INSTRUCTIONS
PART I
PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (*E.g.* Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only.

Circle the seven problems you wish to be graded:

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	**1 2**	**3 4**	**5 6**	**7 8**	**9 10**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, etc.) in a corner of the exam room.

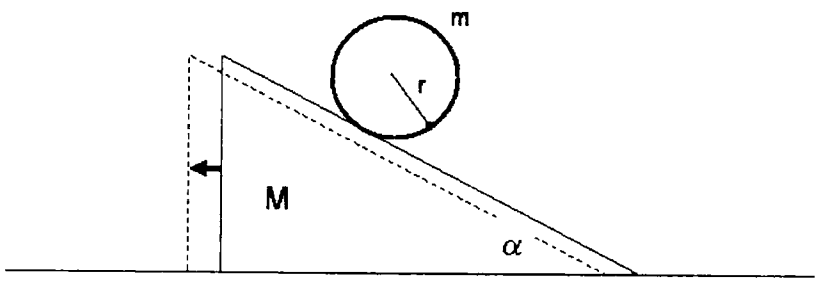
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

*Colored scratch paper is provided and may be discarded when the examination is over.

At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 (SECTION 1)

PROBLEM: A hoop of mass m and radius r rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations for the hoop and the plane if the plane can slide without friction along a horizontal surface.

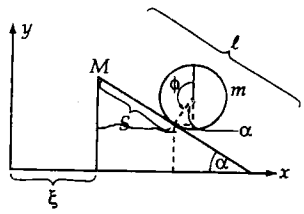


UG 1 SOL

Undergraduate Mechanics Problem 1

A hoop of mass m and radius r rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations for the hoop and the plane if the plane can slide without friction along a horizontal surface.

Solution



Let us choose ξ, S as our generalized coordinates. The x,y coordinates of the center of the hoop are expressed by

$$\left.\begin{aligned} x &= \xi + S\cos\alpha + r\sin\alpha \\ y &= r\cos\alpha + (\ell - S)\sin\alpha \end{aligned}\right] \tag{1}$$

Therefore, the kinetic energy of the hoop is

$$\begin{aligned} T_{\text{hoop}} &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\phi}^2 \\ &= \frac{1}{2}m\left[\left(\dot{\xi} + \dot{S}\cos\alpha\right)^2 + \left(-\dot{S}\sin\alpha\right)^2\right] + \frac{1}{2}I\dot{\phi}^2 \end{aligned} \tag{2}$$

Using $I = mr^2$ and $S = r\phi$, (2) becomes

$$T_{\text{hoop}} = \frac{1}{2}m\left[2\dot{S}^2 + \dot{\xi}^2 + 2\dot{\xi}\dot{S}\cos\alpha\right] \tag{3}$$

In order to find the total kinetic energy, we need to add the kinetic energy of the translational motion of the plane along the x-axis which is

$$T_{\text{plane}} = \frac{1}{2}M\,\dot{\xi}^2 \tag{4}$$

Therefore, the total kinetic energy becomes

$$T = m\dot{S}^2 + \frac{1}{2}(m + M)\dot{\xi}^2 + m\dot{\xi}\dot{S}\cos\alpha \tag{5}$$

The potential energy is

$$U = mgy = mg\left[r\cos\alpha + (\ell - S)\sin\alpha\right] \tag{6}$$

Hence, the Lagrangian is

$$L = m\dot{S}^2 + \frac{1}{2}(m + M)\dot{\xi}^2 + m\dot{\xi}\dot{S}\cos\alpha - mg\left[r\cos\alpha + (\ell - S)\sin\alpha\right] \tag{7}$$

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(cont.)

from which the Lagrange equations for ξ and S are easily found to be

$$\boxed{2m\ddot{S} + m\ddot{\xi}\cos\alpha - mg\sin\alpha = 0} \tag{8}$$

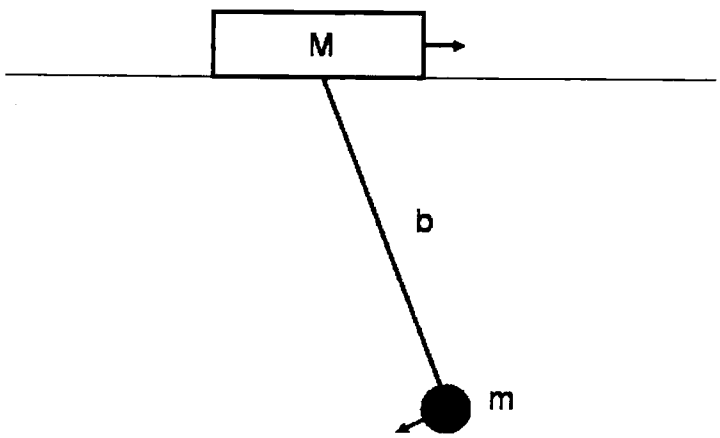
$$\boxed{(m+M)\ddot{\xi} + m\ddot{S}\cos\alpha = 0} \tag{9}$$

