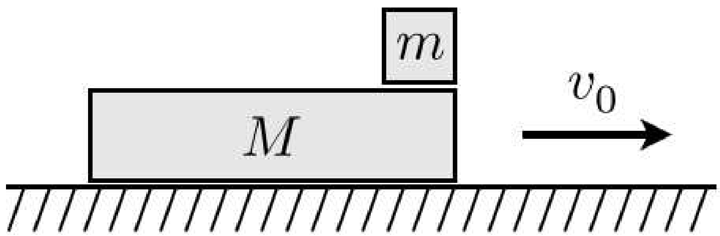


#1 : UNDERGRADUATE MECHANICS

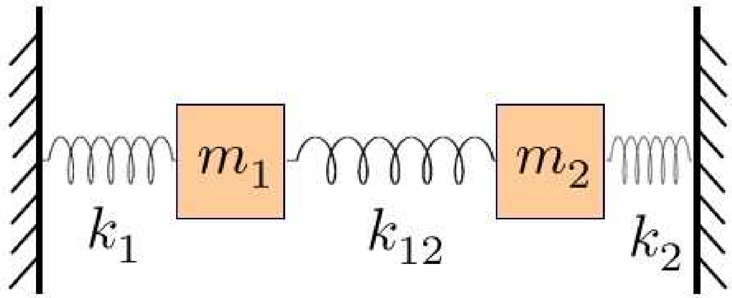
`PROBLEM:` A board of length L and mass M can slide frictionlessly along a horizontal surface. A small block of mass m initially rests on the board at its right end, as shown in the figure. The coefficient of friction between the block and the board is μ. Starting from rest, the board is set in motion to the right with initial speed v_0. What is the smallest value of v_0 such that the block ends up sliding off the left end of the board? Assume the small block is sufficiently narrow relative to L that its width can be neglected.

`SOLUTION:` The initial speed of the block is $v = 0$ and the initial speed of the board is $V = v_0$. The total momentum of the system is conserved, because the surface is frictionless. Thus, the total momentum is $P = MV + mv = Mv_0$ at all times. Now while the total momentum P is conserved, the total energy E is not, due to the friction between the board and the block. The kinetic energy of the system is given by $E = \frac{1}{2}MV^2 + \frac{1}{2}mv^2$ and must be equal to $E_0 - W$, where $E_0 = \frac{1}{2}Mv_0^2$ is the initial kinetic energy of the system and $W = \mu mgd$ is the work done against friction for the block to slide a distance d to the left relative to the board. The minimum value of v_0 must occur when $V = v$ and $d = L$. Thus, we have two equations in the two unknowns (v_0, v):

$$(M+m)v = Mv_0$$
$$\tfrac{1}{2}(M+m)v^2 = \tfrac{1}{2}Mv_0^2 - \mu mgL \ .$$

The solution is

$$v_0 = \sqrt{2\mu gL\left(1 + \frac{m}{M}\right)} \ .$$



#2 : UNDERGRADUATE MECHANICS

`PROBLEM:` Two blocks of masses m_1 and m_2 and three springs with spring constants k_1, k_2, and k_{12} are arranged as shown in the figure. All motion is purely horizontal.
(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.
(b) Find the T and V matrices.
(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_{12} = k \quad , \quad k_2 = 2k \; ,$$

Find the frequencies of small oscillations.
(d) Find the normal modes of oscillation. You do not need to normalize them.

`SOLUTION:`
(a) The Lagrangian is

$$L = \tfrac{1}{2} m_1 \, \dot{x}_1^2 + \tfrac{1}{2} m_2 \, \dot{x}_2^2 - \tfrac{1}{2} k_1 \, x_1^2 - \tfrac{1}{2} k_{12} \, (x_2 - x_1)^2 - \tfrac{1}{2} k_2 \, x_2^2$$

(b) The T and V matrices are

$$\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad , \qquad \mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i \, \partial x_j} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_2 \end{pmatrix}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_{12} = k$, and $k_2 = 2k$. Let us write $\omega^2 \equiv \lambda \, \omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \, \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \; .$$

The determinant is

$$\begin{aligned}\det\left(\omega^2\mathrm{T}-\mathrm{V}\right) &= \left(2\lambda^2-11\lambda+14\right)k^2\\ &= (2\lambda-7)\,(\lambda-2)\,k^2\ .\end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\omega_- = \sqrt{\frac{2k}{m}} \qquad , \qquad \omega_+ = \sqrt{\frac{7k}{2m}}$$

(d) The normal modes are determined from $(\omega_a^2\mathrm{T}-\mathrm{V})\,\vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

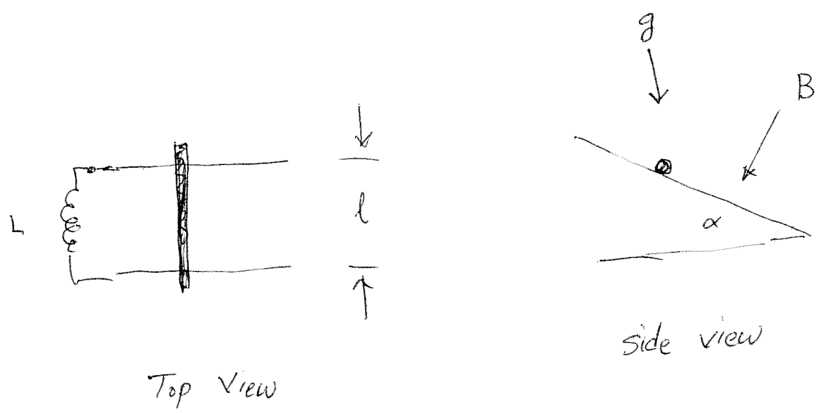
$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\psi_1^{(-)}\\ \psi_2^{(-)}\end{pmatrix} = 0 \qquad \Rightarrow \qquad \vec{\psi}^{(-)} = \mathcal{C}_-\begin{pmatrix}1\\ 1\end{pmatrix}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix}2 & 1\\ 1 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi_1^{(+)}\\ \psi_2^{(+)}\end{pmatrix} = 0 \qquad \Rightarrow \qquad \vec{\psi}^{(+)} = \mathcal{C}_+\begin{pmatrix}1\\ -2\end{pmatrix}$$

The standard normalization $\psi_i^{(a)}\,\mathrm{T}_{ij}\,\psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \qquad , \qquad \mathcal{C}_+ = \frac{1}{\sqrt{6m}}\ .$$



#3 : UNDERGRADUATE E & M

`PROBLEM:` A homogeneous magnetic field B is perpendicular to a track of width l, which is inclined at an angle α to the horizontal. A frictionless conducting rod of mass m can move along the two rails of the track as shown in the figure. The resistance of the rod and the rails is negligible. The rod is released from rest. The circuit formed by the rod and the rails is closed by a coil of inductance L. Find the position of the rod as a function of time.

