(n, B = 0)$ of Bose-Einstein condensation at zero field in terms of the number density n.

#19: GRADUATE MATH

`PROBLEM:` Find the first term in the asymptotic expansion of the following integral (i.e. the behavior of the integral in the limit $x \to \infty$).

$$I = \frac{1}{\pi} \int_0^\pi (t^4 + 2t^6)^{1/2} e^{x \cos t} \cos(nt) dt,$$

where n is a constant. You may want definition of the Gamma function: $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$, with $\Gamma(1/2) = \sqrt{\pi}$.

#20: GRADUATE GENERAL

`PROBLEM:`

a) We examine the distribution of bubbles that form along the path of a charged particle in a bubble chamber and we find that bubbles are apparently randomly distributed with a uniform probability of occurance per unit length. This is equivalent to the following statements:

(i) There is at most one bubble in an infinitesimal interval of length $[l, l+\Delta l]$.

(ii) The probability $P_1(\Delta l)$ of finding one bubble in this interval is proportional to Δl (as long as $P_1(\Delta l)$ remains small).

(iii) The occurrence of a bubble in the interval $[l, l+\Delta l]$ is independent of the occurrence of bubbles in any other non-overlapping interval.

Derive an equation that gives the probability $P_o(l)$ of zero bubbles in the finite interval of length l, assuming that the average density of bubbles per unit length is g and that bubbles have negligible size.

b) Derive the probability density (per unit length) $f(l)$ that the first bubble on a track is at a distance l from some arbitrary origin.

c) If we count one bubble in a length of $l = 1$ mm, what is (1) the uncertainty in the observation and (2) the 68.3% confidence interval (one-sigma uncertainty) associated with our best estimate for g?

INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1: UNDERGRADUTE MECHANICS

`PROBLEM:` Two particles of mass m_1 and m_2, move about each other in circular orbits under the influence of gravitational forces with a period τ. The motion is instantly stopped and the particles are then released and allowed to fall into each other. Find the time after which they collide. Express your answer in terms of τ.

`SOLUTION:`
From energy conservation, we have

$$\frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_1m_2}{r} = -\frac{Gm_1m_2}{r_0}, \tag{1}$$

where G is the gravitational constant, r the relative coordinate, $\mu = m_1m_2/(m_1+m_2)$ the reduced mass. The initial relative coordinate r_0 is obtained from the centrifugal equation $\frac{Gm_1m_2}{r_0^2} = \mu\frac{v^2}{r_0}$, which gives

$$r_0^3 = \frac{G(m_1+m_2)\tau^2}{4\pi^2} \tag{2}$$

(Kepler's Third Law). From Eq. (1) the time t at which the particles collide is

$$t = C\int_0^{r_0} \frac{dr}{(1/r-1/r_0)^{1/2}}, \tag{3}$$

where $C = \sqrt{\mu/(2Gm_1m_2)}$.
Let $r = r_0 \sin^2\theta$. Then

$$t = 2r_0^{3/2}C\int_0^{\pi/2} \sin^2\theta d\theta = 2r_0^{3/2}C\frac{\pi}{4} = \frac{\pi}{2}\sqrt{\frac{\mu r_0^3}{2Gm_1m_2}}. \tag{4}$$

Using Eq. (2), we obtain $t = \tau/(4\sqrt{2})$.

#2: UNDERGRADUTE MECHANICS

`PROBLEM:` A bead of mass m_1 can slide on a horizontal frictionless rod; call its position X. A (massless) string connects the bead to another mass, m_2, which hangs down because of gravity; call its angle relative to vertical ϕ.

(a) Write the Lagrangian for the above system, in the coordinates given above. Please simplify the expression as much as you can (e.g. use basic trig identities).

(b) Compute the generalized momenta p_X and p_ϕ, and associated generalized forces.

(c) Write the Euler-Lagrange equations of motion.

(d) Find expressions for all conserved quantities.

`SOLUTION:`

(a) The bob has $x_{bob} = X + \ell \sin\phi$ and $y_{bob} = -\ell\cos\phi$, so its velocity is $(\dot{x}_{bob}, \dot{y}_{bob}) = (\dot{X} + \ell\cos\phi\dot{\phi}, \ell\sin\phi\dot{\phi})$ and so $v_{bob}^2 = \dot{X}^2 + 2\dot{X}\ell\cos\phi\dot{\phi} + \ell^2\dot{\phi}^2$. We thus have

$$\mathcal{L} = \tfrac{1}{2}(m_1+m_2)\dot{X}^2 + m_2\ell\cos\phi\dot{X}\dot{\phi} + \tfrac{1}{2}m_2\ell^2\dot{\phi}^2 + m_2 g\ell\cos\phi.$$

(b)

$$p_X = \frac{\partial\mathcal{L}}{\partial\dot{X}} = (m_1+m_2)\dot{X} + m_2\ell\cos\phi\dot{\phi}, \qquad p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m_2\ell\cos\phi\dot{X} + m_2\ell^2\dot{\phi}.$$

$$F_X = \frac{\partial\mathcal{L}}{\partial X} = 0, \qquad F_\phi = \frac{\partial\mathcal{L}}{\partial\phi} = -m_2\ell\sin\phi\dot{X}\dot{\phi} - m_2 g\ell\sin\phi.$$

(c)

$$\frac{d}{dt}p_X = F_X, \qquad \frac{d}{dt}p_\phi = F_\phi.$$

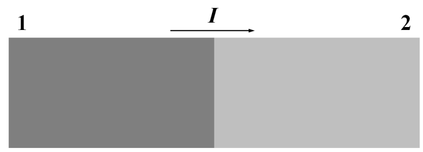
Let's write these out and simplify them:

$$p_X = (m_1+m_2)\dot{X} + m_2\ell\cos\phi\dot{\phi} = \text{constant}.$$

$$m_2\ell\cos\phi\frac{d^2X}{dt^2} + m_2\ell^2\frac{d^2\phi}{dt^2} = -m_2 g\ell\sin\phi.$$

(d) Because $\mathcal{L}$ does not depend on X, p_X is conserved. Because it does not depend explicitly on t, $\frac{\partial\mathcal{L}}{\partial t} = 0$, the Hamilton is also conserved:

$$\mathcal{H} = p_X\dot{X}+p_\phi\dot{\phi}-\mathcal{L} = \frac{1}{2}(m_1+m_2)\dot{X}^2+m_2\ell\cos\phi\dot{X}\dot{\phi}+\frac{1}{2}m_2\ell^2\dot{\phi}^2-m_2g\ell\cos\phi.$$



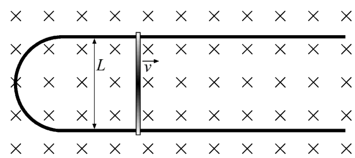
#3: UNDERGRADUTE E&M

`PROBLEM:` A current I flows through a wire made of a piece of material 1 and a piece of material 2 of identical cross-sections A welded end-to-end as shown in the figure. The resistivity of material 1 is ρ_1, the resistivity of material 2 is ρ_2. How much electric charge accumulates at the boundary between the two materials?

`SOLUTION:`
Ohm's law for a wire of length l gives the voltage $V = I\rho l/A$. Therefore, the electric field strength in the wire is $E = V/l = I\rho/A$. Due to the difference in the resistivity of the materials the electric field strength has to be different in material 1 and material 2. According to Gauss's law, the difference in the electric field strengths implies an accumulation of charge at the boundary of the two materials. The net accumulated charge is

$$Q = \varepsilon_0 A(E_2 - E_1) = \varepsilon_0 I(\rho_2 - \rho_1), \tag{5}$$

where ε_0 is the electric constant.



#4: UNDERGRADUTE E&M

`PROBLEM:` A cylindrical rod made of a material with a density ρ, and electrical conductivity σ, has a small diameter and a length L and is moving with a speed v along an infinite U-shaped wire in a uniform magnetic field B, which is applied perpendicularly to the wire and the rod. The distance between the wires is the same as the rod length. The electrical resistance of the wire and the friction between the wide and the rod are negligible, and there is no gravity. Find the distance the wire will move before it stops.

