INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

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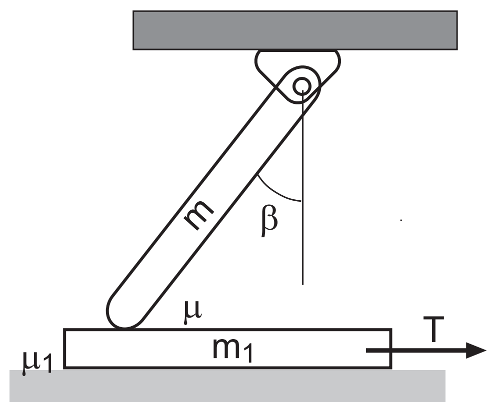
#1 : UNDERGRADUATE Mechanics

An astronaut travels to a nearby star system, a distance of 12 light years away, then immediately turns around and returns; assume an instantaneous turn around. Both legs of the trip are made at a velocity of $0.6c$ so that the trip takes a total travel time of 40 years in the frame of the Earth. The astronaut's spaceship sends out a radio ping once per second which is monitored by the astronaut's twin sister on earth.
(a) What frequency of pings does she hear initially and for how long.
(b) The frequency shifts after some time. What is the second frequency of pings and for how long do they last on Earth.
(c) Assuming that there are approximately 3.156×10^7 seconds per year, how many total pings does she hear?

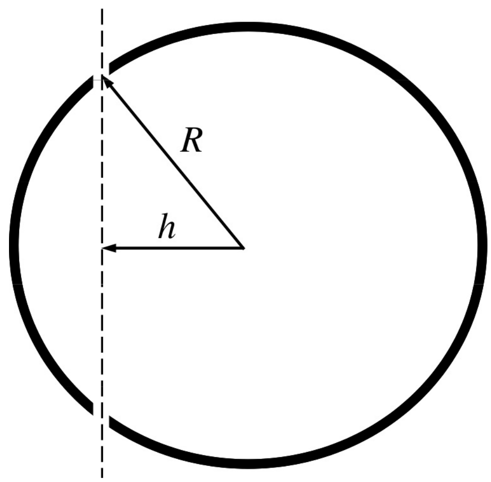
#2 : UNDERGRADUATE Mechanics

`PROBLEM:` A beam with a mass m can freely rotate in a plane about a pivot attached to the ceiling, as depicted in the sketch below. Let β denote the angle between the beam and the vertical. A board with mass m_1 lying on the floor is being pulled to the right at a constant speed with force $\boldsymbol{T}$. The coefficients of friction between the beam and the board and between the board and the floor are μ and μ_1, respectively. What is the value of $\boldsymbol{T}$? Under what conditions does the board get jammed, so that pulling the board becomes impossible?



#3 : UNDERGRADUATE E & M

`PROBLEM:` A hollow sphere of radius R is uniformly charged by an electric charge Q. The sphere is cut in two parts along a plane whose minimum distance from the sphere's center is h (see figure). The cutting does not redistribute the charge. What force is necessary to hold the two parts of the sphere together?



#4 : UNDERGRADUATE EM (II)

`PROBLEM:` Charged particles moving through a region of space filled with a uniform magnetic field travel on curved rather than stright paths.

(a) Given two otherwise identical particles, which is deflected more, the one with more momentum (p) or the one with less momentum? Explain your answer in words.

(b) Given two otherwise identical particles, which is deflected more, the one with more charge (q) or the one with less charge? Explain your answer in words.

(c) For a relativistic nucleus of charge $q = Ze$ (Z is the number of protons, e the charge on the electron), the behavior in a magnetic field depends on $R = pc/q$ where p is the relativistic momentum. Explain why a wide variety of nuclei less massive than iron (with A nucleons) have similar R values when their energy per nucleon are the same.

(d) Consider cosmic rays with $p = 10^{11}$ GeV/c that come from outside our Galaxy. The radius of curvature of their motion in the disk of our Galaxy (where the magnetic fields are strongest) is given by

$$r\,[\mathrm{m}] = \frac{p\,[\mathrm{GeV/c}]}{0.3\,B\,[\mathrm{T}]}\ ,$$

Here the square brackets denote the units for the various quantities in the equation. $B = 3 \times 10^{-10}$ T is the interstellar magnetic field in the disk. The disk has a thickness of $h = 10^{19}$ m (*i.e.* they travel at least that distance to reach the sun that is in the middle of the disk). Do we expect to identify the sources that emitted these particles? Explain your answer.

(e) Over what range of cosmic ray energy do we obtain useful information on the direction to the sources?

#5 : UNDERGRADUATE StatMech

Find the mean energy density u, particle density n, and entropy density s for black-body radiation at temperature T.

1. Write the answers in terms of dimensionless integrals.
2. Evaluate the integrals using a series expansion in terms of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

#6 : UNDERGRADUATE StatMech

1.5 mol of H_2 considered as an ideal diatomic gas with 5 degrees of freedom per molecule is initially at a temperature of 300 K and has a volume of 15 liters. It undergoes an isothermal expansion to a volume of 30 liters and then an adiabatic contraction to the initial volume. By what factor does the number of gas molecules with the x-component of velocity between 200 m/s and 200.1 m/s change?

Various constants are (in SI units): Avogadro's number is $N_{\rm A} = 6.03 \times 10^{23}$; Boltzmann's constant $k_{\rm B} = 1.38 \times 10^{-23}$; mass of a proton $m_{\rm p} = 1.67 \times 10^{-27}$; molar mass of H_2 is 0.002 kg; universal gas constant 8.31 J/mol K; charge of electron (in magnitude) $e = 1.6 \times 10^{-19}$.

You may use the forumla

$$\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$$

#7 : UNDERGRADUATE QM

PROBLEM:

Harmonic oscillator coupled to qbit, with a hidden symmetry

Consider a quantum system consisting of a harmonic oscillator with a spin-$\frac{1}{2}$ degree of freedom. A basis for the Hilbert space is made up states of the form

$$\Big\{ |\, n\,, \uparrow\rangle \quad , \quad |\, n\,, \downarrow\rangle \quad , \quad n = 0, 1, 2, \ldots \Big\} \; .$$

where n is the eigenvalue of $a^\dagger a$, where a and $a^\dagger$ are the ladder operators. Acting on the spin variable in this basis, the Pauli matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

The Hamiltonian is

$$\hat{H}_0 \equiv \hbar\omega\, \sigma^z + \hbar\omega \left(a^\dagger a + \tfrac{1}{2} \right) \; ,$$

where ω is a parameter[1].

1) Describe the spectrum of $\hat{H}_0$, beginning with the state of lowest energy, the ground state $|\,\mathrm{G}\,\rangle$. More precisely, find the five lowest energy eigenstates along with their eigenvalues.

To understand this spectrum better, consider the operator

$$\hat{Q} \equiv \sigma^- \otimes a^\dagger \; ,$$

where $\sigma^\pm \equiv \frac{1}{2}\,(\sigma^x \pm i\,\sigma^y)$.

2) What is $\big[\sigma^\pm\,,\, \sigma^z\big]$?

3) What is $\big[a^\dagger a\,,\, a^\dagger\big]$?

4) What is $\hat{Q}\,|\,\mathrm{G}\,\rangle$? What is $\hat{Q}^\dagger\,|\,\mathrm{G}\,\rangle$?

[1] A more precise expression for the Hamiltonian is

$$\hat{H}_0 \equiv \hbar\omega\, \sigma^z \otimes \mathbb{1} + \hbar\omega \mathbb{1} \otimes \left(a^\dagger a + \tfrac{1}{2} \right) .$$

5) Show that $\left[\hat{H}_0\,,\,\hat{Q}\right] = 0$.

This result shows that $\hat{Q}$ generates a symmetry of $\hat{H}_0$. In the next parts of this problem, we try to understand the nature of this symmetry.

6) Show that $\hat{Q}^2 = 0$.

7) Show that

$$\left(\hat{Q} + \hat{Q}^\dagger\right)^2 = \frac{1}{\hbar\omega}\,\hat{H}_0\ .$$

[Cultural remarks: This relationship, which is called a *supersymmetry algebra*, explains the value of the ground state energy given the result of part (4). The generator $\hat{Q}$ is called the *supercharge*. The symmetry it generates is called supersymmetry and is weird because the generator squares to zero. It is therefore sometimes called a *fermionic* symmetry.]

#8 : UNDERGRADUATE QM (II)

PROBLEM:

A crude model of the H_2^+ ion.

Consider a system consisting of two (heavy) nuclei each with charge $+e$ and one (light) electron, with charge $-e$. We wish to understand how these objects might form a H_2^+ ion, using an approximation (Born-Oppenheimer), where we treat the nuclei classically. In particular, we would like to develop a model for $V(R)$, the potential for the separation of the two nuclei.

In the absence of the electron, the two nuclei would experience Coulomb repulsion: $V_\mathrm{C}(R) = +e^2/R$.

Let $E_B < 0$ denote the binding energy of the electron in the Hydrogen atom in its groundstate. Let $\Delta(R)$ denote the amplitude for an electron to tunnel from one nucleus to the other, when the nuclei are at fixed separation R; that is, the electron Hamiltonian contains a term of the form

$$\hat{H}_{\mathrm{tunnel}} = -\Delta(R)\left(|\,2\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,2\,|\right) ,$$

where $|\,1\,\rangle$ and $|\,2\,\rangle$ denote the groundstates of the electron near each of the two nuclei.

1) Treating R as fixed, and ignoring excited hydrogen states, find the groundstate energy of the electron, $E_0(R)$, in terms of Δ, E_B.

2) Supposing that $\Delta(R) = AR^{-1/2}$ for some constant A (not actually a very good model), find the size of the H_2^+ ion predicted by this model.

In this final part of the problem, we make a better estimate for $\Delta(R)$. Suppose the electron moves in the Coulomb potential produced by the two nuclei at $x = 0$ and $x = R$. Treat the problem as one-dimensional.

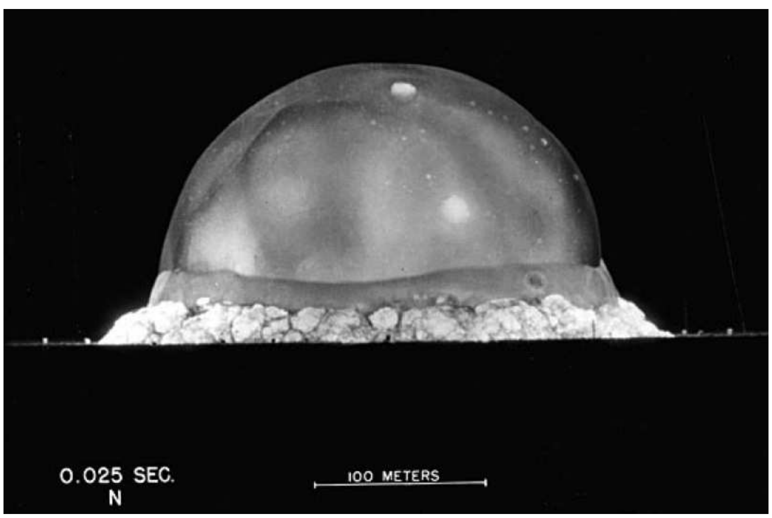
3) Using WKB, find an approximate expression for the amplitude for an electron with energy $E_B \equiv -e^2/a < 0$ to tunnel from one nucleus to the other. *Do not do the integral*, but estimate its leading dependence on R in the limit $a \ll R$.

#9 : UNDERGRADUATE General Physics

One of the great physicists of the 20th century, G. I. Taylor, is known for estimating the energy released by the explosion of an atomic bomb from a series of timelapse pictures from the Trinity test in New Mexico in 1945 (see below for the photograph). Taylor assumed that the explosion started at a point and propagated radially outward as a spherical shock wave.

