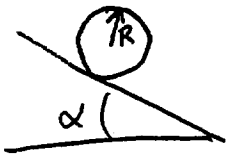
PHYSICS DEPARTMENTAL EXAM

SPRING 2001

EXAM AND SOLUTIONS

Undergraduate Mechanics

A uniform wheel, of total mass M and radius R, rolls down an incline plane without slipping. The angle of the plane with respect to the horizon is α and gravity is g, pointing downward as usual. Find the acceleration of the drum down the incline.



$$Mg\sin\theta - f = Ma$$

torque about wheel axis: $Rf = I\dot{\omega}$

$$I = \frac{1}{2}MR^2 \qquad \dot{\omega} = a/R$$

So $f = \frac{1}{2}Ma \implies \frac{3}{2}a = g\sin\alpha$

$$a = \frac{2}{3}g\sin\alpha$$

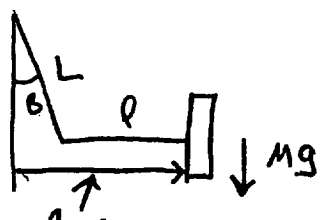
I - sol 2

Undergraduate Mechanics

A uniform wheel, of total mass M, and radius R spins at angular velocity ω_s, at the end of a shaft of length ℓ. The wheel and shaft are suspended by a string of length L, which is tied to the other end of the shaft. The shaft and string have negligible mass. There is gravity g, pointing down as usual. The spinning wheel exhibits gyroscopic precession, with the shaft horizontal. The string makes an angle β with respect to the vertical. Assume that β is small enough so that approximations like $\sin\beta \approx \beta$ are valid. Assume also that $\beta \ll \ell/L$

a) How long does it take the gyro and shaft to precess all the way around and return to its starting point?

b) What is the angle β?



$\ell + L\sin\beta \approx \ell$. $\vec{\tau} = Mg\ell\hat{\Theta}$ torque (pointing into paper)

$$\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow |\vec{\tau}| = \Omega|\vec{L}|$$

$$|\vec{L}| = I\omega_s = \frac{1}{2}MR^2\omega_s$$

$$T_{precess} = \frac{2\pi}{\Omega} = 2\pi\frac{|\vec{L}|}{|\vec{\tau}|} = \frac{\pi \cancel{M} R^2\omega_s}{\cancel{M}g\ell} = \frac{\pi R^2\omega_s}{g\ell}$$

) $F\cos\beta = Mg$, $F\sin\beta = M\Omega^2\ell$

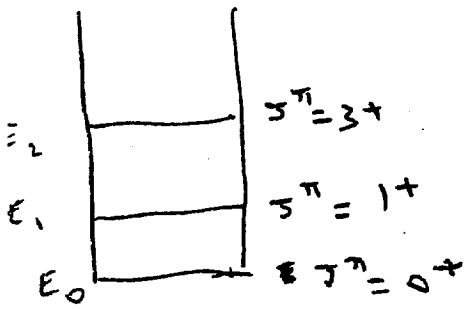
$F \approx Mg$, $\beta \approx \frac{\Omega^2\ell}{g} = \frac{\ell}{g}\left(\frac{2g\ell}{R^2\omega_s}\right)^2 = \frac{4g\ell^3}{R^4\omega_s^2}$

Undergrad Stat Mech

Consider a ficticious three-level nucleus in thermal equilibrium with a heat bath at temperature T. In particular, all of the three levels of this nucleus have population ratios in accord with thermal equilibrium. The ground state energy of this nucleus is E_0 and the ground state spin and parity are $J^\pi = 0^+$. (The spin and parity quantum numbers label the overall angular momentum and parity of the wave function for a given level.) The first excited state has energy E_1 and spin/parity $J^\pi = 1^+$, while the third state has energy E_2 and spin/parity $J^\pi = 3^+$. The laboratory (zero temperature) decay rate of the ground state is measured to be λ_0, while the laboratory individual decay rates of the first and second excited states are measured to be λ_1 and λ_2, respectively.

What is the decay rate of this nucleus in the thermal bath?

<u>solution</u> U.G. stat mech problem



~~Population~~

partition function

$$Z = \sum_i (2J_i+1)\, e^{-E_i/kT}$$

$$= e^{-E_0/T} + 3\, e^{-E_1/T} + 7\, e^{-E_2/T}$$

population probability for state i

$$p_i = \frac{(2J_i+1)\, e^{-E_i/T}}{Z}.$$

decay rate

$$\lambda = \sum_i \cancel{2J_i+1}\; p_0 p_i\, \lambda_i = \sum_i \frac{(2J_i+1)\, e^{-E_i/T}}{Z}\, \lambda_i$$

$$= \frac{\lambda_0 e^{-E_0/T} + 3\lambda_1 e^{-E_1/T} + 7\lambda_2 e^{-E_2/T}}{e^{-E_0/T} + 3\, e^{-E_1/T} + 7\, e^{-E_2/T}}$$

I- sol. 4

Consider the Sterling cycle, with the following four steps:

1) Isothermal expansion at temperature T_H from volume V_A to volume $V_B > V_A$.