`SOLUTION:`
The rods' equation of motion is

$$ma = mg\sin\alpha - BlI, \tag{1}$$

where a is acceleration of the rod and I is the current through the rod. The induced voltage in the rod $V = Blv$, where v is velocity of the rod. The relationship between the induced voltage and the current is

$$L\tfrac{dI}{dt} = Blv = Bl\tfrac{dx}{dt}. \tag{2}$$

Since $I = 0$ and $x = 0$ at the start of the motion, the above formula gives $LI = Blx$. Substituting the current $I = Blx/L$ into the equation of motion gives

$$ma = mg\sin\alpha - \tfrac{B^2l^2}{L}x. \tag{3}$$

This equation is similar to the equation of motion for a body on a spring. The rod makes harmonic oscillations about the equilibrium position

$$x_0 = \tfrac{mgL\sin\alpha}{B^2L^2}. \tag{4}$$

The amplitude of the oscillations is $A = x_0$ and the frequency of the oscillations is $\omega^2 = \frac{B^2 l^2}{mL}$. The position of the rod as a function of time

$$x(t) = A(1 - \cos \omega t) = \frac{mgL \sin \alpha}{B^2 L^2}(1 - \cos \frac{Bl}{\sqrt{mL}} t). \tag{5}$$

#4 : UNDERGRADUATE E & M

`PROBLEM:`

Most planets in our solar system have a magnetic field that extends well beyond their atmosphere and contains trapped charged particles. Consider a small volume where the field **B** can be considered uniform.

(a) Give an equation for the force **F** experienced by a particle of charge q moving with velocity $\mathbf{v} \ll c$ in the field **B**. What is the direction of the force?

(b) The particle in (a) will move in a circle of radius r. Given an expression for r for a particle of mass m.

(c) Give an equation for the frequency $\nu = v/(2\pi r)$ of the rotation. What name do we give this frequency?

(d) Discuss how we could detect keV electrons trapped in a 1 Gauss planetary field.

`SOLUTION:`

(a) The charged particles experiences the Lorentz force

$$F = qv \times B \tag{1}$$

that is perpendicular to both **v** and **B**.

(b) We equate the centripetal magnetic Lorentz force and the centrifugal force mv^2/r giving $r = mv/(qB)$.

(c) The cyclotron frequency is

$$\nu = \frac{qB}{2\pi m}. \tag{2}$$

(d) The electrons will emit long wavelength radio waves at the cyclotron frequency of 2.8 MHz, where we use $q = 1.6\times10^{-19}$C and $m_e = 9.11\times10^{-28}$g. This radiation can be intense. It was first detected from Jupiter (B=4 G) in 1955.

The cyclotron radiation is polarized. Viewed form the pole of the motion of the electron this radiation will be circularly polarized, and viewed from the plane of the circular motion it is linearly polarized. From other directions it is elliptically polarized.

This radiation can only escape the region of origin if the cyclotron frequency exceeds the electron plasma frequency.

We can also detect these electrons if they interact with species in the outer atmosphere to produce auroral emissions over a broad range of wavelengths from X-rays to infrared, and especially UV.

#5 : UNDERGRADUATE STAT MECH

`PROBLEM:`

The heat capacities for gases defined at fixed pressure and fixed volume are denoted as C_P and C_V, respectively.

1) Can you explain which one should be larger for ideal gases without calculation?

2) Prove that

$$C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P \tag{1}$$

3) For ideal gases, what is the value of $C_P - C_V$? Express the result in terms of the number of gas molecules N and the Boltzmann constant k_B.

4) Define the constant $\gamma = C_P/C_V$. Prove that PV^γ is a constant for the adiabatic process.

`SOLUTION:`

1) C_p is larger than C_v. With increasing temperature from T to $T+\delta T$, during the iso-pressure process, the volume of the gas expands, and thus the gas does work, while for the iso-thermal process, the work is zero. For ideal gases, the internal energy only depends on temperature, and thus the change of internal energies are the same for both processes. According to $\Delta Q = \Delta W + \Delta U$, $\Delta Q/\Delta T$ is larger in the iso-pressure process, i.e., C_p is larger.

2)

$$C_P \;=\; T\left(\frac{\partial S}{\partial T}\right)_p, \quad C_V = T\left(\frac{\partial S}{\partial T}\right)_V \tag{2}$$

From the relation $S(T,P) = S(T, V(T,P))$, we have

$$\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P. \tag{3}$$

From $dF = -SdT - PdV$, we get the Maxwell relation

$$(\frac{\partial S}{\partial V})_T = (\frac{\partial P}{\partial T})_V, \tag{4}$$

thus $C_P - C_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P$.

3) For the idea gas $PV = Nk_BT$,

$$\left(\frac{\partial P}{\partial T}\right)_V \;=\; \frac{Nk_B}{V} \tag{5}$$

$$\left(\frac{\partial V}{\partial T}\right)_P \;=\; \frac{Nk_B}{P}, \tag{6}$$

and thus $C_p - C_v = T(Nk_B)^2/(PV) = Nk_B$.

4) During the adiabatic process

$$dS = (C_v dT + PdV)/T = 0, \quad PdV + VdP = Nk_B dT, \tag{7}$$

thus $-PdV = C_v dT$ and $-PdV + Nk_B dT = C_p dT = \gamma C_v dT = -\gamma PdV$. We have

$$VdP + \gamma PdV = 0 \tag{8}$$

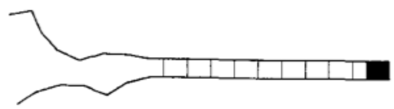
$$-\frac{dP}{P} = \gamma \frac{dV}{V}, \tag{9}$$

and thus PV^γ is a constant.

#6 : UNDERGRADUATE STAT MECH

PROBLEM:

A zipper has $N-1$ links. Each link is either open with energy ε or closed with energy 0. We require, however, that the zipper can only open from the left end, and that link ℓ can only unzip if all links to the left $(1, 2, , \ell - 1)$ are already open. Each *open* link has G degenerate states available to it (it can flop around).



1. Compute the partition function of the zipper at temperature T.

   Notational request from the grader: please use the name $x \equiv Ge^{-\varepsilon/k_BT}$.

2. Find the average number of open links at low temperature, *i.e.*, in the limit $\varepsilon \gg k_BT$.

3. Find the average number of open links at high temperature, *i.e.*, in the limit $k_BT \gg \varepsilon$.

   [Note: You can do this part of the problem independently of part 1.]

4. Is there a special temperature at which something interesting happens at large N? What happens there?

[Cultural remark: this is a very simplified model of the unwinding of two-stranded DNA molecules – see C. Kittel, Amer. J. Physics, **37** 917 (1969).]

SOLUTION:

1. The state of the system is completely specified by the number of open links n, which can go from $n = 0, 1, 2, ...N - 1$. The partition function can be summed to give

$$Z = \sum_{n=0}^{N-1} G^n e^{-n\varepsilon/k_BT} = \sum_{n=0}^{N-1} x^n = \frac{1 - x^N}{1 - x}$$

where $x \equiv Ge^{-\beta\varepsilon}$.