or, if we rewrite these equations in the form of uncoupled equations by substituting for $\ddot{\xi}$ and $\ddot{S}$, we have

$$\left.\begin{aligned} \left[2 - \frac{m\cos^2\alpha}{m+M}\right]\ddot{S} - g\sin\alpha &= 0 \\ \ddot{\xi} = -\frac{mg\sin\alpha\cos\alpha}{2(m+M) - m\cos^2\alpha} \end{aligned}\right] \tag{10}$$

#2 (SECTION 1)

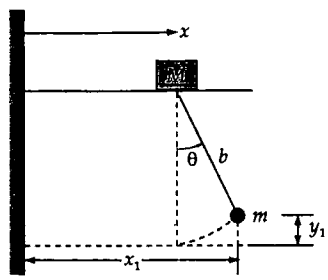
PROBLEM: A mass M moves horizontally along a frictionless rail. A pendulum is hung from M with a weightless rod and mass m at its end. The length of the rod is b. Find the eigenfrequencies and describe the normal modes for small oscillations.



Undergraduate Mechanics Problem 2

A mass M moves horizontally along a frictionless rail. A pendulum is hung from M with a weightless rod and mass m at its end. Find the eigenfrequencies and describe the normal modes.

Solution



$$x_1 = x + b\sin\theta; \quad \dot{x}_1 = \dot{x} + b\dot{\theta}\cos\theta$$

$$y_1 = b - b\cos\theta; \quad \dot{y}_1 = b\dot{\theta}\sin\theta$$

Thus

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + b^2\dot{\theta}^2 + 2b\dot{x}\dot{\theta}\cos\theta\right)$$

$$U = mgy_1 = mgb\left(1 - \cos\theta\right)$$

For small θ, $\cos\theta \approx 1-\frac{\theta^2}{2}$. Substituting and neglecting the term of order $\theta^2\dot{\theta}$ gives

$$T=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m\left(b^2\dot{\theta}^2+2b\dot{x}\dot{\theta}\right)$$

$$U=\frac{mgb}{2}\theta^2$$

Thus

$$\mathbf{m}=\begin{bmatrix} M+m & mb \\ mb & mb^2 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} 0 & 0 \\ 0 & mgb \end{bmatrix}$$

We must solve

$$\begin{vmatrix} -\omega^2(M+m) & -\omega^2 mb \\ -\omega^2 mb & mgb-\omega^2 mb^2 \end{vmatrix}=0$$

which gives

$$\omega^2(M+m)\left(\omega^2 mb^2-mgb\right)-\omega^4 m^2 b^2=0$$

$$\omega^2\left[\omega^2 Mb^2-mgb(m+M)\right]=0$$

Thus

$$\boxed{\begin{aligned} &\omega_1=0 \\ &\omega_2=\sqrt{\frac{g}{mb}(M+m)} \end{aligned}}$$

$$\sum_j\left(A_{jk}-\omega_r^2 m_{jk}\right)a_{jk}=0$$

Substituting into this equation gives

$$a_{21}=0 \qquad (k=2, r=1)$$

$$a_{12}=-\frac{bm}{(m+M)}a_{22} \qquad (k=2, r=2)$$

Thus the equations

$$x=a_{11}\eta_1+a_{12}\eta_2$$

$$\theta=a_{21}\eta_1+a_{22}\eta_2$$

become

$$x = a_{11}\,\eta_1 - \frac{m\,b}{(m+M)}\,a_{22}\,\eta_2$$

$$\theta = a_{22}\,\eta_2$$

Solving for η_1, η_2:

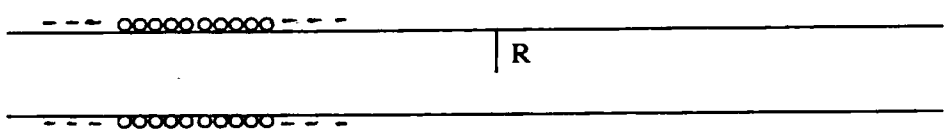
$$\eta_2 = \frac{\theta}{a_{22}}$$

$$n_1 = \frac{x + \frac{bm}{(m+M)}\theta}{a_{11}}$$

$$\boxed{\begin{array}{l} n_1 \text{ occurs when } n_2 = 0; \text{or } \theta = 0 \\ \\ n_2 \text{ occurs when } n_1 = 0; \text{or } x = -\frac{bm}{(m+M)}\theta \end{array}}$$

#3 (SECTION 2)

PROBLEM: A long cylindrical solenoid of radius R is tightly wound with a single layer of wire (see figure). The number of turns per unit length is N/L. The wire breaks when the tension in the wire is greater than T. Determine the maximum current that can be carried by the wire.



N/L turns/length

UG 3 SOL

SOLUTION

The magnetic field in the solenoid is given by

$$LB = \frac{4\pi}{c} NI \, , \, B = \frac{4\pi}{c} \frac{N}{L} I$$

From the stress tensor, the transverse pressure (radial force/area) on the coil is $B^2/8\pi$.

Net force on top half is given by

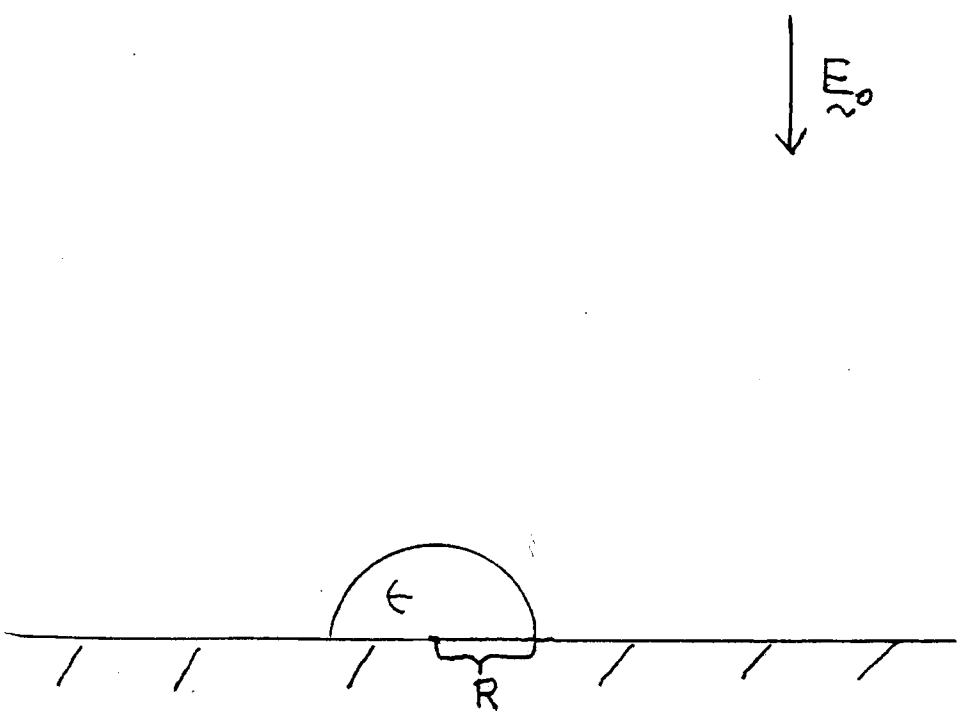
$$\begin{aligned} F_y &= L \int_0^\pi R d\theta \frac{B^2}{8\pi} \sin\theta \\ &= LR \frac{B^2}{8\pi} [-\cos\theta] \int_0^\pi = 2LR \frac{B^2}{8\pi} \end{aligned}$$

The tension in each wire is

$$\begin{aligned} T &= \frac{F_y}{2N} - \frac{2LR}{2N} \frac{B^2}{8\pi} = \frac{LR}{N8\pi} \left[\frac{4\pi}{c} \frac{N}{L} I \right]^2 \\ T &= \frac{2\pi R}{c^2} \frac{N}{L} I^2 \\ I^2 &= \frac{c^2 LT}{2\pi RN} \end{aligned}$$

#4 (SECTION 2)

PROBLEM: A solid piece of dielectric is characterized by a permittivity epsilon and has the shape of a hemisphere of radius R. The hemisphere is glued flat side down to a grounded conducting plane (see figure). High above the plane, a uniform electric field E_0 is directed vertically down. Determine the electric potential in and around the piece of dielectric.



