INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:

   a. Write the problem number and your ID number on each sheet;

   b. Write only on one side of the paper;

   c. Start each problem on the attached examination sheets;

   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 : UNDERGRADUATE MECHANICS

`PROBLEM:` The radial geodesic equation for an orbit around a black hole is given by:

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{r_S}{r}\right)\left(mc^2 + \frac{l^2}{mr^2}\right),$$

where $r_S = 2GM/c^2$ is the Schwarzschild radius, r is the radial coordinate, τ is the proper time along the orbit, E is the conserved energy along the orbit, and l is the conserved angular momentum.

`(a)` Write this equation using an effective potential V_{eff}.
`(b)` Using this V_{eff} find the circular orbits around the black hole.
`(c)` The larger orbit is stable and the smaller one is unstable. Using this information find the condition on l for the smallest stable orbit.
`(d)` Find the radius of this smallest stable orbit. Express your answer in terms of r_S alone.

`SOLUTION:` The solution is hand written.

#2 : UNDERGRADUATE MECHANICS

`PROBLEM:` A 1.2-kg block rests on a frictionless surface and is attached to a horizontal spring of constant $k = 23$ N/m (see Figure). The block is oscillating with amplitude $A_1 = 10$ cm and with phase constant $\phi_1 = -\pi/2$. A block of mass 0.80 kg is moving from the right at 1.7 m/s. It strikes the first block when the latter is at the rightmost point in its oscillation. The collision is perfectly inelastic. Determine the frequency, amplitude, and phase constant (relative to the *original* t=0) of the resulting motion.

`SOLUTION:`

The simple harmonic motion with just the first block on the spring can be described by

$$x(t) = A_1 \cos(\omega_1 t + \phi_1) \tag{1}$$

with the given amplitude A_1 and phase constant ϕ_1, where the angular frequency $\omega_1 = \sqrt{k/m_1} = \sqrt{(23N/m)/(1.2kg)} = 4.38s^{-1}$.

This equation holds up to the time of the collision, i.e., for $t < t_c$. Since for the rightmost point of oscillation, $\cos(\omega_1 t_c + \phi_1) = 1$, we find that $t_c = (\pi/2)/\omega_1$.

March 2010

[redacted]

Qual Solution

UC Classical Mechanics

$$m\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - \left(1-\frac{r_s}{r}\right)\left(mc^2 + \frac{\ell^2}{mr^2}\right)$$

(a) Effective potential write $E = T + V$.

$$\frac{E^2}{mc^2} = m\left(\frac{dr}{d\tau}\right)^2 + V_{eff} \quad n$$

$$V_{eff} = \left(1-\frac{r_s}{r}\right)\left(mc^2 + \frac{\ell^2}{mr^2}\right)$$

$$V_{eff} = mc^2 - \frac{r_s}{r}mc^2 + \frac{\ell^2}{mr^2} - \frac{r_s\ell^2}{mr^3}$$

(b) Circular orbits have $dr/d\tau = 0$

$$\frac{dV_{eff}}{dr} = \frac{r_s mc^2}{r^2} - \frac{2\ell^2}{mr^3} + \frac{3r_s\ell^2}{mr^4} = 0$$

or $r^2 r_s mc^2 - \frac{2\ell^2 r}{m} + \frac{3r_s\ell^2}{m} = 0$

$$r_\pm = \frac{\frac{2\ell^2}{m} \pm \sqrt{\frac{4\ell^4}{m^2} - 12r_s^2\ell^2c^2}}{2r_s mc^2}$$

$$\boxed{r_\pm = \frac{\ell^2}{r_s m^2c^2} \pm \sqrt{\frac{\ell^4}{r_s^2m^4c^4} - \frac{3\ell^2}{m^2c^2}}}$$

$$\boxed{r_\pm = \frac{\ell^2}{r_s m^2c^2}\left(1 \pm \sqrt{1 - \frac{3r_s^2m^2c^2}{\ell^2}}\right)}$$

(c) r_+ is the larger orbit, so smallest stable orbit is smallest value of r_+. This occurs when $\sqrt{\ } = 0$

or $1 = \frac{3r_s^2m^2c^2}{\ell^2}$ or $\boxed{\ell^2 = 3r_s^2m^2c^2}$ or $\boxed{\ell = \sqrt{3}\, r_s mc}$

or $\boxed{r_+ = \frac{\ell^2}{r_s m^2c^2} = \frac{3r_s^2m^2c^2}{r_s m^2c^2} = 3r_s}$

Collision is perfectly inelastic, so the two blocks stick together. The simple harmonic motion after the collision is described by

$$x(t) = A\cos(\omega t + \phi) \tag{2}$$

for $t > t_c$, where $\omega = \sqrt{k/(m_1 + m_2)} = 3.39s^{-1}$ is the angular frequency when both blocks oscillate on the spring. The frequency, as asked in the problem, is

$$f = \omega/(2\pi) = 0.540Hz.$$

It follows from Eq.(2) that

$$v(t) = -\omega A\sin(\omega t + \phi) \tag{3}$$

The amplitude A and phase constant ϕ of the resulting motion can be determined from Eqs.(2) and (3) evaluated just after the collision, essentially at t_c, if we assume that the collision takes place almost instantaneously. Conservation of momentum during the collision can then be applied.