`SOLUTION:` Take the rod cross-section as S. Then its resistance is $R = l/(\sigma S)$ and the mass is $m = LS\rho$. The product of the resistance and mass $Rm = L^2\rho/\sigma$. The emf induced in the loop made by the wire and the rod is $EMF = vLB$, and the current in the wire is $I = vLB/R$. So it experiences force $F = BIL = v(LB)^2/R$, and acceleration $a = -F/m = -v(LB)^2/(RM) = -vB^2\sigma/\rho$. The velocity is $v(t) = v\exp(-B^2\sigma t/\rho)$. The displacement is $\Delta x(t) = v\frac{\rho}{B^2\sigma}[1 - \exp(-B^2\sigma t\rho)]$. The final displacement is $\Delta x = \frac{v\rho}{B^2\sigma}$.

#5: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` Assume particles A,B, and C are bosonic states with angular momentum $J = 0$ and parity $P = -1$. Particle D is a bosonic state with $J = 1$ and $P = -1$, and particle E is boson with $J = 0$ and $P = +1$. Assume that J and P are conserved. Which of the following transitions are allowed? Please state your answer with reasons for each case.

1. $A \rightarrow B + C$
2. $D \rightarrow B + C$
3. $E \rightarrow B + C$
4. $D \rightarrow A + A$

`SOLUTION:` $A \rightarrow B{+}C$ is not allowed because the final state must be $L_{tot} = 0$ to conserve angular momentum, and thus has P=+1. ($P = (-1)^L$.) The initial state has P=-1. The transition violates parity, and is thus not allowed.

$D \rightarrow B + C$ is allowed. Here the final state is $L_{tot} = 1$ to conserve angular momentum, and thus P=-1, just like the initial state.

$E \rightarrow B + C$ is allowed. The final state is L=0, and thus P=+1. The initial state is also P=+1, and the transition is thus allowed.

$D \rightarrow A{+}A$ is not allowed because angular momentum conservation requires $L_{tot} = 1$. However, such a state can not exist for two identical bosons due to Bose statistics.

#6: UNDERGRADUTE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in a time dependent electric field $\overrightarrow{E} = E(t)\hat{k}$. Consider transitions between the n=1 ground state, and the n=2 excited states given the perturbation $H' = eEz$.

(a) Use symmetry to show that for all n=1 and n=2 states the matrix element $H'_{ii} = 0$.

(b) Use symmetry arguments to show that only one of the n=2 states can be reached from the n=1 ground state.

(c) Calculate the one remaining transition matrix element H'_{ij} between n=1 and n=2 states that is not zero.

You may need: $\psi_{nlm} = R_{nl}Y_l^m$ with $\psi_{100} = \frac{1}{\sqrt{\pi a^3}}e^{-r/a}$, $\psi_{200} = \frac{1}{\sqrt{8\pi a^3}}(1 - \frac{r}{2a})e^{-r/2a}$, $\psi_{210} = \frac{1}{\sqrt{32\pi a^3}}\frac{r}{a}e^{-r/2a}\cos\theta$, $\psi_{21\pm1} = \mp\frac{1}{\sqrt{64\pi a^3}}\frac{r}{a}e^{-r/2a}\sin\theta e^{\pm i\phi}$,

---

`SOLUTION:`

(a) With $r\cos\theta = z$ and $r\sin\theta e^{\pm i\phi} = x \pm iy$ it is clear that $|\psi|^2$ is an even function of z in all cases, and hence $\int z|\psi|^2 dxdydz = 0$, and thus $H'_{ii} = 0$

(b) ψ_{100} is even in z, and so are all n=2 states except ψ_{210}. The only non-zero transition is thus 100 to 210.

(c) Let's now calculate the one transition that is not zero:

$H'_{100,210} = -eE\frac{1}{\pi a^3}\frac{1}{32\pi a^3}\frac{1}{a}\int e^{-r/a}e^{-r/2a}z^2 d^3r$

$H'_{100,210} = -\frac{eE}{4\sqrt{2}\pi a^4}\int e^{-3r/2a}r^2\cos^2\theta r^2\sin\theta dr d\theta d\phi$

$H'_{100,210} = -\frac{eE}{4\sqrt{2}\pi a^4}\int_0^\infty e^{-3r/2a}r^4 dr \int_0^\pi \cos^2\theta\sin\theta d\theta \int_0^{2\pi} d\phi$

$H'_{100,210} = -\frac{eE}{4\sqrt{2}\pi a^4}4!(\frac{2a}{3})^5\frac{2}{3}2\pi = -(\frac{2^8}{3^5\sqrt{2}})eEa = -0.7449eEa$

Solution. For Helium $\gamma = 5/3$. For adiabatic contraction from $T_1 = 300$ K and $V_1 = 30$ L to $V_2 = 15$ L we have $\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}$, $T_2 = T_1\left(\frac{V_1}{V_2}\right)^{2/3} = T_1 \cdot 2^{2/3} = 476$ K

Maxwell-Boltzmann distribution for a single component of gas velocity:

$$f(v) = \sqrt{m/(2\pi kT)}\exp[-\frac{mv^2}{2kT}]$$

$$\frac{f_2(v)}{f_1(v)} = \sqrt{T_1/T_2}\exp[-\frac{mv^2}{2kT_2} + \frac{mv^2}{2kT_1}] = 2^{-1/3}\exp[-\frac{4 \cdot 1.67 \cdot 10^{-27} kg \cdot (200m/s)^2}{2 \cdot 1.38 \cdot 10^{-23} J/K \cdot 300K}(2^{-2/3} - 1)]$$

$$\frac{f_2(v)}{f_1(v)} = 2^{-1/3}\exp(0.012) = 0.80$$

#7: UNDERGRADUTE STAT MECH/THERMO

`PROBLEM:`

1.5 mol of Helium considered as an ideal gas is initially at a temperature of 300 K and has a volume of 15 liters. It undergoes an isothermal expansion to a volume of 30 liters and then an adiabatic contraction to the initial volume. By what factor does the number of gas molecules with the x-component of velocity between 200 m/s and 200.1 m/s change? Various constants are (in SI units) Avogadro number: 6.03×10^{23}; Boltzmann constant: 1.38×10^{-23}; mass of proton: 1.67×10^{-27}; molar mass of Helium 0.004; universal gas constant: 8.31; charge of electron: 1.6×10^{-19}. You might find the following integral useful: $\int_{-\infty}^{+\infty} e^{(-\lambda x^2)} = \sqrt{\pi/\lambda}$.

`SOLUTION:`

#8: UNDERGRADUTE STAT MECH/THERMO

`PROBLEM:` The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice, at the melting point (absolute temperature T_0). We want to use an ideal, maximally efficient, cyclic (reversible) refrigerator (powered by some external mechanical work) to freeze a mass m amount more of the liquid water in the bucket into ice. The refrigerator rejects all heat to a finite, external reservoir, of constant heat capacity C, that is initially also at temperature T_0.

(a) What is the change in the entropies of (i) the contents of the bucket where the water is changed to ice, (ii) the refrigerator, and (iii) the external reservoir? Write your answers as inequalities if appropriate.

(b) What is the change in the Gibbs function of the ice-water mixture in this process?

(c) What is the minimum mechanical work required to run the refrigerator for this process?

`SOLUTION:`

The heat removed from the bucket is Lm, and the temperature stays at T_0.

(a) $\Delta S_{bucket} = -Lm/T_0$, $\Delta S_{fridge} = 0$, $\Delta S_{body} = \int dQ/T = C\ln(T_f/T_0) = C\ln(1 + Q_R/CT_0)$, where $T_f = T_0 + Q_R/C$.

(b) $\Delta G = 0$ along a phase transition.

(c) Assuming maximal efficiency, since $\Delta S_{bucket} + \Delta S_{fridge} + \Delta S_{body} = 0$, the minimum heat rejected to the reservoir is

$$Q_R = CT_0\left(\exp(Lm/CT_0) - 1\right).$$

By conservation of energy, the work needed to run the refrigerator is $Q_R - Lm$, so the minimum is

$$W \geq CT_0\left(\exp(Lm/CT_0) - 1\right) - Lm.$$

#9: UNDERGRADUTE MATH

`PROBLEM:` Evaluate the integral

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}dz$$

around a circle C with equation $|z| = 3$.