2) Reversibly decrease temperature to $T_C < T_H$ at constant volume V_B.

3) Isothermal compression at temperature T_C from V_B back to V_A.

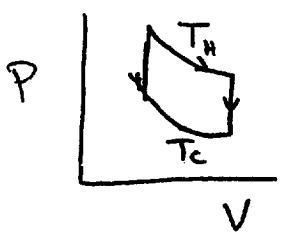
4) Reversibly increase temperature from T_C back to T_H at constant volume V_A.

The gas inside the piston is ideal, with N polyatomic molecules (three rotational degrees of freedom).

a) What is the total work done by the engine in one cycle?

b) What is the change of entropy of the gas in the piston in step (2)? (Write your answer as positive if the entropy increases or negative if it decreases.)

c) . . .



$$W = \int p\,dV = N k_B T \int \frac{dV}{V}$$

$$\Delta W_1 = N k_B T_H \ln(V_B/V_A)$$

$$\Delta W_2 = \Delta W_4 = 0$$

$$\Delta W_3 = -N k_B T_C \ln(V_B/V_A)$$

$$\Delta W_{total} = N k_B (T_H - T_C) \ln(V_B/V_A)$$

$$dE = T\,dS - p\,dV \qquad dV = 0$$

$$= \frac{1}{2} f N k_B\, dT$$

So $dS = \frac{1}{2} f N k_B \frac{dT}{T}$ $\qquad f = \overset{\text{trans}}{3} + \overset{\text{rot}}{3} = 6$

$$\Delta S = -3 N k_B \ln\left(\frac{T_H}{T_C}\right)$$

c) $\Delta S = k_B \ln\left(\frac{\Omega_{after}}{\Omega_{before}}\right)$ $\qquad$ so $\frac{\Omega_{after}}{\Omega_{before}} = \left(\frac{T_H}{T_C}\right)^{-3N}$

Consider a theory with

$$H = (a^\dagger a + \tfrac{1}{2})\hbar\omega + Ca^3 a^{\dagger 3},$$

with $a^\dagger$ and a the usual harmonic oscillator creation and annhiliation operators. Find the energy eigenvalues of this theory.

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\therefore \quad a^3 a^{\dagger 3}|n\rangle = \sqrt{(n+1)(n+2)(n+3)(n+3)(n+2)(n+1)}\,|n\rangle$$

$$= (n+1)(n+2)(n+3)|n\rangle$$

$|n\rangle$ are still energy eigenstates

Eigenvals $\quad E_n = (n+\tfrac{1}{2})\hbar\omega + C(n+1)(n+2)(n+3)$

$$n = 0, 1, 2, \ldots$$

Undergraduate quantum mechanics

A particle of mass m feels the force of gravity, with $\vec{g} = -g\hat{z}$. The floor, which the particle can not penetrate (not even a little), is at $z = 0$. Classically the particle sits on the floor, with energy $E = 0$. Estimate the quantum mechanical energy of the particle in its lowest energy state.

$$V(z) = \begin{cases} mgz & z > 0 \\ \infty & z < 0 \end{cases}$$

$$E = \frac{p^2}{2m} + mgz \longrightarrow E_{min} = \frac{(\Delta p)^2}{2m} + mg\Delta z$$

$$\Delta p \, \Delta z \sim \hbar$$

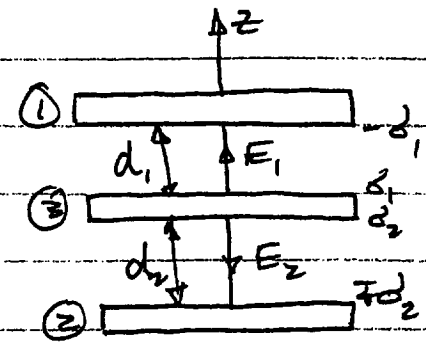
so $$E \sim \frac{(\Delta p)^2}{2m} + \frac{mg\hbar}{\Delta p}$$

minimize w.r.t. Δp: $$\frac{\Delta p}{m} = \frac{mg\hbar}{(\Delta p)^2}$$

$$\Delta p \sim (m^2 g\hbar)^{1/3}$$

$$E \sim \frac{(m^2 g\hbar)^{2/3}}{m} = \underline{\underline{m^{1/3}(g\hbar)^{2/3}}}$$

Problem #7: UG E&M



a) Suppose σ_1 and σ_2 are charge densities on 2 sides of middle plate,

i.e. $A[\sigma_1 + \sigma_2] = Q$.

