INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:

   a. Write the problem number and your ID number on each sheet;

   b. Write only on one side of the paper;

   c. Start each problem on the attached examination sheets;

   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

m_1	m_2	J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2		
J		1	
M		+1	
+1/2	+1/2	1	
J		1	0
M		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
J		1	
M		−1	
−1/2	−1/2	1	

1×1/2

m_1	m_2		
J		3/2	
M		+3/2	
+1	+1/2	1	
J		3/2	1/2
M		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
J		3/2	1/2
M		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
J		3/2	
M		−3/2	
−1	−1/2	1	

2×1

m_1	m_2			
J		3		
M		+3		
+2	+1	1		
J		3	2	
M		+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
J		3	2	1
M		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
J		3	2	1
M		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
J		3	2	1
M		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
J		3	2	
M		−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
J		3		
M		−3		
−2	−1	1		

1×1

m_1	m_2			
J		2		
M		+2		
+1	+1	1		
J		2	1	
M		+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
J		2	1	0
M		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
J		2	1	
M		−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
J		2		
M		−2		
−1	−1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

m_1	m_2			
J		5/2		
M		+5/2		
+3/2	+1	1		
J		5/2	3/2	
M		+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
J		5/2	3/2	1/2
M		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
J		5/2	3/2	1/2
M		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
J		5/2	3/2	
M		−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
J		5/2		
M		−5/2		
−3/2	−1	1		

2×1/2

m_1	m_2		
J		5/2	
M		+5/2	
+2	+1/2	1	
J		5/2	3/2
M		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
J		5/2	3/2
M		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
J		5/2	3/2
M		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
J		5/2	3/2
M		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
J		5/2	
M		−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
J		2	
M		+2	
+3/2	+1/2	1	
J		2	1
M		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
J		2	1
M		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
J		2	1
M		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
J		2	
M		−2	
−3/2	−1/2	1	

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

m_1	m_2				
J		7/2			
M		+7/2			
+2	+3/2	1			
J		7/2	5/2		
M		+5/2	+5/2		
+2	+1/2	3/7	4/7		
+1	+3/2	4/7	−3/7		
J		7/2	5/2	3/2	
M		+3/2	+3/2	+3/2	
+2	−1/2	1/7	16/35	2/5	
+1	+1/2	4/7	1/35	−2/5	
0	+3/2	2/7	−18/35	1/5	
J		7/2	5/2	3/2	1/2
M		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10
J		7/2	5/2	3/2	1/2
M		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5
J		7/2	5/2	3/2	
M		−3/2	−3/2	−3/2	
0	−3/2	2/7	18/35	1/5	
−1	−1/2	4/7	−1/35	−2/5	
−2	+1/2	1/7	−16/35	2/5	
J		7/2	5/2		
M		−5/2	−5/2		
−1	−3/2	4/7	3/7		
−2	−1/2	3/7	−4/7		
J		7/2			
M		−7/2			
−2	−3/2	1			

3/2×3/2

m_1	m_2				
J		3			
M		+3			
+3/2	+3/2	1			
J		3	2		
M		+2	+2		
+3/2	+1/2	1/2	1/2		
+1/2	+3/2	1/2	−1/2		
J		3	2	1	
M		+1	+1	+1	
+3/2	−1/2	1/5	1/2	3/10	
+1/2	+1/2	3/5	0	−2/5	
−1/2	+3/2	1/5	−1/2	3/10	
J		3	2	1	0
M		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4
J		3	2	1	
M		−1	−1	−1	
+1/2	−3/2	1/5	1/2	3/10	
−1/2	−1/2	3/5	0	−2/5	
−3/2	+1/2	1/5	−1/2	3/10	
J		3	2		
M		−2	−2		
−1/2	−3/2	1/2	1/2		
−3/2	−1/2	1/2	−1/2		
J		3			
M		−3			
−3/2	−3/2	1			

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

m_1	m_2					
J		4				
M		+4				
+2	+2	1				
J		4	3			
M		+3	+3			
+2	+1	1/2	1/2			
+1	+2	1/2	−1/2			
J		4	3	2		
M		+2	+2	+2		
+2	0	3/14	1/2	2/7		
+1	+1	4/7	0	−3/7		
0	+2	3/14	−1/2	2/7		
J		4	3	2	1	
M		+1	+1	+1	+1	
+2	−1	1/14	3/10	3/7	1/5	
+1	0	3/7	1/5	−1/14	−3/10	
0	+1	3/7	−1/5	−1/14	3/10	
−1	+2	1/14	−3/10	3/7	−1/5	
J		4	3	2	1	0
M		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5
J		4	3	2	1	
M		−1	−1	−1	−1	
+1	−2	1/14	3/10	3/7	1/5	
0	−1	3/7	1/5	−1/14	−3/10	
−1	0	3/7	−1/5	−1/14	3/10	
−2	+1	1/14	−3/10	3/7	−1/5	
J		4	3	2		
M		−2	−2	−2		
0	−2	3/14	1/2	2/7		
−1	−1	4/7	0	−3/7		
−2	0	3/14	−1/2	2/7		
J		4	3			
M		−3	−3			
−1	−2	1/2	1/2			
−2	−1	1/2	−1/2			
J		4				
M		−4				
−2	−2	1				

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

#1 : UNDERGRADUATE MECHANICS

Two balls of masses m_1 and m_2 are placed on top of each other (m_1 on top of m_2, with a small gap between them) and then dropped from height h onto the ground. (a) What is the ratio m_1/m_2 for which the top ball of mass m_1 receives the largest possible fraction of the total energy of the system after the collision? What is the height of the bounce for the top ball in this case? (b) What is the maximum possible height of the bounce for the top ball, and what is the mass ratio m_1/m_2 in that case? Consider the collision elastic and neglect air resistance.

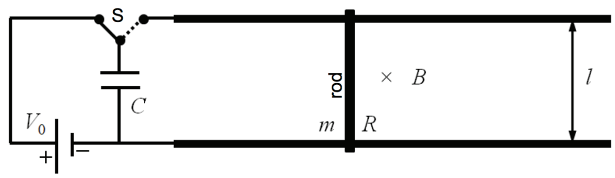
#2 : UNDERGRADUATE MECHANICS

PROBLEM: Masses m_1 and m_2 interact with potential energy $U = k|\vec{r}_1 - \vec{r}_2|$. The system has angular momentum $\vec{L} = \ell\widehat{z}$ in the CM frame. Express your answers in terms of m_1, m_2, k, and ℓ.

(a) What is the radius $r_0 = |\vec{r}_1 - \vec{r}_2|$ of circular orbits?

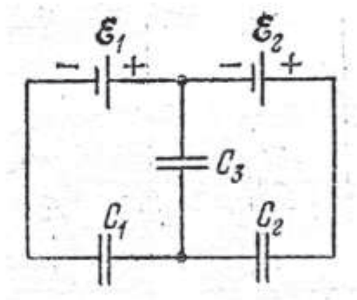
(b) What is the period T of circular orbits.

(c) Suppose that the circular orbit is slightly perturbed. What is the frequency of small radial oscillations about the circular orbit?



#3 : UNDERGRADUATE E+M

One end of a horizontal track of width l and negligible resistance is connected to a capacitor of capacitance C charged to voltage V_0 of polarity shown in the figure. The inductance of the assembly is negligible. The system is placed in a homogeneous vertical magnetic field B pointing into the page. A frictionless conducting rod of mass m and resistance R is placed perpendicular onto the track. After the capacitor is fully charged the position of the switch S is changed from the position indicated by the full line to the position indicated by the dotted line, and the rod starts moving. (a) in which direction does the rod move, and why? (b) What is the maximum velocity that the rod acquires?



#4 : UNDERGRADUATE E+M

For the circuit shown above, find the voltages on all three capacitors

#5 : UNDERGRADUATE QUANTUM MECHANICS

Consider a system of two spin 1/2 particles that are coupled via an spin-spin interaction $W = a\ \mathbf{S_1} \cdot \mathbf{S_2}$. Here a is a measure of the coupling strength.

How many states does the two particle system have, and what are their energy levels? Show how you calculate this.