UG 4 SOL

(4) A solid piece of dielectric is characterized by a permittivity epsilon and has the shape of a hemisphere of radius R. The hemisphere is glued flat side down to a grounded conducting plane (see figure). High above the plane, a uniform electric field $\mathbf{E}_0$ is directed vertically down. Determine the electric potential in and around the piece of dielectric.

$$\left.\begin{aligned} \varphi_2 &= E_0 r\cos\theta + \frac{B}{r^2}\cos\theta \\ \varphi_1 &= C r\cos\theta \end{aligned}\right\} \quad \begin{aligned} \varphi_1(r,\pm\pi/2) &= 0 \\ \varphi_2(r,\pm\pi/2) &= 0 \end{aligned}$$

$$\varphi_1(R,\theta) = \varphi_2(R,\theta) \Rightarrow E_0 R + \frac{B}{R^2} = CR$$

$$\epsilon \frac{\partial\varphi_1}{\partial r}\Big|_R = \epsilon_0 \frac{\partial\varphi_2}{\partial r}\Big|_R \Rightarrow \cancel{\epsilon_0}\left[E_0 - \frac{2B}{R^3}\right] = \frac{\epsilon}{\epsilon_0} C$$

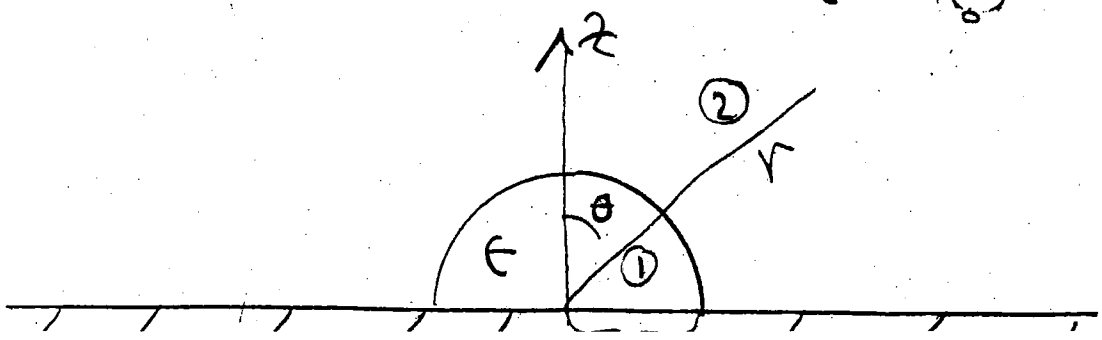
$$\therefore \quad 3E_0 = C\left[2 + \epsilon/\epsilon_0\right], \qquad B = R^3\left[C - E_0\right]$$

$$C = \frac{3E_0}{2+\epsilon/\epsilon_0} \qquad \text{"} = R^3\frac{\left[1-\epsilon/\epsilon_0\right]}{\left[2+\epsilon/\epsilon_0\right]}E_0$$

$$\downarrow \underset{\sim}{E_0} \qquad \downarrow \underset{\sim}{E_0}$$

$$\varphi_1(r,\theta) = \frac{3E_0}{2+\epsilon/\epsilon_0} r\cos\theta$$

$$\varphi_2(r,\theta) = E_0 r\cos\theta + \frac{E_0 R^3}{r^2}\frac{(1-\epsilon/\epsilon_0)}{(2+\epsilon/\epsilon_0)}\cos\theta$$



#5 (SECTION 3)

PROBLEM: A monatomic classical gas is contained in a rectangular box of base size $w_x \times w_y$ and height h. The N atoms with mass m that make up the gas are subject to the linear potential

$$U(z) = b \cdot z$$

where z is the vertical coordinate. Assume that the temperature T is uniform in the box.

(a) Show that the free energy of this system is

$$F = -NkT\left[\ln\left(\frac{w_x w_y kT}{bN}\right) + \frac{3}{2}\ln\left(\frac{mkT}{2\pi\hbar^2}\right) + \ln\left(1 - e^{-bh/kT}\right)\right].$$

(b) From the free energy, find the total force exerted on one of the sidewalls of the box.