Then: $\bar{E}_1 = \frac{\sigma_1}{\epsilon_0}\hat{z}$, $\bar{E}_2 = \frac{\sigma_2}{\epsilon_0}(-\hat{z})$

But plates 1 & 2 are connected: $\Delta V_{1-0-2} = 0$

$$\therefore \Delta V = 0 = \int_1^2 \bar{E}\cdot d\bar{s} = -E_2 d_2 + E_1 d_1$$

Hence: $\frac{\sigma_2}{\epsilon_0} d_2 = \frac{\sigma_1}{\epsilon_0} d_1$ or $\frac{\sigma_1}{\sigma_2} = \frac{d_2}{d_1}$

$$\therefore \sigma_1 = \frac{d_2}{d_1+d_2}\frac{Q}{A} \; ; \; \sigma_2 = \frac{d_1}{d_1+d_2}\frac{Q}{A}$$

b) Energy density: $\mu_1 = \frac{\epsilon_0}{2}\bar{E}_1^2 = \frac{\sigma_1^2}{2\epsilon_0}$, $\mu_2 = \frac{\sigma_2^2}{2\epsilon_0}$

$$U_e = \int_V \mu_e d\tau = A d_1 \mu_1 + A d_2 \mu_2$$

$$= \frac{A}{2\epsilon_0}\left[d_1\sigma_1^2 + d_2\sigma_2^2\right]$$

$$= \frac{A}{2\epsilon_0}\left(\frac{Q}{A}\right)^2 \frac{d_1 d_2^2 + d_2 d_1^2}{(d_1+d_2)^2}$$

$$\therefore U_e = \frac{Q^2}{2\epsilon_0 A}\frac{d_1 d_2}{d_1+d_2}$$

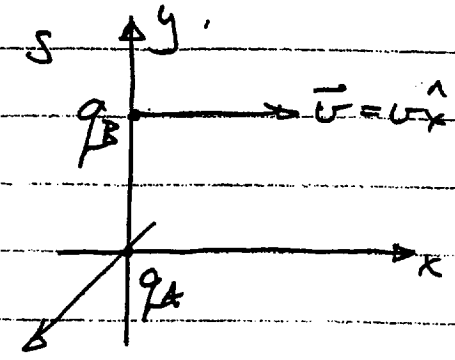
With $d = d_1 + d_2$ fixed, $U_{e,max}$ occurs at $d_1 = d_2 = d/2$ where $\frac{dU_e}{dd_1} = 0$

and $U_{e,max} = (Q^2 d / 8\epsilon_0 A)$

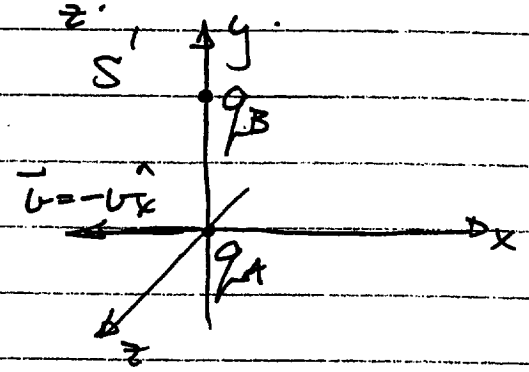
Problem #8: UG E&M

a) $\vec{F}_{A\to B} = q_B \vec{E}_A = \frac{q_A q_B}{4\pi\epsilon_0 d^2}\hat{y}$

since q_A is stationary in frame S



b) (i) $\vec{u} = +v\hat{x}$ is velocity of q_B in S



Transforming Forces:

$F_y' = \frac{F_y}{\gamma(1-\beta v/c)} = \gamma F_y$

$F_z' = \frac{F_z}{\gamma(1-\beta v/c)} = 0$ since $F_z = 0$

$F_x' = \frac{F_x^{\,0} - \beta(\vec{u}\cdot\vec{F})/c}{1-\beta v/c} = 0$ $\quad\therefore F'_{A\to B} = \frac{\gamma q_A q_B}{4\pi\epsilon_0 d^2}\hat{y}$

where $\gamma = \frac{1}{(1-\beta^2)^{1/2}}$

(ii) $\vec{E}_A' = \gamma\vec{E}_A = \frac{\gamma q_A}{4\pi\epsilon_0 d^2}\hat{y}$ since transverse to direction of relative motion.

$\vec{B}_A' = \frac{\gamma v}{c^2}\frac{q_A}{4\pi\epsilon_0 d^2}(-\hat{z})$ since $\vec{B} = 0$ in frame S.

$\therefore \vec{F}'_{A\to B} = q_B(\vec{E}_A' + \vec{u}_B' \times \vec{B}_A') = \frac{\gamma q_A q_B}{4\pi\epsilon_0 d^2}\hat{y}$ as in part (i).

9

I - SOL. 9

(1.) Order of Magnitude Estimates

(a) How long does it take a photon created at the center of the sun to random walk its way to the surface? The mass of the sun is $M_\odot \approx 2 \times 10^{33}\,\mathrm{g}$ and the solar radius is $R_\odot \approx 7 \times 10^{10}\,\mathrm{cm}$. (Hint: Estimate the mean free path of the photon using the Thomson cross section $\sigma_\mathrm{T} \approx 10^{-24}\,\mathrm{cm}^2$.)

(b) Say that all sources of nuclear energy in the sun were suddenly shut off. At its current luminosity ($L_\odot \approx 4 \times 10^{33}\,\mathrm{erg\,s^{-1}}$), how long would it take the sun to radiate away all its internal energy? The central temperature of the sun is $\sim 1\,\mathrm{keV}$.