`SOLUTION:` Use the residue theorem $I = 2\pi i \sum$ residues inside contour.

There are 3 singularities inside the contour. A double pole at $z = 0$, two single poles at $z = -1 \pm i$.

Residue at $z = 0$ is:

$$\lim_{z\to 0}\frac{d}{dz}\frac{z^2e^{zt}}{z^2(z^2+2z+2)} = \lim_{z->0}\left(\frac{te^{zt}}{z^2+2z+2} - \frac{e^{zt}(2z+2)}{(z^2+2z+2)^2}\right) = \frac{t}{2}-\frac{2}{4} = \frac{t-1}{2}$$

Residue at $z = -1 + i$ is:

$$\lim_{z\to -1+i}\frac{(z+1-i)e^{zt}}{z^2(z^2+2z+2)} = \frac{e^{t(-1+i)}}{(-1+i)^2}\frac{(z+1+i)}{z^2+2z+2} = \frac{e^{t(-1+i)}}{(-2i)(2i)} = \frac{e^{t(-1+i)}}{4}$$

Residue at $z = -1 - i$ is:

$$\lim_{z\to -1-i}\frac{(z+1+i)e^{zt}}{z^2(z^2+2z+2)} = \frac{e^{t(-1-i)}}{(-1-i)^2}\frac{(z+1+i)}{z^2+2z+2} = \frac{e^{t(-1-i)}}{(2i)(-2i)} = \frac{e^{t(-1-i)}}{4}$$

Adding the residues:

$$\oint_C = 2\pi i\left(\frac{t-1}{2}+\frac{e^{-t}e^{it}}{4}+\frac{e^{-t}e^{-it}}{4}\right) \Rightarrow \frac{1}{2\pi i}\oint_C = \frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t.$$

#10: UNDERGRADUTE GENERAL

`PROBLEM:` a) Estimate the life time in years of a star that has a total mass 2×10^{30} kg, assuming that it life ends when the nuclear reactions consume 10% of all of the hydrogen, and that the star radiates at a constant rate that gives a flux, $f = 1360$ W/m^2, at $a = 1.5 \times 10^{11}$ m from the star.

b) The center of a star is dense and fully ionized. Estimate the time in years for a photon to escape from the center of the star out to a radius of $r = 10^8$ m, using a mean (plasma) density of $\rho = 10^5$ kg/m^3. Assume only elastic scattering off electrons and that the photon wavelength is much smaller than the mean free path.

The mass of a proton is $m_H = 1.673 \times 10^{-27}$ kg, the mass of an electron is $m_e = 9.11 \times 10^{-31}$ kg and the mass of a helium nucleus is $m_{He} = 6.646 \times 10^{-27}$ kg. The Thomson cross-section is $\sigma_T = 6.65 \times 10^{-29}$ m^2.

`SOLUTION:`

a) The majority of the energy released by a star comes from the fusion reactions that convert four protons into a helium nucleus. One such reaction releases binding energy $E = (4m_H - m_{He})c^2 = 4.13 \times 10^{-12}$ J. In its lifetime the star will convert $0.1 \times 2 \times 10^{30}$ kg of H into He (ignoring the mass in electrons), making $2 \times 10^{29}/m_{He} = 3.01 \times 10^{55}$ He nuclei, and releasing $E_L = 1.24 \times 10^{44}$ J.

The energy released by the star per second (its luminosity) is $L = 4\pi a^2 f = 3.845 \times 10^{26}$ W. The star can radiate at this rate for $E_L/L = 3.22 \times 10^{17}$ s, or 1.02×10^{10} years.

b) The photons random walk as they scatter on the electrons. We need the mean free path, $l = 1/\sigma_T n_e = 2.51 \times 10^{-5}$ m, where the number density of electrons is $n_e = \rho/(m_p + m_e) = 5.98 \times 10^{32}$ m^{-3}. The time per step is $t_1 = l/c = 8.37 \times 10^{-14}$ s. The number of steps is $N_s = (r/l)^2 = (10^8/2.51 \times 10^{-5})^2 = 1.59 \times 10^{25}$, and this many steps take $t_N = t_1 N_s = 1.33 \times 10^{12}$ s, or 42,100 years.

#11: GRADUATE MECHANICS

`PROBLEM:` A non-uniform spherical mass M, of radius a and moment of inertia I rolls on the outside of a fixed hemi-spherical surface of radius R. The sphere starts at rest at the top of the hemi-sphere, $\theta_1 = 0$, and then rolls down, without slipping, until it loses contact with the hemisphere. Here θ_1 is the coordinate for the position of the CM of the sphere.

(a) Write the Lagrangian, including the constraints via Lagrange multipliers.

(b) Find the value of θ_1 where the sphere first loses contact with the hemi-sphere, for general moment of inertia I.

(c) Which would lose contact first: a uniform solid ball, or a thin spherical shell?

`SOLUTION:`

$$L = \tfrac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}_1^2) + \tfrac{1}{2}I\theta_2^2 - Mgr\cos\theta_1 + \lambda_1(r - R - a) + \lambda_2(R\theta_1 + a\theta_1 - a\theta_2)$$

where λ_i are Lagrange-multipliers to enforce the constraints. The sphere loses contact where the normal force$N_r = Mg\cos\theta_1 - M(R+a)\dot{\theta}_1^2$ vanishes. The conserved energy is

$$\tfrac{1}{2}M(R+a)^2(1+\alpha)\dot{\theta}_1^2 + Mg(R+a)\cos\theta_1$$

where $\alpha \equiv I/Ma^2$, which gives

$$\dot{\theta}_1^2 = \frac{2g}{R+a}\frac{1-\cos\theta_1}{1+\alpha}.$$

The normal force is thus $N_r = \frac{Mg}{1+\alpha}((3+\alpha)\cos\theta_1 - 2)$. So the sphere loses contact at

$$\theta_1 = \cos^{-1}\left(\frac{2}{3+\alpha}\right).$$

The uniform solid ball loses contact first, since it has a smaller α, which corresponds to a smaller θ_1.

#12: GRADUATE MECHANICS

`PROBLEM:`

The Hamiltonian for a charged particle moving in a uniform magnetic field $B\hat{z}$ is

$$H = \frac{1}{2m}\left(p_x + \frac{eB}{c}y\right)^2 + \frac{1}{2m}p_y^2.$$

(a) Find Hamilton's equations of motion for this Hamiltonian.

(b) Find at least three constants of the motion.

(c) Solve Hamilton's equations of motion of part (a) for $x(t)$, $p_x(t)$, $y(t)$ and $p_y(t)$ in terms of initial value data $x(0)$, $p_x(0)$, $y(0)$ and $p_y(0)$.

(d) Find the Lagrangian corresponding to this Hamiltonian. Find the Euler-Lagrange equations of this Lagrangian. Check that your solutions for $x(t)$ and $y(t)$ from part (a) satisfy the Euler-Lagrange equations.