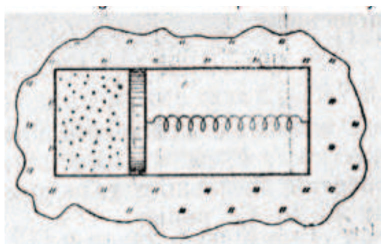
#6 : UNDERGRADUATE QUANTUM MECHANICS

A one-dimensional non-relativistic particle interacts with the potential

$$V(x) \quad = \quad \lambda \tfrac{\hbar^2}{2m} \delta(x) \tag{1}$$

where $\hbar^2/(2m)$ has been factored out to simplify the algebra.

1. Calculate the reflection and transmission coefficients (probabilities) as a function of the incident particle wavenumber k.
2. Calculate the scattering and bound states for $\lambda < 0$. Show that there is a single bound state, and that it is orthogonal to the scattering states.



#7 : UNDERGRADUATE THERMO/STAT MECH

In a horizontal cylinder with a wedged piston (see the picture), there is a monatomic ideal gas to the left of the piston and vacuum to the right of the piston. The cylinder is thermally insulated and the spring (which connects the piston with the right wall) is initially relaxed. After the piston is released and equilibrium is reached, the volume of the gas is doubled. What is the ratio of the final temperature to the initial temperature? What is the ratio of the final pressure to the initial pressure? Neglect the heat capacities of the cylinder, piston and spring.

#8 : UNDERGRADUATE THERMO/STAT MECH

The rotational energies for *planar rotation* of a molecule are given by

$$E_n \quad = \quad \frac{\hbar^2}{2I} n^2, \qquad n = 0, \pm 1, \pm 2, \ldots$$

where I is the moment of inertia.

1. Compute the rotational partition function in the low and high temperature limits. Keep at least the leading temperature dependent term.
2. Compute the specific heat per molecule (at constant volume) in the low and high temperature limits

#9 : UNDERGRADUATE MATH METHODS

Develop a series solution that is regular at $x = 0$ to the equation:

$$x^2 y'' = xy' + (x^2 - m^2)y = 0,$$

where m is an integer. That is, find a recursion relation that gives the terms of the series solution.

#10 : UNDERGRADUATE EXP-GENERAL

(a) A telescope with a primary mirror with a diameter of 20m records 314 photons / second from a star. Assume that no photons are lost in our atmosphere and that the telescope plus detector system is 1% efficient. What is the flux of photons/s/m^2?

b) What is the energy flux in W/m^2 if the effective wavelength is 500 nm and $h = 6.63 \times 10^{-34}$ Js?

c) You observe this star for a year and see that it inscribes a small circle with diameter 1/3600 degrees relative to the fainter and much more distant stars. The circular motions repeats with exactly one year period. What is the distance to this star? You need to know that the Earth is 150 million km from the sun.

d) How much energy does this star emit in all directions per second into the waveband that was detected (in part a)?

#11 : GRADUATE MECHANICS

`PROBLEM:` For a one dimensional system with Hamiltonian $H = \frac{1}{2}(p^2 - q^{-2})$:

(a) Show that $D = \alpha pq - Ht$ is a constant of the motion for a particular value of the constant α, which you should determine.

(b) Suppose that at time $t = 0$, $p = 0$ and $q = 1$. Find $p(t)$ and $q(t)$ for $t > 0$.

#12 : GRADUATE MECHANICS

A circular cone of height h and angle 2α rolls without slipping inside a fixed cone of angle 2β, where $\alpha < \beta$. The axis of the cone rotates about the axis of the outer cone with constant angular speed Ω. (a) Find the angular velocity of the cone, and show that $\dot{\psi}$ is a constant (ϕ, θ, ψ are the usual Euler angles for the orientation of a rigid body). (b) Find the kinetic energy of the cone. (The moments of inertia of a circular cone of radius R and height h about its tip are $I_1 = I_2 = \frac{3}{5}M(\frac{1}{4}R^2 + h^2)$ and $I_3 = \frac{3}{10}MR^2$.)

#13 : GRADUATE E+M

Consider a uniform external magnetic field with the magnetic inductance $\vec{B} = B\hat{z}$ along the z-direction. Now we put a superconducting ball of radius R at the origin. The superconductor is a perfect diamagnet, which means that the B-field inside the superconductor vanishes as $\vec{B}_{inside} = 0$. Introducing the superconducting ball changes the distribution of $\vec{B}$ outside the ball. Find $\vec{B}(r, \theta, \phi)$ for all $r > R$.

Hint: the relation between $\vec{B}$ and the magnetic field strength $\vec{H}$ and the magnetic moment density $\vec{M}$ is $\vec{B} = \vec{H} + 4\pi\vec{M}$. We assume that $\nabla \cdot \vec{M} = 0$ inside the superconductor. You can use the method of the magnetic scalar potential $\vec{H}(\vec{r}) = -\nabla W(\vec{r})$ to solve for the distribution of the magnetic inductance $\vec{B}(\vec{r})$ outside the superconducting ball. You need to determine the correct boundary conditions.

Hint: the general solution to the Laplace equation in spherical coordinates (r, θ, ϕ) with axial symmetry can be expressed as

$$W(r, \theta, \phi) = \sum_{l=0}^{+\infty} (a_l r^l + \tfrac{b_l}{r^{l+1}}) P_l(\cos\theta). \tag{2}$$

#14 : GRADUATE E+M

A charge density ρ_0 is placed at time $t = 0$ in a small region in the interior of a homogeneous charge-neutral material that has electrical conductivity σ.

(a) Derive an expression for the time evolution of the charge density in that region, $\rho_c(t)$, with $\rho_c(0) = \rho_0$. Hint: use a continuity equation.

(b) Estimate how long it will take (in seconds) for the charge density to decrease to 1/1000 of its initial value if the material is (i) copper with conductivity $\sigma = 1/(2\mu\Omega cm)$ and (ii) quartz with conductivity $\sigma = 1/(10^{24}\mu\Omega cm)$.

Use $\epsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$.

#15 : GRADUATE QUANTUM MECHANICS

Consider two spin-$\frac{1}{2}$ particles interacting through a Heisenberg Hamiltonian $H = J\vec{S}_1 \cdot \vec{S}_2$, where $\vec{S}_{1,2} = \frac{1}{2}\vec{\sigma}_{1,2}$ and $\vec{\sigma}_{1,2}$ are Pauli matrices for particles 1 and 2, respectively.

(a) Solve for the eigenenergy of each eigenstate, and express each eigenstate in the basis of eigenstates of $\sigma_{1,z}$ and $\sigma_{2,z}$.

(b) Consider an initial two-spin state at $t = 0$ $|\Psi(t = 0)\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are σ_z-eigenstates with eigenvalues 1 and -1, respectively. Calculate the time evolution of the expectation value of $S_{1,z}$ of the first particle.

#16 : GRADUATE QUANTUM MECHANICS

Consider scattering of two relativistic spin 1/2 particles, e.g. electron positron, via a $J^{PC} = 1^{--}$ resonance, e.g. the Υ(4S) resonance, that decays into two spin 0 particles, e.g. two B mesons. What is the angular distribution of the outgoing spin 0 particles with regard to the axis of the incoming spin 1/2 particles in the center of mass frame of the spin 1 resonance assuming that total angular momentum (spin and orbital) is conserved.

#17 : GRADUATE STAT MECH

Consider a relativistic gas of N indistinguishable massless particles. Assume that the particles are classical, not quantum, particles.

(a) Calculate the canonical partition function $Z(T, V, N)$.

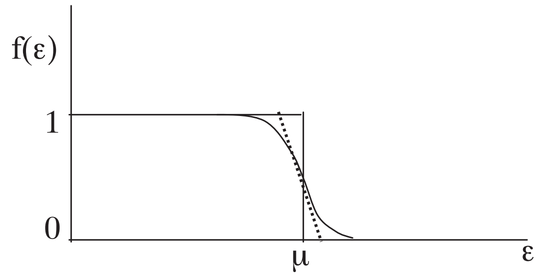
(b) Find the Helmholtz free energy $F(T, V, N)$.

(c) Determine the pressure p, the entropy S and the chemical potential μ for this relativistic gas.

(d) Determine the energy $U(T, V, N)$ and the heat capacity at constant volume C_V.