(c.) Why is there such a large disparity between the timescales in parts (a) and (b)?

Solution

) average density: $\frac{4}{3}\pi R_\odot^3 \bar{\rho} = M_\odot$

$$\Rightarrow \bar{\rho} \simeq 1.4 \text{ g cm}^{-3}$$
$$\simeq 1 \text{ g cm}^{-3}$$

$$\Rightarrow \bar{n}_e \simeq \bar{\rho} N_A \simeq 8\times10^{23} = 10^{24} \text{ cm}^{-3}$$

an free path $\Rightarrow$

$$\bar{\lambda}_{mfp} = \frac{1}{\bar{n}_e \sigma_T} \simeq \frac{1}{(10^{24}\text{ cm}^{-3})(10^{-24}\text{ cm}^2)} \simeq 1 \text{ cm}$$

Random walk $\Rightarrow R_\odot = \sqrt{N}\,\bar{\lambda}_{mfp}$

N = # of scatterings

$$N = \left(R_\odot / \bar{\lambda}_{mfp}\right)^2 \simeq 4.9\times10^{20}$$
$$\simeq 5\times10^{21}$$

travel speed of light between scatterings

$$\Rightarrow \tau_{\text{Random walk}} = N\frac{\bar{\lambda}_{mfp}}{c} \simeq (5\times10^{21})\left(\frac{1\text{ cm}}{3\times10^{10}\text{ cm/s}}\right)$$

$$\simeq \frac{5}{3}\times10^{11}\text{ s} \sim 5000 \text{ yrs}$$

) internal energy $U \sim \frac{3}{2} N kT \sim N kT$

$$\sim (2\times10^{33}\text{ g})\left(6\times10^{23}\frac{\text{cm}^2}{\text{s}^2}\right)(1\text{ keV})$$
$$\sim 1.2\times10^{57}\ (1\text{ keV})$$
$$\sim 1.2\times10^{57}\ (1\times10^{-9}$$

$$\tau_{\text{Kelvin}} \simeq \frac{U}{L_\odot}$$

10^{48} ... $\simeq$

or, one could apply virial theorem

$$U \sim \text{Grav Pot.} \sim \frac{G M_\odot^2}{R_\odot} \sim \frac{1}{R_\odot}\left(\frac{M_\odot}{m_{pl}}\right)^2 \hbar c$$

$$\sim \frac{1}{7\times 10^{23}\,\text{fm}}\left(\frac{10^{60}\,\text{MeV}}{10^{22}\,\text{MeV}}\right)^2 200\ \text{MeV-fm}$$

$$\sim 2\times 10^{54}\,\text{MeV} \sim 10^{48}\,\text{ergs}$$

(c.) The radiation field in the sun accounts for very little of the internal energy.

(energy density of photons $\sim aT^4$

$$\sim \frac{\pi^2}{15}T^4 \sim T^4 \sim \frac{(10^{-3}\,\text{MeV})^4}{(200\ \text{MeV-f})^3}$$

$$\sim 10^{14}\,\text{erg cm}^{-3}$$

$$\Rightarrow U_{\text{int for }\gamma} \sim 10^{46}\,\text{erg} \ll \frac{1}{10}U_{\text{int}}^{\text{total}}\ )$$

much less because we have used $T = 1\,\text{keV}$ whereas "mean" temperature is [illegible]

Problem 10: UG Experiment

a) ^{14}C decays according to $N = N_0 e^{-\lambda t}$, yielding a counting rate:

$$A = -\frac{dN}{dt} = \lambda N_0 e^{-\lambda t} = \lambda N .$$

Hence: $\Delta A = \left|\frac{dA}{dt}\right| \Delta t = |-\lambda A| \Delta t$

or $\frac{\Delta A}{A} = \lambda \Delta t$.

With $\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730\,yr} = 1.21 \times 10^{-4}\,yr^{-1}$,

and $\Delta t = 50$ years; assuming A is Poisson distributed:

$$\frac{\Delta A}{A} = \frac{\sqrt{A}}{A} = \frac{1}{\sqrt{A}} = 50\lambda$$

or $A = \frac{1}{(50 \times 1.21 \times 10^{-4})^2} = 2.73 \times 10^4\,h^{-1}$.