`SOLUTION:`

(a)

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eB}{c}y\right)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{eB}{mc}\left(p_x + \frac{eB}{c}y\right)$$

(b)

$$p_x(t), \qquad p_y(t) + \frac{eB}{c}x(t), \qquad H$$

(c)

$$\dot{p}_x = 0 \qquad \Rightarrow \qquad p_x(t) = p_x(0)$$

$$\left(\dot{p}_y + \frac{eB}{c}\dot{x}\right) = 0 \quad \Rightarrow \quad p_y(t) + \frac{eB}{c}x(t) = p_y(0) + \frac{eB}{c}x(0)$$

$$\ddot{p}_y = -\omega^2 p_y, \quad \omega = \frac{eB}{mc} \qquad \Rightarrow$$

$$p_y(t) = p_y(0)\cos\omega t + B\sin\omega t$$

$$\dot{y} = \frac{1}{m}p_y(t) = \frac{p_y(0)}{m}\cos\omega t + \frac{B}{m}\sin\omega t \qquad \Rightarrow$$

$$y(t) = C + \frac{p_y(0)}{m\omega}\sin\omega t - \frac{B}{m\omega}\cos\omega t \qquad \Rightarrow \qquad y(0) = C - \frac{B}{m\omega}$$

$$\dot{p}_y = -\omega\left(p_x + m\omega y\right) = -\omega\left(p_x(0) + m\omega C + p_y(0)\sin\omega t - B\cos\omega t\right) = -\omega p_y(0)\sin\omega t + \omega B\cos\omega t$$

$$\Rightarrow -\omega\left(p_x(0) + m\omega C\right) = 0 \quad \Rightarrow \quad C = -\frac{p_x(0)}{m\omega}$$

$$\Rightarrow \qquad B = m\omega\left(C - y(0)\right) = -p_x(0) - m\omega y(0)$$

$$x(t) = x(0) + \frac{c}{eB}\left(p_y(0) - p_y(t)\right)$$

Finally, solution in terms of initial value data is

$$p_x(t) = p_x(0)$$

$$p_y(t) = p_y(0)\cos\omega t - [p_x(0) + m\omega y(0)]\sin\omega t$$

$$y(t) = -\frac{p_x(0)}{m\omega} + \frac{p_y(0)}{m\omega}\sin\omega t + \left(\frac{p_x(0)}{m\omega} + y(0)\right)\cos\omega t$$

$$x(t) = x(0) + \frac{p_y(0)}{m\omega} - \frac{p_y(0)}{m\omega}\cos\omega t + \left[\frac{p_x(0)}{m\omega} + y(0)\right]\sin\omega t$$

(d)

$$\mathcal{L}(x,\dot{x},y,\dot{y}) = p_x\dot{x} + p_y\dot{y} - H(x,p_x,y,p_y)$$

where

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left(p_x + \frac{eB}{c}y\right)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$\mathcal{L}(x,\dot{x},y,\dot{y}) = \left(m\dot{x} - \frac{eB}{c}y\right)\dot{x} + m\dot{y}^2 - \frac{1}{2m}(m\dot{x})^2 - \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \frac{eB}{c}y\dot{x}$$

$$\frac{d}{dt}m\left(\dot{x}-\frac{eB}{mc}y\right)=0 \quad \Rightarrow \quad \frac{d}{dt}\left(\dot{x}-\omega y\right)=0$$

$$(\dot{x}-\omega y)=\text{constant}=C_y$$

$$\frac{d}{dt}\left(m\dot{y}\right)=-\left(\frac{eB}{c}\right)\dot{x} \quad \Rightarrow \quad \frac{d}{dt}\left(\dot{y}+\omega x\right)=0$$

$$(\dot{y}+\omega x)=\text{constant}=C_x$$

#13: GRADUATE E&M

PROBLEM:

The electric potential that results when a metallic sphere of radius R is placed in an external uniform electric field E_0 pointing in the z direction is of the form

$$\phi(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-(l+1)})P_l(cos\theta) \tag{6}$$

where P_l are Legendre polynomials.

(a) Give the expression $\phi_0(r,\theta)$, for the electric potential due to the external electric field only in spherical coordinates.

(b) By considering the potential $\phi(r,\theta)$ for large values of r, deduce which coefficients A_l, B_l in the expression for ϕ must vanish.

(c) By considering $\phi(r,\theta)$ on the surface of the sphere, deduce the values of all the coefficients A_l, B_l in the formula for ϕ, and hence an expression for $\phi(r,\theta)$.

(d) Find the magnitude of the electric field at $\theta = 0$ right outside the surface of the sphere.

Show all steps in your derivations and justify all steps.

SOLUTION:

Solution grad E & M

(a) $\Phi_0(z) = -E_0 z$ so that $\vec{E}_0 = E_0 \hat{z} = -\vec{\nabla}\Phi_0$

so in spherical coordinates: $\boxed{\Phi_0(r,\theta) = -E_0 r\cos\theta}$

(b) For large r, $\Phi \to \Phi_0 \Rightarrow$

$$\phi(r,\theta) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \to -E_0 r\cos\theta$$

since $P_1(\theta) = \cos\theta$, that term ($l=1$) and $l=0$ are only non-zero terms

$\Rightarrow$ $\boxed{A_l = 0 \text{ for } l \geq 2}$ and $\boxed{A_1 = -E_0}$

(c) On the surface of a metallic sphere, the potential is constant. Let's say its value is V_0. We have then:

$$\phi(R,\theta) = A_0 - E_0 R\cos\theta + \sum_{l=0}^{\infty} B_l R^{-(l+1)} P_l(\cos\theta) = V_0$$

By orthogonality of Legendre polynomials we conclude that:

(i) $A_0 = V_0$, (ii) $\frac{B_1}{R^2} = E_0 R \Rightarrow \boxed{B_1 = E_0 R^3}$

Hence, $\boxed{\phi(r,\theta) = V_0 + \left(-r + \frac{R^3}{r^2}\right) E_0 \cos\theta}$

(d) For $\theta = 0$, $\phi(r,0) = V_0 + \left(-r + \frac{R^3}{r^2}\right) E_0$

electric field is in the r direction, by symmetry

$$E(R,0) = -\left.\frac{\partial\phi(r,0)}{\partial r}\right|_R = \left.\left(1 + \frac{2R^3}{r^3}\right) E_0\right|_{r=R} = \boxed{3E_0}$$

#14: GRADUATE E&M

`PROBLEM:`

The radiation fields $\vec{E}, \vec{B}$ of an oscillating electric dipole $\vec{P}(t)$ are expressed in the SI unit as

$$\begin{aligned} \vec{B}(\vec{r}, t) &= -\tfrac{\mu_0}{4\pi rc} \hat{e}_r \times \tfrac{\partial^2}{\partial_t^2} \vec{P}(t - \tfrac{r}{c}) \\ \vec{E}(\vec{r}, t) &= -c\hat{e}_r \times \vec{B}(\vec{r}, t), \end{aligned} \tag{7}$$

where the dipole is put at the origin, μ_0 is the vacuum permeability; c is the light velocity; $\hat{e}_r$ is the unit vector along the radial direction. Using these formulae, solve the following problems.

a) Put a point charge q at the origin. It is driven by a planar polarized electromagnetic plane wave $\vec{E} = \hat{x} E_0 e^{i(kz - \omega t)}$ where $k = \frac{\omega}{c}$. What is the radiation field $\vec{E}$ and $\vec{B}$ of this point charge.

b) Use the spherical coordinate, derive the angular distribution of the radiation power intensity (Poynting vector) $\vec{S}(\theta, \phi)$.

`SOLUTION:`

(a)

$$\tfrac{d^2}{dt^2} x = qE_0 e^{-i\omega t}, \tag{8}$$

thus $x = -\frac{qE_0}{m\omega^2} e^{-i\omega t}$ up to a phase. The electric dipole moment read

$$\vec{P}(t) = qx\hat{x} = -\tfrac{q^2 E_0}{m\omega^2} e^{-i\omega t}, \tag{9}$$

and

$$\tfrac{\partial^2}{\partial_t^2} \vec{P}(t - \tfrac{r}{c}) = \tfrac{q^2 E_0}{m} e^{i(kr - \omega t)}. \tag{10}$$

Thus

$$\begin{aligned} \vec{B}(\vec{r}, t) &= -\tfrac{\mu_0 q^2 E_0}{4\pi mrc} e^{i(kr - \omega t)} \hat{e}_r \times \hat{x} \\ \vec{E}(\vec{r}, t) &= \tfrac{\mu_0 q^2 E_0}{4\pi mr} e^{i(kr - \omega t)} \hat{e}_r \times (\hat{e}_r \times \hat{x}) \\ &= \tfrac{\mu_0 q^2 E_0}{4\pi mr} e^{i(kr - \omega t)} [(\hat{x} \cdot \hat{e}_r)\hat{e}_r - \hat{x}]. \end{aligned} \tag{11}$$

From

$$\begin{aligned}
\hat{e}_r &= \cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} \\
\hat{e}_\theta &= -\sin\theta\hat{z} + \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} \\
\hat{e}_\phi &= -\sin\phi\hat{x} + \cos\phi\hat{y}, \qquad (12)
\end{aligned}$$

we have $\hat{x} = \sin\theta\cos\phi\hat{e}_r + \cos\theta\cos\phi\hat{e}_\theta - \sin\phi\hat{e}_\phi$, thus $\hat{e}_r \times \hat{x} = \cos\theta\cos\phi\hat{e}_\phi - \sin\phi\hat{e}_\theta$.