(c) Find the number density of the system as a function of z.

(d) Using the ideal gas law, compute the pressure as a function of z. Show the relation between your result here and that of part (b).

UG 5 SOL

Solution:

(a) The molecules in the gas are not interacting, but they do see the external potential. Since they are monatomic, there is no internal energy. Hence the partition function is given by

$$Z = \frac{1}{N!} Z_1^N$$

where Z1 is the partition function for a single molecule, which is

3

$$Z_1 = \int \frac{d^3x\ d^3p}{(2\pi\hbar)^3} \exp\left[-\beta(p^2/2m + bz)\right]$$

$$= \frac{w_x w_y}{b}\left(\frac{m}{2\pi\hbar^2}\right)^{3/2} (kT)^{5/2}\left(1 - e^{-bh/kT}\right)$$

This yields the free energy

$$F = -kT \ln Z = -NkT\left[\ln\left(\frac{w_x w_y kT}{bN}\right) + \frac{3}{2}\ln\left(\frac{mkT}{2\pi\hbar^2}\right) + \ln\left(1 - e^{-bh/kT}\right)\right]$$

(b) The total force on one of the sidewalls, e.g., in the x direction, is given by

2

$$f_x = -\frac{\partial F}{\partial w_x} = \frac{NkT}{w_x}.$$

(c) The number density is proportion to the probability of finding a molecule at height between z and $z + dz$. This probability is

2

$$p(z) = -\frac{e^{-bz/kT} dz}{\int_0^h dz\ e^{-bz/kT}} = \frac{\beta b e^{-bz/kT}}{1 - e^{-bh/kT}}$$

The number density is then by the mean number between z and $z + dz$ divided by the volume $A\,dz$:

$$n(z) = -\frac{Np(z)dz}{w_x w_y dz} = \frac{\beta b e^{-bz/kT}}{w_x w_y (1 - e^{-bh/kT})}$$

(d) Using the ideal gas law, we have that the pressure as a function of z is

3

$$P(z) = n(z)kT = \frac{b e^{-bz/kT}}{w_x w_y (1 - e^{-bh/kT})}.$$

The *average* pressure is then

$$\overline{P} = \frac{1}{h}\int_0^h P(z)dz = \frac{NkT}{w_x w_y h}.$$

The total force is on the wall is just the average pressure times the area of the wall. E.g., in the x-direction, $f_x = \overline{P} \cdot w_y \cdot h$ which recovers the result of (b).

#6 (SECTION 3)

PROBLEM: A man takes a series of steps along a one dimensional line. His motion depends on a series of completely random events: the throws of an ordinary six-sided die. Each time he throws the die, if the number is odd (*i.e.* 1, 3, or 5), he takes that many steps to the left. If the number is even (*i.e.* 2, 4, or 6), he takes that many steps to the right. All steps are of length a.

(a) After N throws of the die, what is his average displacement, $\langle X_N \rangle$?

(b) How many throws of the die are necessary before the magnitude of $\langle X_N \rangle$ becomes greater than the root-mean-square fluctuation,

$$\Delta_N = \langle X_N^2 \rangle - \langle X_N \rangle^2 \ .$$

UG 6 SOL

SOLUTION

(a) We have

$$X_N = a \sum_{j=1}^{N} \sigma_j \ ,$$

with $\sigma_j \in \{-1, +2, -3, +4, -5, +6\}$. Thus,

$$\langle \sigma_j \rangle = \tfrac{1}{6}\big(-1+2-3+4-5+6\big) = +\tfrac{1}{2} \ ,$$

and thus $\langle X_N \rangle = \frac{1}{2} N a$.

(b) The root-mean-square fluctuation in X_N is $\sqrt{\Delta_N}$, where

$$\begin{aligned} \Delta_N &= \langle X_N^2 \rangle - \langle X_N \rangle^2 \\ &= a^2 \sum_{i,j}^{N} \Big\{ \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \Big\} \\ &= N a^2 \Big\{ \langle \sigma^2 \rangle - \langle \sigma \rangle^2 \Big\} \ . \end{aligned}$$

Now compute

$$\begin{aligned} \langle \sigma \rangle &= \tfrac{1}{2} \\ \langle \sigma^2 \rangle &= \tfrac{1}{6}\big(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\big) = \tfrac{91}{6} \ , \end{aligned}$$

and thus

$$\Delta_N = \Big(\tfrac{91}{6} - \tfrac{1}{4}\Big) N a^2 = \tfrac{179}{12} N a^2 \ .$$

In order for $|X_N| > \Delta_N$, we must have

$$\tfrac{1}{2} N a > \sqrt{\tfrac{179}{12}}\, \sqrt{N}\, a \quad \Rightarrow \quad N > \tfrac{179}{3} \ .$$

Thus, we must have $N \geq 60$.