For x grams of ^{14}C:

$$A = \frac{\lambda x N_A}{12} \times 10^{-12} e^{-5000\lambda}$$

Hence: $x = \frac{12 A \times 10^{12} \times e^{5000\lambda}}{N_A \lambda}$

$$= \frac{12 \times 2.73 \times 10^4 \times 365 \times 24}{6.023 \times 10^{23} \times 1.21 \times 10^{-4}} \times 10^{12} \times e^{5000\lambda}$$

$= 72.1\,g$.

b) With background counting A_B, the total rate is $A+A_B \pm \sqrt{A+A_B}$, where A_B is known precisely ($\Delta A_B = 0$)

$$\therefore \Delta(A+A_B) = \Delta A = \sqrt{A+A_B}$$

or $$\frac{\Delta A}{A} = \sqrt{\frac{1}{A} + \frac{A_B}{A^2}}$$

Yields quadratic eq'n for A:

$$(\lambda \Delta t)^2 A^2 - A - A_B = 0$$

with solution:

$$A = \frac{1}{2(\lambda\Delta t)^2}\left(1 + \sqrt{1+4(\lambda\Delta t)^2 A_B}\right)$$

$$= \frac{1}{2\times(1.21\times10^{-4}\times50)^2}\left[1+\sqrt{1+4\times(1.21\times10^{-4}\times50)^2\times4000}\right]$$

$$= 3.09\times10^4\ h^{-1}.$$

In this case, the sample mass required is:

$$x = \frac{3.09\times10^4}{2.73\times10^4}\times 72.1 = 81.6\,g.$$

PHYSICS DEPARTMENTAL EXAM
SPRING 2001

PART II - SOLUTIONS

Graduate mechanics

The Hamiltonian is

$$H = q_1 p_1 - q_2 p_2 + a q_2^2 - b q_1^2$$

with a and b constants and the q_i and p_i canonical coordinates and momenta. Show that

$$F_1 = \frac{p_1 - b q_1}{q_2}, \quad \text{and} \quad F_2 = q_1 q_2$$

are constants of the motion. Are there any additional constants generated by Jacobi's identity?

Poisson Brackets: $[q_i, p_j] = \delta_{ij}$ (others vanish)

$$[H, F_1] = [q_1(p_1 - b q_1), F_1] - q_2 (p_1 - b q_1) \underbrace{[p_2, q_2^{-1}]}_{= \frac{1}{q_2^2}}$$

$$= \frac{p_1 - b q_1}{q_2} - \frac{(p_1 - b q_1)}{q_2} = 0 \quad \checkmark$$

$$[H, F_2] = q_1 q_2 [p_1, q_1] - q_2 q_1 [p_2, q_2] = 0 \checkmark$$

$$[F_1, F_2] = -1$$

$\therefore$ no new sol'n.

Graduate mechanics

A particle moves along the curve

$$x = \ell(2\phi + \sin 2\phi) \qquad y = \ell(1 - \cos 2\phi)$$

in a uniform gravitational field in the negative y direction. Find the oscillation period using action angle variables. (Assume that the maximum value of $\phi < \pi/2$.)

Grad Mechanics action-angle.

A particle moves along the curve

$$x = l\,(2\phi + \sin 2\phi) \qquad y = l(1-\cos 2\phi)$$

in a uniform gravitational field in the -ve y direction. Find the oscillation period using action angle variables, [Assume that the maximum value of $\phi < \pi/2$]

soln:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2} m l^2 \dot{\phi}^2 \left[(2 + 2\cos 2\phi)^2 + (2\sin 2\phi)^2\right]$$

$$= 2 m l^2 \dot{\phi}^2 \left[1 + 2\cos 2\phi + \cos^2 2\phi + \sin^2 2\phi\right]$$

$$= 8 m l^2 \dot{\phi}^2 \cos^2\phi$$

$$V = -mgy = -mgl\,(1-\cos 2\phi)$$

$$= -2mgl \sin^2\phi$$

$$\therefore\ L = 8 m l^2 \cos^2\phi\, \dot{\phi}^2 + 2mgl \sin^2\phi$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = 16 m l^2 \cos^2\phi\, \dot{\phi}$$

$$H = \frac{p_\phi^2}{32 m l^2 \cos^2\phi} + 2mgl \sin^2\phi$$

$$\therefore\ p_\phi = \sqrt{32 m l^2 \cos^2\phi\,(H - 2mgl \sin^2\phi)}$$

$$= \pm \sqrt{32 m l^2}\, \cos\phi \sqrt{H - 2mgl \sin^2\phi}$$

$$J = \oint p\,dq = \oint p_\phi\, d\phi = 4 \int_0^{\phi_{max}} \sqrt{32 m l^2}\, \cos\phi \sqrt{H - 2mgl \sin^2\phi}$$

where $\sin^2\phi_{max} = H/(2mgl)$

let $\sin\phi = u$

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$$J = 4\sqrt{32 m l^2} \int_0^{\sqrt{H/2mgl}} \sqrt{H - 2mgl u^2}\, du$$

$$= 4\sqrt{32 m l^2} \cdot \left\{ \frac{1}{2} u \sqrt{H - 2mgl u^2} + \frac{H}{2\sqrt{2mgl}} \sin^{-1}\sqrt{\frac{2mgl}{H}}\, u \right\}_0^{\sqrt{H/2mgl}}$$

$$= 4\sqrt{32 m l^2} \cdot \frac{H}{2\sqrt{2mgl}} \cdot \frac{\pi}{2}$$

$$J = \frac{\pi H}{4}\sqrt{\frac{l}{g}}$$

$$\tau = \frac{\partial J}{\partial H} = \frac{\pi}{4}\sqrt{\frac{l}{g}}$$

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Graduate quantum mechanics

Consider the scattering of a particle of mass m off a spherically symmetric delta function shell of radius R.

$$V(r) = \gamma\delta(r - R)$$

(a) What is the s-wave wave function and phase shift δ_0 for general wave vector k?