The average radiation intensity

$$\begin{aligned}
\bar{S} &= \frac{1}{2}Re(\vec{E}^* \times \vec{H}) \\
&= \frac{1}{2\mu_0}Re(-c(\hat{e}_r \times \vec{B}^*) \times \vec{B}). \qquad (13)
\end{aligned}$$

Because $\hat{e}_r \cdot \vec{B} = 0$, we have

$$\bar{S} = \frac{c}{2\mu_0}|B|^2\hat{e}_r = \frac{\mu_0 q^4 E_0^2}{32\pi^2 c m^2 r^2}(\cos^2\theta\cos^2\phi + \sin^2\phi)\hat{e}_r \qquad (14)$$

#15: GRADUATE QUANTUM MECHANICS

`PROBLEM:`

A particle with charge q and mass M is confined in the 2D xy-plane and moves in the presence of an external magnetic field $\vec{B} = B\hat{z}$. Its Hamiltonian is written as

$$H = \frac{(-i\hbar\vec{\nabla} - q\vec{A})^2}{2M}, \tag{15}$$

Use the symmetric gauge $\vec{A} = \frac{B}{2}\hat{z} \times \vec{r}$.

(a) Solve for the semiclassical motion. According to the electron's classic equation of motion and Bohr-Sommerfeld quantization condition, what is the radius l_B of the smallest cyclotron orbit? What is the angular frequency ω associated with this smallest cyclotron orbit?

(b) The "mechanical momentum" is defined as $\vec{P}_m = -i\hbar\vec{\nabla} - q\vec{A}$. Define operators a and $a^\dagger$ as $a = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} + iP_{m,y})$ and $a^\dagger = \frac{l_B}{\sqrt{2}\hbar}(P_{m,x} - iP_{m,y})$. Work out their commutation relations $[a, a^+]$.

(c) Show that the Hamiltonian can be written in terms of a and a^+ and find all its eigenvalues.

(d) Define the "guiding center coordinates" $\vec{R}_g = \vec{r} - \hat{z} \times \frac{\vec{P}_m}{M\omega}$ and the operators $b = \frac{1}{\sqrt{2}l_B}(R_{g,x} - iR_{g,y})$ and $b^\dagger = \frac{1}{\sqrt{2}l_B}(R_{g,x} + iR_{g,y})$. Work out the commutation relation $[b, b^\dagger]$. Prove that $[a, b] = [a^\dagger, b^\dagger] = 0$, and $[a, b^\dagger] = [a^\dagger, b] = 0$.

(e) Express the canonical angular momentum $L_z = (\vec{r} \times -i\hbar\vec{\nabla}) \cdot \hat{z}$ in terms of a, $a^\dagger$, b and $b^\dagger$ and find all its eigenvalues.

(f) For each energy level of Eq. 15, figure out its allowed eigenvalues of L_z.

`SOLUTION:`

(a) From

$$M\frac{v^2}{l_B} = qvB, \quad Mvl_B = \hbar, \tag{16}$$

we arrive at $l_B = \sqrt{\frac{\hbar}{qB}}$, and $\omega = \frac{qB}{M}$.

(b) We have

$$
\begin{aligned}
P_{m,x} &= -i\hbar\partial_x + \tfrac{qB}{2}y, = -i\hbar\partial_x + \tfrac{\hbar y}{2l_B^2}, \\
P_{m,y} &= -i\hbar\partial_y - \tfrac{qB}{2}x = -i\hbar\partial_y - \tfrac{\hbar x}{2l_B^2}.
\end{aligned} \tag{17}
$$

Then

$$
\begin{aligned}
a &= \tfrac{l_B}{\sqrt{2}\hbar}(P_{m,x} + iP_{m,y}) = \tfrac{-i}{\sqrt{2}}\{l_B(\partial_x + i\partial_y) + (x+iy)/2l_B\} \\
a^\dagger &= \tfrac{l_B}{\sqrt{2}\hbar}(P_{m,x} + iP_{m,y}) = \tfrac{i}{\sqrt{2}}\{l_B(-\partial_x + i\partial_y) + (x-iy)/2l_B\}, \\
[a, a^\dagger] &= \tfrac{1}{4}([\partial_x, x] + [\partial_y, y] - [x, \partial_x] - [y, \partial_y]) = 1
\end{aligned} \tag{18}
$$

(c) The Hamiltonian is

$$
\begin{aligned}
H &= \tfrac{1}{2M}\{(P_{m,x} + iP_{m,y})(P_{m,x} - iP_{m,y}) + (P_{m,x} - iP_{m,y})(P_{m,x} + iP_{m,y})\} \\
&= \tfrac{\hbar\omega}{2}(aa^\dagger + a^\dagger a) = \hbar\omega(a^\dagger a + \tfrac{1}{2})
\end{aligned} \tag{19}
$$

The energy spectrum is

$$
E_{n_a} = (n + \tfrac{1}{2})\hbar\omega. \tag{20}
$$

with $n_a = 0, 1, 2, ...$

(d) We have

$$
\begin{aligned}
R_{g,x} &= x + \tfrac{1}{M\omega}(-i\hbar\partial_y - \tfrac{\hbar x}{2l_B^2}) = \tfrac{x}{2} - il_B^2\partial_y, \\
R_{g,y} &= y - \tfrac{1}{M\omega}(-i\hbar\partial_x + \tfrac{\hbar y}{2l_B^2}) = \tfrac{y}{2} + il_B^2\partial_x,
\end{aligned} \tag{21}
$$

then

$$
\begin{aligned}
b &= \tfrac{1}{\sqrt{2}l_B}(R_{g,x} - iR_{g,y}) = \tfrac{1}{\sqrt{2}}\left\{l_B(\partial_x - i\partial_y) + \tfrac{x-iy}{2l_B}\right\} \\
b^\dagger &= \tfrac{1}{\sqrt{2}l_B}(R_{g,x} + iR_{g,y}) = \tfrac{1}{\sqrt{2}}\left\{l_B(-\partial_x - i\partial_y) + \tfrac{x+iy}{2l_B}\right\} \\
[b, b^\dagger] &= \tfrac{1}{4}\{[\partial_x, x] + [\partial_y, y] + [x, -\partial_x] + [y, -\partial_y]\} = 1 \\
[a, b] &= \tfrac{-i}{2}\{[\partial_x, x] + [\partial_y, y] + [x, \partial_x] + [y, \partial_y]\} = 0 \\
[a^\dagger, b^\dagger] &= -\{[a, b]\}^\dagger = 0 \\
[a, b^\dagger] &= \tfrac{-i}{2}\{[\partial_x, x] - [\partial_y, y] + [x, -\partial_x] + [y, \partial_y]\} = 0 \\
[a^\dagger, b] &= -\{[a, b^\dagger]\}^\dagger = 0
\end{aligned} \tag{22}
$$

e)

$$L_z \;=\; -i\hbar\big(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\big) \tag{23}$$

We have

$$\begin{aligned} a + a^\dagger &= -i\sqrt{2}l_B\partial_x + \tfrac{1}{\sqrt{2}l_B}y \\ a - a^\dagger &= \sqrt{2}l_B\partial_y - \tfrac{i}{\sqrt{2}l_B}x \end{aligned} \tag{24}$$

and

$$\begin{aligned} b + b^\dagger &= -i\sqrt{2}l_B\partial_y + \tfrac{1}{\sqrt{2}l_B}x \\ b - b^\dagger &= \sqrt{2}l_B\partial_x - \tfrac{i}{\sqrt{2}l_B}y, \end{aligned} \tag{25}$$

thus

$$\begin{aligned} x &= \tfrac{l_B}{\sqrt{2}}(b + b^\dagger + i(a - a^\dagger)), \quad -i\partial_y = \tfrac{1}{2\sqrt{2}l_B}(b + b^\dagger - i(a - a^\dagger)) \\ y &= \tfrac{l_B}{\sqrt{2}}(a + a^\dagger + i(b - b^\dagger)), \quad -i\partial_x = \tfrac{1}{2\sqrt{2}l_B}(a + a^\dagger - i(b - b^\dagger)). \end{aligned} \tag{26}$$