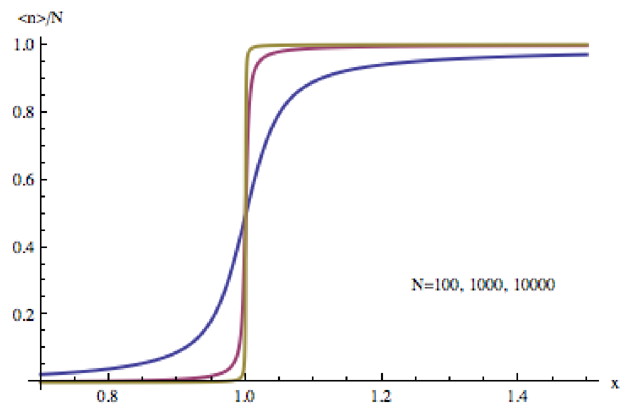
2. At any temperature,

$$\langle n \rangle = x\partial_x \ln Z = -\frac{Nx^N}{1-x^N} + \frac{x}{1-x} \ .$$

$$k_B T \ll \epsilon \implies x \ll 1.$$

In this limit, we can Taylor expand in the Boltzmann factor x. The leading term is linear in x, and comes from one open link.

$$\langle n \rangle \overset{k_B T \gg \epsilon}{\approx} x + \mathcal{O}(x^2)$$



3. At high temperature, every configuration is equally probable, and $x \to G$. The probability of any given state is $\frac{1}{Z}$ where Z is just the number of states; this is still not so easy to evaluate. For general G, the answer is

$$\langle n \rangle = x\partial_x \ln Z|_{x=G} = -\frac{NG^N}{1-G^N} + \frac{G}{1-G} \ .$$

In the large-G limit $\langle n \rangle$ simplifies to $\langle n \rangle \overset{G\to\infty}{\to} N-1$.

4. The partition function has a denominator which goes to zero at $x = 1$, that is when

$$1 = Ge^{-\beta\varepsilon} \qquad \Leftrightarrow \qquad k_B T = \frac{\varepsilon}{\ln G}.$$

At any finite N, the actual pole in Z is cancelled by the numerator. As $N \to \infty$, this is not the case and there is a real singularity – it becomes a sharp phase transition. When $N = \infty$, at $x = 1$ we reach the radius of convergence about $x = 0$ of the partition sum. Physically: above this temperature, the entropy of the open bonds wins out over the energetic cost of opening them, and there is a transition between a mostly-zippered state and a mostly-unzippered state. This is strikingly visible in the plot of the fraction of open links $\frac{\langle n \rangle}{N}$ above.

#7 : UNDERGRADUATE QUANTUM

`PROBLEM:`

A one-dimensional particle of mass m and energy E is incident on the δ-function potential $V(x) = V_0\delta(x)$.

1. Find the reflection and transmission coefficients.
2. Find the phase shift δ of the transmitted wave, and the difference $\delta(E \to \infty) - \delta(E \to 0)$.
3. The scattering amplitudes have a pole at a complex value of momentum $\hbar k$. Find the location k_0 of the pole. What is the physical interpretation of this pole?

`SOLUTION:`

1.

$$E = \frac{\hbar^2 k^2}{2m} \tag{1}$$

The wavefunctions are

$$\psi_< = e^{ikx} + Re^{-ikx} \qquad\qquad \psi_> = Te^{ikx} \tag{2}$$

ψ continuous gives

$$1 + R = T \tag{3}$$

and

$$\psi'_>(0) - \psi'_<(0) = \frac{2mV_0}{\hbar^2}\psi(0) \tag{4}$$

gives

$$ikT - ik(1 - R) = \frac{2mV_0}{\hbar^2}(1 + R) \tag{5}$$

so that

$$R = -\frac{c}{c - 2ik} \qquad T = -\frac{2ik}{c - 2ik} \qquad c = \frac{2mV_0}{\hbar^2} \tag{6}$$

2.

$$T = |T|\, e^{i\delta}$$
$$\tan\delta = -\frac{c}{2k} \tag{7}$$

As $E \to \infty$, $k \to \infty$ and

$$\delta(E \to \infty) = 0. \tag{8}$$

As $E \to 0$, $k \to 0$ and

$$\delta(E \to 0) = -\frac{\pi}{2} \tag{9}$$

so

$$\delta(E \to \infty) - \delta(E \to 0) = \frac{\pi}{2} \tag{10}$$

3. The pole is at

$$k_0 = -i\frac{c}{2} \tag{11}$$

The pole corresponds to a bound state with energy

$$E = -\frac{\hbar^2}{2m}\frac{c^2}{4} = -\frac{mV_0^2}{2\hbar^2} \tag{12}$$

which is present when $c < 0$, i.e. $V_0 < 0$.

#8 : UNDERGRADUATE QUANTUM

`PROBLEM:` Consider an electron constrained to move in the xy plane under the influence of a uniform magnetic field of magnitude B oriented in the $+\hat{z}$ direction. The Hamiltonian for this electron is

$$\mathbf{H} = \frac{1}{2m}\left(\left(\mathbf{p}_x - \frac{e}{c}A_x\right)^2 + \left(\mathbf{p}_y - \frac{e}{c}A_y\right)^2\right)$$

where m and e are the mass and charge of the electron, and c is the speed of light.

(a) Find a suitable expression for $\vec{A}$ so that $\mathbf{p}_x$ is a constant of motion for the above Hamiltonian.

(b) With this choice for $\vec{A}$, show that the eigenfunctions of $\mathbf{H}$ can be written in the form

$$\Psi(x,y) = e^{\frac{\mathrm{i}}{\hbar}p_x x}\Phi(y)$$

where $\Phi(y)$ satisfies the Schrödinger equation for a one-dimensional harmonic oscillator whose equilibrium position is $y = y_0$. Find the effective spring constant k for this oscillator and the shift of the origin y_0 in terms of p_x, B, m, e, c.

(c) Find the energy eigenvalues for this system, and indicate degeneracies.

(d) For the remainder of the problem, suppose we further restrict the particles to live in a square of side length L. Suppose we demand periodic boundary conditions. What are the possible values of p_x?

`SOLUTION:`

(a) We can choose a gauge for $\vec{A}$ with $\vec{\nabla}\times\vec{A} = B\hat{z}$, uniform, so that $\mathbf{x}$ does not appear:

$$A_x = -B\mathbf{y}, A_y = 0.$$

(b) The Schrödinger equation is

$$\mathbf{H}\Psi(x,y) = E\psi(x,y) \tag{1}$$

and with the choice of gauge from part (a) we have

$$\mathbf{H} = \frac{1}{2m}\left(\left(\mathbf{p}_x + \frac{e}{c}B\mathbf{y}\right)^2 + \mathbf{p}_y^2\right).$$

Plugging in the given ansatz turns $\mathbf{p}_x$ into a number, and (1) becomes:

$$\frac{1}{2m}\left(\left(p_x + \frac{e}{c}By\right)^2 + \mathbf{p}_y^2\right)e^{\frac{\mathrm{i}}{\hbar}p_x x}\Phi(y) = Ee^{\frac{\mathrm{i}}{\hbar}p_x x}\Phi(y)$$

or

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{1}{2m}\left(p_x + \frac{e}{c}By\right)^2\right)\Phi(y) = E\Phi(y)$$

or

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(y + \frac{c}{eB}p_x\right)^2\right)\Phi(y) = E\Phi(y)$$

which is the Schrödinger equation for a simple harmonic oscillator (SHO)

$$\left(\frac{\mathbf{p}_y^2}{2m} + \frac{k}{2}\left(y - y_0\right)^2\right)\Phi(y) = E\Phi(y)$$

with $k = m\left(\frac{eB}{mc}\right)^2$ and $y_0 = -\frac{cp_x}{eB}$.