(b) What is the s-wave partial scattering cross section? (You can write it in terms of known functions and quantities; you needn't spend too much time simplifying the expression.)

s wave soln: $\psi = u(r)/r$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + Vu = \frac{\hbar^2 k^2}{2m}u$$

$$u(r) = \begin{cases} A\sin(kr) & 0 \le r \le R \\ B\sin(kr+\delta_0) & r > R \end{cases}$$

$u(r)$ continuous @ $R \Rightarrow$ $A\sin kR = B\sin(kR+\delta_0)$

integrate above (*) $\int_{R-\varepsilon}^{R+\varepsilon} \Rightarrow u'\Big|_{R-\varepsilon}^{R+\varepsilon} = \frac{2m\gamma}{\hbar^2}u(0)$

$$\Rightarrow \tan(kR+\delta_0) = \tan kR \Big/ \left(1 - \frac{2\gamma m}{\hbar^2 k}\tan(kR)\right)$$

$$\delta_0 = -kR + \tan^{-1}\left(\frac{\tan kR}{1 - \frac{2\gamma m}{\hbar^2 k}\tan kR}\right)$$

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0$$

Graduate Quantum mechanics

A non-relativistic electron, with magnetic moment $\vec{\mu} = \mu_o \vec{\sigma}$, is in a constant magnetic field $\vec{B} = B_0 \hat{x}$. At time $t = 0$, the spin $\vec{S} = \frac{1}{2}\vec{\sigma}$ is measured to be $+\frac{1}{2}$ along the $\hat{z}$ axis. Find the probability $P(t)$ for measuring the spin to be $+\frac{1}{2}$ along the $\hat{y}$ axis at a later time t. Recall that the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$i\hbar \frac{d}{dt} |\psi\rangle = -\mu_0 B_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} |\psi\rangle \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\hookrightarrow \quad |\psi(t)\rangle = \begin{pmatrix} \cos(\omega t) \\ i\sin(\omega t) \end{pmatrix} \qquad \omega \equiv \mu_0 B/\hbar$$

σ_y eigenvectors $\quad |y+\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\qquad |y-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(\cos\omega t + \sin\omega t \right) |y+\rangle +$$

$$+ \frac{1}{\sqrt{2}} \left(\cos\omega t - \sin\omega t \right) |y-\rangle$$

$$P(t) = \frac{1}{2} \left(1 + \sin\left(\frac{2\mu_0 B t}{\hbar} \right) \right)$$

Graduate electrodynamics

Consider a solid sphere, of radius R, which is made of a substance of permeability μ, with no intrinsic magnetization (i.e. $\vec{B} = \mu\vec{H}$). Outside the sphere is vacuum, with permeability μ_0. There is a uniform, external magnetic field $\vec{B}_{ext} = B\hat{z}$ filling space. Find the magnetic field everywhere in space, both inside and outside the sphere.

$$\vec{H} = -\nabla\Phi_M \qquad \nabla^2\Phi_M = 0 \quad \text{inside \& out}$$

$$\Phi_M = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} \left(-Br\,\delta_{\ell,1} + \frac{C_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) & r > R \end{cases}$$

match $\vec{B}\cdot\hat{r}$ & $\vec{H}\times\hat{r}$ @ $r = R$

$$\Rightarrow \quad \mu\,\ell A_\ell R^{\ell-1} = \mu_0\left(-B\,\delta_{\ell,1} - \frac{(\ell+1)C_\ell}{R^{\ell+2}}\right)$$

$$A_\ell R^\ell = -BR\,\delta_{\ell,1} + \frac{C_\ell}{R^{\ell+1}}$$

$$\Rightarrow \quad A_{\ell\neq 1} = C_{\ell\neq 1} = 0$$

$$A_1 = -\frac{3B}{2+\frac{\mu}{\mu_0}} \qquad C_1 = \left(\frac{\frac{\mu}{\mu_0}-1}{\frac{\mu}{\mu_0}+2}\right) BR^3$$

$$\Rightarrow \quad \vec{B} = \frac{3B\hat{z}}{\left(\frac{\mu}{\mu_0}+2\right)} \qquad r < R$$

$$\vec{B} = B\hat{z} + \frac{BR^3}{r^3}\left(\frac{\frac{\mu}{\mu_0}-1}{\frac{\mu}{\mu_0}+2}\right)\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)$$

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Charges q and $-q$ are separated by a vector $\vec{d}(t)$, which rotates about its center (which we take to be the origin of our coordinate system) in a plane. Find the leading term in the vector potential $\vec{A}(\vec{r},t)$ at a large distance $|\vec{r}| \equiv r \gg |\vec{d}|$. Be sure to take the finite speed of light into account. Find the leading term in the magnetic field at large r. What is the radiated power $dP(\vec{r},t)$ into solid angle $d\Omega$. (You should find that dP depends only on $\hat{r} = \vec{r}/r$, independent of the magnitude $r = |\vec{r}|$ as is reasonable for radiated, conserved radiation energy.) Write your answers in terms of $\vec{d}$ and its various time derivatives.