Then we have

$$\begin{aligned} -ix\partial_y &= \tfrac{1}{4}[(b + b^\dagger)^2 + (a - a^\dagger)^2] \\ -iy\partial_x &= \tfrac{1}{4}[(a + a^\dagger)^2 + (b - b^\dagger)^2], \end{aligned} \tag{27}$$

thus

$$L_z = \hbar(-ix\partial_y + iy\partial_x) = \tfrac{1}{2}(bb^\dagger + b^\dagger b - aa^\dagger - a^\dagger a) = b^\dagger b - a^\dagger a. \tag{28}$$

f) Since $[H, L_z] = [a^\dagger a + \frac{1}{2}, b^\dagger b - a^\dagger a] = 0$, we can choose the quantum number of L_z to lable each state in each LL. For each LL with $E_{n_a} = (n_a + \frac{1}{2})\hbar\omega$, L_z takes the values $(n_b - n_a)\hbar = -n_a\hbar, (-n_a + 1)\hbar, ..., 0, \hbar, 2\hbar, ...$

#16: GRADUATE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom is placed in an external perturbing potential

$$V = f(r)\left(x^2 + y^2\right)$$

You are given the matrix elements

$$\begin{aligned}
\langle 2s, m=0|f(r)x^2|2s, m=0\rangle &= v_x \\
\langle 2s, m=0|f(r)y^2|2s, m=0\rangle &= v_y \\
\langle 2s, m=0|f(r)z^2|2s, m=0\rangle &= v_z \\
\langle 2p, m=0|f(r)x^2|2p, m=0\rangle &= w_x \\
\langle 2p, m=0|f(r)y^2|2p, m=0\rangle &= w_y \\
\langle 2p, m=0|f(r)z^2|2p, m=0\rangle &= w_z
\end{aligned}$$

(a) There are two relations between v_x, v_y and v_z. Find these to write v_x and v_y in terms of v_z.

(b) There is one relation between w_x, w_y and w_z. Find this to eliminate w_y.

(c) Find the first order shift in the energy levels of the $n = 2$ states in terms of v_z and $w_{x,z}$.

`SOLUTION:`

- The s-wave states are spherically symmetric, and so $v_x = v_y = v_z$.
- The $2p, m = 0$ state is symmetric under rotations about the z axis, so $w_x = w_y$.
- The $2s$ and $2p$ $m = \pm 1, 0$ states are degenerate, so we have to use first order degenerate perturbation theory, and consider matrix elements of V between these four states.

$$\begin{aligned}
V &= V_0(r) + V_2(r) \\
V_0 &= \tfrac{2}{3} f(r)\left(x^2 + y^2 + z^2\right) \\
V_2 &= \tfrac{1}{3} f(r)\left(x^2 + y^2 - 2z^2\right)
\end{aligned}$$

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J M	J M	...
m_1 m_2	Coefficients		
m_1 m_2			
⋮ ⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

1/2×1/2

m_1 m_2	1 +1		
+1/2 +1/2	1		
m_1 m_2	1 0	0 0	
+1/2 −1/2	1/2	1/2	
−1/2 +1/2	1/2	−1/2	
m_1 m_2	1 −1		
−1/2 −1/2	1		

1×1/2

m_1 m_2	3/2 +3/2	
+1 +1/2	1	
m_1 m_2	3/2 +1/2	1/2 +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3
m_1 m_2	3/2 −1/2	1/2 −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3
m_1 m_2	3/2 −3/2	
−1 −1/2	1	

2×1/2

m_1 m_2	5/2 +5/2	
+2 +1/2	1	
m_1 m_2	5/2 +3/2	3/2 +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5
m_1 m_2	5/2 +1/2	3/2 +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5
m_1 m_2	5/2 −1/2	3/2 −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5
m_1 m_2	5/2 −3/2	3/2 −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5
m_1 m_2	5/2 −5/2	
−2 −1/2	1	

3/2×1/2

m_1 m_2	2 +2	
+3/2 +1/2	1	
m_1 m_2	2 +1	1 +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4
m_1 m_2	2 0	1 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2
m_1 m_2	2 −1	1 −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4
m_1 m_2	2 −2	
−3/2 −1/2	1	

2×1

m_1 m_2	3 +3		
+2 +1	1		
m_1 m_2	3 +2	2 +2	
+2 0	1/3	2/3	
+1 +1	2/3	−1/3	
m_1 m_2	3 +1	2 +1	1 +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10
m_1 m_2	3 0	2 0	1 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10
m_1 m_2	3 −1	2 −1	1 −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5
m_1 m_2	3 −2	2 −2	
−1 −1	2/3	1/3	
−2 0	1/3	−2/3	
m_1 m_2	3 −3		
−2 −1	1		

3/2×1

m_1 m_2	5/2 +5/2		
+3/2 +1	1		
m_1 m_2	5/2 +3/2	3/2 +3/2	
+3/2 0	2/5	3/5	
+1/2 +1	3/5	−2/5	
m_1 m_2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6
m_1 m_2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2
m_1 m_2	5/2 −3/2	3/2 −3/2	
−1/2 −1	3/5	2/5	
−3/2 0	2/5	−3/5	
m_1 m_2	5/2 −5/2		
−3/2 −1	1		

1×1

m_1 m_2	2 +2		
+1 +1	1		
m_1 m_2	2 +1	1 +1	
+1 0	1/2	1/2	
0 +1	1/2	−1/2	
m_1 m_2	2 0	1 0	0 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3
m_1 m_2	2 −1	1 −1	
0 −1	1/2	1/2	
−1 0	1/2	−1/2	
m_1 m_2	2 −2		
−1 −1	1		

3/2×3/2

m_1 m_2	3 +3			
+3/2 +3/2	1			
m_1 m_2	3 +2	2 +2		
+3/2 +1/2	1/2	1/2		
+1/2 +3/2	1/2	−1/2		
m_1 m_2	3 +1	2 +1	1 +1	
+3/2 −1/2	1/5	1/2	3/10	
+1/2 +1/2	3/5	0	−2/5	
−1/2 +3/2	1/5	−1/2	3/10	
m_1 m_2	3 0	2 0	1 0	0 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4
m_1 m_2	3 −1	2 −1	1 −1	
+1/2 −3/2	1/5	1/2	3/10	
−1/2 −1/2	3/5	0	−2/5	
−3/2 +1/2	1/5	−1/2	3/10	
m_1 m_2	3 −2	2 −2		
−1/2 −3/2	1/2	1/2		
−3/2 −1/2	1/2	−1/2		
m_1 m_2	3 −3			
−3/2 −3/2	1			

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

m_1 m_2	7/2 +7/2			
+2 +3/2	1			
m_1 m_2	7/2 +5/2	5/2 +5/2		
+2 +1/2	3/7	4/7		
+1 +3/2	4/7	−3/7		
m_1 m_2	7/2 +3/2	5/2 +3/2	3/2 +3/2	
+2 −1/2	1/7	16/35	2/5	
+1 +1/2	4/7	1/35	−2/5	
0 +3/2	2/7	−18/35	1/5	
m_1 m_2	7/2 +1/2	5/2 +1/2	3/2 +1/2	1/2 +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10
m_1 m_2	7/2 −1/2	5/2 −1/2	3/2 −1/2	1/2 −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5
m_1 m_2	7/2 −3/2	5/2 −3/2	3/2 −3/2	
0 −3/2	2/7	18/35	1/5	
−1 −1/2	4/7	−1/35	−2/5	
−2 +1/2	1/7	−16/35	2/5	
m_1 m_2	7/2 −5/2	5/2 −5/2		
−1 −3/2	4/7	3/7		
−2 −1/2	3/7	−4/7		
m_1 m_2	7/2 −7/2			
−2 −3/2	1			