(c) The energy spectrum of the SHO is

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

where

$$\omega = \sqrt{\frac{k}{m}} = \frac{eB}{mc} = \omega_c,$$

the cyclotron frequency. Notice that p_x drops out of the expression for the energy and so there is a big degeneracy, approximately linear in the system size.

(d) Using the boundary condition that the wavefunction should be the same at the end points ($\psi(x = L) = \psi(x = 0)$), we have

$$p_x = \frac{2\pi\ell}{L}, \quad \ell \in \mathbb{Z} \quad .$$

#9 : UNDERGRADUATE GENERAL/MATH

`PROBLEM:`

Imagine a long cylindrical tube of radius R at temperature T_{wall}. Fluid flows through the tube at velocity v. The temperature of the fluid when it enters the tube is T_{fluid}. What is the length L that the fluid must travel in the tube so that its temperature reaches T_{wall}?

1. Write down the equation you would use to solve the problem, and the boundary conditions.

2. Write down a back-of-the-envelope estimate of L using dimensional analysis. Find L for water if $R = 0.5\,\text{mm}$, $v = 1\,\text{mm/s}$, D, the thermal diffusivity is $0.15\,\text{mm}^2/\text{s}$, $T_{\text{fluid}} = 21^\circ$ C, and $T_{\text{wall}} = 37^\circ$ C.

`SOLUTION:`

Basically, all these problems work by dimensional analysis.

(1)

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \tag{1}$$

(2) From (1), the diffusion coefficient has units [length]^2/[time]. If the fluid were stationary, it would take a time $\sim R^2/D$ to warm up.

Now, it's moving at velocity v, so the length you have to travel would be $v * R^2/D$. For the given parameters, the characteristic travel distance for thermal equilibration would be order of 1 mm. Since these are estimates for factors of e, it would be safer to multiply by 10, so the safe estimate would be about 10 mm.

#10 : GENERAL/MATH

`PROBLEM:`

(a) Recall that $\Gamma(n) \equiv \int_0^\infty e^{-t} t^n \frac{dt}{t} \equiv (n-1)!$. Verify the second equality in

$$\mathcal{I} \equiv \int_{-\infty}^{\infty} e^{-x^2} dx = \Gamma(1/2).$$

(b) Explicitly evaluate the Gaussian integral $\mathcal{I}$ above. Hint: consider $\mathcal{I}^2$.

(c) The volume element in D-dimensional polar coordinates is $dV = r^{D-1} dr d\Omega_{D-1}$, where $d\Omega_{D-1}$ is an angle element. A sphere S^{D-1} of radius r in D dimensions, given by $x_1^2 + \ldots x_D^2 = r^2$, has surface area $\Omega_{D-1} r^{D-1}$, where $\Omega_{D-1} = \int d\Omega_{D-1}$ is the total solid angle, e.g. $\Omega_1 = 2\pi$ for a circle in $D = 2$ and $\Omega_2 = 4\pi$ for a sphere in $D = 3$. Evaluate Ω_{D-1}, for general D, in terms of the Gamma function. Hint: consider $\mathcal{I}^D$.

`SOLUTION:`

(a) Substitute $t = x^2$, so $dt = 2xdx$ and $\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx = \int_{-\infty}^\infty e^{-x^2} dx$.

(b) Using the hint, going to polar coordinates, and substituting $t = r^2$,

$$I^2 = \int\int dxdy e^{-(x^2+y^2)} = \int\int r dr d\theta e^{-r^2} = \pi \int_0^\infty dt e^{-t} = \pi,$$

so $\mathcal{I} = \Gamma(1/2) = \sqrt{\pi}$.

(c) Follow the suggestion in the question and go to D-dimensional polar coordinates, with $r^2 = x_1^2 + \ldots x_D^2$

$$\mathcal{I}^D = \pi^{D/2} = \int \ldots \int dx_1 \ldots dx_D e^{-(x_1^2+\ldots x_D^2)} = \int r^{D-1} dr \int d\Omega_{D-1} e^{-r^2} =$$

$$= \Omega_{D-1} \int_0^\infty e^{-r^2} r^D \frac{dr}{r} = \Omega_{D-1} \int_0^\infty e^{-t} t^{D/2} \frac{dt}{2t} \equiv \tfrac{1}{2} \Omega_{D-1} \Gamma(D/2),$$

so $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$.

#11 : GRADUATE MECHANICS

`PROBLEM:` A yo-yo of mass M is composed of 2 large disks of radius R and thickness t separated by a distance t with a shaft of radius r. Assume a uniform density throughout. Find the tension in the massless string as the yo-yo descends under the influence of gravity.

`SOLUTION:`

Let the density of the yo-yo be ρ, then its moment of inertia and mass are respectively

$$I = 2 \times \tfrac{1}{2}\pi t\rho R^4 + \tfrac{1}{2}\pi t\rho r^4, \tag{1}$$

$$M = 2 \times \pi t\rho R^2 + \pi t\rho r^2, \tag{2}$$

whence

$$I = \tfrac{1}{2} M \left(\tfrac{2R^4+r^4}{2R^2+r^2}\right). \tag{3}$$

The equations of motion of the yo-yo are

$$M\ddot{x} = Mg - T, \tag{4}$$

$$I\ddot{\theta} = Tr, \tag{5}$$

where T is the tension in the string. We also have the constraint $\ddot{x} = r\ddot{\theta}$. From the above we obtain

$$T = \tfrac{IMg}{I+Mr^2} = \tfrac{(2R^4+r^4)Mg}{2R^4+4R^2r^2+3r^4}. \tag{6}$$

#12 : GRADUATE MECHANICS

`PROBLEM:` A particle under the action of gravity slides on the inside of a smooth paraboloid of revolution whose axis is vertical. Using the distance from the axis, r, and the azimuthal angle ϕ as generalized coordinates, find

(a) The Lagrangian of the system.

(b) The generalized momenta and the corresponding Hamiltonian.

(c) The equation of motion for the coordinate r as a function of time.

(d) If $\frac{d\phi}{dt} = 0$, show that the particle can execute small oscillations about the lowest point of the paraboloid, and find the frequency of these oscillations.

`SOLUTION:` Suppose the paraboloid of revolution is generated by a parabola which in cylindrical coordinates (r, ϕ, z) is represented by

$$z = Ar^2, \tag{1}$$

where A is a positive constant.