Hint:

$$\int dx\ x^2\ e^{-x} = \Gamma(3)$$



#18 : GRADUATE STAT MECH

The figure shows the Fermi function $f(\epsilon)$ (full line) for electrons in a metal at temperature $k_BT << \epsilon_F$ as function of energy ϵ, with ϵ_F the Fermi energy, and a dotted line that is tangent to the Fermi function at the chemical potential μ. The electrons are assumed to be non-interacting. Assume at this low temperature the chemical potential $\mu \sim \epsilon_F$ is independent of temperature. Taking as an approximation to the true Fermi function the horizontal portions joined by the dotted line, derive an expression for the electronic heat capacity of this metal. Assume the density of states in energy $g(\epsilon)$ is a constant g, independent of energy. Proceed as follows:

(a) Find the function describing the dotted line and the points where the dotted line intersects the horizontal lines (values 1 and 0).

(b) Compute the energy of the electrons in the region of energy where $f(\epsilon) = 1$.

(c) Compute the energy of the electrons in the region where $f(\epsilon)$ is given by the dotted line.

(d) Compute the heat capacity. How does it compare to the correct result for the low temperature heat capacity of a metal?

<u>#19 : GRADUATE MATH METHODS</u>

Using contour integration find the value of

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx.$$

Hint: $\ln(x^2+1) = \ln(i-x) + \ln(i+x) - i\pi$.

#20 : GRADUATE EXP-GENERAL

An emission line is observed over a certain range of wavelengths in a spectrum. We suspect that an emission line is present in three adjacent pixels that contain 62, 71 and 69 photons. The emission line is observed on top of a background of with the following number of photons per pixel on either side of the emission line and away from the pixels where the emission is suspected: 52, 48, 60, 42, 45, 59, 51, 61, 43.

(i) What is the mean background per pixel and the uncertainty in this value?

(ii) What is the statistical significance of the emission line? Examine the pixels individually.

(iii) What is the statistical significance of the emission line if you examine the pixels together?

(iv) In the above we assume that we know the pixels in which the emission line might be found. What is the significance of the emission line if instead we know the expected width but not the position of the line prior to collecting the data? Assume that the spectrum contained 1024 pixels and the line could lie anywhere in the spectrum.

It might help to know that for a normal distribution the fraction of the area lying between $-n\sigma$ and $+n\sigma$ is 68.27% for $n = 1$; 95.45% for $n = 2$; 99.73% for $n = 3$; and 99.9937% for $n = 4$.

INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

#1 : UNDERGRADUATE MECHANICS

Two balls of masses m_1 and m_2 are placed on top of each other (m_1 on top of m_2, with a small gap between them) and then dropped from height h onto the ground. (a) What is the ratio m_1/m_2 for which the top ball of mass m_1 receives the largest possible fraction of the total energy of the system after the collision? What is the height of the bounce for the top ball in this case? (b) What is the maximum possible height of the bounce for the top ball, and what is the mass ratio m_1/m_2 in that case? Consider the collision elastic and neglect air resistance.

`SOLUTION:`
(a) The balls reach the ground with speed $v = \sqrt{2gh}$. The bottom ball hits the ground and then collides with the top ball. The top ball receives the largest possible fraction of the total energy if the bottom ball is at rest after the collision. In this case, the equations expressing the conservation of momentum and energy are

$$(m_2 - m_1)v = m_1 u \tag{1}$$

and

$$(m_2 + m_1)v^2/2 = m_1 u^2/2. \tag{2}$$

The speed u of the top ball after the collision and the ratio of the masses calculated from these equations are $u = 2v$ and $m_1/m_2 = 1/3$. The upper ball rises to a height of $h_{bounce} = u^2/2g = 4h$.

(b) For the top ball, the maximum speed after collision is achieved when $m_2 \gg m_1$. In this case, its speed is $3v$ and the height of the bounce is $9h$.

#2 : UNDERGRADUATE MECHANICS

PROBLEM: Masses m_1 and m_2 interact with potential energy $U = k|\vec{r}_1 - \vec{r}_2|$. The system has angular momentum $\vec{L} = \ell\hat{z}$ in the CM frame. Express your answers in terms of m_1, m_2, k, and ℓ.

(a) What is the radius $r_0 = |\vec{r}_1 - \vec{r}_2|$ of circular orbits?

(b) What is the period T of circular orbits.

(c) Suppose that the circular orbit is slightly perturbed. What is the frequency of small radial oscillations about the circular orbit?

SOLUTION:

(a) This is $U = kr$. So $U_{eff} = kr + \frac{\ell^2}{2\mu r^2}$. The orbits are circular at the minimum of the effective potential, $U'_{eff}(r_0) = 0$: thus $U'_{eff} = k - \ell^2/\mu r_0^3 = 0$, so

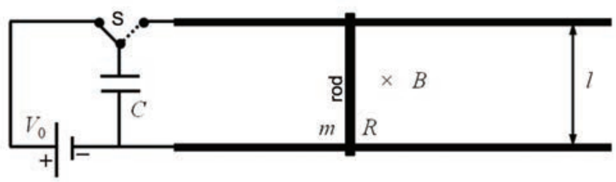
$$r_0 = (\ell^2/\mu k)^{1/3} = (\ell^2(m_1 + m_2)/km_1m_2)^{1/3}.$$

(b) $\ell = \mu r^2\dot{\phi} = \mu r_0^2(2\pi/T)$, so

$$T = \frac{2\pi}{\ell}\mu r_0^2 = 2\pi\left(\frac{m_1m_2}{(m_1 + m_2)\ell k^2}\right)^{1/3}.$$

(c) If the circular orbit is slightly perturbed, the frequency of oscillation is

$$\omega = \sqrt{U''_{eff}/\mu} = \sqrt{3\ell^2/\mu^2 r_0^4} = \sqrt{3}(k^2/\mu\ell)^{1/3}.$$



#3 : UNDERGRADUATE E+M

One end of a horizontal track of width l and negligible resistance is connected to a capacitor of capacitance C charged to voltage V_0 of polarity shown in the figure. The inductance of the assembly is negligible. The system is placed in a homogeneous vertical magnetic field B pointing into the page. A frictionless conducting rod of mass m and resistance R is placed perpendicular onto the track. After the capacitor is fully charged the position of the switch S is changed from the position indicated by the full line to the position indicated by the dotted line, and the rod starts moving. (a) in which direction does the rod move, and why? (b) What is the maximum velocity that the rod acquires?

SOLUTION:
When the capacitor is connected, a current I starts flowing in the rod, which experiences the force $F = BlI$. The charge Q on the capacitor and the voltage across it decrease. Meanwhile the voltage induced in the rod increases, until the two voltages cancel each other. The rod then continues with its maximum velocity given by

$$Blv_{max} = \frac{Q_{min}}{C}. \tag{3}$$

The equation of motion of the rod is

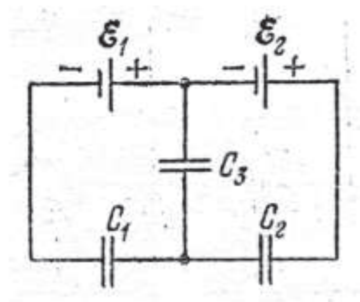
$$m\frac{dv}{dt} = BlI = -Bl\frac{dQ}{dt}. \tag{4}$$

The speed of the rod increases from zero to v_{max}, while the charge of the capacitor decreases from $Q_0 = CV_0$ to Q_{min}. Equation (2) gives

$$mv_{max} = Bl(Q_0 - Q_{min}). \tag{5}$$

The maximum velocity can be calculated using Eq. (1) and (3)

$$v_{max} = \frac{BlCV_0}{m+B^2l^2C}. \tag{6}$$



#4 : UNDERGRADUATE E+M

For the circuit shown above, find the voltages on all three capacitors

SOLUTION:

We can assume that the left plate of C_1, left plate of C_2, and bottom plate of C_3 are all charged negatively. The equation for conservation of charge then gives

$Q_1 = Q_2 + Q_3$, giving $V_1C_1 = V_2C_2 + V_3C_3$

By the definition of emf of an ideal battery, battery

$V_1 + V_2 = \varepsilon_1 + \varepsilon_2$ and $V_1 + V_3 = \varepsilon_1$

This system of three linear equations has solutions

$$V_1 = \frac{(\varepsilon_1 + \varepsilon_2)C_2 + \varepsilon_1 C_3}{C_1 + C_2 + C_3}$$

$$V_1 = \frac{(\varepsilon_1 + \varepsilon_2)C_1 + \varepsilon_1 C_3}{C_1 + C_2 + C_3}$$

$$V_3 = \frac{\varepsilon_1 C_1 - \varepsilon_2 C_2}{C_1 + C_2 + C_3}$$

.