#7 (SECTION 4)

PROBLEM: Two atoms are bound together in a potential that only depends on the distance between them,

$$V(r) = \frac{1}{2}\mu\omega^2 r^2$$

where μ is the reduced mass of the two atoms. Due to the spherical symmetry of the problem, the energy eigenstates can also be eigenstates of the L^2 and L_z operators. One can solve the radial equation in the usual way, using a power series and obtain a recursion relation.

$$\psi(\vec{r}) = R(r)Y_{\ell m}(\theta, \phi)$$

$$y \equiv \sqrt{\frac{\mu\omega}{\hbar}}r$$

$$R(y) = \sum_{k=0}^{\infty} a_k y^{\ell+k} e^{-y^2/2}$$

$$a_{k+2} = \frac{(2\ell + 2k + 3) - \frac{2E}{\hbar\omega}}{(k+2)(2\ell + k + 3)} a_k$$

$$a_1 = 0$$

The last equation above is derived along with the recursion relation and indicates that all the odd terms in the series must be zero.

(a) Show that the power series must terminate.
Now let a_{2n} be the last nonzero coefficient in the power series (which has only even terms).

(b) Show that the energies are quantized and derive an expression for E in terms of the quantum numbers n and ℓ.

(c) Now consider the four lowest energies of this system. What are these energies and how many degenerate states are there at each energy.

(d) Give the quantum numbers n, ℓ, and m for all of the degenerate states at each energy.

UG 7 SOL

Solution

The wavefunction must be normalizable for any physical solution to the problem, however, the recursion relation gives coefficients that do no fall off rapidly enough for large k.

$$a_{k+2} = \frac{(2\ell + 2k + 3) - \frac{2E}{\hbar\omega}}{(k+2)(2\ell + k + 3)} a_k \approx \frac{2}{k} a_k$$

Note that this is the power series for e^{+y^2} so the wavefunction will diverge at large r. To avoid this divergence we must terminate the series at some k. We can do this simply by making the numerator zero in the recursion relation.

$$(2\ell + 2k + 3) - \frac{2E}{\hbar\omega} = 0$$

$$\frac{2E}{\hbar\omega} = (2\ell + 2k + 3)$$

$$E = \left(\ell + k + \frac{3}{2}\right)\hbar\omega$$

$$E = \left(\ell + 2n + \frac{3}{2}\right)\hbar\omega$$

2

The value of k at which we terminate is $2n$ according to the problem so we make that replacement in the last step.

The ground state energy is $E = \frac{3}{2}\hbar\omega$. We can get this if $\ell = 0$ and $n = 0$. Of course $m = 0$. This would be called the $0s$ state.

The first excited state energy is $E = \frac{5}{2}\hbar\omega$. We can get this if $\ell = 1$ and $n = 0$. There are three states since $m = -1, 0, +1$ are allowed. This would be called the $0p$ state. There are 3 degenerate states.

The second excited state energy is $E = \frac{7}{2}\hbar\omega$. We can get this if $\ell = 0$ and $n = 1$. Of course $m = 0$. We can also get his if $\ell = 2$ and $n = 0$. There are five states since $m = -2, -1, 0, +1, +2$ are allowed. These would be called the $1s$ and $0d$ states. There are 6 degenerate states.

The third excited state energy is $E = \frac{9}{2}\hbar\omega$. We can get this if $\ell = 1$ and $n = 1$. There are three states since $m = -1, 0, +1$ are allowed. We can also get his if $\ell = 3$ and $n = 0$. There are seven states since $m = -3, -2, -1, 0, +1, +2, +3$ are allowed. These would be called the $1p$ and $0f$ states. There are 10 degenerate states.

E: 1

#8 (SECTION 4)

PROBLEM: HCl is a diatomic molecule with a distance between the atoms of about 1.3 Angstroms. Calculate the rotational absorption spectrum for photons incident upon HCl at room temperature. You may assume the molecule remains in the vibrational ground state and that the distance between the atoms remains constant. Only consider the transitions which are lowest order in α and hence appear as strong lines in the spectrum. Give the lowest 6 absorption energies in eV. Assume the HCl molecule is made of regular Hydrogen with a mass of 939 MeV/c^2 and Chlorine 35 with a mass of 32872 MeV/c^2.