A. Taylor assumed, $R = f(\rho, E, t)$, where R is the radius of the fireball, $\rho =$ the density of surrounding air, $t =$ time since the explosion, $E =$ energy released. Find the functional form of f. *Hint: use dimensional analysis.*

B. One ton of TNT releases of approximately 4.2×10^9 J of energy. Using the density of air, $\rho = 1.2\,\mathrm{kg/m^3}$, the result from B, and the picture below, estimate how many equivalent tons of TNT energy was released from the explosion at the Trinity test. Assume f involves no dimensionful physical constants.



#10 : UNDERGRADUATE Mathematical Physics

Find the general solution to the differential equation

$$\frac{d^2x}{dt^2} + \omega_0^2\, x = A\,\delta(t - t_0) \ ,$$

for all $t \geq 0$. You should assume $t_0 > 0$, and that the initial conditions $x(0)$ and $\dot{x}(0)$ are unspecified (and hence must appear in your answer).

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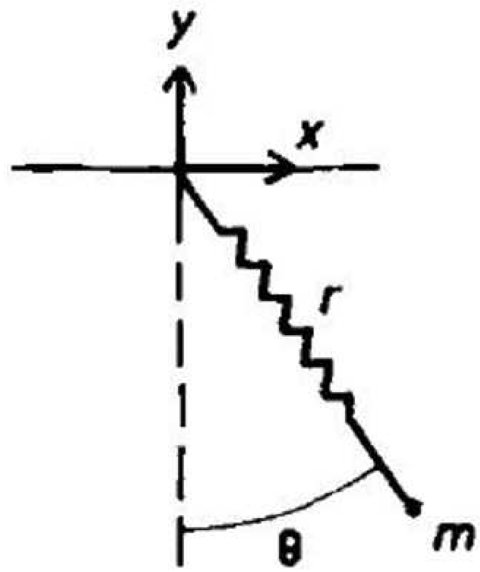
#11 : GRADUATE MECHANICS

`PROBLEM:` A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is l. Assume that the motion is confined to a vertical plane. Derive the equations of motion. Solve the equations of motion in the approximation of small angular and radial displacements from equilibrium.



#12 : GRADUATE MECHANICS

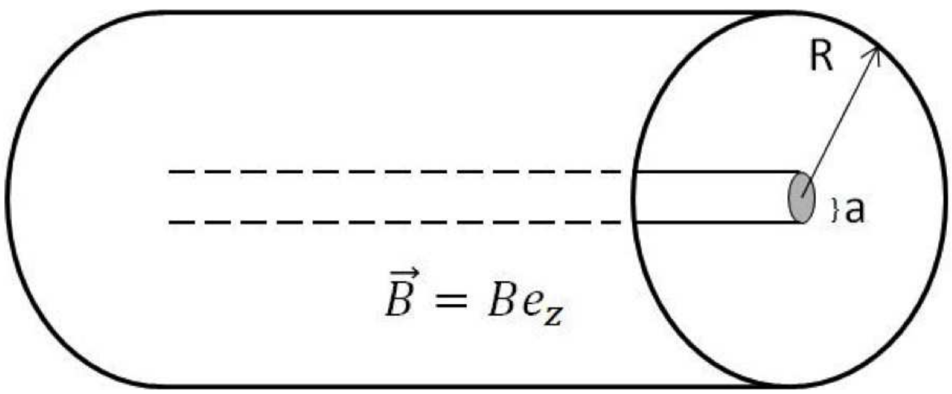
`PROBLEM:` A non-relativistic electron of mass m, charge $-e$ in a cylindrical magnetron moves between a wire of radius a at a negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius R at zero potential. There is a uniform constant magnetic field B parallel to the axis. Use cylindrical coordinates r, θ, z. The electric and magnetic vector potentials can be written as

$$\phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \quad , \quad \boldsymbol{A} = \tfrac{1}{2} B r \, \hat{\mathbf{e}}_\theta \ .$$

Here, $\hat{\mathbf{e}}_\theta$ is a unit vector in the direction of increasing azimuthal angle θ.

(a) Write the Lagrangian and Hamiltonian functions.

(b) Show that there are three constants of the motion. Write them down, and discuss the kinds of motion which can occur.



#13 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Consider two long concentric cylindrical shells, of radii $r = a$ and $r = b$ (with $b > a$ and we use cylindrical coordinates (r, θ, z), with r the distance perpendicular to the z-axis). Both cylinders are centered along the z axis. Each cylinder is wrapped with a wire (winding in the $\hat{\mathbf{e}}_\theta$ direction) with N turns of the wire per unit length. The wire wrapping the outer cylinder (radius b) carries current $I(t)$, and that wrapping the inner cylinder (radius a) carries current $-I(t)$. (Imagine that the wire connects from the outer to the inner cylinder at $z = +\infty$, with a current source at $z = -\infty$.) The current $I(t)$ is slowly varying in time; so you can work to leading non-trivial order in an expansion in terms of time-derivatives of $I(t)$.

(a) Find $\boldsymbol{B}$ (to order $(d/dt)^0$) and $\boldsymbol{E}$ (to order $(d/dt)^1$) everywhere.

(b) Find the magnetic field energy per unit length, and the associated self-inductance per unit length of the system.

(c) Find the energy flux through the boundaries of the region between the shells, and verify energy conservation.

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Consider a cylindrically symmetric setup, with two conducting cylindrical shells and long wire along the $\hat{\mathbf{e}}_z$ symmetry axis of the cylinders. The wire carries current I. One conducting cylindrical shell, at $r = a$ and of length L, carries charge Q. The other conducting cylindrical shell, at $r = b$ (with $b > a$), also of length L, carries charge $-Q$. These charges are uniformly distributed, and treat $L \gg b - a$, so edge effects can be neglected.

(a) Find the total electromagnetic momentum $\boldsymbol{p}_{\text{field}}$.

(b) Now suppose that the current I in the wire slowly drops to zero. Directly compute the total impulse $\int dt\, \boldsymbol{F}$ delivered to the cylindrical shells due to the induced electric force. Does it agree with momentum conservation?

#15: STATISTICAL MECHANICS

A gas of bosonic particles in $d = 3$ dimensions obeys the single particle dispersion $\varepsilon(k) = \varepsilon_0 \, (ka)^{3/2}$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$.

(b) Find the Bose condensation temperature $T_{\rm c}(n)$, where n is the number density of the bosons. You may express numerical prefactors in terms of dimensionless integrals, but it is useful to recall the Riemann zeta function $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$.

(c) Suppose instead that the particles are spin-$\frac{1}{2}$ fermions with number density n. What is the Fermi energy at $T = 0$?

#16: STATISTICAL MECHANICS

Consider a spin-2 Ising model with Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j} J_{ij}\, S_i\, S_j - H \sum_i S_i$$

where $S_i \in \{-2, -1, 0, 1, 2\}$. The system is on a simple cubic lattice, with nearest neighbor coupling $J_1/k_{\mathrm{B}} = 40\,\mathrm{K}$ and next-nearest neighbor coupling $J_2/k_{\mathrm{B}} = 10\,\mathrm{K}$. Every site has six nearest neighbors and 12 next-nearest neighbors.

(a) Derive the mean field Hamiltonian by writing $\langle S_i \rangle = m + \delta S_i$ and then neglecting terms quadratic in fluctuations.

(b) Find the mean field free energy $F(T, H, N, m)$.

(c) Find the mean field equation for m.

(d) Find the mean field transition temperature T_{c} when $H = 0$.

#17 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the low energy scattering problem in the central potential $V(r)$ by using the partial-wave method. Assume that $V(r)$ is a short range potential with the interaction range d beyond which $V(r) = 0$. The particle energy is E, and the wavevector k is defined as $k = \sqrt{2mE/\hbar^2}$ where m is the mass of the particle.

1) In the low energy limit, *i.e.*, $k \to 0$, the s-wave channel scattering dominates. The scattering wave is approximated by an isotropic outgoing spherical wave as $f_0 \, e^{ikr}/r$, where f_0 is the s-wave scattering amplitude. Prove the relation between f_0 and the s-wave phase shift δ_0,

$$f_0 = \frac{1}{k} \, e^{i\delta_0} \sin \delta_0 \ .$$

Hint: You may use the asymptotic expansion

$$e^{ikz} \simeq \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} \, j_l(kr) \, Y_{l0}(\theta, \phi) \ ,$$

where $j_l(u)$ are the spherical Bessel functions. You only need to extract the s-wave component, for which $j_{l=0}(u) = \frac{\sin u}{u}$.

2) The s-wave scattering is often described by the scattering length defined as follows. Show that the radial wavefunction $R(r)$ in the s-wave channel at $k \to 0$ can be approximate as

$$R(r) \longrightarrow \frac{1}{r} \left(1 - \frac{r}{a_0} \right) \ ,$$

for $d < r \ll 2\pi/k$. Here a_0 is a constant known as the *scattering length*.

Prove that δ_0 satisfies

$$\lim_{k \to 0} k \cot \delta_0(k) = -\frac{1}{a_0} \ .$$

3) Express f_0 and the total cross section $\sigma_{\text{tot}} = 4\pi |f_0|^2$ in the s-wave approximation in terms of a_0 and k.

#18 : GRADUATE QUANTUM MECHANICS

A spin-$\frac{1}{2}$ particle in a magnetic field has Hamiltonian

$$H = -\mu\, \boldsymbol{\sigma} \cdot \boldsymbol{B}$$

1. Write down the Hamiltonian in matrix form in the basis $\Big\{ |\uparrow\rangle\, ,\, |\downarrow\rangle \Big\}$ of states with respect to the spin states along the z-axis.
2. At $t = 0$ the particle initially has spin along the direction $\hat{\boldsymbol{n}}$, which is given by angles (θ_0, ϕ_0) in spherical polar coordinates. Write the wavefunction $|\, \psi(t = 0)\, \rangle$ in the $|\uparrow\rangle$ and $|\downarrow\rangle$ basis.
3. Find the state $|\, \psi(t)\, \rangle$ of the particle at time t after evolution using the Hamiltonian above, assuming the magnetic field is in the z direction.
4. Find the expectation $\langle \boldsymbol{\sigma} \rangle$ of the spin operator in the state $|\, \psi(t)\, \rangle$. Find the polar and azimuthal angles $\theta(t)$ and $\phi(t)$ which describe this vector on the Bloch sphere.

#19 : GRADUATE GENERAL

`PROBLEM:` Evaluate the inverse Fourier transform

$$f(x,y) = \int \frac{d^2q}{(2\pi)^2} \, \frac{2\pi \, e^{-|\boldsymbol{q}|a}}{|\boldsymbol{q}|} \, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \ ,$$

where $a > 0$ and $\boldsymbol{r} = (x,y)$ in Cartesian coordinates. The $\boldsymbol{q}$ integral is over the entire two-dimensional plane $(q_x \, , \, q_y)$.

#20 : GENERAL

`PROBLEM:` F. Dyson described in his 1968 article a hydrogen-bomb-powered spaceship. If each explosion adds w to the velocity of the ship, and the explosions occur at equal time intervals τ, such a ship would move with the average acceleration w/τ. The performance of the ship is restricted by the capacity of shock absorbers to transfer momentum from an impulsively accelerated pusher plate to the smoothly accelerated ship. Let m be the total mass of the ship, fm the mass of the pusher plate, and sm the mass of the shock absorbers. Following Dyson, we assume $f = 1/3$ and $s = 1/50$.

a) Based on momentum conservation, what is the change in velocity of the pusher after each explosion?

b) What is the amount of energy that needs to be absorbed after each explosion?

c) Graphene — the strongest material currently known — can handle elastic energy density up to $8 \times 10^6\,\mathrm{J/m^3}$. What is the maximum admissible velocity increment w that can be achieved using shock absorbers made of graphene?