#5 : UNDERGRADUATE QUANTUM MECHANICS

Consider a system of two spin 1/2 particles that are coupled via an spin-spin interaction $W = a\ \mathbf{S_1} \cdot \mathbf{S_2}$. Here a is a measure of the coupling strength.

How many states does the two particle system have, and what are their energy levels? Show how you calculate this.

`SOLUTION:`

There are 4 states, three of which have S=1 and are degenerate in energy at $\frac{a\hbar^2}{4}$, the 4th is S=0 at an energy of $-\frac{3a\hbar^2}{4}$. You calculate this as:

$(H_0 + W)|S, M >= \frac{a\hbar^2}{2}[S(S+1) - 3/2]|S, M >$

#6 : UNDERGRADUATE QUANTUM MECHANICS

A one-dimensional non-relativistic particle interacts with the potential

$$V(x) = \lambda \tfrac{\hbar^2}{2m}\delta(x) \tag{7}$$

where $\hbar^2/(2m)$ has been factored out to simplify the algebra.

1. Calculate the reflection and transmission coefficients (probabilities) as a function of the incident particle wavenumber k.
2. Calculate the scattering and bound states for $\lambda < 0$. Show that there is a single bound state, and that it is orthogonal to the scattering states.

SOLUTION:

1. The solution is

$$\begin{aligned} \psi(x) &= e^{ikx} + Re^{-ikx}, \qquad x < 0 \\ \psi(x) &= Te^{ikx} \qquad x > 0 \end{aligned} \tag{8}$$

where the incident energy is

$$E = \tfrac{\hbar^2 k^2}{2m} \tag{9}$$

The Schrödinger equation is

$$\begin{aligned} -\tfrac{\hbar^2}{2m}\tfrac{d^2\psi}{dx^2} + V(x)\psi &= E\psi \\ -\tfrac{d^2\psi(x)}{dx^2} + \lambda\delta(x)\psi(x) &= k^2\psi(x) \end{aligned} \tag{10}$$

The boundary condition at $x = 0$ is $\psi(x)$ is continuous, and integrating the Schrödinger equation across $x = 0$ gives

$$\left.\tfrac{d\psi(x)}{dx}\right|_{x-\epsilon} - \left.\tfrac{d\psi(x)}{dx}\right|_{x+\epsilon} + \lambda\psi(0) = 0 \tag{11}$$

This gives

$$\begin{aligned} 1 + R &= T \\ ik(1-R) + \lambda(1+R) &= ikT \end{aligned} \tag{12}$$

The solution is

$$\begin{aligned} R &= -\tfrac{i\lambda}{2k+i\lambda} \\ T &= \tfrac{2k}{2k+i\lambda} \end{aligned} \tag{13}$$

The reflection and transmission probabilities are

$$\begin{aligned} |R|^2 &= \tfrac{\lambda^2}{4k^2+\lambda^2} \\ T &= \tfrac{4k^2}{4k^2+\lambda^2} \end{aligned} \tag{14}$$

2. The scattering states have already been computed, since the first part does not depend on the sign of λ. The bound states have wave functions which are exponentially damped.

$$\begin{aligned} \psi(x) &= Ae^{\kappa x}, \qquad x < 0 \\ \psi(x) &= Be^{-\kappa x} \qquad x > 0 \end{aligned} \tag{15}$$

where the energy is

$$E = -\tfrac{\hbar^2\kappa^2}{2m} \tag{16}$$

The same boundary conditions as before lead to

$$\begin{aligned} A &= B \\ \kappa A + \lambda A &= -\kappa B \end{aligned} \tag{17}$$

with solution

$$2\kappa = -\lambda \tag{18}$$

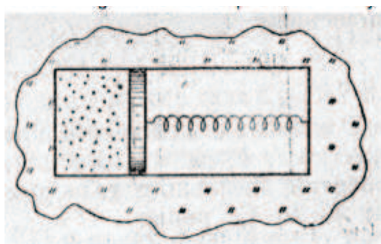
and $A = B$. This is only a sensible solution for $\kappa > 0$, i.e. for $\lambda < 0$. A and B are fixed by normalizing the bound state,

$$\psi(x) = \sqrt{\kappa}e^{-\kappa|x|} \tag{19}$$

The bound state is unique, since there is a single solution. The overlap of the bound state ψ_B with a scattering state ψ_k with wavenumber k is

$$\begin{aligned} \int_{-\infty}^{\infty} dx\ \psi_B(x)^* \psi_k(x) &= \int_{-\infty}^{0} \sqrt{\kappa}e^{\kappa x}\left(1 + Re^{-ikx}\right) + \int_0^{\infty} \sqrt{\kappa}e^{-\kappa x}\left(Te^{ikx}\right) \\ &= \sqrt{\kappa}\left[\tfrac{R-T}{\kappa-ik} + \tfrac{1}{\kappa}\right] = 0 \end{aligned} \tag{20}$$

using Eq. (12).



#7 : UNDERGRADUATE THERMO/STAT MECH

In a horizontal cylinder with a wedged piston (see the picture), there is a monatomic ideal gas to the left of the piston and vacuum to the right of the piston. The cylinder is thermally insulated and the spring (which connects the piston with the right wall) is initially relaxed. After the piston is released and equilibrium is reached, the volume of the gas is doubled. What is the ratio of the final temperature to the initial temperature? What is the ratio of the final pressure to the initial pressure? Neglect the heat capacities of the cylinder, piston and spring.

`SOLUTION:`

.

Solution:

Conservation of energy:

$\Delta U + \dfrac{kx^2}{2} = 0$, where x is the displacement of the piston from equilibrium

$$\Delta U = nC_v(T_2 - T_1)$$

$$P_2 A = kx$$

$$x = (V_2 - V_1)/A$$

$$nC_v(T_1 - T_2) = k(V_2 - V_1)^2/(2A^2)$$

$$P_2 V_2 = nRT_2$$

$$P_2 A/k = (V_2 - V_1)/A$$

$$nRT_2/V_2 = k(V_2 - V_1)/A^2$$

$$\frac{C_v(T_1 - T_2)V_2}{RT_2} = (V_2 - V_1)/2$$

$$\frac{2C_v}{R}\frac{T_1 - T_2}{T_2} = \frac{V_2 - V_1}{V_2}$$

$$\frac{2C_v}{R}\left(\frac{T_1}{T_2} - 1\right) = \left(1 - \frac{V_1}{V_2}\right)$$

$$\frac{T_1}{T_2} = 1 + \frac{R}{2C_v}\left(1 - \frac{V_1}{V_2}\right)$$

We have got $\dfrac{V_1}{V_2} = 1/2$ and $C_v = 1.5R$, giving $\dfrac{T_1}{T_2} = 1 + \dfrac{R}{2C_v}\left(1 - \dfrac{V_1}{V_2}\right) = 7/6$ and $\dfrac{P_1}{P_2} = 3/7$

#8 : UNDERGRADUATE THERMO/STAT MECH

The rotational energies for *planar rotation* of a molecule are given by

$$E_n = \frac{\hbar^2}{2I} n^2, \qquad n = 0, \pm 1, \pm 2, \ldots$$

where I is the moment of inertia.