HW 8 Sol

Solution

Under the assumptions given, energy due to rotation is simply $\frac{L^2}{2I}$ classically. Similarly the angular term in the Schödinger equation is $\frac{L^2}{2\mu r^2}$ which amounts to the same thing. We know the eigenvalues of the L^2 operator so the rotational energies are simple.

$$E = \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}$$

At room temperature, many of the rotational levels are populated, however, the strong absorption lines are for $\Delta\ell = 1$ since they are the Electric Dipole transitions. So the photon energies corresponding to the strong absorption lines are

$$E = \frac{[(\ell+1)(\ell+2) - \ell(\ell+1)]\hbar^2}{2\mu r^2} = \frac{2(\ell+1)\hbar^2}{2\mu r^2}$$

for the ℓ of the initial state 0,1,2,3,4...

We now need to plug in the numbers. The reduced mass is about 913 MeV/c^2.

$$E = \frac{(\ell+1)(\hbar c)^2}{(\mu c^2)r^2} = \frac{(\ell+1)(1973eV\ A)^2}{(913\times 10^6)(1.3A)^2} = 0.0025(\ell+1)\ eV$$

(Note that at room temperature, these exitations are about a factor of 10 less than kt.)

The lowest energy absorption lines are then at 0.0025, 0.0050, 0.0076, 0.0101, 0.0126 and 0.0151 eV. They are evenly spaced.

#9 (SECTION 5)

PROBLEM: An early experiment to accurately measure the speed of light was based on a rapidly rotating mirror. As shown in the somewhat simplified diagram below, a light source emits light through a narrow, vertical slit, the image of which is viewed in the experiment. Light from the slit is reflected by the rotating mirror shown at the right of the diagram. A key element of the experiment is the lens L which has a focal length $f = 10$ m and a diameter $d = 0.1$ m. The distance between it and the rotating mirror is set to be f as is the distance between the slit and the rotating mirror. A fixed mirror is placed at a distance of $2f$ from L so that there is an image of the slit at the location of the fixed mirror. The fixed mirror is carefully alligned to be perpendicular to the line from the center of the mirror to the center of the lens.

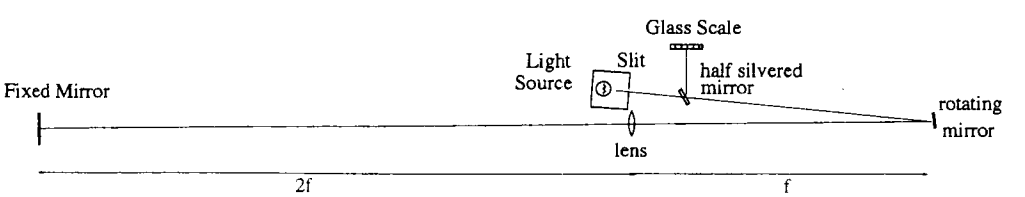
All together, with the rotating mirror held fixed, light travels from the slit to the the rotating mirror, through L, to the fixed mirror, back through L, back to the rotating mirror, and finally it is reflected by a half silvered mirror onto a fine glass scale which is used to measure the position of the image. The scale is at the same distance from the half silvered surface as is the slit. With the rotating mirror fixed, there is an image of the slit on the glass scale.

(a) As the rotating mirror is turned back and forth slightly by hand, the image of the slit on the glass scale does not move. Explain how this can be true.
The mirror is now set to rotate with a frequency of rotation set to 440 Hz using a tuning fork.

(b) How does the image of the slit on the glass scale change as the mirror rotates rapidly?

(c) What is the intensity of the image relative to the original intensity I_0 when the mirror is not rotating?

(d) Derive a formula for the speed of light in terms of the parameters given in the problem and a quantity measurable on the glass scale.