2×2

m_1 m_2	4 +4				
+2 +2	1				
m_1 m_2	4 +3	3 +3			
+2 +1	1/2	1/2			
+1 +2	1/2	−1/2			
m_1 m_2	4 +2	3 +2	2 +2		
+2 0	3/14	1/2	2/7		
+1 +1	4/7	0	−3/7		
0 +2	3/14	−1/2	2/7		
m_1 m_2	4 +1	3 +1	2 +1	1 +1	
+2 −1	1/14	3/10	3/7	1/5	
+1 0	3/7	1/5	−1/14	−3/10	
0 +1	3/7	−1/5	−1/14	3/10	
−1 +2	1/14	−3/10	3/7	−1/5	
m_1 m_2	4 0	3 0	2 0	1 0	0 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5
m_1 m_2	4 −1	3 −1	2 −1	1 −1	
+1 −2	1/14	3/10	3/7	1/5	
0 −1	3/7	1/5	−1/14	−3/10	
−1 0	3/7	−1/5	−1/14	3/10	
−2 +1	1/14	−3/10	3/7	−1/5	
m_1 m_2	4 −2	3 −2	2 −2		
0 −2	3/14	1/2	2/7		
−1 −1	4/7	0	−3/7		
−2 0	3/14	−1/2	2/7		
m_1 m_2	4 −3	3 −3			
−1 −2	1/2	1/2			
−2 −1	1/2	−1/2			
m_1 m_2	4 −4				
−2 −2	1				

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\,\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

V_0 is a $J = 0$ operator, and V_2 is a $J = 2$, $J_z = 0$ operator, and so V is a $J_z = 0$ operator. By the Wigner-Eckart theorem,

$$\begin{aligned}
\langle 2s|V_0|2s\rangle &= r_0 \\
\langle 2s|V_2|2s\rangle &= 0 \\
\langle 2s|V_0|2p,m\rangle &= 0 \\
\langle 2s|V_2|2p,m\rangle &= 0 \\
\langle 2p,m'|V_0|2p,m\rangle &= u_0\delta_{m'm} \\
\langle 2p,m'|V_2|2p,m\rangle &= u_2\,\langle 1m,20|1m'\rangle
\end{aligned}$$

where r_0, u_0 and u_2 are the reduced matrix element.

Then we have the matrix elements of V in the $n = 2$ sector:

$$\begin{bmatrix}
r_0 & & & \\
 & u_0 - \sqrt{\frac{2}{5}}u_2 & & \\
 & & u_0 + \frac{1}{\sqrt{10}}u_2 & \\
 & & & u_0 + \frac{1}{\sqrt{10}}u_2
\end{bmatrix}$$

where the states are in the order $2s$, $2p, m = 0$, $2p, m = 1$, $2p, m = -1$. The perturbation is already diagonal, so the energies are the diagonal elements.

From the matrix elements given in the problem, we see that

$$\begin{aligned}
\langle 2s|V_0|2s\rangle &= r_0 = \tfrac{2}{3}\left(v_x + v_y + v_z\right) = 2v_z \\
\langle 2p, m = 0|V_0|2p, m = 0\rangle &= u_0 = \tfrac{2}{3}\left(w_x + w_y + w_z\right) = \tfrac{2}{3}\left(2w_x + w_z\right) \\
\langle 2p, m = 0|V_2|2p, m = 0\rangle &= -\sqrt{\tfrac{2}{5}}u_2 = \tfrac{1}{3}\left(w_x + w_y - 2w_z\right) = \tfrac{2}{3}\left(w_x - w_z\right)
\end{aligned}$$

so the energies are

$$\begin{aligned}
&2v_z \\
&w_x \\
&w_x + w_z \text{ (twice)}
\end{aligned}$$

#17: GRADUATE STAT MECH

`PROBLEM:`

In the Ising model of ferromagnetism, the spin at each lattice site σ_i can take the values ± 1. The energy for each configuration of spins is

$$E = -J \sum_{<ij>} \sigma_i \sigma_j \tag{29}$$

where $J > 0$ and the sum $< ij >$ is over nearest neighbor sites. Assume each site has z nearest neighbors.

Call $m =< \sigma_j >$ the thermal average of the spin σ_j at temperature T. In the mean field approximation, the interaction of the spin σ_i with a neighbor σ_j is approximated by replacing σ_j by $< \sigma_j >= m$.

(a) Set up a self-consistency condition for the value of m using the fact that all lattice sites are equivalent, that will determine the value of m as function of J, z, and the temperature T. Hint: Use the canonical ensemble.

(b) Show that the self-consistency condition has a solution with $m \neq 0$ only for T lower than a critical value T_c, and find an expression for T_c in terms of z and J.

(c) By expanding the self-consistency condition for small m, show that m for T close to T_c can be written as

$m = C(T_c - T)^\beta$

with β and C constants, and find the values of β and C.

Show all steps in your derivations and justify all steps.

`SOLUTION:`

Solution Grad Stat Mech

The interaction of spin σ_i with its neighbors is described by

$$E_i(\sigma_i) = -J\sigma_i \sum_{j=\text{neighbr of } i} \sigma_j \xrightarrow[\text{field}]{\text{mean}} -J\sigma_i \underbrace{\sum \langle\sigma_j\rangle}_{z \text{ terms in sum}}$$

$$\langle\sigma_j\rangle \equiv m$$

$$\Rightarrow \quad E_i(\sigma_i) = -Jzm\,\sigma_i$$

The thermal average of σ_i is

$$\langle\sigma_i\rangle = \sum_{\sigma_i=\pm1} e^{-\beta E_i(\sigma_i)}\sigma_i \Big/ \sum_{\sigma_i=\pm1} e^{-\beta E_i(\sigma_i)}$$

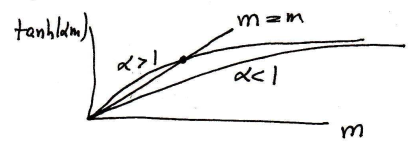
Hence $\langle\sigma_i\rangle = \left(e^{\beta Jzm} - e^{-\beta Jzm}\right) / \left(e^{\beta Jzm} + e^{-\beta Jzm}\right)$

$$\Rightarrow \quad \langle\sigma_i\rangle = \tanh\left(\frac{Jzm}{k_BT}\right) \quad ; \text{ self-consistency}$$

$$\Rightarrow \langle\sigma_i\rangle = m \quad \Rightarrow \quad \boxed{m = \tanh\left(\frac{Jz}{k_BT}\right)m}$$

(b) Let $\alpha = \dfrac{Jz}{k_BT}$; $m = \tanh(\alpha m)$ has no solution with

$m \neq 0$ if $\alpha \leq 1$, see graph :



So : $\dfrac{Jz}{k_BT} > 1 \Rightarrow T < \dfrac{Jz}{k_B} \Rightarrow$ the critical

temperature is $\boxed{T_c = \dfrac{Jz}{k_B}}$

(c) So we have:

$$m = \tanh\left(\frac{T_c}{T} m\right)$$

The expansion of $\tanh(x)$ for small x is

$$\tanh(x) = x - \frac{x^3}{3} \Rightarrow$$

$$m = \frac{T_c}{T} m - \left(\frac{T_c}{T}\right)^3 \frac{m^3}{3} \Rightarrow 1 = \frac{T_c}{T} - \left(\frac{T_c}{T}\right)^3 \frac{m^2}{3} \Rightarrow$$

$$\Rightarrow m^2 = 3\left(\frac{T}{T_c}\right)^3 \left(\frac{T_c}{T} - 1\right) = \frac{3T^2}{T_c^3}(T_c - T)$$

$$\Rightarrow m = \sqrt{3}\frac{T}{T_c^{3/2}}(T_c - T)^{1/2}$$

For T close to T_c,

$$\boxed{m = \frac{\sqrt{3}}{T_c^{1/2}}(T_c - T)^{1/2}}$$

hence $m = C(T_c - T)^\beta$ with $\boxed{\beta = \frac{1}{2}}$, $\boxed{C = \frac{\sqrt{3}}{T_c^{1/2}}}$

#18: GRADUATE STAT MECH

`PROBLEM:` Consider a gas of non-interacting, non-relativistic spin-1 bosons in an external magnetic field $B\hat{z}$. The one-particle Hamiltonian is

$$\mathcal{H}(\vec{p}, s_z) = \frac{\vec{p}^{\,2}}{2m} - \mu_0 s_z B,$$

where $\mu_0 \equiv \frac{e\hbar}{mc}$ and s_z, the spin quantum number in the $\hat{z}$ direction, can take three possible values $-1, 0, 1$.