Just after the collision, $x(t_c) = 10$ cm (given) and $v(t_c) = (m_1v_1+m_2v_2)/(m_1+m_2)$, where just before the collision, $v_1 = 0$ (given m_1 at rightmost point of its original motion) and v_2 = -1.7 m/s (also given). Numerically,

$$v(t_c) = (-1.7m/s)0.8kg/(0.8kg + 1.2kg) = -68cm/s.$$

Solving Eqs.(2) and (3) for A (using $\sin^2 x + \cos^2 x = 1$), we find

$$A = \sqrt{x(t_c)^2 + [-v(t_c)/\omega]^2} = \sqrt{(10cm)^2 + (68cm/3.39)^2} = 22.4cm$$

Solving for ϕ (using $\sin x/\cos x = \tan x$), we find

$$\phi = \tan^{-1}[-v(t_c)/(\omega x(t_c))] - \omega t_c,$$

$$\phi = \tan^{-1}[68/(3.39 \times 10)] - 69.7^\circ = -6.20^\circ = -0.108radians.$$

#3 : UNDERGRADUATE E&M

`PROBLEM:` 1) An inductor and capacitor are in series forming an L-C circuit consisting of a capacitor with $C = 3.4 \times 10^{-6}$ F and an inductor with $L = 0.080H$. At $t = 0$ the capacitor has charge $5.4 \times 10^{-6}C$ and the current in the inductor is zero. The circuit oscillates at its resonant frequency.

(a) What is the resonant frequency in Hz?

(b) How long after t = 0 will the current in the circuit be maximum?

(c) What is the maximum amount of energy that the solenoid will store after closing the switch?

(d) What will be this maximum current?

`SOLUTION:`

(a) What is the resonant frequency in Hz?

From Kirchoff's voltage law we have for a 2-element series circuit:

$LdI/dt + Q/C = 0$ implying $Ld^2I/dt^2 + 1/CI = 0$. With a sinusoidal, oscillatory current of the form $I = I_o cos\omega t$, we find $\omega^2 LI = I/C$.

So, $\omega = 1/\sqrt{LC}$ and since $\nu = \omega/2\pi$ we get $\nu = (2\pi\sqrt{LC})^{-1} = 304$ Hz.

(b) The current in the circuit be maximum one-quarter of a period after it is zero. Since $\nu = 304Hz, T = 0.0033$s, and one-quarter period is 8.2×10^{-4}s.

(c) What is the maximum amount of energy that the solenoid will store at any point? Assuming no dissipation, the energy is exchanged between capacitor and inductor and $1/2CV^2 = 1/2LI^2$.

V can be obtained from $Q/C = V = 5.4 \times 10^{-6}C/3.4 \times 10^{-6} = 1.6$V. $E = 4.4 \times 10^{-6}$ J

(d) What will be the maximum current flowing in the circuit?

From above, energy balance we have $I = V\sqrt{C/L} = V \cdot 0.007\Omega^{-1}$. With $V = 1.6$V we obtain $I = 1.0 \times 10^{-2}A$.

#4 : UNDERGRADUATE E&M

`PROBLEM:` A metallic cylinder rotates with angular velocity ω about its axis. The cylinder is in a homogeneous magnetic field B parallel to its axis. Find the charge distribution inside the cylinder. Consider both magnetic field directions.

`SOLUTION:`

For the free electron moving on a circular track with the cylinder, the equation of motion is

$$eE \pm er\omega B = mr\omega^2,$$

where e is the electron charge, m is its mass, r is the distance from the axis of the rotation, E is the electrostatic field produced in the cylinder by the charge distribution. The $\pm$ sign shows that the Lorents force can be directed inwards or outwards, depending on the magnetic field direction. From this we obtain

$$E = \left(\tfrac{m\omega^2}{e} \pm \omega B\right) r = Kr.$$

Consider a thin cylindrical shell at distance r from the axis. Electric flux $2\pi r L E(r)$ enters the shell and a flux $2\pi(r+\delta r)LE(r+\delta r)$ exits it, where L is the cylinder length. According to Gauss's law

$$2\pi(r+\delta r)LE(r+\delta r) - 2\pi r L E(r) = \tfrac{1}{\epsilon_0}\rho(r) 2\pi r L \delta r,$$

where ρ is the electric charge density in the cylinder. Entering $E(r) = Kr$ we obtain

$$2\pi(r+\delta r)LK(r+\delta r) - 2\pi r L K r = \tfrac{1}{\epsilon_0}\rho(r) 2\pi r L \delta r,$$

From this we obtain

$$\rho(r) = 2K\epsilon_0 = 2\left(\tfrac{m\omega^2}{e} \pm \omega B\right)\epsilon_0.$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a particle of mass m and charge Q in a simple 1-D harmonic oscillator potential with a constant electric field E in the x direction.
`(a)` Show that there is no first order perturbation theory correction to the energy of the simple harmonic oscillator states, $(n+1/2)\hbar\omega$.
`(b)` Using second order perturbation theory show that the energy shift is $-Q^2E^2/(2m\omega^2)$, independent of n.

`SOLUTION:` The solution is hand written.

#6 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A particle of mass m moves in d-dimensional space with position vector $\vec{r} = (x_1, x_2, ..., x_d)$. Its potential energy is

$$V(r) = \frac{1}{2}kr^2 \tag{4}$$

with

$$r^2 = x_1^2 + x_2^2 + ... + x_d^2 \tag{5}$$

and k a constant.

(a) As in the Bohr atom, assume the particle moves in a circular orbit and find the radii and energies of the allowed orbits assuming the angular momentum is quantized according to $L = n\hbar$. Give your answers as function of m and $\omega = \sqrt{k/m}$

(b) Give the exact energies of the particle resulting from solution of the Schrodinger equation and compare with the result in (a).

(c) Give the ground state wave function of the particle.