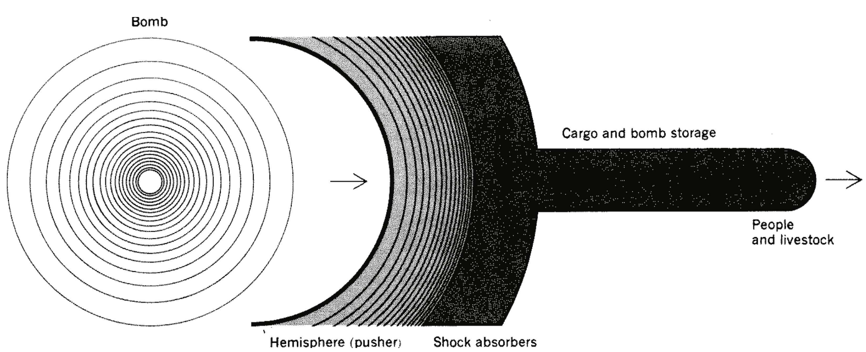


Figure 1: Bomb-propelled spaceship. Debris from the exploding bombs transfer momentum to the shock absorbers and hence to the payload.

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#1 : UNDERGRADUATE Mechanics

An astronaut travels to a nearby star system, a distance of 12 light years away, then immediately turns around and returns; assume an instantaneous turn around. Both legs of the trip are made at a velocity of $0.6c$ so that the trip takes a total travel time of 40 years in the frame of the Earth. The astronaut's spaceship sends out a radio ping once per second which is monitored by the astronaut's twin sister on earth.
(a) What frequency of pings does she hear initially and for how long.
(b) The frequency shifts after some time. What is the second frequency of pings and for how long do they last on Earth.
(c) Assuming that there are approximately 3.156×10^7 seconds per year, how many total pings does she hear?

`SOLUTION:` Let the spaceship system be the unprimed system.

$$\beta = 0.6$$

$$\gamma = \frac{1}{\sqrt{1 - 0.6^2}} = 1.25$$

$$\nu_1' = 1\,\text{Hz} \times \sqrt{\frac{1 - 0.6}{1 + 0.6}} = 0.50\,\text{Hz}$$

$$t_1' = 20\,\text{yr} + 12\,\text{yr} = 32\,\text{yr}$$

$$n_1' = (32\,\text{yr}) \times (3.156 \times 10^7\,\text{sec/yr}) \times (0.50\,\text{Hz}) = 5.05 \times 10^8$$

$$\nu_2' = 1\,\text{Hz} \times \sqrt{\frac{1 + 0.6}{1 - 0.6}} = 2.00\,\text{Hz}$$

$$t_2' = 40\,\text{yr} - 32\,\text{yr} = 8\,\text{yr}$$

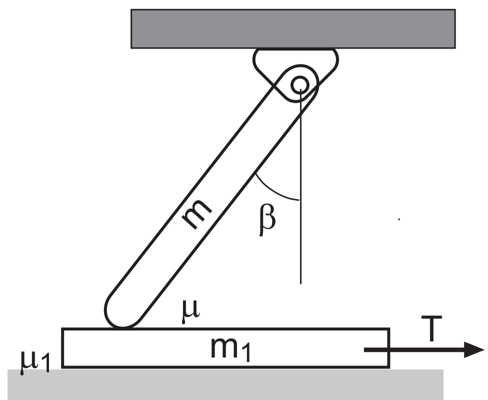
$$n_2' = (8\,\text{yr}) \times (3.156 \times 10^7\,\text{sec/yr}) \times (2.00\,\text{Hz}) = 5.05 \times 10^8$$

$$t_{\text{out}} = t_{\text{back}} = \frac{12\,\text{yr}}{\beta\gamma} = 16.0\,\text{yr}$$

$$n_{\text{out}} = n_{\text{back}} = (16.0\,\text{yr}) \times (3.156 \times 10^7\,\text{sec/yr}) \times (1\,\text{Hz}) = 5.05 \times 10^8 \text{ for a check.}$$

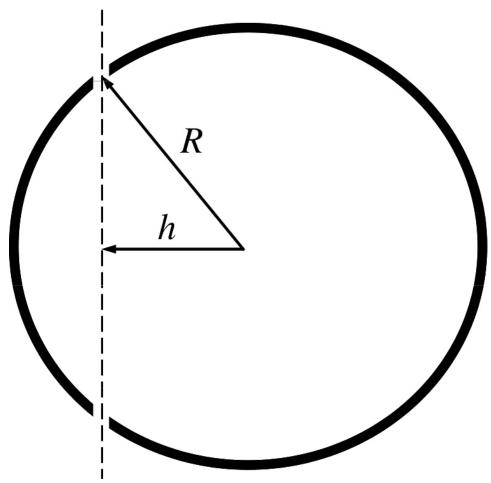
#2 : UNDERGRADUATE Mechanics

`PROBLEM:` A beam with a mass m can freely rotate in a plane about a pivot attached to the ceiling, as depicted in the sketch below. Let β denote the angle between the beam and the vertical. A board with mass m_1 lying on the floor is being pulled to the right at a constant speed with force $\boldsymbol{T}$. The coefficients of friction between the beam and the board and between the board and the floor are μ and μ_1, respectively. What is the value of $\boldsymbol{T}$? Under what conditions does the board get jammed, so that pulling the board becomes impossible?



#3 : UNDERGRADUATE E & M

`PROBLEM:` A hollow sphere of radius R is uniformly charged by an electric charge Q. The sphere is cut in two parts along a plane whose minimum distance from the sphere's center is h (see figure). The cutting does not redistribute the charge. What force is necessary to hold the two parts of the sphere together?



`SOLUTION:`
At the outer surface of a charged sphere the electric field strength is

$$E = \frac{Q}{4\pi\varepsilon_0 R^2}.$$

The electric charge per unite surface area is

$$\sigma = \frac{Q}{4\pi R^2}.$$

This electric field exerts a force $\delta F = \frac{1}{2} E\, \delta Q$ on the charge $\delta Q = \sigma \delta A$ on a surface area δA. The factor 1/2 comes from the fact that we should not include the self-action of δQ on itself. The total electric field is composed of the self part E_{self}, which is produced by the charge δQ, and external part E_{ext}, which is produced by the rest of the charges. The electric field is $E = E_{\text{ext}} + E_{\text{self}}$ at the outer surface and $0 = E_{\text{ext}} - E_{\text{self}}$ at the inner surface that gives $E_{\text{ext}} = \frac{1}{2} E$. The force per unit area of the sphere is therefore

$$\frac{\delta F}{\delta A} = \tfrac{1}{2} E\sigma = \frac{Q^2}{32\pi^2 \varepsilon_0 R^4} = p\ .$$

The component of the force δF along the direction normal to the plane per the projection of the surface area δA on the plane is p. Therefore, the total repulsive force between the parts of the sphere $F = pA_i$, where $A_i = \pi(R^2 - h^2)$ is the cross-section area of the intersection of the plane and sphere. This gives

$$F = \frac{Q^2(R^2 - h^2)}{32\pi\varepsilon_0 R^4} \ .$$

#4 : UNDERGRADUATE EM (II)

`PROBLEM:` Charged particles moving through a region of space filled with a uniform magnetic field travel on curved rather than stright paths.

(a) Given two otherwise identical particles, which is deflected more, the one with more momentum (p) or the one with less momentum? Explain your answer in words.

(b) Given two otherwise identical particles, which is deflected more, the one with more charge (q) or the one with less charge? Explain your answer in words.

(c) For a relativistic nucleus of charge $q = Ze$ (Z is the number of protons, e the charge on the electron), the behavior in a magnetic field depends on $R = pc/q$ where p is the relativistic momentum. Explain why a wide variety of nuclei less massive than iron (with A nucleons) have similar R values when their energy per nucleon are the same.

(d) Consider cosmic rays with $p = 10^{11}$ GeV/c that come from outside our Galaxy. The radius of curvature of their motion in the disk of our Galaxy (where the magnetic fields are strongest) is given by

$$r\,[\mathrm{m}] = \frac{p\,[\mathrm{GeV/c}]}{0.3\,B\,[\mathrm{T}]}\ ,$$

Here the square brackets denote the units for the various quantities in the equation. $B = 3 \times 10^{-10}$ T is the interstellar magnetic field in the disk. The disk has a thickness of $h = 10^{19}$ m (*i.e.* they travel at least that distance to reach the sun that is in the middle of the disk). Do we expect to identify the sources that emitted these particles? Explain your answer.

(e) Over what range of cosmic ray energy do we obtain useful information on the direction to the sources?

`SOLUTION:`

(a) The one with less momentum is deflected more. The Lorentz force $F = qv \times B$ that acts to change the direction of motion of the particle is smaller for smaller velocity v. However, the centrifugal force $mv^2/r = pv/r$ required for the particle to bend with a radius of curvature r is more sensitive to the momentum. When we equate these two, we find the non-relativistic gyroradius $r = mv/(qB)$ that gives the radius of curvature of the particle path. This is smaller (more deflection) for a smaller momentum.

(b) The one with more charge is deflected more since it has a stronger coupling to the magnetic field. The Lorentz force $F = qv \times B$ is proportional to the charge q, and the gyroradius is smaller for larger q.

(c) Similar relativistic kinetic energy (K) per nucleon means similar velocity and relativistic γ factor. In equations, the energy per nucleon $K/A = (\gamma - 1)Am_pc^2/A$ depends on γ alone. Relativistic momentum $p = \gamma A m_p v$, where m_p is the proton rest mass, hence $R = pc/Ze = \gamma A m_p vc/Ze = (A/Z)(\gamma m_p vc/e)$. The R values are similar for different nuclei because A/Z is about 2 for nuclei less massive than iron. In short, the effects of the mass and charge differences in (a) and (b) largely cancel for nuclei.

(d) The radius of curvature $r = 10^{11}/(0.3 \times 3 \times 10^{-10}) = 10^{21}$m. The angle of deflection is then about $r/h = 0.01$ radians or 0.5 degrees. This means that the direction from which the particles are detected is within 0.5 degrees of the direction to the source, sufficiently small that we might identify the source if they are in some other way conspicuous and rare enough to have less than about 1 per square degree on the sky.

(e) At low energies, say $E < 10^{17}$ eV, the radius of curvature is much less than the path length in the disk, and particles spiral around in the disk before arriving from nearly random directions. At the highest high energies the spiraling in the magnetic field of the disk is minimal.

#5 : UNDERGRADUATE StatMech

Find the mean energy density u, particle density n, and entropy density s for black-body radiation at temperature T.

1. Write the answers in terms of dimensionless integrals.
2. Evaluate the integrals using a series expansion in terms of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

SOLUTION:

The density of states is $2\,d^3\!k/(2\pi)^3$, where the initial factor of 2 accounts for both photon polarizations.