$$\vec{A}(\vec{r},t) = \int d^3\vec{x}' \, \frac{\vec{J}(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c})}{c\,|\vec{x}-\vec{x}'|}$$

$$\rightarrow \quad \vec{A}(\vec{r},t) \approx \frac{q}{rc}\,\dot{\vec{d}}(t - r/c) + O(1/r^2)$$

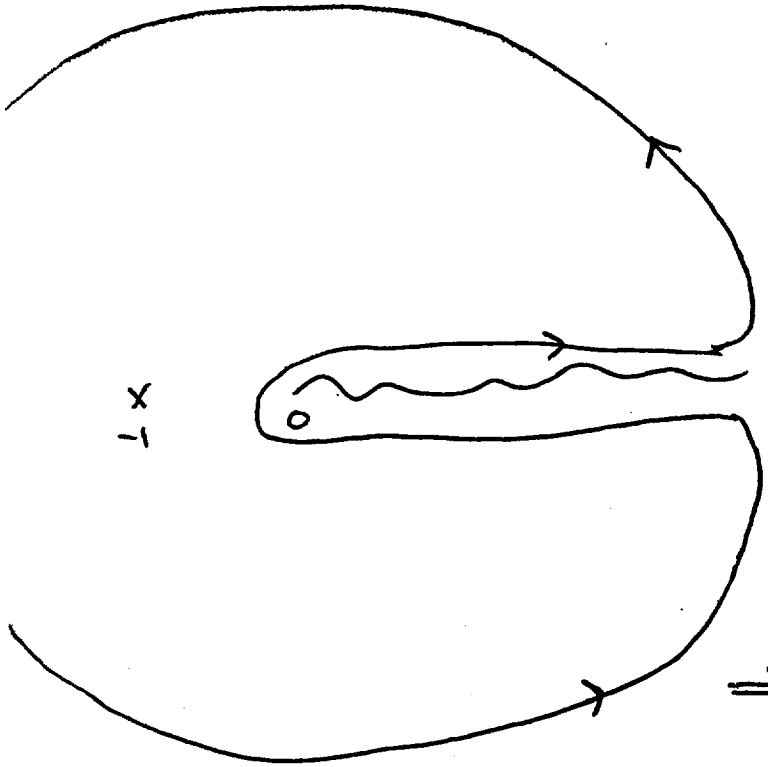
$$\vec{B} = \nabla \times \vec{A} = \frac{q}{rc^2}\,\ddot{\vec{d}}(t - r/c) \times \hat{r} + O(1/r^2)$$

$$\vec{S} = \frac{c}{4\pi}\left(\frac{q}{rc^2}\right)^2 |\ddot{\vec{d}} \times \hat{r}|\,\hat{n}$$

$$dP = \frac{q^2}{4\pi c^3}\,|\ddot{\vec{d}} \times \hat{r}|^2\,d\Omega$$

Evaluate the function

$$I(z) = \int_0^\infty \frac{t^{z-1}dt}{1+t}.$$



$$\oint = I\left(1 - e^{2\pi i z}\right)$$

$$= 2\pi i\left(e^{i\pi(z-1)}\right)$$

$$\Rightarrow I = \frac{2\pi i}{e^{i\pi z} - e^{-i\pi z}}$$

$$\therefore \quad I(z) = \frac{\pi}{\sin \pi z}$$

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n quantities $x_1, \ldots x_n$ are statistically independent; each has a Gaussian distribution with zero mean and common variance: $\langle x_i \rangle = 0$, $\langle x_i x_j \rangle = \sigma^2 \delta_{ij}$ Define the variable $r \equiv \sqrt{\sum_{i=1}^n x_i^2}$. Find the probability distribution $P(r)$ for this variable, i.e.

$$\langle f(r) \rangle = \int_0^\infty dr f(r) P(r)$$

for any function $f(r)$. Fix the normalization of $P(r)$ so that $\langle 1 \rangle = 1$. Now compute $\langle r^p \rangle$ for general p (for $p = 2$ your answer should reduce to an answer which can easily be obtained directly from the above $\langle x_i x_j \rangle$). Your answer might involve $\Gamma(a) = \int_0^\infty \frac{dt}{t} e^{-t} t^a$.