(d) Find the most probable value of r for the particle in the ground state. Determine for which (if any) value(s) of the space dimensionality d does the most probable value of r coincide with the radius of the lowest Bohr orbit found in (a).

Hint 1: the ground state wavefunction for the one-dimensional harmonic oscillator is

$$\Psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \tag{6}$$

Hint 2: the volume of a d-dimensional sphere of radius R is proportional to R^d.

`SOLUTION:` The solution is hand written.

#7 :UNDERGRADUATE STAT MECH

`PROBLEM:` Consider a 2-level system with energy states ϵ and $\epsilon + \delta$ ($\delta \geq 0$). Compute the partition function and the free energy of the system. Derive an expression for the specific heat $C(T)$. Obtain the low-T and high-T limits of this expression. Make a sketch of your result.

`SOLUTION:`

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Qual Solution

UG Quantum

$V = -\int \vec{E} \cdot d\vec{x} = -Ex$

Perturbation is $V_1 = -QEx$

(a) SHO has eigen states $|n\rangle$

$$\Delta E_n^{(1)} = \langle n | V | n \rangle = -QE \langle n | x | n \rangle$$

write $x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a + a^{\dagger})$ with $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\Delta E_n^{(1)} = -QE\left(\frac{\hbar}{2m\omega}\right)^{1/2} \Big(\underbrace{\langle n | a | n \rangle}_{0} + \underbrace{\langle n | a^{\dagger} | n \rangle}_{0} \Big) = 0$$

No first order correction

(b) 2nd order energy correction

$$\Delta E_n^{(2)} = \sum_{k \neq n} \frac{|\langle n | V | k \rangle|^2}{E_n - E_k}$$

only get non-zero when $k = n+1$ or $k = n-1$

$$\Delta E_n^{(2)} = \frac{Q^2 E^2 \hbar}{2m\omega} \left\{ \frac{|\langle n | (a+a^{\dagger}) | n-1 \rangle|^2}{E_n - E_{n-1}} + \frac{|\langle n | a+a^{\dagger} | n+1 \rangle|^2}{E_n - E_{n+1}} \right\}$$

$$\Delta E_n^{(2)} = \frac{Q^2 E^2 \hbar}{2m\omega} \left(\frac{(\sqrt{n})^2}{\hbar\omega} + \frac{(\sqrt{n+1})^2}{-\hbar\omega} \right)$$

$$\Delta E_n^{(2)} = \hbar \frac{Q^2 E^2}{2m\omega} \left(\frac{n - n - 1}{\hbar\omega} \right) = -\frac{Q^2 E^2}{2m\omega^2}$$

`(a)` Partition function:

$$Z = \sum_j e^{-\beta\epsilon} = e^{-\beta\epsilon} + e^{-\beta(\epsilon+\delta)}$$

where $\beta = 1/k_B T$.

Free energy:

$$F = -k_B T \ln Z$$

$$F = -k_B T \ln(e^{-\beta\epsilon} + e^{-\beta(\epsilon+\delta)})$$

`(b)` To calculate the specific heat, we first need to compute the energy:

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\epsilon e^{-\beta\epsilon} + (\epsilon+\delta)e^{-\beta(\epsilon+\delta)}}{e^{-\beta\epsilon} + e^{-\beta(\epsilon+\delta)}} = \epsilon + \frac{\delta}{1+e^{\delta/k_B T}}$$

The specific heat is then

$$C(T) = \frac{\partial E}{\partial T} = \frac{\delta^2 e^{-\delta/k_B T}}{k_B T^2 (1+e^{-\delta/k_B T})^2}$$

Expressions for the low-T and the high-T limits of the specific heat are shown in the figure.

#8 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` In Debye's theory of solid, a crystal of N atoms is described by a set of $3N$ harmonic oscillators, whose angular frequencies ω are distributed according to $D(\omega) = A\omega^2$ for $\omega < \omega_D$ and $D(\omega) = 0$ for $\omega > \omega_D$.

(i) Find the dependence of the average thermal energy E on the temperature T to the leading order; find also the specific heat $C(T)$ to this order in T.

(ii) Again to leading order in T, find the entropy $S(T)$ and express it in terms of $C(T)$.

`SOLUTION:`

(i) The average thermal energy is given by

$$\begin{aligned} E &= \int d\omega D(\omega) \frac{\hbar\omega}{e^{\hbar\omega/kT}-1} \\ &= A\int_0^{\omega_D} d\omega \frac{\hbar\omega^3}{e^{\hbar\omega/kT}-1} \\ &= A\cdot (kT)^4/\hbar^3 \int_0^{\hbar\omega_D/kT} dx \frac{x^3}{e^x-1} \end{aligned}$$

To leading order in T, $\hbar\omega_D/kT \to \infty$ and the integral becomes a number. Thus $E(T) \propto T^4$ and the specific heat is

$$C(T) = \partial E/\partial T \propto T^3.$$

(ii) The entropy is given by

$$S = \int_0^T \frac{dQ}{T} = \int_0^T \frac{C(T)dT}{T}.$$

Let $C(T) = BT^3$, then

$$S = \int_0^T BT^2 dT = \frac{1}{3}BT^3 = C(T)/3.$$

#9 : UNDERGRADUATE MATH

`PROBLEM:` A point force f is applied normally to the center of a circular plate of radius R. The deflection $h = h(r)$ of the plate satisfies the biharmonic equation

$$D\Delta^2 h = f\delta(\mathbf{r}), \quad \Delta \equiv \tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr} \quad \text{(2D Laplacian)},$$

where D is the bending elastic modulus. Find $h(0)$ assuming that the edge of the plate is clamped: $h(R) = h'(R) = 0$.