1. Compute the rotational partition function in the low and high temperature limits. Keep at least the leading temperature dependent term.
2. Compute the specific heat per molecule (at constant volume) in the low and high temperature limits

`SOLUTION:`

1. Let $\beta = 1/(k_B T)$.

$$\begin{aligned} Z &= \sum_{n=-\infty}^{\infty} e^{-\beta E_n} = 1 + 2\sum_{n=1}^{\infty} e^{-\lambda n^2} \\ \lambda &= \frac{\beta \hbar^2}{2I} \end{aligned}$$

In the low temperature limit, $\beta \to \infty$ so $\lambda \to \infty$. The terms are exponentially damped with n, so

$$Z \approx 1 + 2e^{-\lambda} + 2e^{-4\lambda} + \ldots$$

In the high temperature limit, $\beta \ll 1$, and the sum can be estimated by an integral,

$$Z \approx \int_{-\infty}^{\infty} dx\, e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} = \sqrt{\frac{2\pi I}{\beta \hbar^2}}$$

2. The specific heat at constant volume is

$$\begin{aligned} C_V &= \left.\frac{\partial U}{\partial T}\right|_V \\ U &= -\frac{\partial \ln Z}{\partial \beta} \end{aligned}$$

At low temperatures,

$$
\begin{aligned}
Z &\approx 1 + 2e^{-\lambda} \\
\ln Z &\approx 2e^{-\lambda} = 2e^{-\beta\hbar^2/(2I)} \\
U &= \frac{\hbar^2}{I} e^{-\beta\hbar^2/(2I)} = \frac{\hbar^2}{I} e^{-\hbar^2/(2Ik_BT)} \\
C_V &= \left(\frac{\hbar^2}{I}\right)^2 \frac{1}{2k_BT^2} e^{-\hbar^2/(2Ik_BT)}
\end{aligned}
$$

At high temperatures,

$$
\begin{aligned}
\ln Z &\approx \frac{1}{2} \ln \frac{2\pi I}{\beta\hbar^2} \\
U &= \frac{1}{2\beta} = \frac{k_BT}{2} \\
C_V &= \frac{k_B}{2}
\end{aligned}
$$

#9 : UNDERGRADUATE MATH METHODS

Develop a series solution that is regular at $x = 0$ to the equation:

$$x^2 y'' = xy' + (x^2 - m^2)y = 0,$$

where m is an integer. That is, find a recursion relation that gives the terms of the series solution.

`SOLUTION:`

This equation has a regular singular point at $x = 0$, so a solution of the form

$$y = x^s \sum_{n=0}^{\infty} c_n x^n \qquad\qquad (c_0 \neq 0)$$

is guaranteed to exist. From the differential equation we see that a two-term recursion relation will be obtained. Substituing this solution into the equation, the coefficient of x^s is $c_0(s^2 - m^2) = 0$. This is the indicial equation and its roots are $s = \pm m$. Next the coefficient of x^{s+1} is $c_1[(s+1)^2 - m^2] = 0$. Thus $c_1 = 0$, unless $m = 1/2$, which was not allowed. So there are no even terms in the expansion. Thus we write

$$y = x^{\pm m}(c_0 + c_2 x^2 + c_4 x^4 \cdots)$$

and find the recursion relation:

$$\frac{c_{n+2}}{c_n} = \frac{-1}{(s + n + 2)^2 - m^2}.$$

#10 : UNDERGRADUATE EXP-GENERAL

(a) A telescope with a primary mirror with a diameter of 20m records 314 photons / second from a star. Assume that no photons are lost in our atmosphere and that the telescope plus detector system is 1% efficient. What is the flux of photons/s/m^2?

b) What is the energy flux in W/m^2 if the effective wavelength is 500 nm and $h = 6.63 \times 10^{-34}$ Js?

c) You observe this star for a year and see that it inscribes a small circle with diameter 1/3600 degrees relative to the fainter and much more distant stars. The circular motions repeats with exactly one year period. What is the distance to this star? You need to know that the Earth is 150 million km from the sun.

d) How much energy does this star emit in all directions per second into the waveband that was detected (in part a)?

`SOLUTION:` a) The mirror area is $\pi 10^2$ m, and the system has an effective area of πm. The photon flux is then $314/3.14 = 100$ photons/s/m^2.

b) Energy per photon E=hc/$\lambda = 6.63 \times 10^{-34} 3 \times 10^8 / 500 \times 10 - 9 = 3.98 \times 10^{-19} J$. The energy flux is then $100E = 3.98 \times 10^{-17}$ W/s

c) This parallactic motion is caused by the Earth's orbit of the sun, hence the exact one year period. The circular motion has a radius of 1/7200 degrees $= 2.42 \times 10^{-6}$ radians. This is the apex angle of a skinny triangle with base 1.5×10^{11}m the 2 long sides equal to the distance to the star. The distance is then the base/angle $d = 6.19 \times 10^{16}$m.

d) Assume the star emits isotropically. Then energy emitted = flux$\times 4\pi$d^2 = 1.92×10^{18} W.

#11 : GRADUATE MECHANICS

`PROBLEM:` For a one dimensional system with Hamiltonian $H = \frac{1}{2}(p^2 - q^{-2})$:

(a) Show that $D = \alpha pq - Ht$ is a constant of the motion for a particular value of the constant α, which you should determine.

(b) Suppose that at time $t = 0$, $p = 0$ and $q = 1$. Find $p(t)$ and $q(t)$ for $t > 0$.

`SOLUTION:`

(a)

$$\frac{dD}{dt} = [D, H] + \frac{\partial D}{\partial t} = [\alpha pq, H] - H = \alpha(p^2 - q^{-2}) - H,$$

which vanishes for $\alpha = \frac{1}{2}$, so $D = \frac{1}{2}pq - Ht$ is a constant.

(b) At $t = 0$, evaluate $H = -\frac{1}{2}$ and $D = 0$. These are both constants of the motion, so

$$q^{-2} - p^2 = 1, \qquad pq = -t.$$

These give (for $0 \le t \le 1$)

$$q = \sqrt{1 - t^2}, \qquad p = -\frac{t}{\sqrt{1 - t^2}}.$$

#12 : GRADUATE MECHANICS (Jenkins)

A circular cone of height h and angle 2α rolls without slipping inside a fixed cone of angle 2β, where $\alpha < \beta$. The axis of the cone rotates about the axis of the outer cone with constant angular speed Ω. (a) Find the angular velocity of the cone, and show that $\dot{\psi}$ is a constant (ϕ, θ, ψ are the usual Euler angles for the orientation of a rigid body). (b) Find the kinetic energy of the cone. (The moments of inertia of a circular cone of radius R and height h about its tip are $I_1 = I_2 = \frac{3}{5}M(\frac{1}{4}R^2 + h^2)$ and $I_3 = \frac{3}{10}MR^2$.)

`SOLUTION:` Take the space fixed axes to be the symmetry axis of the outer cone $\hat{x}_3$, and its two perpendicular directions $\hat{x}_{1,2}$. The body fixed axes are the symmetry axis of the cone $\hat{x}'_3$ and $\hat{x}'_{1,2}$.

$$\begin{aligned} \theta &= \beta - \alpha = \text{constant} \qquad \Rightarrow \quad \dot{\theta} = 0 \\ \dot{\phi} &= \Omega \end{aligned}$$

$$\begin{aligned} \vec{\omega} &= \dot{\phi}\ \hat{x}_3 + \dot{\theta}\ \hat{n} + \dot{\psi}\ \hat{x}'_3 \\ &= \Omega\ \hat{x}_3 + \dot{\psi}\ \hat{x}'_3 \\ &= \Omega\left[\sin(\beta-\alpha)\ \hat{x}'_2 + \cos(\beta-\alpha)\ \hat{x}'_3\right] + \dot{\psi}\ \hat{x}'_3 \\ &= \Omega\sin(\beta-\alpha)\ \hat{x}'_2 + \left[\Omega\cos(\beta-\alpha) + \dot{\psi}\right]\ \hat{x}'_3 \end{aligned}$$

The instantaneous axis of rotation of the cone is along the line of contact of the inner and outer cones, so

$$\vec{\omega} = A\left[-\sin\alpha\ \hat{x}'_2 + \cos\alpha\ \hat{x}'_3\right]$$

$$\begin{aligned} -A\sin\alpha &= \Omega\sin(\beta-\alpha), \\ A\cos\alpha &= \Omega\cos(\beta-\alpha) + \dot{\psi}, \\ &\Rightarrow \quad A = -\frac{\Omega\sin(\beta-\alpha)}{\sin\alpha} \\ &\Rightarrow \quad \dot{\psi} = -\Omega\left[\sin(\beta-\alpha)\cot\alpha + \cos(\beta-\alpha)\right] = -\Omega\left(\frac{\sin\beta}{\sin\alpha}\right) = \text{constant} \end{aligned}$$