(a) The Lagrangian of the system is

$$\begin{aligned} L = T - V &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - mgz & (2)\\ &= \tfrac{1}{2}m(1 + 4A^2r^2)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 - Amgr^2. & (3) \end{aligned}$$

(b) The generalized momenta are

$$\begin{aligned} p_r &= \tfrac{\partial L}{\partial \dot{r}} = m(1 + 4A^2r^2)\dot{r}, & (4)\\ p_\phi &= \tfrac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}, & (5) \end{aligned}$$

and the Hamiltonian is

$$\begin{aligned} H &= p_r\dot{r} + p_\phi\dot{\phi} - L & (6)\\ &= \tfrac{1}{2}m(1 + 4A^2r^2)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\phi}^2 + Amgr^2 & (7)\\ &= \tfrac{p_r^2}{2m(1+4A^2r^2)} + \tfrac{p_\phi^2}{2mr^2} + Amgr^2. & (8) \end{aligned}$$

(c) Lagrange's equations

$$\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{q}_i}\right) - \tfrac{\partial L}{\partial q_i} = 0 \tag{9}$$

give

$$m(1+4A^2r^2)\ddot{r}+4mA^2r\dot{r}^2-mr\dot{\phi}^2+2Amgr=0, \tag{10}$$
$$mr^2\dot{\phi}=constant. \tag{11}$$

Letting the constant be mh and eliminating $\dot{\phi}$ from Eq. 10, we obtain the equation for r:

$$(1+4A^2r^2)r^3\ddot{r}+4A^2r^4\dot{r}^2+2Agr^4=h^2. \tag{12}$$

(d) If $\dot{\phi}=0$, Eq. 10 becomes

$$(1+4A^2r^2)\ddot{r}+4A^2r\dot{r}^2+2Agr=0. \tag{13}$$

The lowest point of the paraboloid is given by $r=0$. For small oscillations in its vicinity, $r, \dot{r}, \ddot{r}$ are small quantities. Then to first approximation Eq. 13 becomes

$$\ddot{r}+2Agr=0. \tag{14}$$

Since the coefficient of r is positive, the particle executes simple harmonic motion about $r=0$ with angular frequency

$$\omega=\sqrt{2Ag}. \tag{15}$$

#13 :GRADUATE ELECTRODYNAMICS

`PROBLEM:` A long, straight cylindrical wire, of radius a carries a uniformly distributed current I. It emits an electron from $r = a$, with initial, relativistic velocity v_0 parallel to its axis. Find the maximum distance r_{max} from the axis of the wire which the electrons can reach, treating everything relativistically.

`SOLUTION:` Find $\vec{B} = \widehat{\phi}2I/rc$ and then $\vec{A} = -\widehat{z}(2I/c)\ln(r/a)$, so the electrons have

$$L = -mc^2\sqrt{1 - v^2/c^2} + (2I|e|/c^2)v_z \ln(r/a).$$

The energy and p_z are conserved:

$$p_z = \frac{\partial L}{\partial v_z} = \gamma m v_z + (2I|e|/c^2)\ln(r/a) = \gamma_0 m v_0.$$

$$H = \gamma mc^2 = \gamma_0 mc^2$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $\gamma_0 \equiv 1/\sqrt{1 - v_0^2/c^2}$. So $\gamma = \gamma_0$ and r_{max} is where $\dot{r} = 0$, which means that $v_z = -v_0$ (half-period of cyclotron rotation), which gives

$$r_{max} = a\exp(\gamma_0 m v_0 c^2/I|e|).$$

#14 :GRADUATE ELECTRODYNAMICS

`PROBLEM:` Consider a hollow spherical shell of radius a and surface charge density σ.

(a) Derive the magnetic field, $\vec{B}$ both inside and outside the shell, if it is spinning at frequency ω around an axis through its center. One way to solve this is to write $\vec{B} = -\nabla\phi_{mag}$, note that ϕ_{mag} solves Laplace's equation, and that the rotational symmetry is broken only by the vector $\vec{\omega}$, so only the $\ell = 1$ harmonic contributes. If you solve the question this way, be sure to note and use all the matching conditions above and below $r = a$.

(b) How much work is required to get the shell spinning at frequency ω, starting from the shell at rest? Show that your answer fits with assigning an additional moment of inertia associated with electrodynamics, $I_{total} = I_{mech} + I_{E\&M}$, with I_{mech} the usual moment of inertia of a hollow sphere. Don't bother to work out I_{mech}, it's $2mr^2/3$ and not relevant for computing the quantity of interest, $I_{E\&M}$.

`SOLUTION:`

(a) Following the hints in the question, we have

$$\phi^{out}_{mag} = C\cos\theta/r^2, \qquad \phi^{in}_{mag} = -Dr\cos\theta = -Dz.$$

Gauss' law implies that $\vec{B}\hat{r}$ must be continuous at the surface, and the curl $\vec{B}$ Maxwell equation implies that $\hat{r} \times (\vec{B}_{out} - \vec{B}_{in}) = 4\pi\vec{K}/c$, with $\vec{K} = \sigma\vec{v} = \sigma\vec{\omega} \times a\hat{r}$. It follows that $\vec{B}^{out}$ is that of a magnetic dipole, and $\vec{B}^{in}$ is a constant:

$$\vec{B}_{in} = \frac{2\vec{m}}{a^3}, \qquad \vec{B}_{out} = \frac{3(\hat{r}\cdot\vec{m})\hat{r} - \vec{m}}{r^3},$$

with $\vec{m} = \frac{4\pi}{3}\frac{\vec{\omega}\sigma a^4}{c}$.

(b) The work required is the additional energy associated with the magnetic field of the spinning sphere,

$$W_{E\&M} = \Delta U_{field} = \int d^3x\vec{B}^2/8\pi$$

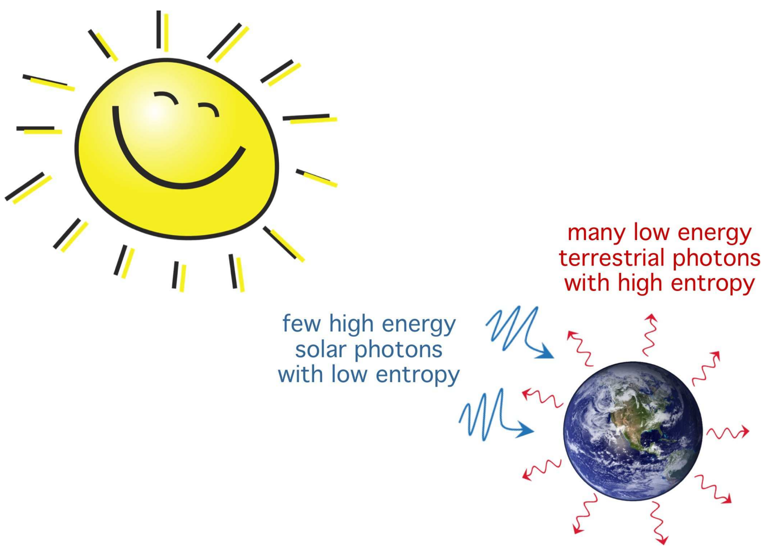
Plugging in the above $\vec{B}$ and doing the volume integral gives

$$W_{E\&M} = \frac{1}{2}a^{-3}\vec{m}^2 = \tfrac{1}{2}(4\pi/3)^2\vec{\omega}^2\sigma^2a^5/c^2 \equiv \tfrac{1}{2}I_{E\&M}\vec{\omega}^2,$$

with

$$I_{E\&M} = (4\pi/3)^2\sigma^2 a^5/c^2.$$

(Easy to check that the units are correct.)