Hint: seek the solution in the form of series $h(r) = \sum c_{mn} r^m \ln^n r$ where $m \geq 0$ and $n = 0$ or 1.

`SOLUTION:`

Everywhere except the point $r = 0$ the right-hand side of the biharmonic equation vanishes, which means

$$h(r) = a + br^2 + cr^2 \ln r\,, \quad a, b, c = \text{const}\,.$$

Note that another possible term of the series — $\ln r$ — is not physically allowed because it diverges at $r = 0$. The boundary conditions further restrict $h(r)$ to the form

$$h(r) = c\left[\tfrac{1}{2}(R^2 - r^2) - r^2 \ln \tfrac{R}{r}\right]\,.$$

To determine c, we integrate both sides of the biharmonic equation over the area. Applying Gauss' theorem, we get

$$\tfrac{f}{D} = \int d^2r \Delta^2 h = \int_{r=R} dl \tfrac{d}{dr}\, \Delta h = 2\pi R\, \tfrac{4c}{R} = 8\pi c\,.$$

This gives $c = f/(8\pi D)$, and so

$$h(0) = fR^2/(16\pi D)\,.$$

#10 : UNDERGRADUATE GENERAL

`PROBLEM:` Resonant Cavities and Radiation

An electromagnetic standing wave with mode number $n = 1$ is resonating in a metallic empty cavity. The nodal plane – the plane containing the nodes of the E field – and the closest nodal planes of the B field are separated by ± 0.87 m.

(a) Find the length of the resonating cavity. (b) What is the frequency of the electromagnetic wave and what region of the electromagnetic spectrum does it reside in?

Next, consider a laser beam emanating from a laser cavity which has a wavelength of 633 nm and a power of 0.500 mW spread uniformly over a circular area 1.20 mm in diameter. This beam falls perpendicularly on a perfectly reflecting mirror having twice the diameter of the laser beam and a mass of 1.50 milligrams.

(c) What are the amplitudes of the electric (in Coulombs/meter) and magnetic fields (in Tesla) in this laser beam?

(d) What acceleration does the laser beam give to the mirror?

Finally, consider a perfectly black sphere 18.0 cm in diameter is at a temperature of 215°C.

(e) If all the photons are radiated at the wavelength where the sphere radiates most strongly how many photons would the sphere emit each second?

(Note: $\epsilon_o = 8.9 \times 10^{-12} Coulombs/N/m^2$, $h = 6.626 \times 10^{-34} J \cdot s$, the constant in Wiens displacement law is $0.00290 m \cdot K$, and $c = 2.99 \times 10^8 m/s$, and the Stefan-Boltzmann constant is $5.67 \times 10^{-8} W/m^2 \cdot K^4$.)

`SOLUTION:` (a) The nodal plane of the fundamental n=1 mode resides halfway between the ends of the cavity, since the boundary conditions on the standing wave and transverse electric field require it to vanish at the walls of the cavity (where the B field's nodal planes lie. Therefore, $L = 1.74m$.

(b) $L = \lambda/2$ and $\nu = c/\lambda = 2.99 \times 10^8 m/s/2 \times 1.74m = 8.6 \times 10^7$ Hz, or 86 MHz, which is in the (FM) radio regime of the electromagnetic spectrum.

(c) The rate of energy transport per unit area, the Poynting vector S, is perpendicular to both E and B and in the direction of propagation of the wave. A condition of the wave solution for a plane wave is $B_{max} = E_{max}/c$ so that the average intensity for a plane wave can be written $S = \epsilon_o c \frac{E^2}{2}$ since the average of the square of a sinusoidal function over a whole number of periods is 1/2.

Power/unit area $= 0.0005W/(0.0006^2 m^2 \pi) = 442 \times W/m^2$

$E_{max} = \sqrt{2S/c\epsilon_o}$

$E_{max} = 577N/C, B_{max} = E_{max}/c = 1.92\mu T.$

(d) $S = Pc$ where the pressure $P = F/area = ma/A$. $P = 442Wm^{-2}/2.99 \times 10^8 = 1.47 \times 10^{-6} N/m^2$. $F = P \cdot Area, A = 1.1 \times 10^{-6} m^2$, is the area of the laser spot, implying the acceleration of the paper is $F/mass = 1.6 \times 10^{-12} N/1.5 \times 10^{-6} kg = 1.1 \times 10^{-6} m/s^2$

(e) The Wien law reads: $\lambda_{max} T = 0.00290 m\cdot$K, then λ_{max} converts to frequency via $\nu = c/\lambda$and $Energy = h\nu$.

$$\lambda = 5.94 \times 10^{-6} m, \nu = 50.3 \times 10^{12} Hz, h\nu = 334.2 \times 10^{-22} J.$$

Stefan Boltzmann Law is $S = \sigma_B T^4$ = power/area = Energy/time/area = Number of photons/sec $\times$ energy per photon/area. Temperature must be in Kelvin. Area = $4\pi(D/2)^2$. So, power $P = 325$W and if each photon has the same wavelength $N/s = 9.79 \times 10^{21}$ photons/sec.

#11 :GRADUATE MECHANICS

`PROBLEM:` A particle moves along the curve

$$x = \ell(2\phi + \sin 2\phi), \qquad y = \ell(1 - \cos 2\phi),$$

in a uniform gravitational field in the negative y direction. Find the oscillation period using action angle variables. (Assume that the maximum value of $\phi < \pi/2$).

`SOLUTION:`

This problem is recycled from the Fall 2001 qual. The solution can be found there, on the web.