The mean value of $f(k)$ for any function of k is

$$\langle f \rangle = 2 \int \frac{d^3\!k}{(2\pi)^3} \, \frac{f(k)}{\exp(\hbar c k/k_{\rm B}T) - 1} = \frac{1}{\pi^2} \int_0^\infty dk \, \frac{k^2 \, f(k)}{\exp(\hbar c k/k_{\rm B}T) - 1}$$

since the photon energy is $E = \hbar\omega = \hbar c k$. Let $\hbar c k \equiv x k_{\rm B} T$,

$$\langle f \rangle = \frac{1}{\pi^2} \left(\frac{k_{\rm B}T}{\hbar c} \right)^3 \int_0^\infty dx \, \frac{x^2 f(x k_{\rm B} T/\hbar c)}{\exp(x) - 1} \ .$$

For a power-law, $f(k) = k^n$, we have

$$\langle k^n \rangle = \frac{1}{\pi^2} \left(\frac{k_{\rm B}T}{\hbar c} \right)^3 \int_0^\infty dx \, \frac{x^{n+2}}{\exp(x) - 1} \ .$$

The integral can be evaluated by expansion,

$$\begin{aligned} I_n = \int_0^\infty dx \, \frac{x^{n+2}}{\exp(x) - 1} &= \int_0^\infty dx \, x^{n+2} \sum_{j=1}^{\infty} e^{-jx} \\ &= (n+2)! \sum_{j=1}^{\infty} \frac{1}{j^{n+3}} = (n+2)! \, \zeta(n+3) \ . \end{aligned}$$

Therefore the average is

$$\langle k^n \rangle = \frac{(n+2)!\,\zeta(n+3)}{\pi^2} \left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^3 .$$

The particle number density is $\langle 1 \rangle$ and the energy density is $\langle \hbar c k \rangle$, so

$$n = \frac{I_0}{\pi^2} \left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^3 = \frac{2\,\zeta(3)}{\pi^2} \left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^3$$

$$u = \frac{I_1}{\pi^2} \left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^4 \hbar c = \frac{6\,\zeta(4)}{\pi^2} \left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^4 \hbar c$$

We have

$$u = a\,T^4$$

$$\mathrm{d}u = 4\,a\,T^3\,\mathrm{d}T = T\,\mathrm{d}s$$

so that

$$\mathrm{d}s = 4\,a\,T^2\,\mathrm{d}T$$

and integrating from $T = 0$ to T with $s = 0$ at $T = 0$ gives

$$s = \tfrac{4}{3} a\,T^3 = \frac{4u}{3T} = \frac{8\,\zeta(4)}{\pi^2} \left(\frac{k_{\mathrm{B}}}{\hbar c}\right)^4 \hbar c\,T^3 .$$

no 3 in the denominator by the Arvoas identity 24/3=8

#6 : UNDERGRADUATE StatMech

1.5 mol of H_2 considered as an ideal diatomic gas with 5 degrees of freedom per molecule is initially at a temperature of 300 K and has a volume of 15 liters. It undergoes an isothermal expansion to a volume of 30 liters and then an adiabatic contraction to the initial volume. By what factor does the number of gas molecules with the x-component of velocity between 200 m/s and 200.1 m/s change?

Various constants are (in SI units): Avogadro's number is $N_{\mathrm{A}} = 6.03 \times 10^{23}$; Boltzmann's constant $k_{\mathrm{B}} = 1.38 \times 10^{-23}$; mass of a proton $m_{\mathrm{p}} = 1.67 \times 10^{-27}$; molar mass of H_2 is 0.002 kg; universal gas constant 8.31 J/mol K; charge of electron (in magnitude) $e = 1.6 \times 10^{-19}$.

You may use the forumla

$$\int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$$

#7 : UNDERGRADUATE QM

PROBLEM:

Harmonic oscillator coupled to qbit, with a hidden symmetry

Consider a quantum system consisting of a harmonic oscillator with a spin-$\frac{1}{2}$ degree of freedom. A basis for the Hilbert space is made up states of the form

$$\Big\{ |\, n\,,\!\uparrow\rangle \quad , \quad |\, n\,,\!\downarrow\rangle \quad , \quad n = 0, 1, 2, \ldots \Big\} \; .$$

where n is the eigenvalue of $a^\dagger a$, where a and $a^\dagger$ are the ladder operators. Acting on the spin variable in this basis, the Pauli matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

The Hamiltonian is

$$\hat{H}_0 \equiv \hbar\omega\, \sigma^z + \hbar\omega \left(a^\dagger a + \tfrac{1}{2} \right) \; ,$$

where ω is a parameter[1].

1) Describe the spectrum of $\hat{H}_0$, beginning with the state of lowest energy, the ground state $|\,\mathrm{G}\,\rangle$. More precisely, find the five lowest energy eigenstates along with their eigenvalues.

To understand this spectrum better, consider the operator

$$\hat{Q} \equiv \sigma^- \otimes a^\dagger \; ,$$

where $\sigma^\pm \equiv \frac{1}{2}\,(\sigma^x \pm i\,\sigma^y)$.

2) What is $\big[\sigma^\pm\,,\,\sigma^z\big]$?

3) What is $\big[a^\dagger a\,,\,a^\dagger\big]$?

4) What is $\hat{Q}\,|\,\mathrm{G}\,\rangle$? What is $\hat{Q}^\dagger\,|\,\mathrm{G}\,\rangle$?

[1] A more precise expression for the Hamiltonian is

$$\hat{H}_0 \equiv \hbar\omega\, \sigma^z \otimes \mathbb{1} + \hbar\omega \mathbb{1} \otimes \left(a^\dagger a + \tfrac{1}{2} \right) .$$

5) Show that $\left[\hat{H}_0\,,\,\hat{Q}\right] = 0$.

This result shows that $\hat{Q}$ generates a symmetry of $\hat{H}_0$. In the next parts of this problem, we try to understand the nature of this symmetry.

6) Show that $\hat{Q}^2 = 0$.

7) Show that

$$\left(\hat{Q} + \hat{Q}^\dagger\right)^2 = \frac{1}{\hbar\omega}\,\hat{H}_0\ .$$

[Cultural remarks: This relationship, which is called a *supersymmetry algebra*, explains the value of the ground state energy given the result of part (4). The generator $\hat{Q}$ is called the *supercharge*. The symmetry it generates is called supersymmetry and is weird because the generator squares to zero. It is therefore sometimes called a *fermionic* symmetry.]

SOLUTION:

1. To put the answer in context, we describe the spectrum of the more general Hamiltonian,

$$\hat{H} \equiv \mu\,\sigma^z + \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)\ .$$

For $\mu > 0$, the ground state is

$$|\,\mathrm{G}\,\rangle = |\!\downarrow_z\rangle \otimes |\,0\,\rangle\ .$$

Its energy is given by ε_0, with

$$\hat{H}\,|\,\mathrm{G}\,\rangle = \left(-\mu + \tfrac{1}{2}\hbar\omega\right)|\,\mathrm{G}\,\rangle = \varepsilon_0\,|\,\mathrm{G}\,\rangle\ .$$

If $\mu < \hbar\omega$, the next excited state is obtained by flipping the spin and leaving the oscillator alone:

$$|\!\uparrow_z\rangle \otimes |\,0\,\rangle\ \text{ has energy } \varepsilon_0 + \mu\ .$$

If $2\mu < \hbar\omega$, the next state up in energy is

$$|\!\downarrow_z\rangle \otimes |\,1\,\rangle\ \text{ has energy } \varepsilon_0 + \hbar\omega\ .$$

In general, the states come in pairs where

$$|\downarrow_z\rangle \otimes |\, n+1\,\rangle \quad \text{and} \quad |\uparrow_z\rangle \otimes |\, n\,\rangle$$

have approximately the same energy. So: when $\mu = \hbar\omega$, the ground-state energy is zero, and all the excited states are doubly degenerate.

2. We have

$$\left[\sigma^{\pm}\,,\,\sigma^{z}\right] = \pm\sigma^{\pm}\ .$$

They are raising and lowering operators for σ^z.

$$\sigma^{+}|\uparrow_z\rangle = 0 \quad , \quad \sigma^{-}|\downarrow_z\rangle = |\uparrow_z\rangle \quad , \quad \sigma^{-}|\uparrow_z\rangle = |\downarrow_z\rangle \quad , \quad \sigma^{-}|\downarrow_z\rangle = 0\,.$$

In this basis, they are the matrices

$$\sigma^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \sigma^{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ .$$

3. One finds $\left[a^\dagger a\,,\,a^\dagger\right] = a^\dagger$.

4. Find

$$\hat{Q}\,|\,\mathrm{G}\,\rangle = \sigma^{-}a^{\dagger}\,|\downarrow_z\rangle \otimes |\,0\,\rangle = 0\ ,$$

because $\sigma^{-}\,|\downarrow_z\rangle = 0$, and

$$\hat{Q}^{\dagger}\,|\,\mathrm{G}\,\rangle = \sigma^{+}a\,|\downarrow_z\rangle \otimes |\,0\,\rangle = 0\ ,$$

because $a\,|\,0\,\rangle = 0$.

5. Using parts (2) and (3), we have

$$\left[\hat{Q}\,,\,\mu\,\sigma^{z}+\hbar\omega\,(a^{\dagger}a+\tfrac{1}{2})\right] = \mu\left[\sigma^{-},\sigma^{z}\right]a^{\dagger}+\hbar\omega\,\sigma^{-}\left[a^{\dagger},a^{\dagger}a\right] = (2\mu-\hbar\omega)\,\hat{Q}\ .$$

6. Find

$$\hat{Q}^2 \propto \left(\sigma^{-}\right)^2 = 0\ .$$

7. We have

$$\left(\hat{Q}+\hat{Q}^{\dagger}\right)^2 = \hat{Q}^2+\hat{Q}\hat{Q}^{\dagger}+\hat{Q}^{\dagger}\hat{Q}+\left(\hat{Q}^{\dagger}\right)^2 = 0+\hat{Q}\hat{Q}^{\dagger}+\hat{Q}^{\dagger}\hat{Q}+0$$

$$= \sigma^{-}a^{\dagger}\sigma^{+}a+\sigma^{+}a\,\sigma^{-}a^{\dagger} = \sigma^{-}\sigma^{+}a^{\dagger}a+\sigma^{+}\sigma^{-}aa^{\dagger}$$

Now we can use $a\,a^\dagger = 1 + a^\dagger a$ to find

$$\left(\hat{Q} + \hat{Q}^\dagger\right)^2 = a^\dagger a \left(\sigma^- \sigma^+ + \sigma^+ \sigma^-\right) + \sigma^+ \sigma^- \ .$$

But now

$$\sigma^+ \sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad \sigma^- \sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

So $\sigma^+ \sigma^- + \sigma^- \sigma^+ = \mathbb{1}$ and $\sigma^+ \sigma^- = \frac{1}{2}\left(\mathbb{1} + \sigma^z\right)$. So:

$$\left(\hat{Q} + \hat{Q}^\dagger\right)^2 = a^\dagger a + \tfrac{1}{2}\left(\mathbb{1} + \sigma^z\right) = \frac{1}{\hbar\omega} \hat{H}_0\Big|_{\mu = \hbar\omega/2} \ .$$

#8 : UNDERGRADUATE QM (II)

PROBLEM:

A crude model of the H_2^+ ion.

Consider a system consisting of two (heavy) nuclei each with charge $+e$ and one (light) electron, with charge $-e$. We wish to understand how these objects might form a H_2^+ ion, using an approximation (Born-Oppenheimer), where we treat the nuclei classically. In particular, we would like to develop a model for $V(R)$, the potential for the separation of the two nuclei.

In the absence of the electron, the two nuclei would experience Coulomb repulsion: $V_{\mathrm{C}}(R) = +e^2/R$.

Let $E_B < 0$ denote the binding energy of the electron in the Hydrogen atom in its groundstate. Let $\Delta(R)$ denote the amplitude for an electron to tunnel from one nucleus to the other, when the nuclei are at fixed separation R; that is, the electron Hamiltonian contains a term of the form

$$\hat{H}_{\mathrm{tunnel}} = -\Delta(R)\left(|\,2\,\rangle\langle\,1\,| + |\,1\,\rangle\langle\,2\,|\right) ,$$

where $|\,1\,\rangle$ and $|\,2\,\rangle$ denote the groundstates of the electron near each of the two nuclei.