#15 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The energy flux emitted from the surface of a perfect blackbody is $J_E = \sigma T^4$, where $\sigma = 5.67 \times 10^{-8}\,\mathrm{W/m^2K^4}$ is Stefan's constant and T is the temperature.

(a) Find a corresponding expression for the entropy flux, J_S.

(b) Idealizing the earth as a perfect blackbody, the average absorbed solar energy flux is $\Phi_E = 342\,\mathrm{W/m^2}$. Derive an expression for the entropy flux of the radiation emitted by the earth, and provide an estimate of its value in MKS units.

(c) Derive an expression for the ratio of the entropy flux of radiated terrestrial photons to the entropy flux of solar photons incident on the earth? Express your answer in terms of the surface temperatures of the earth and the sun.

SOLUTION:

(a) The energy flux is

$$J_E = \frac{cE}{4\pi V} \int\limits_0^{\pi/2} \! d\theta \, \sin\theta \, \cos\theta \int\limits_0^{2\pi} \! d\phi = \frac{cE}{4V} \ ,$$

where E is the total energy. *Mutatis mutandis*, we have that $J_S = cS/4V$. We relate E and S using thermodynamics: $dE = T\,dS - p\,dV$. From $J_E = \sigma T^4$ we then have

$$J_S = \tfrac{4}{3}\sigma T^3 \ .$$

(b) The total terrestrial entropy flux is

$$\Phi_S = \frac{4\Phi_E}{3T_\mathrm{e}} \approx \frac{4 \cdot 342\,\mathrm{W/m^2}}{3 \cdot 278\,\mathrm{K}} = 1.64\,\mathrm{W/m^2K} \ ,$$

where we have approximated the mean surface temperature of the earth as $T_\mathrm{e} \approx 278\mathrm{K}$, which is what one finds assuming the earth is a perfect blackbody. (The actual mean surface temperature is about 287 K.)

(c) First, recall the argument for the steady state temperature of the earth. The energy of the absorbed solar radiation is $(\sigma T_\odot^4)(4\pi R_\odot^2)(\pi R_\mathrm{e}^2/4\pi a_\mathrm{e}^2)$, which is obtained by multiplying the total energy output of the sun by the ratio of the earth's cross section πR_e^2 to the area of a sphere whose radius is $a_\mathrm{e} = 1\,\mathrm{AU}$. Setting this to the energy output of the earth, $(\sigma T_\mathrm{e}^4)(4\pi R_\mathrm{e}^2)$ yields the expression $T_\mathrm{e} = T_\odot\sqrt{R_\odot/2a_\mathrm{e}}$. When comparing the entropy flux of the solar radiation to that emitted by the earth, we use the same expressions, except we replace T^4 by $\frac{4}{3}T^3$ throughout. Therefore the ratio of entropy fluxes is

$$\eta = \frac{T_\mathrm{e}^3}{T_\odot^3(R_\odot^2/4a_\mathrm{e}^2)} = \frac{T_\odot}{T_\mathrm{e}} \approx 21 \ ,$$

taking $T_\odot \approx 5780\,\mathrm{K}$.

#16 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The Landau expansion of the free energy density of a particular system is given by

$$f(T,m) = (T - T_0)\, m^2 - T_0\, m^3 + T_0\, m^4 \ ,$$

where m is the order parameter and T_0 is some temperature scale, and where we have set $k_{\rm B} \equiv 1$.

(a) Find the critical temperature $T_{\rm c}$. Is the transition first or second order?

(b) Sketch the equilibrium magnetization $m(T)$.

(c) Find $m(T_{\rm c}^-)$, *i.e.* the value of m just below the critical temperature.

SOLUTION:

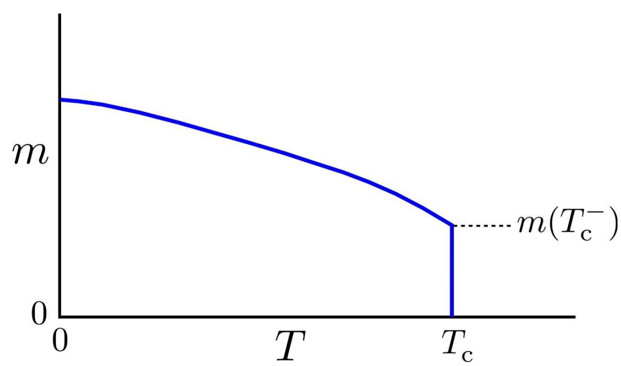


Figure 1: Sketch for part (b).

(a) The free energy is of the general form $f(m) = \frac{1}{2}am^2 - \frac{1}{3}ym^3 + \frac{1}{4}bm^4$, with T dependence implicit in the coefficients:

$$a = 2(T - T_0) \quad , \quad y = 3T_0 \quad , \quad b = 4T_0 \ .$$

Setting $f'(m) = 0$ we have $(a - ym + bm^2)\, m = 0$, yielding three solutions, one of which lies at $m = 0$. The other two solutions are roots of the quadratic factor:

$$m_\pm = \frac{y}{2b} \pm \sqrt{\left(\frac{y}{2b}\right)^2 - \frac{a}{b}} \ .$$

These solutions are both real and positive if $a < y^2/4b$. Since the free energy increases without bound for $m \to \pm\infty$, we must have that m_- is a local maximum and m_+ a minimum. To see if it is the global minimum, we must compare $f(m_+)$ with $f(0)$. To do this, set $f(m) = f(0)$, which yields $2a - \frac{4}{3}ym + bm^2 = 0$, and subtract from the equation $a - ym + bm^2 = 0$ to find $m = 3a/y$. This is the value of m for which the two minima are degenerate. Equating this with the expression for m_+, we obtain $a = 2y^2/9b$, which is our equation for T_c. Solving this equation for T, we obtain $T_c = \frac{5}{4}\,T_0$.

(b) A sketch is provided above. Note that $m(T)$ drops discontinuously to zero at the transition.

(c) Evaluating $m = 3a/y$ at $T = T_c$, we have $m(T_c^-) = \frac{1}{2}$.

#17 : GRADUATE QUANTUM

PROBLEM:

1. An electron gun produces electrons randomly polarized with spins up or down along one of the three possible radomly selected orthogonal axes $1, 2, 3$ (i.e. x, y, z), with probabilities $p_{i,\uparrow}$ and $p_{i,\downarrow}$, $i = 1, 2, 3$. To simplify the final results, it is better to rewrite these in terms of d_i and δ_i defined by

$$p_{i,\uparrow} = \frac{1}{2}d_i + \frac{1}{2}\delta_i \qquad p_{i,\downarrow} = \frac{1}{2}d_i - \frac{1}{2}\delta_i \qquad i = 1, 2, 3 \tag{1}$$

Probabilites must be non-negative, so $d_i \geq 0$ and $|\delta_i| \leq d_i$.

(a) Write down the resultant electron spin density matrix ρ in the basis $|\uparrow\rangle$, $|\downarrow\rangle$ with respect to the z axis.