The equation

$$|\dot{\psi}| = \Omega\left(\frac{\sin\beta}{\sin\alpha}\right)$$

also is easily obtained from the constraint of rolling without slipping,

$$R\,\dot{\psi} = \dot{\phi}\left[\left(\frac{h}{\cos\alpha}\right)\sin\beta\right]\ ,$$

with $R = h\tan\alpha$.

$$\begin{aligned}
T &= \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2,\\
\frac{R}{h} &= \tan\alpha,\\
I_2 &= \frac{3}{5}Mh^2\left(\frac{1}{4}\tan^2\alpha + 1\right) = \frac{3}{20}Mh^2\left(\tan^2\alpha + 4\right),\\
I_3 &= \frac{3}{10}Mh^2\tan^2\alpha,\\
\omega_2 &= \Omega\sin(\beta-\alpha),\\
\omega_3 &= -\Omega\sin(\beta-\alpha)\cot\alpha,\\
T &= \frac{3}{40}Mh^2\Omega^2\sin^2(\beta-\alpha)\left[\left(\tan^2\alpha+4\right) + 2\tan^2\alpha\cot^2\alpha\right]\\
&= \frac{3}{40}Mh^2\Omega^2\sin^2(\beta-\alpha)\left[\tan^2\alpha+6\right]
\end{aligned}$$

#13 : GRADUATE E+M

Consider a uniform external magnetic field with the magnetic inductance $\vec{B} = B\hat{z}$ along the z-direction. Now we put a superconducting ball of radius R at the origin. The superconductor is a perfect diamagnet, which means that the B-field inside the superconductor vanishes as $\vec{B}_{inside} = 0$. Introducing the superconducting ball changes the distribution of $\vec{B}$ outside the ball. Find $\vec{B}(r, \theta, \phi)$ for all $r > R$.

Hint: the relation between $\vec{B}$ and the magnetic field strength $\vec{H}$ and the magnetic moment density $\vec{M}$ is $\vec{B} = \vec{H} + 4\pi\vec{M}$. We assume that $\nabla \cdot \vec{M} = 0$ inside the superconductor. You can use the method of the magnetic scalar potential $\vec{H}(\vec{r}) = -\nabla W(\vec{r})$ to solve for the distribution of the magnetic inductance $\vec{B}(\vec{r})$ outside the superconducting ball. You need to determine the correct boundary conditions.

Hint: the general solution to the Laplace equation in spherical coordinates (r, θ, ϕ) with axial symmetry can be expressed as

$$W(r, \theta, \phi) = \sum_{l=0}^{+\infty} (a_l r^l + \tfrac{b_l}{r^{l+1}}) P_l(\cos\theta). \tag{21}$$

SOLUTION:

1) Outside the sphere, i.e., $r > R$, $M = 0$, $\vec{H} = \vec{B}$, and thus $\nabla \cdot \vec{H} = 0$, which means that $-\nabla^2 W(r, \theta, \phi) = 0$. In this region, we know that $\vec{B}(\vec{r}) = B_0\hat{z}$ as $r \to +\infty$. Thus we have

$$W_{out}(r, \theta, \phi) = a_1 r \cos\theta + \sum_{l=0}^{+\infty} \tfrac{b_l}{r^{l+1}} P_l(\cos\theta). \tag{22}$$

$\vec{B}(\vec{r}) \to B_0\hat{z}$ as $r \to +\infty$ shows that $a_1 = -B_0$.

2) Inside the sphere, i.e. $r < R$. $\vec{H} = -4\pi\vec{M}$, again $\nabla \cdot \vec{H} = -\nabla^2 W(r, \theta, \phi) = 0$. In this region, we have

$$W_{in}(r, \theta, \phi) = \sum_{l=0}^{+\infty} a_l r^l P_l(\cos\theta). \tag{23}$$

3) The boundary condition: W should be continuous at $r = R$, thus

$$W_{in}(R, \theta) = W_{out}(R, \theta), \tag{24}$$

which shows that

$$a_l R^l = \frac{b_l}{R^{l+1}} \tag{25}$$

for $l \neq 1$. and

$$a_1 R = \frac{b_1}{R^2} - B_0 R. \tag{26}$$

From the continuity of the radial component of $\vec{B}$, we have $\vec{B} \cdot \hat{e}_r = 0$ at $r = R$ outside the sphere, which means

$$\frac{\partial W_{out}(r,\theta)}{\partial r}|_{r=R} = 0. \tag{27}$$

From Eq. 27, we arrive at

$$-B_0 P_1(\cos\theta) - \sum_{l=0}^{+\infty}(l+1)\frac{b_l}{R^{l+2}} P_l(\cos\theta) = 0, \tag{28}$$

which shows that for $l \neq 1$, we have

$$b_l = 0, \tag{29}$$

and thus $a_l = 0$ for $l \neq 1$ according to Eq. 25. For $l = 1$, we have

$$B_0 - 2\frac{b_1}{R^3} = 0, \tag{30}$$

thus we have $b_1 = -\frac{1}{2}B_0 R^3$ and $a_1 = -\frac{3}{2}B_0$. Thus the

$$W_{out} = -B_0 r\cos\theta - \frac{B_0}{2r^2}R^3\cos\theta \tag{31}$$

$$= -B_0 z - \frac{B_0 R^3 z}{2r^3}. \tag{32}$$

The $\vec{B}$ field at $r > R$ is

$$\vec{B} = -\nabla W = B_0\hat{z} + \frac{B_0 R^3}{2}\left(\frac{r\hat{z} - 3z\hat{e}_r}{r^4}\right) \tag{33}$$

$$= B_0\hat{z} + B_0\frac{R^3}{2r^3}(\hat{z} - 3\cos\theta\hat{e}_r) \tag{34}$$

Because $\hat{e}_r = \cos\theta\hat{z} + \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y}$, we have

$$\vec{B}(\vec{r}) = B_0\hat{z} + B_0\frac{R^3}{2r^3}\Big\{(1 - 3\cos^2\theta)\hat{z} - \frac{3}{2}\sin 2\theta(\cos\phi\hat{x} + \sin\phi\hat{y})\Big\}, \tag{35}$$

$$\tag{36}$$

for $r > R$.

#14 : GRADUATE E+M

A charge density ρ_0 is placed at time $t = 0$ in a small region in the interior of a homogeneous charge-neutral material that has electrical conductivity σ.

(a) Derive an expression for the time evolution of the charge density in that region, $\rho_c(t)$, with $\rho_c(0) = \rho_0$. Hint: use a continuity equation.

(b) Estimate how long it will take (in seconds) for the charge density to decrease to 1/1000 of its initial value if the material is (i) copper with conductivity $\sigma = 1/(2\mu\Omega cm)$ and (ii) quartz with conductivity $\sigma = 1/(10^{24}\mu\Omega cm)$.

Use $\epsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$.

`SOLUTION:`

(a) Use $\vec{J} = \sigma\vec{E}$, $\vec{\nabla} \cdot \vec{E} = \rho_c/\epsilon_0$ and the continuity equation

$$\vec{\nabla} \cdot \vec{J} + \tfrac{\partial \rho_c}{\partial t} = 0 \tag{37}$$

to get

$$\tfrac{\partial \rho_c}{\partial t} = -\tfrac{\sigma}{\epsilon_0}\rho_c \tag{38}$$

hence

$$\rho_c(t) = \rho_0 e^{-(\sigma/\epsilon_0)t} \equiv \rho_0 e^{-t/\tau} \tag{39}$$

with

$$\tau = \epsilon_0/\sigma \tag{40}$$

(b) For $t = 7\tau$, $\rho_c(t) \sim \rho_0/1000$. We have, with $\epsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$

$$\sigma_{Cu} = 1/(2\mu\Omega cm) = 1/(2 \times 10^{-8}\Omega m) \tag{41}$$

$$t_{Cu} = 7\tau_{Cu} = 7\epsilon_0/\sigma_{Cu} = 1.24 \times 10^{-18} s \tag{42}$$

$$\sigma_{quartz} = 1/(10^{24}\mu\Omega cm) = 1/(10^{16}\Omega m) \tag{43}$$

$$t_{quartz} = 7\tau_{quartz} = 7\epsilon_0/\sigma_{quartz} = 6.2 \times 10^{5} s \tag{44}$$

#15 : GRADUATE QUANTUM MECHANICS

Consider two spin-$\frac{1}{2}$ particles interacting through a Heisenberg Hamiltonian $H = J\vec{S}_1 \cdot \vec{S}_2$, where $\vec{S}_{1,2} = \frac{1}{2}\vec{\sigma}_{1,2}$ and $\vec{\sigma}_{1,2}$ are Pauli matrices for particles 1 and 2, respectively.