1) Treating R as fixed, and ignoring excited hydrogen states, find the groundstate energy of the electron, $E_0(R)$, in terms of Δ, E_B.

2) Supposing that $\Delta(R) = AR^{-1/2}$ for some constant A (not actually a very good model), find the size of the H_2^+ ion predicted by this model.

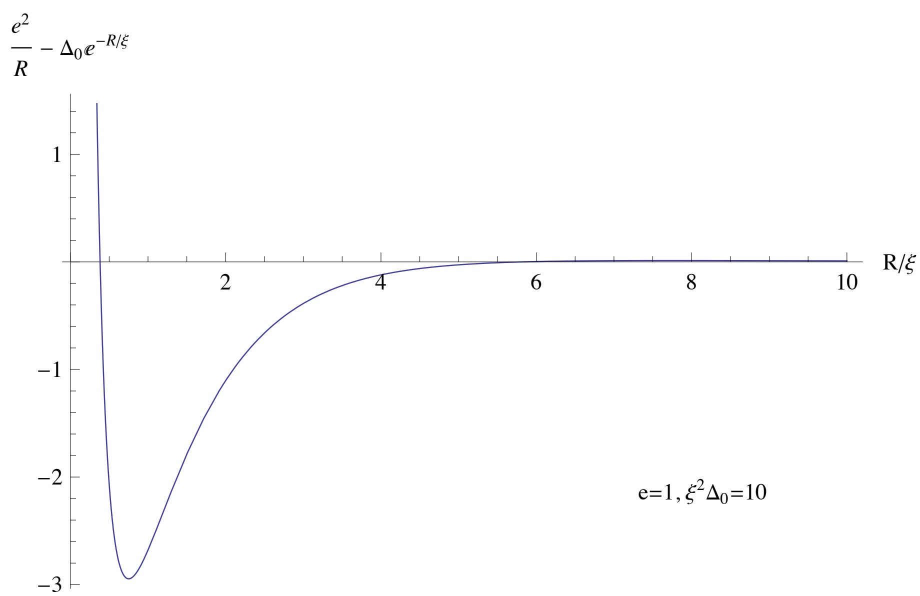
In this final part of the problem, we make a better estimate for $\Delta(R)$. Suppose the electron moves in the Coulomb potential produced by the two nuclei at $x = 0$ and $x = R$. Treat the problem as one-dimensional.

3) Using WKB, find an approximate expression for the amplitude for an electron with energy $E_B \equiv -e^2/a < 0$ to tunnel from one nucleus to the other. *Do not do the integral*, but estimate its leading dependence on R in the limit $a \ll R$.

SOLUTION:

1. The Hamiltonian for the electron is a 2×2 matrix:
$$\hat{H} = E_B \mathbb{1} - \Delta\, \sigma^x \ ,$$
whose eigenvalues are $E_B \pm |\Delta|$, the lower of which is
$$E_0(R) = E_B - |\Delta(R)| \ .$$

2. So the effective potential for R is
$$V(R) = V_{\rm C}(R) + E_0(R) = \frac{e^2}{R} + E_B - |\Delta(R)| \ .$$
With the given form $\Delta(R) = AR^{-1/2}$, we have that $V(R)$ is minimized when
$$0 = \frac{dV}{dR}\bigg|_{R=R_\star} = -\frac{e^2}{R_\star^2} + \frac{A}{2R_\star^{3/2}} \ ,$$
so
$$R_\star = \frac{4e^4}{A^2} \ .$$



3. The tunneling amplitude can be approximated as
$$\Delta(R) \propto \exp\left\{ -i \int\limits_{x_-}^{x_+} dx\, \sqrt{p(x)} \right\} \ ,$$

where $p(x) = \sqrt{2m(E_B - V(x))}$ is the WKB momentum, and $x_\pm$ are the turning points where $p(x_\pm) = 0$. Since $E_B < V(x)$ this is real and small:

$$\Delta(R) \propto \exp\left\{ -\sqrt{2m}\int_{x_-}^{x_+} dx\,\sqrt{\frac{e^2}{a} - \frac{e^2}{x} - \frac{e^2}{x-R}} \right\} \equiv e^{-I}\ .$$

The integral is

$$\begin{aligned} I &= \sqrt{2m}\int_{x_-}^{x_+} dx\,\sqrt{\frac{e^2}{a} - \frac{e^2}{x} - \frac{e^2}{x-R}} \\ &= \sqrt{2me^2}\int_{x_-}^{x_+} dx\,\sqrt{\frac{x(x-R) - a(x-R) - ax}{x(x-R)a}}\ . \end{aligned}$$

The integral can be non-dimensionalized (*i.e.* we can extract the dependence on R) by letting $y = x/R$:

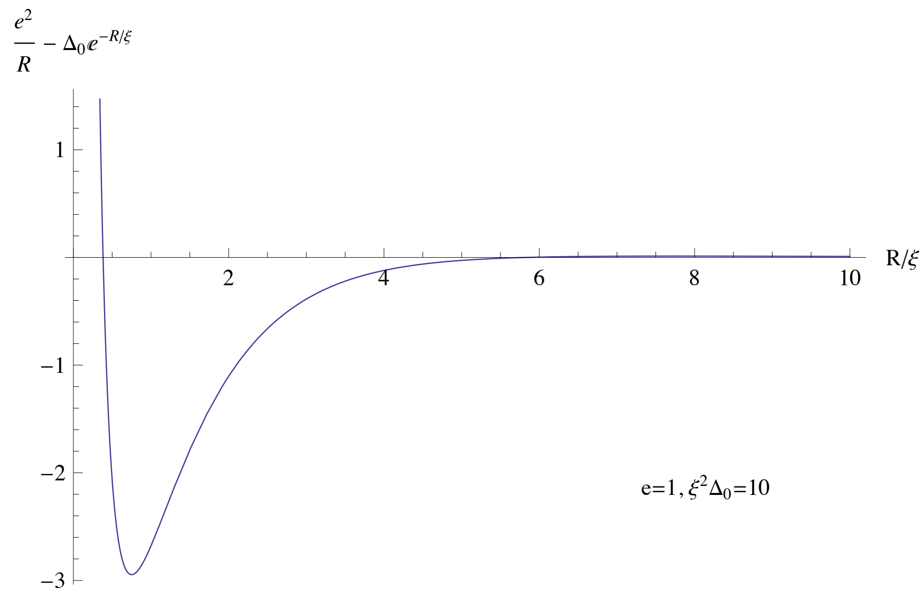
$$\begin{aligned} I &= \sqrt{\frac{2me^2}{a}}\,R\int_{y_-}^{y_+} dy\,\sqrt{\frac{y - y^2 + \frac{a}{R}\,(2y-1)}{y(1-y)}} \\ &= \sqrt{\frac{2me^2}{a}}\,R\int_{y_-}^{y_+} dy\,\sqrt{\frac{(y-y_-)(y_+-y)}{y(1-y)}}\ , \end{aligned}$$

where $y_\pm = x_\pm/R$. Now notice that when $a/R \to 0$, the integrand is 1, and the range of integration is $(y_-, y_+) \to (0,1)$. Therefore, the leading term in the expansion in a/R is $\Delta(R) = \Delta_0\, e^{-R/\xi}$, with $\xi = \sqrt{a/2me^2}$. The R-dependence of the fluctuation contribution, Δ_0, is subleading.

(This last part was not required.) With the potential $E_0(R) = \Delta_0\, e^{-R/\xi} +$ *const*, extremizing $V(R)$ produces a transcendental equation:

$$0 = -\frac{e^2}{R^2} + \frac{\Delta_0}{\xi}e^{-R/\xi} \quad \Rightarrow \quad \frac{e^2R^2}{\xi\Delta_0} = e^{R/\xi}\ .$$

Since the RHS of the second equation varies much more rapidly with R than the LHS, equality is determined mostly by $R \sim \xi$. A solution only exists if $e^2/\xi > \Delta_0$.



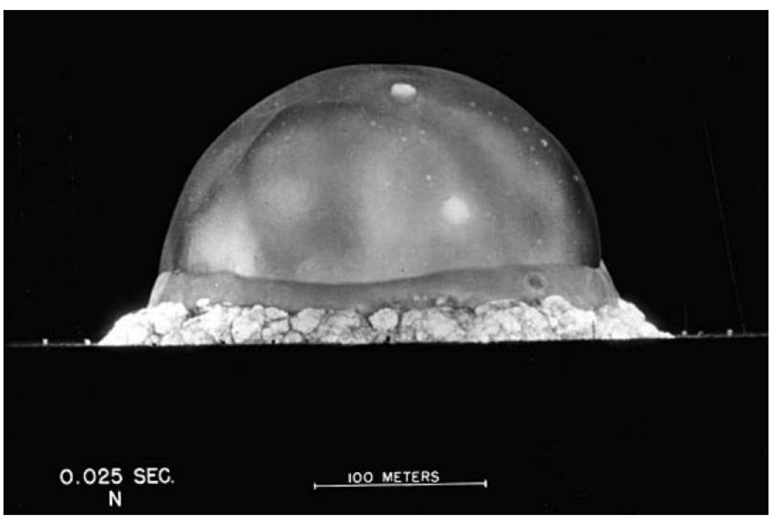
$\frac{e^2}{R} - \Delta_0 e^{-R/\xi}$
1
−1
−2
−3
2
4
6
8
10
R/ξ
e=1, $\xi^2\Delta_0$=10

#9 : UNDERGRADUATE General Physics

One of the great physicists of the 20th century, G. I. Taylor, is known for estimating the energy released by the explosion of an atomic bomb from a series of timelapse pictures from the Trinity test in New Mexico in 1945 (see below for the photograph). Taylor assumed that the explosion started at a point and propagated radially outward as a spherical shock wave.

A. Taylor assumed, $R = f(\rho, E, t)$, where R is the radius of the fireball, ρ = the density of surrounding air, t = time since the explosion, E = energy released. Find the functional form of f. *Hint: use dimensional analysis.*

B. One ton of TNT releases of approximately 4.2×10^9 J of energy. Using the density of air, $\rho = 1.2\,\mathrm{kg/m^3}$, the result from B, and the picture below, estimate how many equivalent tons of TNT energy was released from the explosion at the Trinity test. Assume f involves no dimensionful physical constants.



SOLUTION:

A. $R = \rho^a E^b t^c$, where a, b, c are the exponents to be determined. The

dimensions of the four variables are
$R \sim [L]$
$\rho \sim [ML^{-3}]$
$E \sim [ML^2T^{-2}]$
$t \sim [T]$

Plug these into $R = \rho^a E^b t^c$ and we find

-3a + 2b = 1 (from L)
a + b = 0 (from M)
-2b + c = 0 (from T)

and, thus, $R = \rho^{-1/5}\, E^{1/5}\, t^{2/5}$.

B. from the solution in B, we obtain $E \sim R^5\rho/t^2$. At $t = 0.025\,\mathrm{s}$, $R = 100\,\mathrm{m}$. Plug these and $\rho = 1.2\ \ \mathrm{kg/m^3}$, $E \approx 2 \times 10^{13}\,\mathrm{kg\,m^2/s^2} = 2 \times 10^{13}\,\mathrm{J}$. The explosive energy in one ton of TNT is roughly 4.2 gigajoules, hence our result is about 5000 tons of TNT.

#10 : UNDERGRADUATE Mathematical Physics

Find the general solution to the differential equation

$$\frac{d^2x}{dt^2} + \omega_0^2\, x = A\,\delta(t - t_0) \ ,$$

for all $t \geq 0$. You should assume $t_0 > 0$, and that the initial conditions $x(0)$ and $\dot{x}(0)$ are unspecified (and hence must appear in your answer).