(b) Any 2×2 matrix ρ can be written as

$$\rho = a\,\mathbf{1} + \mathbf{b} \cdot \boldsymbol{\sigma} \tag{2}$$

in terms of the unit matrix and 3 Pauli matrices. Determine a and $\mathbf{b}$ for ρ from part (a).

(c) A second electron gun produces electrons with spins up or down along a single axis in the direction $\hat{\mathbf{n}}$ with probabilities $(1 \pm \Delta)/2$. Find $\hat{\mathbf{n}}$ and Δ so that the electron ensemble produced by the second gun is the same as that produced by the first gun.

SOLUTION:

(a) The density matrix is

$$p_{1,\uparrow}\,\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + p_{1,\downarrow}\,\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + p_{2,\uparrow}\,\frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix} + p_{2,\downarrow}\,\frac{1}{2}\begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$$
$$+\, p_{3,\uparrow}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + p_{3,\downarrow}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}(d_1 + d_2 + d_3) + \delta_3 & \delta_1 - i\delta_2\\ \delta_1 + i\delta_2 & (d_1 + d_2 + d_3) - \delta_3\end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix}1 + \delta_3 & \delta_1 - i\delta_2\\ \delta_1 + i\delta_2 & 1 - \delta_3\end{pmatrix} \tag{3}$$

(b) By inspection,

$$a = \frac{1}{2} \qquad \mathbf{b} = \frac{1}{2}\boldsymbol{\delta} \tag{4}$$

(c) This is the same as the density matrix produced by the second electron gun if $\hat{\mathbf{n}}$ is parallel to $\boldsymbol{\delta}$, and $D = |\boldsymbol{\delta}|$.

A simple way to see this is to a go to a rotated coordinate system with z' axis along $\boldsymbol{\delta}$.

#18 : GRADUATE QUANTUM

`PROBLEM:` The Hamiltonian for a quantum mechanical rigid body is

$$\mathbf{H} = \tfrac{1}{2}\left(\frac{\mathbf{L}_1^2}{I_1} + \frac{\mathbf{L}_2^2}{I_2} + \frac{\mathbf{L}_3^2}{I_3}\right), \tag{1}$$

where I_1, I_2, I_3 are the principal moments of inertia, and $(\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3) \equiv (\mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z)$ are the angular momentum operators, satisfying the commutation relations

$$[\mathbf{L}_i, \mathbf{L}_j] = \mathbf{i}\hbar\epsilon_{ijk}\mathbf{L}_k \ . \tag{2}$$

You may assume the angular momentum takes on only integer values. Hamiltonians of the form (1) describe the rotational spectrum of molecules.

(a) (Very easy) First, consider the case $I_1 = I_2 = I_3 = I$ (a spherical top, such as methane). Write down a formula for the energy levels in terms of an appropriate quantum number.

(b) (Easy) Next, consider the case $I_1 = I_2 = I_\perp \neq I_3$ (a symmetric top, such as ammonia). Show that $\mathbf{L}_3$ is a constant of motion. Write down another constant of motion which commutes with $\mathbf{L}_3$, and express the Hamiltonian as a function of the two constants of motion. Then write down a formula for the energy levels as a function of the two good quantum numbers. Indicate the allowed ranges of these quantum numbers. Also indicate any degeneracies.

(c) (A little harder) Now consider the case of a slightly asymmetric top, *i.e.* one for which

$$I_1 = I_\perp - \epsilon, \quad I_2 = I_\perp + \epsilon \tag{3}$$

where ϵ is small. What are the good quantum numbers in this case? Find the shifts in the energy levels relative to those of the symmetric top, to first order in the small quantity ϵ. List the energy shifts for all values of the two quantum numbers in part (b).

Hints: (1) Express the perturbing Hamiltonian $\mathbf{H} - \mathbf{H}_{\text{symmetric}}$ in terms of raising and lowering operators. (2) A useful formula:

$$\mathbf{L}_+|\ell, m\rangle = \hbar\sqrt{(\ell - m)(\ell + m - 1)}|\ell, m+1\rangle \ . \tag{4}$$

(3) You will need to use degenerate perturbation theory.

`SOLUTION:`

(a) For the spherical case,

$$\mathbf{H} = \frac{1}{2I}\left(\mathbf{L}_1^2 + \mathbf{L}_2^2 + \mathbf{L}_3^2\right) = \frac{\vec{\mathbf{L}}^2}{2I} \tag{5}$$

and so

$$E_\ell = \frac{\hbar^2 \ell(\ell+1)}{2I}. \tag{6}$$

(b) For the symmetric top,

$$\mathbf{H} = \frac{1}{2}\left(\frac{\mathbf{L}_1^2 + \mathbf{L}_2^2}{I_\perp} + \frac{\mathbf{L}_3^2}{I_3}\right) = \frac{1}{2}\left(\frac{\vec{\mathbf{L}}^2 - \mathbf{L}_3^2}{I_\perp} + \frac{\mathbf{L}_3^2}{I_3}\right) \ . \tag{7}$$

Since $[\mathbf{L}_3, \vec{\mathbf{L}}^2] = 0$, both $\mathbf{L}_3$ and $\vec{\mathbf{L}}^2$ are constants of the motion ($[\mathbf{L}_3, \mathbf{H}] = 0 = [\vec{\mathbf{L}}^2, \mathbf{H}]$). For a given ℓ, the eigenvalues of $\mathbf{L}_3$ take values $m = -\ell, -\ell + 1, ...0, ...\ell - 1, \ell$, and the energies are:

$$E_{\ell,m} = \frac{\hbar^2}{2}\left(\frac{\ell(\ell+1) - m^2}{I_\perp} + \frac{m^2}{I_3}\right), \quad \begin{array}{c} \ell = 0, 1, ...\infty \\ m = -\ell, ...0, ...\ell \end{array} \ .$$

(c) The perturbing Hamiltonian is

$$\Delta\mathbf{H} \equiv \mathbf{H} - \mathbf{H}_{\text{symmetric}} = \tfrac{1}{2}\left(\frac{1}{I_\perp - \epsilon}\mathbf{L}_1^2 + \frac{1}{I_\perp + \epsilon}\mathbf{L}_2^2 - \frac{1}{I_\perp}\left(\mathbf{L}_1^2 + \mathbf{L}_2^2\right)\right) \ . \tag{8}$$

Using $\frac{1}{I_\perp \mp \epsilon} = \frac{1}{I_\perp} \pm \frac{\epsilon}{I_\perp^2} + ...$, we have

$$\Delta\mathbf{H} = \frac{\epsilon}{2I_\perp^2}\left(\mathbf{L}_1^2 - \mathbf{L}_2^2\right) \ . \tag{9}$$

Now let's write this in terms of raising and lowering operators:

$$\mathbf{L}_\pm \equiv \mathbf{L}_1 \pm \mathbf{i}\mathbf{L}_2, \mathbf{L}_1 = \tfrac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-), \mathbf{L}_2 = \frac{1}{2\mathbf{i}}(\mathbf{L}_+ - \mathbf{L}_-)$$

which satisfy (according to (2))

$$[\mathbf{L}_3, \mathbf{L}_\pm] = \pm\mathbf{L}_\pm, \quad [\mathbf{L}_+, \mathbf{L}_-] = 2\mathbf{L}_3.$$