(a) Solve for the eigenenergy of each eigenstate, and express each eigenstate in the basis of eigenstates of $\sigma_{1,z}$ and $\sigma_{2,z}$.

(b) Consider an initial two-spin state at $t = 0$ $|\Psi(t = 0)\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are σ_z-eigenstates with eigenvalues 1 and -1, respectively. Calculate the time evolution of the expectation value of $S_{1,z}$ of the first particle.

`SOLUTION:` (a) There are four states

$$\begin{aligned} |\phi_1\rangle &= |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \\ |\phi_{-1}\rangle &= |\downarrow\rangle_1 \otimes |\downarrow\rangle_2 \\ |\phi_0\rangle &= |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \\ |\phi_0'\rangle &= |\downarrow\rangle_1 \otimes |\uparrow\rangle_2. \end{aligned} \tag{45}$$

$H = \frac{J}{2}(S_{1+}S_{2-} + S_{1-}S_{2+}) + JS_{1,z}S_{2,z}$.

$S_+|\uparrow\rangle = 0, \quad S_+|\downarrow\rangle = |\uparrow\rangle, \quad S_-|\uparrow\rangle = |\downarrow\rangle, \quad S_-|\downarrow\rangle = 0.$

It is straightforward to show that $H|\phi_{\pm 1}\rangle = \frac{J}{4}|\phi_{\pm 1}\rangle$.

$$H|\phi_0\rangle = -\frac{J}{4}|\phi_0\rangle + \frac{J}{2}|\phi_0'\rangle \tag{46}$$

$$H|\phi_0'\rangle = -\frac{J}{4}|\phi_0'\rangle + \frac{J}{2}|\phi_0\rangle. \tag{47}$$

After diagonalization, we have the eigenstates $|\phi_0^\pm\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle \pm |\phi_0'\rangle)$. Since $H|\phi_0^+\rangle = \frac{J}{4}|\phi_0^+\rangle$ and $H|\phi_0^-\rangle = -\frac{3J}{4}|\phi_0^-\rangle$, the eigenvalues are $J/4$ and $-3J/4$ respectively.

(b) The intital state can be expressed as

$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|\phi_0^+\rangle + |\phi_0^-\rangle)$, thus its time evolution is

$$\begin{aligned} |\Psi(t)\rangle &= \tfrac{1}{\sqrt{2}}e^{-i\frac{J}{4}t}(|\phi_0^+\rangle + e^{iJt}|\phi_0^-\rangle) \\ &= \tfrac{1}{2}e^{-i\frac{J}{4}t}\Big\{(1+e^{iJt})|\uparrow\rangle_1 \otimes |\downarrow\rangle_2) + (1-e^{iJt})|\downarrow\rangle_1 \otimes |\uparrow\rangle_2)\Big\} \end{aligned} \tag{48}$$

$\langle S_{1,z}(t)\rangle = \frac{1}{4}(|1+e^{iJt}|^2 - |1-e^{iJt}|^2) \times \frac{1}{2} = \frac{1}{2}\cos Jt.$

#16 : GRADUATE QUANTUM MECHANICS

Consider scattering of two relativistic spin 1/2 particles, e.g. electron positron, via a $J^{PC} = 1^{--}$ resonance, e.g. the Y(4S) resonance, that decays into two spin 0 particles, e.g. two B mesons. What is the angular distribution of the outgoing spin 0 particles with regard to the axis of the incoming spin 1/2 particles in the center of mass frame of the spin 1 resonance assuming that total angular momentum (spin and orbital) is conserved.

`SOLUTION:` Two possibilities due to helicity conservation along the beam axis. $e^+_L e^-_R \to \Upsilon(4S) \to B^+B^-$ and $e^+_L e^-_R \to \Upsilon(4S) \to B^+B^-$. In the two cases, $J_Z = -1$ and $+1$ respectively in the initial state. The final state B mesons have no spin, and thus $J_Z = 0$. The momentum distribution is thus given by the two d-functions: d_{0-1} and d_{0+1}, both of which are $\propto sin\theta$. The total cross section is then $\propto |d_{0-1}|^2 + |d_{0+1}|^2 \propto sin^2\theta$. Here θ is the angle with respect to the beam axis.

#17 : GRADUATE STAT MECH (Jenkins)

Consider a relativistic gas of N indistinguishable massless particles. Assume that the particles are classical, not quantum, particles.

(a) Calculate the canonical partition function $Z(T, V, N)$.

(b) Find the Helmholtz free energy $F(T, V, N)$.

(c) Determine the pressure p, the entropy S and the chemical potential μ for this relativistic gas.

(d) Determine the energy $U(T, V, N)$ and the heat capacity at constant volume C_V.

Hint:

$$\int dx\ x^2\ e^{-x} = \Gamma(3)$$

`SOLUTION:` (a)

$$H(q_i, p_i) = \sum_{i=1}^{N} |\vec{p}_i|c$$

$$Z(T, V, N) = \frac{1}{N!\ h^{3N}} \prod_i \int d^3q_i d^3p_i e^{-\beta H(q_i,p_i)} = \frac{1}{N!\ h^{3N}} V^N \left(\int d^3p\ e^{-\beta|p|c} \right)^N$$

$$\int d^3p\ e^{-\beta|p|c} = 4\pi \int_0^\infty dp\ p^2 e^{-\beta cp} = \frac{4\pi}{(\beta c)^3}\Gamma(3) = \frac{8\pi}{(\beta c)^3}$$

$$Z(T, V, N) = \frac{1}{N!\ h^{3N}} \left(\frac{8\pi V}{(\beta c)^3} \right)^N = \frac{1}{N!} \left(8\pi V \left(\frac{kT}{hc} \right)^3 \right)^N$$

(b)

$$F(T, V, N) = -kT \ln Z = -kT \left[N \ln \left(8\pi V \left(\frac{kT}{hc} \right)^3 \right) - \ln N! \right]$$

$$F(T, V, N) = -NkT \left[1 + \ln \left(\frac{8\pi V}{N} \left(\frac{kT}{hc} \right)^3 \right) \right]$$

(c)

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NkT}{V}$$

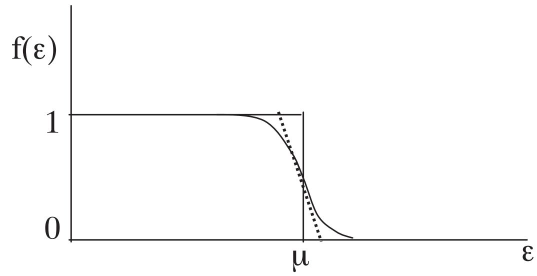
$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = Nk\left[4 + \ln\left(\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right)\right]$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT\ \ln\left(\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right)$$

(d)

$$U(T,V,N) = F + TS = 3NkT$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 3Nk$$



#18 : GRADUATE STAT MECH

The figure shows the Fermi function $f(\epsilon)$ (full line) for electrons in a metal at temperature $k_B T << \epsilon_F$ as function of energy ϵ, with ϵ_F the Fermi energy, and a dotted line that is tangent to the Fermi function at the chemical potential μ. The electrons are assumed to be non-interacting. Assume at this low temperature the chemical potential $\mu \sim \epsilon_F$ is independent of temperature. Taking as an approximation to the true Fermi function the horizontal portions joined by the dotted line, derive an expression for the electronic heat capacity of this metal. Assume the density of states in energy $g(\epsilon)$ is a constant g, independent of energy. Proceed as follows:

(a) Find the function describing the dotted line and the points where the dotted line intersects the horizontal lines (values 1 and 0).

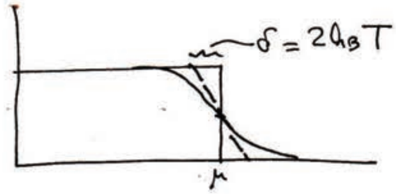
(b) Compute the energy of the electrons in the region of energy where $f(\epsilon) = 1$.