`SOLUTION:`

The Laplace transform of $x(t)$ is defined as

$$\check{x}(z) = \int\limits_0^\infty dt\, x(t)\, e^{-zt} \ .$$

The inverse is

$$x(t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\ \check{x}(z)\, e^{+zt} \ ,$$

where the line $\mathrm{Re}(z) = c$ lies to the right of any singularities of $\check{x}(z)$.

To solve, we multiply the LHS and RHS of the original ODE by e^{-zt} and integrate from $t = 0$ to $t = \infty$. This results in

$$\big(z^2 + \omega_0^2\big)\,\check{x}(z) - \dot{x}(0) - z\,x(0) = A\,e^{-zt_0} \ ,$$

where the terms on the LHS involving $x(0)$ and $\dot{x}(0)$ arise from integration by parts. Thus, we have

$$\check{x}(z) = \frac{A\,e^{-zt_0} + \dot{x}(0) + z\,x(0)}{z^2 + \omega_0^2} \ ,$$

and

$$x(t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i} \left\{ \frac{e^{z(t-t_0)}}{z^2 + \omega_0^2} + \frac{\big[\dot{x}(0) + z\,x(0)\big]\,e^{zt}}{z^2 + \omega_0^2} \right\} .$$

We choose $c > 0$ so the line $\mathrm{Re}(z) = c$ lies to the right of the poles at $z = \pm i\omega_0$. For the first term in the curly brackets on the RHS, we may close in the left half plane provided $t > t_0$. In this case we pick up the residues at $z = \pm i\omega_0$. If, however, $t < t_0$, then we must close in the right half plane, where there are no residues, and the integral of this term then gives zero. Applying Cauchy's theorem, then, we have

$$\begin{aligned} x(t) &= A\left[\frac{e^{i\omega_0(t-t_0)}}{2i\omega_0} + \frac{e^{-i\omega_0(t-t_0)}}{-2i\omega_0}\right]\Theta(t-t_0) + \left[\frac{\dot{x}(0) + i\omega_0\, x(0)}{2i\omega_0}\, e^{i\omega_0 t} + \frac{\dot{x}(0) - i\omega_0\, x(0)}{-2i\omega_0}\, e^{-i\omega_0 t}\right] \\ &= \frac{A}{\omega_0}\, \sin\omega_0(t-t_0)\,\Theta(t-t_0) + x(0)\, \cos\omega_0 t + \frac{\dot{x}(0)}{\omega_0}\, \sin\omega_0 t\ , \end{aligned}$$

where $\Theta(t-t_0)$ is the step function.

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:

   a. Write the problem number and your ID number on each sheet;

   b. Write only on one side of the paper;

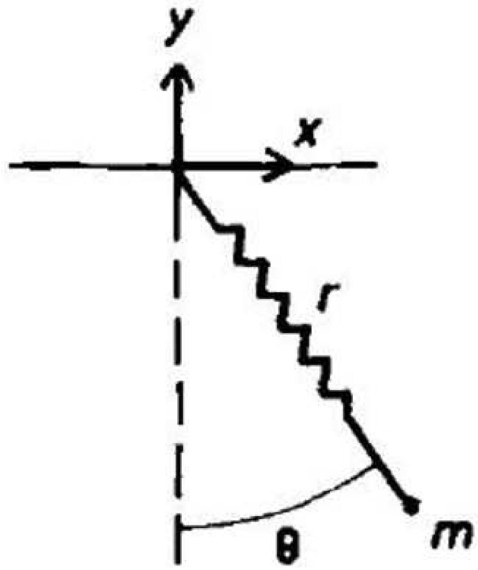
   c. Start each problem on the attached examination sheets;

   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 : GRADUATE MECHANICS

PROBLEM: A spring pendulum consists of a mass m attached to one end of a massless spring with spring constant k. The other end of the spring is tied to a fixed support. When no weight is on the spring, its length is l. Assume that the motion is confined to a vertical plane. Derive the equations of motion. Solve the equations of motion in the approximation of small angular and radial displacements from equilibrium.



SOLUTION:
Use coordinates r and θ as shown above. The mass m has Cartesian coordinates $(r\sin\theta\,,\,-r\cos\theta)$ and velocity components $(r\dot\theta\cos\theta + \dot r\sin\theta\,,\, r\dot\theta\sin\theta - \dot r\cos\theta)$ and hence kinetic energy

$$T = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2),$$

and potential energy

$$V = \tfrac{1}{2}k(r-l)^2 - mgr\cos\theta.$$

The Lagrangian is therefore

$$L = T - V = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - \tfrac{1}{2}k(r-l)^2 + mgr\cos\theta.$$

The Euler-Lagrange equations for the generalized coordinate q_i are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

then give the equations of motion

$$m\ddot{r} - mr\dot{\theta}^2 + k(r-l) - mg\cos\theta = 0,$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0.$$

The equilibrium configuration is $r = r_0$ and $\theta = \theta_0$,

$$r_0 = l + \frac{mg}{k} \quad , \quad \theta_0 = 0 \ .$$

For small oscillations about equilibrium, θ is a small angle. Let $\rho = r - r_0$ with $\rho \ll r_0$ and write the equations of motion as

$$m\ddot{\rho} - m(r_0+\rho)\dot{\theta}^2 + k\rho = 0,$$
$$(r_0+\rho)\ddot{\theta} + 2\dot{\rho}\dot{\theta} + g\theta = 0,$$

or, neglecting higher order terms of the small quantities ρ, $\dot{\rho}$, $\dot{\theta}$,

$$\ddot{\rho} + \frac{k}{m}\rho = 0,$$
$$\ddot{\theta} + \frac{g}{r_0}\theta = 0.$$

Thus both the radial and angular displacements execute simple harmonic motion about equilibrium with angular frequencies $\sqrt{k/m}$ and $\sqrt{g/r_0}$ respectively. The solutions are therefore

$$\rho = A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right),$$

which is equivalent to

$$r = l + \frac{mg}{k} + A\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right) \ ,$$

and

$$\theta = B\cos\left(\sqrt{\frac{kg}{kl+mg}}\,t + \phi_2\right),$$

where the constants A, ϕ_1, B, ϕ_2 are to be determined from the initial conditions.

#12 : GRADUATE MECHANICS

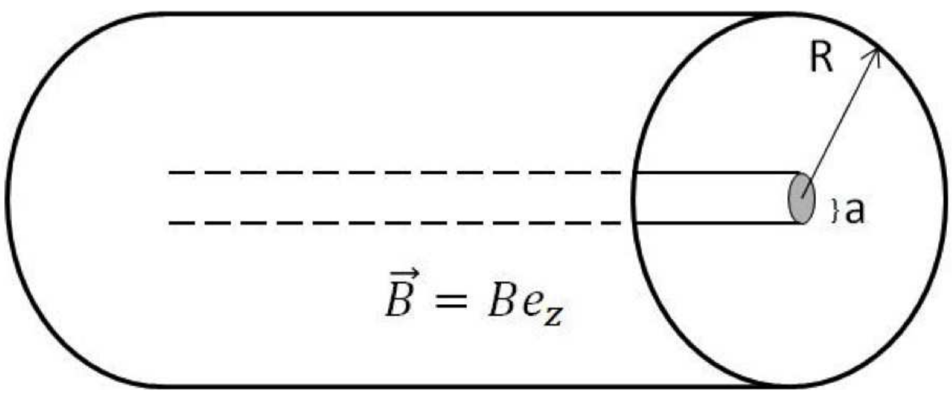
PROBLEM: A non-relativistic electron of mass m, charge $-e$ in a cylindrical magnetron moves between a wire of radius a at a negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius R at zero potential. There is a uniform constant magnetic field B parallel to the axis. Use cylindrical coordinates r, θ, z. The electric and magnetic vector potentials can be written as

$$\phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \quad , \quad \boldsymbol{A} = \tfrac{1}{2} Br\, \hat{\mathbf{e}}_\theta \ .$$

Here, $\hat{\mathbf{e}}_\theta$ is a unit vector in the direction of increasing azimuthal angle θ.

(a) Write the Lagrangian and Hamiltonian functions.

(b) Show that there are three constants of the motion. Write them down, and discuss the kinds of motion which can occur.



SOLUTION: (a) In SI units, the Lagrangian is

$$L = T - V = \tfrac{1}{2} m \dot{\boldsymbol{r}}^2 + e\phi - e\boldsymbol{A} \cdot \dot{\boldsymbol{r}} \ .$$

As

$$\dot{\boldsymbol{r}} = (\dot{r}\,,\, r\dot{\theta}\,,\, \dot{z}) \quad , \quad \boldsymbol{A} = (0\,,\, \tfrac{1}{2} Br\,,\, 0) \ ,$$

the above becomes

$$L = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + e\phi - \tfrac{1}{2} eBr^2\dot{\theta} \ .$$

The generalized momenta are

$$
\begin{aligned}
p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} - \tfrac{1}{2}eBr^2 \\
p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z}\ ,
\end{aligned}
$$

and the Hamiltonian is

$$
\begin{aligned}
H &= p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_z\,\dot{z} - L \\
&= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - e\phi \\
&= \frac{p_r^2}{2m} + \frac{\left(p_\theta + \frac{1}{2}eBr^2\right)^2}{2mr^2} + \frac{p_z^2}{2m} - e\phi \\
&= \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r} + \tfrac{1}{2}eBr\right)^2 + p_z^2\right] - e\phi\ .
\end{aligned}
$$

(b) As H is not an explicit function of time, it is a constant of the motion. Also, as

$$p_i = -\frac{\partial H}{\partial q_i}\ ,$$

if H does not contain q_i explicitly, p_i is a constant of the motion. Hence p_θ and p_z are constants of the motion. Since H is not explicitly time-dependent, $H = E$ is also constant. Thus,

$$
\begin{aligned}
E &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - e\phi \\
p_\theta &= mr^2\dot{\theta} - \tfrac{1}{2}eBr^2 \\
p_z &= m\dot{z}
\end{aligned}
$$

are all constants of the motion. An electron emitted from the inner wire will spiral out conserving its z velocity, and will reach the outer cylinder if the value of B is not too large. If the magnetic field exceeds some critical value B_c the electron will spiral out until it asymptotically reaches a critical radius $r = R_\mathrm{c} < R$.

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: Consider two long concentric cylindrical shells, of radii $r = a$ and $r = b$ (with $b > a$ and we use cylindrical coordinates (r, θ, z), with r the distance perpendicular to the z-axis). Both cylinders are centered along the z axis. Each cylinder is wrapped with a wire (winding in the $\hat{\mathbf{e}}_\theta$ direction) with N turns of the wire per unit length. The wire wrapping the outer cylinder (radius b) carries current $I(t)$, and that wrapping the inner cylinder (radius a) carries current $-I(t)$. (Imagine that the wire connects from the outer to the inner cylinder at $z = +\infty$, with a current source at $z = -\infty$.) The current $I(t)$ is slowly varying in time; so you can work to leading non-trivial order in an expansion in terms of time-derivatives of $I(t)$.

(a) Find $\boldsymbol{B}$ (to order $(d/dt)^0$) and $\boldsymbol{E}$ (to order $(d/dt)^1$) everywhere.

(b) Find the magnetic field energy per unit length, and the associated self-inductance per unit length of the system.

(c) Find the energy flux through the boundaries of the region between the shells, and verify energy conservation.