#12 :GRADUATE MECHANICS

`PROBLEM:` A long straignt grounded wire with radius a carries current I. An electron is emitted from the wire with velocity $v_0 \approx c$ in the (cylindrical) radial direction. Neglecting radiation and image charge effects in the grounded wire, find the maximum distance R that the electron travels from the wire before returning. Assume the motion can be treated with classical mechanics.

`SOLUTION:` The relevant, relativistic lagrangian is

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{q}{c}\vec{v} \cdot \vec{A}$$

$$= -mc^2\sqrt{1 - c^{-2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2} + \frac{2qI}{c^2}\ln(\frac{r_0}{r}).$$

It's independent of t, θ, and z, so energy, angular momentum, and p_z are conserved. These imply that

$$R = R_0 \exp(\frac{\gamma m c^2 v_0}{2qI}).$$

#13 :GRADUATE E&M

`PROBLEM:` A long, hollow cylindrical conducting shell with mass m per unit length, inner radius r_1, outer radius r_2, is placed with its axis of symmetry aligned with a uniform magnetic field B. The shell is set spinning on its axis, on a frictionless bearing, with a rotation frequency $\omega \ll c/r_1$. The magnetic field is then turned off. Find the new rotation frequency ω'.

`SOLUTION:` In the presence of a magnetic field there is an induced (Hall-effect) electric field in the conductor given by $\vec{E} + \frac{1}{c}\vec{v} \times \vec{B} = 0$, which yields $\vec{E} = -\frac{1}{c}\omega r B \hat{r}$. Considering a unit length, the total angular momentum is conserved:

$$p_\theta = I\omega + \int_{r_1}^{r_2} 2\pi r dr \frac{r A_\theta}{c} \rho(r),$$

where $\rho(r)$ is the induced charged density that creates the electric field (including the surface charge density), $I = \frac{1}{2}m(r_2^2 + r_1^2)$ is the moment of inertia of the shell, and $A_\theta = Br/2$. Inside the conductor, $\rho(r)$ is determined via Poisson's equation,

$$4\pi\rho = \frac{1}{r}\frac{\partial}{\partial r} r E_r = -2\omega B/c.$$

On the inner and outer surfaces respectively the surface charge densities are

$$4\pi\sigma_1 = -E_r|_{r=r_1} = \frac{\omega r_1 B}{c}, \qquad 4\pi\sigma_2 = E_r|_{r=r_2} = -\frac{\omega r_2 B}{c}.$$

The conserved angular momentum is then

$$\omega' = \omega \left(1 + \frac{B^2}{4mc^2}(r_2^2 - r_1^2) \right).$$

#14 :GRADUATE E&M

`PROBLEM:` Two halves of a spherical metallic shell of radius R are separated by a small insulating gap. The alternating voltage $V \cos \omega t$ is applied to the top half and $-V \cos \omega t$ to the bottom half. Suppose that $\omega \ll c/R$. Compute the amplitude of the oscillating dipole moment of the system and the time-average power that it radiates.

`SOLUTION:`

This problem is recycled from the 2005 Qual, problem 13. The solution is there, on the web.

#15 :GRADUATE QUANTUM MECHANICS

`PROBLEM:` A spherical well has $V(r \le a) = -V_0$ and $V(r \ge a) = 0$. A (non-relativistic) particle of energy $E > 0$ and mass m is inside, with angular momentum $\ell \neq 0$.

(a) Compute the transition factor T (ratio of flux outside and inside the well) in the WKB approximation. Neglect the angular momentum inside the well, and you need not evaluate the associated integral.

(b) Express the lifetime τ of the particle to be inside the well, in terms of T and the constants of the problem.

`SOLUTION:`

(a)

$$T = \left(\frac{E}{E+V_0}\right)^{1/2} \exp\left(-2\int_a^b dr \sqrt{\frac{\ell(\ell+1)}{r^2} - \frac{2mE}{\hbar^2}}\right),$$

with b given by $\frac{\ell(\ell+1)}{b^2} = 2mE/\hbar^2$. (b) The particle hits the wall at a rate $v/2a$, so

$$\tau^{-1} = T\sqrt{\frac{(E+V_0)}{2ma^2}}.$$

#16 :GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider various numbers of particles, of mass M and spin s, in a 3d central potential $V(\vec{r}) = C|\vec{r}|^2$, where C is a constant.

(a) A single spin s particle is in the first excited state. What is the energy? What is the schematic form of the energy eigenstate? What values of L_z can be measured in this state, where $\vec{L}$ is the orbital angular momentum. What values of $\vec{L}^2$, $\vec{S}^2$, and $\vec{J}^2$ can be measured, where $\vec{J} = \vec{L} + \vec{S}$ is the total angular momentum?

(b) Now consider the case of two identical fermions, with $s = \frac{1}{2}$, which interact with each other only very weakly, with a small repulsive potential of strength $\epsilon \approx 0$. Ignore the potential except to resolve degenerate energy levels. What is the ground state energy? What is the schematic form of the groundstate energy eigenfunction? What values of $\vec{L}^2$, $\vec{S}^2$, and $\vec{J}^2$ can be measured in the groundstate?

(c) Same setup as part (b). Now consider the first excited state. Again, the particles have a slight repulsive potential. What is the first excited state energy? What is the schematic form of its energy eigenfunction? What values of $\vec{L}^2$, $\vec{S}^2$, and $\vec{J}^2$ can be measured in this first excited state?

(d) Same questions as the previous part, but for the case where the two particles have a weakly attractive potential.