In terms of these,

$$\Delta\mathbf{H} = \frac{\epsilon}{4I_\perp^2}\left(\mathbf{L}_+^2 + \mathbf{L}_-^2\right) \ . \tag{10}$$

This operator changes m by ± 2 (*i.e.* $|\Delta m| = 2$). This means that its first order correction to the energies is zero for any state that isn't degenerate with some other state related to it by $|\Delta m| = 2$. The only degeneracy in the spectrum comes from $E_{\ell,m} = E_{\ell,-m}$. So the condition for nonzero energy shift is only met by $m = \pm 1$ states (and any ℓ). So:

$$\Delta E_{\ell,m} = 0, \quad m \neq \pm 1.$$

For $m = \pm 1$, we have to use degenerate perturbation theory, that is, we have to diagonalize the matrix obtained by acting with $\Delta\mathbf{H}$ in the degenerate subspace. For each ℓ, this matrix $\mathbf{h}$ is 2×2, indexed by $m = \pm 1$, and its nonzero matrix elements are:

$$\frac{\epsilon}{4I_\perp^2}\langle \ell, 1|(\mathbf{L}_+^2 + \mathbf{L}_-^2)|\ell, -1\rangle = \frac{\epsilon}{4I_\perp^2}\langle \ell, -1|(\mathbf{L}_+^2 + \mathbf{L}_-^2)|\ell, +1\rangle \equiv \Delta.$$

So within the subspace, the matrix to diagonalize is

$$\mathbf{h} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

whose eigenvalues are $\pm\Delta$. Using (4),

$$\mathbf{L}_+^2|\ell, -1\rangle = \hbar^2 \ell(\ell+1)|\ell, +1\rangle,$$

and $\Delta = \frac{\epsilon\hbar^2}{4I_\perp^2}\ell(\ell+1)$.

The energy shifts are therefore

$$\begin{aligned} \Delta E_{\ell,m} &= \pm\tfrac{\epsilon\hbar^2}{4I_\perp^2}\ell(\ell+1), && \text{for } m = \pm 1 \\ \Delta E_{\ell,m} &= 0 && \text{else.} \end{aligned} \tag{11}$$

#19 : GRADUATE GENERAL/MATH

PROBLEM:

Solve the Laplace equation $\nabla^2 f = 0$ for function $f(r, \theta)$ inside a unit circle that obeys the boundary conditions

$$\begin{aligned} f(1, \theta) &= 1\,, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}\,, \\ &= 0\,, \quad \text{otherwise,} \end{aligned}$$

where $0 \le r \le 1$ and $-\pi < \theta \le \pi$ are the polar coordinates. Assume that f is nonsingular and can be sought in the form of Fourier series

$$f(r, \theta) = \sum_{m=-\infty}^{\infty} c_m r^{|m|} e^{im\theta}.$$

Reduce these series to geometric ones and sum them analytically.

SOLUTION:

We will first consider a more general Dirichlet boundary conditions $f(1, \theta') = g(\theta')$ and then treat the specific case. Inverting the Fourier series, we find

$$c_m = \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} g(\theta') e^{-im\theta'}.$$

Substituting this expression into the direct series and interchanging the order of summation and integration, we get

$$f(r, \theta) = c_0 + \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi} g(\theta') \sum_{m=1}^{\infty} [z^m \zeta^m + (z^* \zeta^*)^m]\,,$$

where we introduced notations $z = re^{i\theta}$ and $\zeta = e^{i\theta'}$. We see that the result is the sum of two geometric series. For real $g(\theta')$, they are complex conjugate of each other. After some algebraic manipulations, we arrive at

$$f(r, \theta) = -c_0 + \frac{1}{\pi} \operatorname{Im} \oint \frac{d\zeta}{\zeta - z} g(\zeta)\,,$$

where the integration path is the unit circle in the complex plane. For our particular $g(\zeta)$ we obtain

$$f(z) = -\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Im}\,\ln\left(\frac{z-i}{z+i}\right).$$

The logarithmic term is assumed to have the branch cut along the arc on which $g(\zeta)$ is nonzero. If desired, this expression can be further reduced to a combination of arctan-functions.

For illustration, consider the diameter that runs along the x-axis, in which case we get $z = x$ and

$$f(x, y = 0) = \frac{1}{2} + \frac{2}{\pi}\arctan x\,.$$

#20 : GRADUATE MATH/OTHER

PROBLEM:

The following problem is motivated by modeling the solar heating of rotating asteroids in space. The surface of a black body is subject to a uniform in space, periodic in time, radiative energy flux $E = E_0 + \delta E \cos \omega t$. The body has thermal conductivity $\varkappa$ and specific heat C. Treat the black body as the (infinite region) $z < 0$, whose surface is the xy plane.

a) Write down the heat diffusion equation and the boundary conditions appropriate for this system.

b) Consider the case of a constant energy flux, $\delta E = 0$, and find the surface temperature as a function of E_0.

c) Assuming δE is a small parameter, use the perturbation theory to find $\delta T = \delta T(E_0, \omega)$, the amplitude of oscillations in the surface temperature. Compute only the term that is of the first order in δE. Give a physical interpretation of the obtained ω-dependence.

SOLUTION:

a) Let z be the coordinate normal to the surface and increasing in the outward direction. Let $z = 0$ be the surface plane. The temperature obeys the diffusion equation

$$C\partial_t T = \varkappa \partial_z^2 T \,.$$

The boundary condition at $z = 0$ is governed by the energy balance:

$$\varkappa \partial_z T + \sigma T^4 \big|_{z=0} = E(t)\,,$$

where σ is the Stefan-Boltzmann constant.

b) For $\delta E = 0$ (time-independent radiation), the solution is $T(z,t) = T_0 =$ const, where

$$T_0 = (E_0/\sigma)^{1/4}.$$

c) For a finite small δE, we seek the solution in the form

$$T(z,t) = T_0 + \mathrm{Re}\, \delta T e^{kz - i\omega t}.$$

The thermal diffusion equation is satisfied if we set

$$k = e^{-i\pi/4}\sqrt{C\omega/\varkappa}\,.$$

The boundary condition expanded to the first order in δT reads

$$(\varkappa k + 4\sigma T_0^3)\delta T = \delta E\,.$$

Solving for δT and doing some algebraic simplifications, we obtain the final result:

$$|\delta T| = \left|\frac{(E_0/\sigma)^{1/4}\delta E}{4E_0 + e^{-i\pi/4}\sqrt{C\omega\varkappa}}\right|\,.$$

The ω-dependence can be interpreted as follows. The system appears to have a characteristic relaxation time $\tau \sim (C\varkappa)/E_0^2$. When the external energy flux variations are slow, $\omega \ll 1/\tau$, the surface temperature tracks them adiabatically, i.e., δT can be computed from the formula for $T_0(E_0)$. In the opposite limit, $\omega \gg 1/\tau$, the temperature variations become suppressed.