(c) Compute the energy of the electrons in the region where $f(\epsilon)$ is given by the dotted line.

(d) Compute the heat capacity. How does it compare to the correct result for the low temperature heat capacity of a metal?

`SOLUTION:`

Solution



$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \;;\quad f(\mu) = \frac{1}{2}$$

$$f'(\varepsilon) = -\beta \frac{e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2} \;;\quad f'(\mu) = -\frac{\beta}{4}$$

The dashed line is a linear function $\boxed{L(\varepsilon) = \frac{1}{2} - \frac{\beta}{4}(\varepsilon-\mu)}$

We have $L(\mu-\delta) = 1$, $L(\mu+\delta) = 0$, $L(\mu) = f(\mu) = \frac{1}{2}$

with $\boxed{\delta = \frac{2}{\beta} = 2k_BT}$

Energy is $U = \int d\varepsilon\, g(\varepsilon)\, \varepsilon f(\varepsilon) = g \int d\varepsilon\, \varepsilon f(\varepsilon)$

So $U = U_0 + U_1$

$$U_0 = g\int_0^{\mu-\delta} d\varepsilon . \varepsilon . 1 \;;\quad U_1 = g\int_{\mu-\delta}^{\mu+\delta} d\varepsilon . \varepsilon . L(\varepsilon)$$

$$U_0 = g \left.\frac{\varepsilon^2}{2}\right]_0^{\mu-\delta} = g\frac{(\mu-\delta)^2}{2} = \frac{g(\mu-2k_BT)^2}{2}$$

$$\boxed{\frac{dU_0}{dT} = -2k_Bg(\mu-2k_BT) = -2k_Bg\mu + 4k_B^2gT}$$

$$U_1 = g \int_{\mu-\delta}^{\mu+\delta} d\varepsilon \; \varepsilon \left(\frac{1}{2} - \frac{\beta}{4}(\varepsilon-\mu) \right)$$

$$\varepsilon - \mu = x \Rightarrow \varepsilon = \mu + x$$

$$U_1 = g \int_{-\delta}^{\delta} dx \; (\mu+x)\left(\frac{1}{2} - \frac{1}{4}\beta x \right) =$$

$$= g \int_{-\delta}^{\delta} dx \left(\frac{\mu}{2} + \frac{x}{2} - \frac{\mu\beta}{4} x - \frac{\beta}{4} x^2 \right) =$$

$$= g \left[\frac{\mu}{\cancel{2}} \cdot \delta \cdot \cancel{2} - \frac{\beta}{6}\delta^3 \right] = 2g\mu k_B T - \frac{4}{3} g k_B^2 T^2$$

$$\boxed{\frac{\partial U_1}{\partial T} = 2g\mu k_B - \frac{8}{3} g k_B^2 T}$$

Heat capacity:

$$\boxed{C = \frac{\partial U}{\partial T} = \frac{\partial U_0}{\partial T} + \frac{\partial U_1}{\partial T} = \left(4 - \frac{8}{3}\right) g k_B^2 T = \frac{4}{3} g k_B^2 T}$$

The exact answer is:

$$\boxed{C_{exact} = \frac{\pi^2}{3} g k_B^2 T}$$

so same functional dependence (linear in T), numerical factor slightly different.

#19 : GRADUATE MATH METHODS

Using contour integration find the value of

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx.$$

Hint: $\ln(x^2+1) = \ln(i-x) + \ln(i+x) - i\pi$.

`SOLUTION:` Consider

$$I = \oint \frac{\ln(z+i)}{z^2+1}dz$$

around a contour in the upper half plane going from $(-R, 0)$ to $(R, 0)$ along the x-axis and returning along a semicircle Γ of radius R in the upper half plane. The only pole is simple at $z = i$ and has residue $\frac{\ln(2i)}{2i}$. Thus by the residue theorem $I = 2\pi i \ln(2i)/(2i) = \pi \ln(2) + i\pi^2/2$.

Now write the contour integral in three parts:

$$I = \int_{-R}^0 \frac{\ln(x+i)}{x^2+1}dx + \int_0^R \frac{\ln(x+i)}{x^2+1}dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}dz.$$

Replacing x by $-x$ in the first integral, the first two integrals can be combined using the Hint:

$$I = \int_0^R \frac{\ln(x^2+1)}{x^2+1}dx + \int_0^R \frac{i\pi}{(x^2+1)}dx + \int_\Gamma \frac{\ln(z+i)}{z^2+1}dz.$$

Finally take the limit $R \to \infty$ and show that the contribution from the Γ path is zero. Then equating just the real parts of the residue solution, one gets

$$\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}dx = \pi \ln 2.$$

#20 : GRADUATE EXP-GENERAL

An emission line is observed over a certain range of wavelengths in a spectrum. We suspect that an emission line is present in three adjacent pixels that contain 62, 71 and 69 photons. The emission line is observed on top of a background of with the following number of photons per pixel on either side of the emission line and away from the pixels where the emission is suspected: 52, 48, 60, 42, 45, 59, 51, 61, 43.

(i) What is the mean background per pixel and the uncertainty in this value?

(ii) What is the statistical significance of the emission line? Examine the pixels individually.

(iii) What is the statistical significance of the emission line if you examine the pixels together?

(iv) In the above we assume that we know the pixels in which the emission line might be found. What is the significance of the emission line if instead we know the expected width but not the position of the line prior to collecting the data? Assume that the spectrum contained 1024 pixels and the line could lie anywhere in the spectrum.

It might help to know that for a normal distribution the fraction of the area lying between $-n\sigma$ and $+n\sigma$ is 68.27% for $n = 1$; 95.45% for $n = 2$; 99.73% for $n = 3$; and 99.9937% for $n = 4$.

`SOLUTION:`

(i) Assume the background is constant across the pixels and that the fluctuations are due to Poisson statistics. Mean background per pixel is the sum of the values/9, or $461/9 = 51.2$. The standard deviation of the 9 background values is 7.4, which is consistent with the assumption of that the distribution is Poisson, since the standard deviation of a Poisson distribution is $\sqrt{mean}$ or $\sqrt{51.2} = 7.2$. The uncertainty in the mean background per pixel is then $\sqrt{sum}/9 = 21.5/9 = 2.4$, or $7.4/\sqrt{9}$ because the background is averaged over 9 pixels.

(ii) Is there an excess over backgound in any of the 3 pixels? In the central pixel, the excess is $71 - 51.2 = 19.8$. We want to know the $Prob(\geq 71$ given mean $= 51)$, under the assumption (null hypothesis) that there is no excess, only random fluctuation in the background. We could calculate this using the Poisson distribution. Since 71 >> 10, we can use the Gaussian approximation with mean 51.2 and standard deviation 7.2. 19.8 is then 2.75 sigma above the mean, which is expected less than 5% of the time. The excess in each of the other two pixels is less significant: 1.5 sigma and 2.5 sigma. It is customary (with no justification) to call results $> 2\sigma$ significant.

(iii) In 3 pixels we expect 153.6 photons and the uncertainty in this expected value (not in the actual number of background photons in those 3 pixels that is not known) is $\sqrt{sum}/3 = 7.2$. The total counts in the 3 pixels is 202 photons. The excess is 48.4. We now want $Prob(\geq 202$ given mean $153.6)$, where 1-standard deviation is $\sqrt{153.6} = 12.4$, and 48.4 is then 3.9 sigma. We do not take $\sqrt{202}$ because there is no uncertainty in the counted number of photons.

(iv) This is now an example of *a posteriori* statistics, an example of which was discussed in the Physics Colloquium on HLC and the Higgs boson in March 2012.

First let us examine pixel by pixel, and assume each pixel is independent. We need to find $Prob(\geq$ one excess $> 2.7\sigma$, given 1024 trials). We recall that we expect one excess $> 2\sigma$ in 2.1% of trials (with another 2.1% giving deficits $> 2\sigma$), and $> 3\sigma$ in 0.13% of trials. We then expect about 1.3 events $> 3\sigma$ in 1024 independent pixels.

Next let us look at the 3-pixel $3.9 - \sigma$ excess. The probability of one excess $> 4\sigma$ in one pixel is 0.000031. There are about 341 independent places where 3-pixels can be located, so the chance of one 3-pixel region somewhere in the spectum exceeding 3.9σ is about 1%, which we call significant.