`SOLUTION:`

(a) It's a 3d SHO, with $\omega = \sqrt{\frac{2C}{M}}$, so the energy eigenvalues are $(n+\frac{3}{2})\hbar\omega$, and the schematic form of the energy eigenstates for a single particle is $|n_x, n_y, n_z\rangle = |n_x\rangle \times |n_y\rangle \times |n_z\rangle$, with $n = n_x + n_y + n_z$. A first excited state energy is e.g. $|1, 0, 0\rangle$, with energy $\frac{5}{2}\hbar\omega$. There are 3 such states, corresponding to $\ell = 1$, i.e. $L_z = \hbar,\ 0, -\hbar$. The value of $\vec{L}^2$ is $2\hbar^2$, $\vec{S}^2 = \hbar^2 s(s+1)$, and $\vec{J}^2 = \hbar^2 j(j+1)$, with $j = 1+s, s, |s-1|$.

(b) The groundstate eigenfunction is of the form $(|\vec{0}\rangle \times |\vec{0}\rangle) \otimes |S = 0\rangle$, where the first are the spatial parts, and the latter is the spin part, $|S = 0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$, so the entire wavefunction is appropriately antisymmetric under particle interchange. Since the particles interact only weakly, the energy is approximately the decoupled sum, $E = 3\hbar\omega$. The values of $\vec{L}^2$, $\vec{S}^2$, and $\vec{J}^2$ are all zero.

(c) Thanks to the repulsion, there is lower energy if the spatial part is antisymmetric under particle interchange, e.g. $\frac{1}{\sqrt{2}}(|1,0,0\rangle|\vec{0}\rangle - |\vec{0}\rangle|1,0,0\rangle)$. Consequently, the spin part must be symmetric, i.e. $s = 1$. The energy is $\frac{7}{2}\hbar\omega + \mathcal{O}(\epsilon)$. The total orbital angular momentum is $\ell = 1$, so $\vec{L}^2 = 2\hbar^2$. Since $s = 1$, $\vec{S}^2 = 2\hbar^2$. The total $\vec{J}^2 = \hbar^2 j(j+1)$, with $j = 2,\ 1,\ 0$.

(d) If it's weakly attractive, now the spatial part should be symmetric, e.g. $\frac{1}{\sqrt{2}}(|1,0,0\rangle|0,0,0\rangle + |0,0,0\rangle|1,0,0\rangle)$. Consequently, the spin part must be antisymmetric, i.e. $s = 0$. The energy is $\frac{7}{2}\hbar\omega - \mathcal{O}(\epsilon)$. The total orbital angular momentum is $\ell = 1$, so $\vec{L}^2 = 2\hbar^2$. Since $s = 0$, $\vec{S}^2 = 0$. The total $\vec{J}^2 = \hbar^2 j(j+1)$, with $j = 1$.

#17 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` An antiferromagnet consists of two different types of atoms, referred to as A and B, arranged in a checkerboard pattern on a square lattice of N total sites. The magnetization direction is perpendicular to the plane of the lattice, and both types of atoms have magnetic moments μ. There is an antiferromagnetic interaction of strength J between nearest neighbors (which are always A-B pairs). The magnetic moments also interact with a constant external magnetic field of strength H.

Introducing the spin variables $\sigma_A(\mathbf{x})$ and $\sigma_B(\mathbf{y})$, which take on values ± 1, to describe the magnetization state of the A and B atoms respectively, and with the lattice vectors $\mathbf{x}$ and $\mathbf{y}$ ranging over the A and B sites respectively, we can write down the following energy for a configuration of spins,

$$E = J \sum_{NN} \sigma_A(\mathbf{x})\sigma_B(\mathbf{y}) - \mu H \left(\sum_{\mathbf{x}} \sigma_A + \sum_{\mathbf{y}} \sigma_B \right),$$

with 'NN' indicating sum over nearest neighbors.

(i) Using the mean-field approximation, derive equation(s) describing the temperature dependence of the average spin for the A and B atoms, $\overline{\sigma}_A$ and $\overline{\sigma}_B$, respectively.

(ii) For $H = 0$, show that spontaneous magnetizations for the two types of atoms arise for temperature below a critical temperature T_c, i.e., solution(s) with $\overline{\sigma}_A \neq 0$ and $\overline{\sigma}_B \neq 0$ exist for $T < T_c$. It is sufficient to show this graphically. Express T_c in terms of the parameters of the system. What is the total magnetization $M \equiv N\mu\left(\overline{\sigma}_A + \overline{\sigma}_B\right)/2$ for $T < T_c$?

(iii) At $T > T_c$ and for small external field H, work out the dependence of $\overline{\sigma}_A$ and $\overline{\sigma}_B$ on H and T. Find an expression for the magnetic susceptibility $\chi \equiv \lim_{H\to 0} \partial M/\partial H$ for $T > T_c$. How does χ behave as T approaches T_c from above?