(a) In the grand canonical ensemble, what are the average occupation numbers $\langle n_+(\vec{k})\rangle$, $\langle n_0(\vec{k})\rangle$ and $\langle n_-(\vec{k})\rangle$ of one-particle states with wavenumber $\vec{k} = \frac{\vec{p}}{\hbar}$ and with $s_z = -1,\ 0,\ +1$?

Use these average occupation numbers to find the average total numbers N_+, N_0, N_- of bosons with $s_z = -1,\ 0,\ +1$ in terms of the functions

$$f_m^+(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty dx\ \frac{x^{m-1}}{z^{-1}e^x - 1},$$

where $z \equiv e^{\beta\mu}$.

(b) The total number density is equal to

$$n = \frac{N_+ + N_0 + N_-}{V}.$$

Find the temperature $T_c(n, B = 0)$ of Bose-Einstein condensation at zero field in terms of the number density n.

`SOLUTION:`

(a)

$$n_s(\vec{k}) = \frac{1}{e^{\beta\left[\mathcal{H}(\vec{k},s)-\mu\right]} - 1} = \frac{1}{\exp\left[\beta\left(\frac{\hbar^2k^2}{2m} - \mu_0 sB - \mu\right)\right] - 1}, \qquad s = -1, 0, +1$$

$$N_s = \frac{V}{(2\pi)^3} \int d^3k \frac{1}{\exp\left[\beta\left(\frac{\hbar^2k^2}{2m} - \mu_0 sB - \mu\right)\right] - 1} = \frac{V}{\lambda^3}\ f_{3/2}^+\left(ze^{\beta\mu_0 sB}\right)$$

where

$$\lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}, \qquad z \equiv e^{\beta\mu}$$

(b) At $B = 0$, $N_+ = N_0 = N_- = \frac{V}{\lambda^3} f^+_{3/2}(z)$, so

$$n = \frac{3}{\lambda^3} f^+_{3/2}(z),$$

which is bounded since $f^+_{3/2}(z) \leq f^+_{3/2}(1) \equiv \zeta_{3/2} \approx 2.61$.

Bose-Einstein condensation occurs when $z = 1$ and $n \geq \frac{3}{\lambda^3}\zeta_{3/2} \Rightarrow$

$$T \leq T_c(n) = \frac{h^2}{2\pi m k_B}\left(\frac{n}{3\zeta_{3/2}}\right)^{2/3}$$

#19: GRADUATE MATH

`PROBLEM:` Find the first term in the asymptotic expansion of the following integral (i.e. the behavior of the integral in the limit $x \to \infty$).

$$I = \frac{1}{\pi}\int_0^\pi (t^4 + 2t^6)^{1/2} e^{x\cos t} \cos(nt) dt,$$

where n is a constant. You may want definition of the Gamma function: $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$, with $\Gamma(1/2) = \sqrt{\pi}$.

`SOLUTION:` The basic idea in asymptotic expansions involving exponentials is that only the region near where the maximum of $x \cos t$ lies contributes in the limit $x \to \infty$. This occurs at $t = 0$. So the integral can be written

$$I \sim \frac{1}{\pi}\int_0^\epsilon (t^4 + 2t^6)^{1/2} e^{x\cos t} \cos(nt) dt,$$

where ϵ is small, and the final answer must not depend on it. Next, Taylor expand everything around $t = 0$, keeping only enough terms so that the answer doesn't depend on ϵ.

$$I \sim \frac{1}{\pi}\int_0^\epsilon t^2 e^{x(1-t^2/2)} dt = \frac{e^x}{\pi}\int_0^\epsilon t^2 e^{-xt^2/2} dt.$$

Note, we needed two terms in the exponential, since if we approximated $e^{x\cos t} \approx e^x$, the integral would have depended explicitly on ϵ. With the 2nd term included, the integral has a term that, in the limit, does not depend upon ϵ. For the asymptotic limit, we can perform the integral using any upper limit, so replace ϵ with ∞.

$$I \sim \frac{e^x}{\pi}\int_0^\infty t^2 e^{-xt^2/2} dt.$$

Changing variables to $t = \sqrt{2/x}u^{1/2}$, or $dt = u^{-1/2}/\sqrt{2x}du$, the integral becomes

$$I \sim \frac{e^x\sqrt{2}}{\pi x^{3/2}}\int_0^\infty u^{1/2} e^{-u} dt = \frac{e^x\sqrt{2}}{\pi x^{3/2}}\Gamma(3/2).$$

Finally, using $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$, we have

$$I \sim \frac{e^x}{\sqrt{2\pi x^3}}.$$

#20: GRADUATE GENERAL

`PROBLEM:`

a) We examine the distribution of bubbles that form along the path of a charged particle in a bubble chamber and we find that bubbles are apparently randomly distributed with a uniform probability of occurance per unit length. This is equivalent to the following statements:

(i) There is at most one bubble in an infinitesimal interval of length $[l, l+\Delta l]$.

(ii) The probability $P_1(\Delta l)$ of finding one bubble in this interval is proportional to Δl (as long as $P_1(\Delta l)$ remains small).

(iii) The occurrence of a bubble in the interval $[l, l+\Delta l]$ is independent of the occurrence of bubbles in any other non-overlapping interval.

Derive an equation that gives the probability $P_o(l)$ of zero bubbles in the finite interval of length l, assuming that the average density of bubbles per unit length is g and that bubbles have negligible size.

b) Derive the probability density (per unit length) $f(l)$ that the first bubble on a track is at a distance l from some arbitrary origin.

c) If we count one bubble in a length of $l = 1$ mm, what is (1) the uncertainty in the observation and (2) the 68.3% confidence interval (one-sigma uncertainty) associated with our best estimate for g?

`SOLUTION:`

a) From (i) and (ii) the probability of one bubble in interval $[l, l+\Delta l]$ is $P_1(\Delta l) = g\Delta l$, while the probability of zero bubbles in the interval is $P_o(\Delta l) = 1 - P_1(\Delta l) = 1 - g\Delta l$. By (iii) we can break $P_o(l, l+\Delta l)$ into two independent factors, $P_o(l, l+\Delta l) = P_o(l)P_o(\Delta l) = P_o(l)(1 - g\Delta l)$. Rearranging, $[P_o(l, l+\Delta l) - P_o(l)]/(\Delta l) = -gP_o(l)$. In the limit $\Delta l \to 0$, this is the derivative $dP_o(l)/dl = -gP_o(l)$. Since $P_o(0) = 1$, $P_o(l) = \exp(-gl)$, which is the Poisson formula for zero events in length l given that the expected number is gl.

b) We start with the joint-probability of zero bubbles in $[0, l)$ and one bubble in the non-overlapping interval $[l, l+\Delta l]$. These two terms are $\exp(-gl)$

and $g\Delta l$. $f(l)$ is the probability per unit length (probability density), hence $f(l) = g \exp(-gl)$.

c) (1) There is no uncertainty in the observation, since we observed one, exactly. (2) Our best estimate for g is one bubble per mm, and the usual one-sigma confidence interval (with equal probability tails) for g is 0.17 to 3.3.

Our (one-sigma) lower limit on the true value of g is the value g_w that gives a probability 15.8% of observing one or more bubbles. From part (a) we want $P_o(l) = \exp(-g_w l) = 1 - 0.158$, or $g_w = 0.17$. Since we saw a bubble, $g_w = 0$ cannot be true. Hence, no credit for $g = 1 \pm 1$.

The upper limit g_u is the value for g that gives a 15.8% probability of observing one or zero bubbles. Here we want to find g by iteration such that $P_o(l) + P_1(l) = (1 + gl)\exp(-gl) = 0.158$, where $P_1(l) = f(l)l = gl \exp(-gl)$ from part (b). By iteration, $g_u = 3.3$.