SOLUTION:

(a) As a warmup, consider the case of a single shell of radius $r = b$, which has $\boldsymbol{B}_{\rm in} = (4\pi IN/c)\hat{z}$, and $\boldsymbol{B}_{\rm out} = 0$. The EMF around a circle of radius r is $\oint \boldsymbol{E} \cdot d\boldsymbol{r} = 2\pi r E_\theta = -\dot{\Phi}/c$, with $\Phi = \pi B_{\rm in} r^2$ for $r < b$ and $\Phi = \pi B_{\rm in} b^2$ for $r > b$. This gives $\boldsymbol{E}_{\rm in} = -(2\pi \dot{I} N r/c^2)\,\hat{\mathbf{e}}_\theta$, and $\boldsymbol{E}_{\rm out} = -(2\pi \dot{I} N b^2/rc^2)\,\hat{\mathbf{e}}_\theta$.

Obtain the two shell solution by superposition. In the region $r < a$, we have $\boldsymbol{E} = \boldsymbol{B} = 0$. In the region $a \le r \le b$, we have $\boldsymbol{B}_{\rm in} = (4\pi IN/c)\,\hat{\mathbf{e}}_z$, and $\boldsymbol{E}_{\rm in} = (2\pi \dot{I} N/c^2)(a^2 r^{-1} - r)\,\hat{\mathbf{e}}_\theta$. In the region $r > b$, we have $\boldsymbol{B}_{\rm out} = 0$, and $\boldsymbol{E}_{\rm out} = -(2\pi \dot{I} N/c^2)(b^2 - a^2)\, r^{-1}\,\hat{\mathbf{e}}_\theta$.

(b) The magnetic field energy is $U_{\rm field} = \int B^2 dV/8\pi$, so per unit length of the cylinder we have $U_{\rm field}/L = (b^2 - a^2)B_{\rm in}^2/8 = 2\pi^2 I^2 N^2 (b^2 - a^2)/c^2$. Writing this as $\frac{1}{2}\mathcal{L} I^2$, with $\mathcal{L}$ the self-inductance per unit length is given by $\mathcal{L} = 4\pi^2 N^2 (b^2 - a^2)/c^2$.

(c) The Poynting vector is $\boldsymbol{S} = c\boldsymbol{E} \times \boldsymbol{B}/4\pi$, which is radial and vanishes at the $r = a$ surface. Integrating $\boldsymbol{S}$ over the surface just inside $r = b$ gives $\int_{\partial V} \boldsymbol{S} \cdot \hat{\boldsymbol{n}}\, dA/L = -4\pi^2 I \dot{I} N^2 (b^2 - a^2)/c^2$, where $\hat{\boldsymbol{n}} = \hat{\mathbf{e}}_r$ is radially outward.

This agrees with

$$\frac{dU_{\text{field}}}{dt} + \int_{\partial V} \boldsymbol{S} \cdot \hat{\boldsymbol{n}}\, dA = 0 \ .$$

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Consider a cylindrically symmetric setup, with two conducting cylindrical shells and long wire along the $\hat{\mathbf{e}}_z$ symmetry axis of the cylinders. The wire carries current I. One conducting cylindrical shell, at $r = a$ and of length L, carries charge Q. The other conducting cylindrical shell, at $r = b$ (with $b > a$), also of length L, carries charge $-Q$. These charges are uniformly distributed, and treat $L \gg b - a$, so edge effects can be neglected.

(a) Find the total electromagnetic momentum $\boldsymbol{p}_{\text{field}}$.

(b) Now suppose that the current I in the wire slowly drops to zero. Directly compute the total impulse $\int dt\, \boldsymbol{F}$ delivered to the cylindrical shells due to the induced electric force. Does it agree with momentum conservation?

`SOLUTION:`

The magnetic field is $\boldsymbol{B} = (2I/rc)\,\hat{\mathbf{e}}_\theta$. The electric field is non-zero only in the region $a < r < b$, where $\boldsymbol{E} = (2Q/rL)\,\hat{\mathbf{e}}_r$. The initial field momentum is

$$\boldsymbol{p}_{\text{field}} = \frac{1}{4\pi c} \int \boldsymbol{E} \times \boldsymbol{B}\, dV = \frac{2IQ}{c^2}\, \ln(b/a)\, \hat{\mathbf{e}}_z \ .$$

The impulse is

$$\boldsymbol{p}_{\text{impulse}} = \int\limits_0^\infty dt \left\{ \int\limits_{r=a} dA_a\, \sigma_a\, \boldsymbol{E}^{\text{ind}}(r = a, t) + \int\limits_{r=b} dA_b\, \sigma_b\, \boldsymbol{E}^{\text{ind}}(r = b, t) \right\} ,$$

where $\sigma_a = Q/2\pi a L$ is the charge density on the inner cylinder, and $\sigma_b = -Q/2\pi b L$ is the charge density on the outer cylinder. The induced field $\boldsymbol{E}^{\text{ind}}$ is in the $\hat{\mathbf{e}}_z$ direction: $\boldsymbol{E}^{\text{ind}} = E_z^{\text{ind}}\, \hat{\mathbf{e}}_z$. The area differentials are $dA_a = 2\pi\, dl\, dz$, where l is a circumference coordinate ranging over $[0, 2\pi a]$ on the inner cylinder and over $[0, 2\pi b]$ on the outer cylinder. Thus,

$$\begin{aligned} \boldsymbol{p}_{\text{impulse}} &= Q \int\limits_0^\infty dt\, \Big(E_z^{\text{ind}}(a,t) - E_z^{\text{ind}}(b,t) \Big) = Q \int\limits_0^\infty dt \int\limits_a^b dr\, \frac{\partial E_z^{\text{ind}}}{\partial r} \\ &= -\frac{Q}{c} \int\limits_0^\infty dt \int\limits_a^b dr\, \frac{\partial B_\theta(r,t)}{\partial t} = \frac{Q}{c} \int\limits_a^b dr\, B_\theta(r, t = 0) = \frac{2IQ}{c^2}\, \ln(b/a)\, \hat{\mathbf{e}}_z \ , \end{aligned}$$

which agrees with the initial field momentum, as expected by momentum conservation.

#15: STATISTICAL MECHANICS

A gas of bosonic particles in $d = 3$ dimensions obeys the single particle dispersion $\varepsilon(k) = \varepsilon_0\,(ka)^{3/2}$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$.

(b) Find the Bose condensation temperature $T_{\rm c}(n)$, where n is the number density of the bosons. You may express numerical prefactors in terms of dimensionless integrals, but it is useful to recall the Riemann zeta function $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$.

(c) Suppose instead that the particles are spin-$\frac{1}{2}$ fermions with number density n. What is the Fermi energy at $T = 0$?

SOLUTION:

(a) We have

$$g(\varepsilon)\,d\varepsilon = \frac{d^3\!k}{(2\pi)^3} = \frac{k^2}{2\pi^2}\,dk$$

and hence

$$g(\varepsilon) = \frac{k^2}{2\pi^2\,\varepsilon'(k)} = \frac{(ka)^{3/2}}{3\pi^2 a^3 \varepsilon_0} = \frac{\varepsilon}{3\pi^2 a^3 \varepsilon_0^2} \quad .$$

(b) The number density in terms of fugacity z and temperature $T > T_{\rm c}$ is

$$n(z,T) = \int\limits_0^\infty \! d\varepsilon\, \frac{g(\varepsilon)}{z^{-1}\,\exp(\varepsilon/k_{\rm B}T) - 1} \quad ,$$

To find $T_{\rm c}$, set $z = 1$:

$$\begin{aligned} n &= \int\limits_0^\infty \! d\varepsilon\, \frac{g(\varepsilon)}{\exp(\varepsilon/k_{\rm B}T) - 1} \\ &= \frac{1}{3\pi^2 a^3 \varepsilon_0^2} \int\limits_0^\infty \! d\varepsilon\, \varepsilon \left\{ e^{-\varepsilon/k_{\rm B}T} + e^{-2\varepsilon/k_{\rm B}T} + \ldots \right\} = \frac{\zeta(2)}{3\pi^2 a^3} \left(\frac{k_{\rm B}T}{\varepsilon_0}\right)^2 \quad , \end{aligned}$$

where $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$. One has $\zeta(2) = \frac{\pi^2}{6}$, hence

$$T_{\rm c} = 3\sqrt{2}\,(na^3)^{1/2}\,\varepsilon_0/k_{\rm B} \quad .$$

It is OK to leave your result in terms of $\zeta(2)$.

(c) The number density is

$$n = 2 \times \frac{1}{8\pi^3} \times \frac{4}{3}\pi k_{\mathrm{F}}^3 \quad \Rightarrow \quad n = \frac{k_{\mathrm{F}}^3}{3\pi^2} \quad .$$

Sticking this into the dispersion, we have

$$\varepsilon_{\mathrm{F}} = (3\pi^2 n a^3)^{1/2} \varepsilon_0 \quad .$$

#16: STATISTICAL MECHANICS

Consider a spin-2 Ising model with Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j} J_{ij}\, S_i\, S_j - H \sum_i S_i$$

where $S_i \in \{-2,-1,0,1,2\}$. The system is on a simple cubic lattice, with nearest neighbor coupling $J_1/k_{\rm B} = 40\,{\rm K}$ and next-nearest neighbor coupling $J_2/k_{\rm B} = 10\,{\rm K}$. Every site has six nearest neighbors and 12 next-nearest neighbors.

(a) Derive the mean field Hamiltonian by writing $\langle S_i\rangle = m + \delta S_i$ and then neglecting terms quadratic in fluctuations.

(b) Find the mean field free energy $F(T,H,N,m)$.

(c) Find the mean field equation for m.

(d) Find the mean field transition temperature $T_{\rm c}$ when $H=0$.

`SOLUTION:`

(a) The mean field Hamiltonian is

$$\mathcal{H}_{\rm MF} = \tfrac{1}{2}N\hat{J}(0)\,m^2 - \big(H + \hat{J}(0)\,m\big)\sum_i S_i\ ,$$

where

$$\hat{J}(0) = \sum_j J_{ij} = z_1 J_1 + z_2 J_2 = 360\,{\rm K}\cdot k_{\rm B}\ ,$$

since there are $z_1 = 6$ nearest neighbors and $z_2 = 12$ next nearest neighbors on the simple cubic lattice.

(b) Computing the partition function and taking the logarithm, we find the mean field free energy

$$F = \tfrac{1}{2}N\hat{J}(0)\,m^2 - Nk_{\rm B}T\,\ln\!\left(1 + 2\cosh\left(\frac{H+\hat{J}(0)\,m}{k_{\rm B}T}\right) + 2\cosh\left(\frac{2H + 2\hat{J}(0)\,m}{k_{\rm B}T}\right)\right)$$

(c) The mean field equation for m is obtained by setting $\frac{\partial F}{\partial m} = 0$. Thus,

$$m = \frac{2\sinh\left(\frac{m+h}{\theta}\right) + 4\sinh\left(\frac{2m+2h}{\theta}\right)}{1 + 2\cosh\left(\frac{m+h}{\theta}\right) + 2\cosh\left(\frac{2m+2h}{\theta}\right)}\ ,$$

with $\theta = k_\mathrm{B}T/\hat{J}(0)$ and $h = H/\hat{J}(0)$.

(d) Setting the slopes of the LHS and RHS of the above equation to be identical at $m = 0$ and $H = 0$ yields the equation for the critical temperature T_c. The slope of the LHS is obviously 1, and that of the RHS is easily found to be $\frac{2}{\theta}$. Thus, $\theta_\mathrm{c} = 2$, and $T_\mathrm{c} = 2\hat{J}(0) = 720\,\mathrm{K}$.

#17 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider the low energy scattering problem in the central potential $V(r)$ by using the partial-wave method. Assume that $V(r)$ is a short range potential with the interaction range d beyond which $V(r) = 0$. The particle energy is E, and the wavevector k is defined as $k = \sqrt{2mE/\hbar^2}$ where m is the mass of the particle.