`SOLUTION:`

(i) In the mean-field approximation, the average value of the spin A is calculated by replacing the spin values of its four nearest neighbors by their (yet unknown) average values. Thus,

$$\overline{\sigma}_A = \frac{\sum_{\sigma_A=\pm 1} \sigma_A \, e^{-4\beta J \overline{\sigma}_B \sigma_A + \beta\mu H \sigma_A}}{\sum_{\sigma_A=\pm 1} e^{-4\beta J \overline{\sigma}_B \sigma_A + \beta\mu H \sigma_A}}.$$

Similarly,

$$\overline{\sigma}_B = \frac{\sum_{\sigma_B=\pm 1} \sigma_B \, e^{-4\beta J \overline{\sigma}_A \sigma_B + \beta\mu H \sigma_B}}{\sum_{\sigma_B=\pm 1} e^{-4\beta J \overline{\sigma}_A \sigma_B + \beta\mu H \sigma_B}}.$$

These two equations can be rewritten as

$$\begin{aligned} \overline{\sigma}_A &= \tanh\left[\beta\left(\mu H - 4J\overline{\sigma}_B\right)\right], && (7) \\ \overline{\sigma}_B &= \tanh\left[\beta\left(\mu H - 4J\overline{\sigma}_A\right)\right]. && (8) \end{aligned}$$

(ii) For $H = 0$, it is clear that if $\overline{\sigma}_A > 0$, then $\overline{\sigma}_B < 0$ and vice versa. Assuming the symmetry $\overline{\sigma}_A = -\overline{\sigma}_B$, then

$$\overline{\sigma}_A = \tanh\left(4\beta J \overline{\sigma}_A\right), \qquad (9)$$

and $\overline{\sigma}_B$ satisfies the same equation.

As Eq. (**??**) admits multiple solutions for $4\beta J = 4J/(kT) < 1$, the critical temperature is $T_c = 4J/k$.

Since $\overline{\sigma}_A = -\overline{\sigma}_B$, the total magnetization is always $M = 0$ in the absence of an external magnetic field.

(iii) In the presence of a small external field H and for $T > T_c$, we can expand Eqs. (**??**) and (**??**) to obtain

$$\begin{aligned} \overline{\sigma}_A &\approx \beta(\mu H - 4J\overline{\sigma}_B), && (10) \\ \overline{\sigma}_B &\approx \beta(\mu H - 4J\overline{\sigma}_A). && (11) \end{aligned}$$

Adding up Eqs. (**??**) and (**??**) yields

$$(\overline{\sigma}_A + \overline{\sigma}_B) \cdot (1 + 4\beta J) \approx 2\beta\mu H,$$

or

$$M \equiv \mu N \frac{\overline{\sigma}_A + \overline{\sigma}_B}{2} \approx \frac{\beta \mu H}{1 + 4\beta J}.$$

Hence the magnetic susceptibility is

$$\chi \equiv \lim_{H \to 0} \frac{\partial M}{\partial H} = \frac{\beta \mu}{1 + 4\beta J} = \frac{\mu}{k \cdot (T + T_c)}$$

for $T > T_c$. Note that χ does *not* diverge as T approaches T_c from above, in contrast to the behavior of ferromagnets.

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a system of non-interacting free spinless fermions of mass m in a two-dimensional box of area A.

(a) Find an expression for the energy per particle at zero temperature, $\bar{\epsilon}$, in terms of m and $n = N/A$, with N the number of spinless fermions in the box.

(b) Assume the system is in contact with a reservoir with which it can exchange particles, and that the chemical potential of the system is independent of temperature. The number of particles then depends on temperature. Find the temperature for which the number of particles is twice the number of particles at T=0. Give your answer as $k_B T = x\bar{\epsilon}$, with $\bar{\epsilon}$ found in (a), k_B Boltzmann's constant, and x a numerical factor accurate to four digits.

`SOLUTION:` Hand written.

#19 : GRADUATE MATH METHODS

`PROBLEM:` Evaluate the integral by contour integration

$$I = \int_0^\infty \frac{dx}{(x + \beta)(x + \gamma)}$$

where β and γ are real, positive numbers.

`SOLUTION:` Hand written. Get by including $\ln z$ (or z^α) in integral and integrate around its branch cut, above and below the positive real axis and around a little circle at zero and a big one at infinity, to get $I = (\ln \beta - \ln \gamma)/(\beta - \gamma)$

$N(T=0) \equiv N_0 = A\int_0^{p_F} \frac{d^2p}{(2\pi\hbar)^2}$

$p_F = 2\hbar\sqrt{\pi N_0/A}$

$\frac{E}{N} = \bar{\epsilon} = \frac{\int_0^{p_F} d^2p \frac{p^2}{2m}}{\int_0^{p_F} d^2p} = \pi\hbar\frac{n_0}{m}$

Or $n_0 = \frac{m\bar{\epsilon}}{\pi\hbar}$

Now in the case where $\mu \neq 0$:

$N(\beta) = A\int_0^{\infty} \frac{d^2p}{(2\pi\hbar)^2}\frac{1}{z^{-1}e^{\beta p^2/2m}+1} = \frac{2\pi A}{(2\pi\hbar)^2}\frac{m}{\beta}\int_0^{\infty}\frac{dx}{z^{-1}e^x+1}$

Where $x = \beta p^2/2m$

The integral can be done analytically:

$\int_0^{\infty}\frac{dx}{z^{-1}e^x+1} = -\ln(e^{-x}+z^{-1})|_0^{\infty} = \ln(1+z)$

So combining all the results:

$\frac{N(\beta)}{A} = n_\beta = \frac{m}{2\pi\hbar^2\beta}\ln(1+z)$

Require: $n_\beta = 2n_0$

$\beta^{-1}\ln(1+e^{\beta\mu}) = 4\bar{\epsilon}$

But what is μ? It's given that μ is temprature independent. At $T=0$ $\mu=\bar{\epsilon}$ so $\mu = \bar{\epsilon}$ for all T.