$$P(x_1 \ldots x_n) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-r^2/2\sigma^2}$$

integrated $\quad dx_1 \ldots dx_n \sim r^{n-1} dr$

$$\therefore P(r) = A r^{n-1} e^{-r^2/2\sigma^2} \qquad 1 = \int_0^\infty dr\, P(r)$$

$$\Rightarrow A^{-1} = \int_0^\infty \frac{dr}{r} r^n e^{-r^2/2\sigma^2} = \frac{1}{2} \int_0^\infty \frac{dt}{t} (2\sigma^2 t)^{n/2} e^{-t}$$

$$t = r^2/2\sigma^2$$

$$\therefore P(r) = \frac{2}{(2\sigma^2)^{n/2}\Gamma(n/2)} r^{n-1} e^{-r^2/2\sigma^2}$$

$$\langle r^p \rangle = \frac{1}{(2\sigma^2)^{n/2}\Gamma(n/2)} \int_0^\infty \frac{dt}{t} (2\sigma^2 t)^{\frac{n+p}{2}} e^{-t}$$

$$= (2\sigma^2)^{p/2} \Gamma\left(\frac{n+p}{2}\right) / \Gamma(n/2)$$

for $\langle r^2 \rangle = 2\sigma^2 \dfrac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})} = n\sigma^2$ ✓

Solution

(a) From elementary statistical mechanics of the free Fermi gas, one has at $T = 0$ the Fermi energy $\varepsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$, where $n = N/V$ is the density. Thus for our system we have

$$E_e(N_e, V, P) = \frac{\hbar^2}{2m}(\tfrac{3}{5}\, k_F^2 - a^{-2})\, N_e P^2 \ , \quad .$$

where $k_F = (3\pi^2 N_e/V)^{1/3}$.

(b) The nuclear entropy is

$$\begin{aligned} S_n &= -N_n k_B \operatorname{Tr}(\varrho \ln \varrho) \\ &= -N_n k_B \left(\frac{1+P}{2} \ln \frac{1+P}{2} + \frac{1-P}{2} \ln \frac{1-P}{2} \right) \ . \end{aligned}$$

(c) Writing $F = E_e - TS_n$ and setting $\partial F/\partial P = 0$, we obtain

$$\frac{\hbar^2}{m}(\tfrac{3}{5}\, k_F^2 - a^{-2})\, N_e P + \tfrac{1}{2} N_n\, k_B T \ln \frac{1+P}{1-P} = 0 \ ,$$

which is equivalent to

$$P = \tanh \left(\frac{\hbar^2}{m k_B T}(a^{-2} - \tfrac{3}{5}\, k_F^2)\, \frac{N_e}{N_n}\, P \right) \ .$$

This is the usual equation from Ising mean field theory. The critical temperature may be read off:

$$k_B T_c = \frac{\hbar^2}{m}(a^{-2} - \tfrac{3}{5}\, k_F^2)\, \frac{N_e}{N_n} \ .$$

Thus, T_c vanishes for densities greater than n_c, where

$$n_c = \frac{(\frac{5}{3})^{2/3}}{3\pi^2 a^3} \ .$$

<u>Solution</u>

Along the coexistence curve, the Gibbs free energy per particle, $g = G/N$, must be the same for both phases. (Incidentally, the Gibbs-Duhem relation says $g = \mu$, the chemical potential.) Expanding the differential of g, we obtain

$$\begin{aligned} dg &= \frac{\partial g}{\partial T}\,dT + \frac{\partial g}{\partial p}\,dp + \frac{1}{2}\frac{\partial^2 g}{\partial T^2}(dT)^2 + \frac{\partial^2 g}{\partial T\,\partial p}\,dT\,dp + \frac{1}{2}\frac{\partial^2 g}{\partial p^2}(dp)^2 + \ldots \\ &= -s\,dT + v\,dp - \frac{c_p}{2T}(dT)^2 + v\alpha\,dT\,dp - \frac{1}{2}v\kappa\,(dp)^2 + \ldots\,. \end{aligned}$$

Setting $dg_s = dg_l$ we obtain the result

$$0 = -\Delta s\,dT + \Delta v\,dp - \frac{\Delta c_p}{2T}(dT)^2 + \Delta(v\alpha)\,dT\,dp - \frac{1}{2}\Delta(v\kappa)\,(dp)^2 + \ldots\,.$$

When Δs or Δv is nonzero (*i.e.* a latent heat or a discontinuous volume change accompanies the transition), we may stop at first order and we obtain the usual Clapeyron relation. In our case, $\Delta s = \Delta v = 0$ and we must go to second order. This yields a quadratic equation for dp/dT,

$$\Delta\kappa\left(\frac{dp}{dT}\right)^2 - 2\,\Delta\alpha\,\frac{dp}{dT} + \frac{\Delta c_p}{vT} = 0\,,$$

the solution of which is

$$\left(\frac{dp}{dT}\right)_{\text{coex}} = \frac{\Delta\alpha}{\Delta\kappa} - \sqrt{\left(\frac{\Delta\alpha}{\Delta\kappa}\right)^2 - \frac{\Delta c_p}{vT\Delta\kappa}}\,.$$

(That one takes the above root of the quadratic equation is apparent from consideration of the case $\Delta\kappa = 0$, when the equation becomes linear.)

Evaluating this expression with the data given yields

$$\begin{aligned} \left(\frac{dp}{dT}\right)_{\text{coex}} &= (1-\sqrt{2}) \times 10^5\,\text{Pa/K} \\ &= -4.14 \times 10^4\,\text{Pa/K}\,. \end{aligned}$$