1) In the low energy limit, *i.e.*, $k \to 0$, the s-wave channel scattering dominates. The scattering wave is approximated by an isotropic outgoing spherical wave as $f_0 \, e^{ikr}/r$, where f_0 is the s-wave scattering amplitude. Prove the relation between f_0 and the s-wave phase shift δ_0,

$$f_0 = \frac{1}{k} e^{i\delta_0} \sin \delta_0 \ .$$

Hint: You may use the asymptotic expansion

$$e^{ikz} \simeq \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} \, j_l(kr) \, Y_{l0}(\theta, \phi) \ ,$$

where $j_l(u)$ are the spherical Bessel functions. You only need to extract the s-wave component, for which $j_{l=0}(u) = \frac{\sin u}{u}$.

2) The s-wave scattering is often described by the scattering length defined as follows. Show that the radial wavefunction $R(r)$ in the s-wave channel at $k \to 0$ can be approximate as

$$R(r) \longrightarrow \frac{1}{r}\left(1 - \frac{r}{a_0}\right) \ ,$$

for $d < r \ll 2\pi/k$. Here a_0 is a constant known as the *scattering length*.

Prove that δ_0 satisfies

$$\lim_{k\to 0} k \cot \delta_0(k) = -\frac{1}{a_0} \ .$$

3) Express f_0 and the total cross section $\sigma_{\text{tot}} = 4\pi |f_0|^2$ in the s-wave approximation in terms of a_0 and k.

`SOLUTION:`

1) In the s-wave channel, the wavefunction $R(r)$ at $r \to \infty$ is

$$\begin{aligned}\frac{\sin kr}{kr} + f_0\,\frac{e^{ikr}}{r} &= \frac{1}{2ikr}\Big((1+i2kf_0)\,e^{ikr} - e^{-ikr}\Big)\\ &= \frac{1}{kr}\,\sin(kr+\delta_0)\,e^{i\delta_0}\ ,\end{aligned}$$

where $1 + 2ikf_0 = e^{2i\delta_0}$, *i.e.* the scattering amplitude f_0 is related to the scattering phase shift δ_0 by

$$f_0 = \frac{e^{2i\delta_0} - 1}{2ik} = \frac{1}{k}\,e^{i\delta_0}\sin\delta_0.$$

2) In the s-wave channel, we write the wavefunction as

$$\psi(r) = \frac{R_0(r)}{\sqrt{4\pi}} = \frac{u(r)}{\sqrt{4\pi}\,r}\ ,$$

where $u(r)$ satisfies the radial Schrödinger equation as

$$\frac{d^2u}{dr^2} + \Big(k^2 - \tfrac{2m}{\hbar^2}\,V(r)\Big)u = 0\ .$$

For $r > R$, $V(r) = 0$, hence $k \to 0$ and $r > R$, we have $u''(r) = 0$, which says that $u(r)$ is a linear function parameterized as $(1 - r/a_0)$, where a_0 is a constant. That is,

$$u(r) \propto 1 - \frac{r}{a_0}\ .$$

On the other hand, the solution for $r > R$ may also be expressed as a sinusoidal function with a phase shift as

$$u(r) \propto \sin(kr+\delta_0) = \sin\delta_0\left(1 + kr\cot\delta_0 + \mathcal{O}(k^2)\right)\ .$$

Compare these two results, we have

$$\lim_{k\to 0} k\cot\delta_0(k) = -\tfrac{1}{a_0}\ .$$

3) We have

$$f = \tfrac{1}{k}e^{i\delta_0}\sin\delta = \frac{1}{k\cot\delta_0 - ik} = -\frac{a_0}{1+ika_0}\ ,$$

in the long wavelength limit $k \to 0$. Thus, the total scattering cross section is

$$\sigma_{\rm tot} = 4\pi|f_0|^2 = \frac{4\pi a_0^2}{1+(ka_0)^2}\ .$$

#18 : GRADUATE QUANTUM MECHANICS

A spin-$\frac{1}{2}$ particle in a magnetic field has Hamiltonian

$$H = -\mu\, \boldsymbol{\sigma} \cdot \boldsymbol{B}$$

1. Write down the Hamiltonian in matrix form in the basis $\Big\{ |\uparrow\rangle \,,\, |\downarrow\rangle \Big\}$ of states with respect to the spin states along the z-axis.
2. At $t = 0$ the particle initially has spin along the direction $\hat{\boldsymbol{n}}$, which is given by angles (θ_0, ϕ_0) in spherical polar coordinates. Write the wavefunction $|\,\psi(t=0)\,\rangle$ in the $|\uparrow\rangle$ and $|\downarrow\rangle$ basis.
3. Find the state $|\,\psi(t)\,\rangle$ of the particle at time t after evolution using the Hamiltonian above, assuming the magnetic field is in the z direction.
4. Find the expectation $\langle \boldsymbol{\sigma} \rangle$ of the spin operator in the state $|\,\psi(t)\,\rangle$. Find the polar and azimuthal angles $\theta(t)$ and $\phi(t)$ which describe this vector on the Bloch sphere.

SOLUTION:

1. The Hamiltonian is

$$H = -\mu \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

2. The spin state satisfies

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}\, |\,\psi\,\rangle = |\,\psi\,\rangle$$

and can be given by projecting an arbitrary state,

$$\begin{aligned} |\,\psi(0)\,\rangle &\propto \tfrac{1}{2}\big(1 + \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}\big)\,|\,\Psi_{\text{arb}}\,\rangle \\ &= \tfrac{1}{2} \begin{pmatrix} \cos\theta_0 & \sin\theta_0\, e^{-\phi_0} \\ \sin\theta_0\, e^{i\phi_0} & -\cos\theta_0 \end{pmatrix} |\,\Psi_{\text{arb}}\,\rangle \ , \end{aligned}$$

where $|\,\Psi_{\text{arb}}\,\rangle$ is arbitrary. Taking $|\,\Psi_{\text{arb}}\,\rangle = |\uparrow\rangle$, and normalizing the resultant state, we obtain

$$|\,\psi(0)\,\rangle = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0\, e^{i\phi_0} \end{pmatrix} \ .$$

3. When B is in the z direction the states $|\pm \boldsymbol{z}\rangle$ have energy $\mp\mu B$ so the state at time t is

$$|\,\psi(t)\,\rangle = \begin{pmatrix} \cos\theta_0 e^{i\omega t} \\ \sin\theta_0\, e^{i\phi_0}\, e^{-i\omega t} \end{pmatrix} = e^{i\omega t} \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0\, e^{i\phi_0}\, e^{-2i\omega t} \end{pmatrix} \;,$$

   with $\omega = \mu B/\hbar$

4. Either compute the matrix element explicitly, or note that the second form of the state is an overall phase times the state in (b) with

$$\theta(t) = \theta_0 \quad , \quad \phi(t) = \phi_0 - 2\omega t \;.$$

#19 : GRADUATE GENERAL

`PROBLEM:` Evaluate the inverse Fourier transform

$$f(x,y) = \int \frac{d^2q}{(2\pi)^2} \, \frac{2\pi \, e^{-|\boldsymbol{q}|a}}{|\boldsymbol{q}|} \, e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \ ,$$

where $a > 0$ and $\boldsymbol{r} = (x, y)$ in Cartesian coordinates. The $\boldsymbol{q}$ integral is over the entire two-dimensional plane $(q_x\,,\,q_y)$.

`SOLUTION:`

The evaluation can be done in 2D polar coordinates. Let θ be the angle between vectors $\vec{r} = (x, y)$ and $\vec{q} = (q_x, q_y)$. In addition, let us denote

$$r = \sqrt{x^2 + y^2} \quad , \quad q = \sqrt{q_x^2 + q_y^2} \ .$$

The Fourier transform becomes

$$f(x,y) = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{\infty} \frac{dq\,q}{2\pi} \, \frac{2\pi}{q} e^{iqr\cos\theta - qa} = \int_0^{2\pi} \frac{d\theta}{2\pi} \, \frac{1}{a - ir\cos\theta} \ .$$

The remaining angular integration is done in the standard way, by changing variables to $z = e^{i\theta}$, which leads to the contour integral over the unit circle:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{a - ir\cos\theta} = \oint_{|z|=1} \frac{dz}{2\pi i z} \, \frac{1}{a - \frac{i}{2}\, r\,(z + z^{-1})} = \frac{2i}{r} \oint_{|z|=1} \frac{dz}{2\pi i} \, \frac{1}{(z - z_1)(z - z_2)} \ ,$$

where $z_{1,2} = (i/r)\big(-a \pm \sqrt{a^2 + r^2}\big)$. Only the pole at $z = z_1$ lies within the contour. Evaluating the corresponding residue, we obtain

$$f(x,y) = \frac{1}{\sqrt{a^2 + r^2}} = \frac{1}{\sqrt{a^2 + x^2 + y^2}} \ .$$

#20 : GENERAL

PROBLEM: F. Dyson described in his 1968 article a hydrogen-bomb-powered spaceship. If each explosion adds w to the velocity of the ship, and the explosions occur at equal time intervals τ, such a ship would move with the average acceleration w/τ. The performance of the ship is restricted by the capacity of shock absorbers to transfer momentum from an impulsively accelerated pusher plate to the smoothly accelerated ship. Let m be the total mass of the ship, fm the mass of the pusher plate, and sm the mass of the shock absorbers. Following Dyson, we assume $f = 1/3$ and $s = 1/50$.

a) Based on momentum conservation, what is the change in velocity of the pusher after each explosion?

b) What is the amount of energy that needs to be absorbed after each explosion?

c) Graphene — the strongest material currently known — can handle elastic energy density up to $8 \times 10^6\,\mathrm{J/m^3}$. What is the maximum admissible velocity increment w that can be achieved using shock absorbers made of graphene?

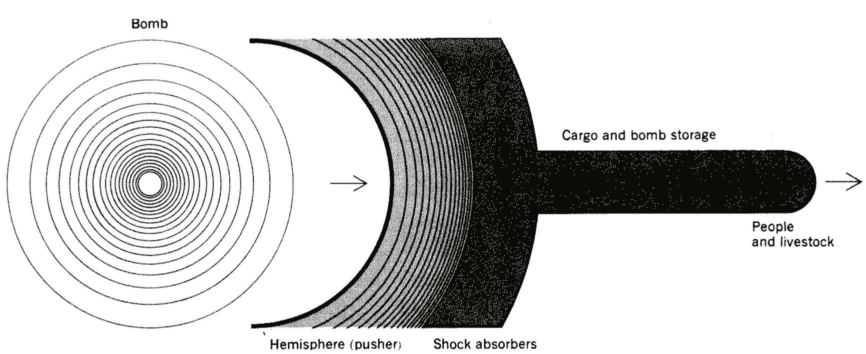


Figure 1: Bomb-propelled spaceship. Debris from the exploding bombs transfer momentum to the shock absorbers and hence to the payload.

SOLUTION:

a) From momentum conservation, the impulsive velocity given to the pusher by each explosion is w/f.

b) The internal energy of the relative motion of the pusher and the ship is $(mw^2/2)(1-f)/f$.

c) The elastic strength of the shock absorbers imposes the inequality

$$\frac{mw^2}{2}\,\frac{1-f}{f} \le sm\varepsilon\,,$$

where $\varepsilon = 8\times 10^6\,\mathrm{J/m^3}$. Hence, the maximum velocity increment is

$$w = \left(\tfrac{2f}{1-f}\,s\varepsilon\right)^{1/2} \approx 400\,\mathrm{m/s}\,.$$

This greatly exceeds 30 m/s estimate of Dyson, who envisioned in 1968 shock absorbers made of steel or nylon.