Now need to solve the equation: $\beta^{-1}\ln(1+e^{\beta\bar{\epsilon}}) = 4\bar{\epsilon}$

Or, $4q = \ln(1+e^q)$ for $q \equiv \beta\bar{\epsilon}$

Rearrange to get: $e^{4q} - e^q = 1$

expanding to 2nd order in q gives the quadratic: $\frac{15}{2}q^2 + 3q - 1 = 0$

with one positive solution: $q = 0.2163$

$k_bT = 4.622\bar{\epsilon}$

PROBLEM:

Using the calculus of residues, evaluate the integral

$$I = \int_0^\infty \frac{dx}{(x+\beta)(x+\gamma)},$$

where β and γ are real, positive numbers.

SOLUTION:

Unfortunately, the given integral CANNOT be evaluated *via* the contour integral

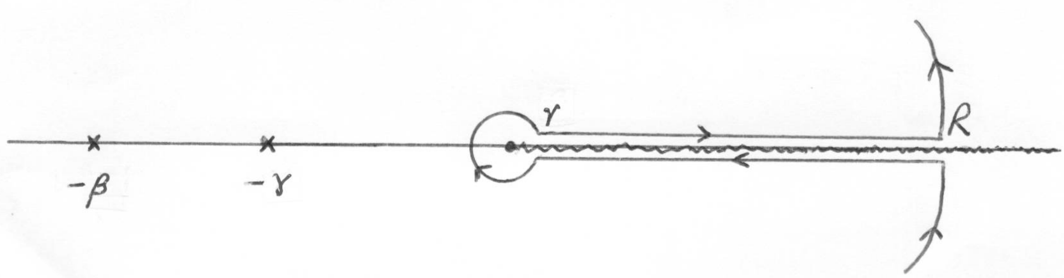
$$I_1 = \int_C \frac{dz}{(z+\beta)(z+\gamma)}$$

for the reason that no contour exists that will allow us to extract I out of I_1.

We may instead employ the contour integral

$$I_2 = \int_C \frac{\ln z \, . \, dz}{(z+\beta)(z+\gamma)},$$

with C as shown in the accompanying figure.



With $z = \rho e^{i\varphi}$, we have

$$I_2 = \int_r^R \frac{\ln \rho \, . \, d\rho}{(\rho+\beta)(\rho+\gamma)} + I_R + \int_R^r \frac{(\ln \rho + i2\pi) \, . \, d\rho}{(\rho+\beta)(\rho+\gamma)} + I_r$$

which, in the limit $r \to 0$ and $R \to \infty$, becomes

$$I_2 = -i2\pi I . \qquad (1)$$

At the same time, by the residue theorem,

$$I_2 = 2\pi i \left[\frac{\ln \beta + i\pi}{-\beta+\gamma} + \frac{\ln \gamma + i\pi}{-\gamma+\beta} \right] = -2\pi i \, \frac{\ln \beta - \ln \gamma}{\beta - \gamma} . \qquad (2)$$

Equating (1) and (2), we obtain the desired result:

$I = (\ln \beta - \ln \gamma)/(\beta - \gamma)$.

Alternatively, we may replace $\ln z$ by z^α (with $-1 < \alpha < 1$) and, using the same contour C, obtain

$$I_3 = \int_0^\infty \frac{x^\alpha \, dx}{(x+\beta)(x+\gamma)} = \frac{\pi}{\sin(\pi\alpha)} \, \frac{\beta^\alpha - \gamma^\alpha}{\beta - \gamma} .$$

Now, letting $\alpha \to 0$, we end up with the same result for I as the one derived above.

#20 : GRADUATE GENERAL

PROBLEM:

Consider a simplified model of lunar tides in which the ocean is assumed to be of a constant depth, there is no land, the Earth does not rotate, and the water achieves the equilibrium in the gravitational field instantaneously. Calculate the difference in the heights of the high and the low tides using the following input parameters: the mass of the Moon is $\mu = 0.012$ of Earth's mass, the radius of the Earth is $R_0 = 6400\,\mathrm{km}$, the distance between the Earth and the Moon is $R = 380,000\,\mathrm{km}$.

SOLUTION:

The perturbation $\Delta h(\mathbf{r})$ of the water height at a given point $\mathbf{r}$ is related to the local perturbation of the potential energy ΔU per unit mass:

$$\Delta h(\mathbf{r}) = -\Delta U(\mathbf{r})/g\,.$$

Denote the Earth's mass by M, then the mass of the Moon is μM. Let us choose a reference frame with the origin at the center of the Earth and the z-axis directed towards the Moon. The position of the Moon is therefore $\mathbf{R} = (0, 0, R)$. The potential energy in question is given by

$$U(\mathbf{r}) = -\tfrac{G\mu M}{|\mathbf{r}-\mathbf{R}|} + za\,.$$

Here the first term is the gravitational potential created by the Moon. The second term, linear in z, has the following origin. In the Earth-Moon center-of-mass frame, the Earth falls onto the Moon with acceleration $a = G\mu M/R^2$. However, in our reference frame the Earth is stationary but there exists an apparent acceleration $-a$, which is equivalent to an effective gravitational field a directed away from the Moon.

Expanding U to the leading nontrivial order in $\mathbf{r}$, we get

$$U(\mathbf{r}) \approx -\tfrac{G\mu M}{R} + \tfrac{G\mu M}{2R^3}(r^2 - 3z^2)\,,$$

which has the expected dipolar form. Using $r = R_0$, $z = R_0 \cos\theta$, where θ is the polar angle, and also $g = GM/R_0^2$, we get

$$\Delta h(\theta) = \tfrac{\mu R_0^4}{2R^3}(3\cos^2\theta - 1)\,, \quad h_{\max} - h_{\min} = \tfrac{3\mu R_0^4}{2R^3} \approx 55\,\mathrm{cm}\,.$$