UG 9 Sol

Solution

Light from the rotating mirror that passes through the lens is returned to the rotating mirror going in exactly the opposite direction from which it came. Since the rotating mirror is one focal length from the lens, light with different angles from the rotating mirror is made parellel by the lens, returns parallel from the fixed mirror, and is focused back to the point from which it came on the rotating mirror. 4

a) Because of this property, the image of the slit remains fixed as the rotating mirror is turned by hand. 2

b) If the mirror rotates rapidly, all the light passing through the lens is still focussed onto a single image on the glass scale, however, the image is displaced because the rotating mirrors angle changes between the first and second reflections from that mirror. The size of the displacement can be calculated. The image also becomes less intense because light is only returned when the rotating mirror is pointing at the lens. 2

c) The new intensity is $\frac{d}{2\pi f} I_0$. (It could be twice that if the rotating mirror is two sided.) 2

d) During the time the light is traveling to and from the fixed mirror, the rotating mirror's angle changes. This light reflected back toward the scale, changes by twice the angle of the mirror. We will compute the displacement δ at the glass scale. 3

$$\theta = \omega t$$
$$t = \frac{6f}{c}$$
$$\delta = 2\theta f$$
$$\delta = \frac{24\pi(440)(10)^2}{c} = \frac{3.3175 \times 10^6 \frac{m^2}{s}}{c}$$

The displacement should be about 1 cm.

We can then calculate the speed of light in terms of the displacement.

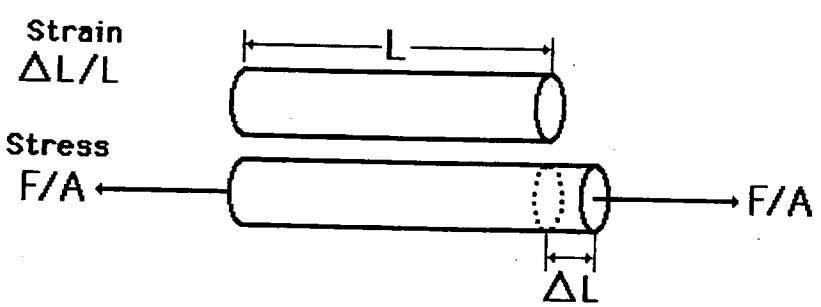
$$c = \frac{3.3175 \times 10^6 \frac{m^2}{s}}{\delta}$$

#10 (SECTION 5)

PROBLEM: In this question, you are asked to make order of magnitude estimates. Do any three of the four parts (any three of parts a, b, c, or d), stating which three you want graded. You may need the numbers:
Boltzmann constant ($k = 1.4 \times 10^{-23}$ J/K),
mass of a proton ($m_p = 1.7 \times 10^{-27}$ kg),
Avogadro's number ($N = 1$ gm$/m_p$),
electron charge ($e = 1.6 \times 10^{-19}$ C),
permittivity of free space ($\epsilon_0 = 8.8 \times 10^{-12}$ F/m),
permeability of free space ($\mu_0 = 4\pi \times 10^{-7}$ H/m),
the energy in an electron volt ($eV = 1.6 \times 10^{-19}$ J),
and the intensity of the solar radiation incident at the upper atmosphere of the earth ($I = 1$ kW/m^2).

(a) Using a simple microscopic picture, estimate Young's modulus for steel. Recall that the modulus is defined as

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{(F/A)}{(\Delta L/L)}$$



(b) Estimate the collision frequency for the molecules in the air of this room.

(c) Estimate the number of solar neutrinos that pass through your body during the course of this exam.

(d) In an amusing demonstration, a frog is floated (suspended against gravity) in the diverging magnetic field at the upper end of a vertical solenoid. Given that the field strength is about 10T and the bore of the solenoid is about 3 inches, estimate the magnetic susceptibility of the frog (i.e., χ_m where $\mathbf{M} = \chi_m \mathbf{H}$). Be sure to state the sign of χ_m, that is, state whether the frog is diamagnetic or paramagnetic.

Sol'ns for 10

(a)

$$Y = \frac{F/A}{\Delta l / l}$$

$$\frac{F}{A} \simeq \frac{\frac{\partial}{\partial a}\frac{e^2}{4\pi\epsilon_0 a^2}\Delta a}{a^2} \sim \frac{e^2}{4\pi\epsilon_0 a^2}\frac{\Delta a}{a}$$

$$\frac{\Delta l}{l} \sim \frac{\Delta a}{a}$$

$$Y \sim \frac{e^2}{4\pi\epsilon_0 a^4} \sim \frac{(1.6\times10^{-19})^2}{10^{-10}\, 2^4\, 10^{-40}} \sim 10^{11}\,\frac{N}{m^2}$$

$a \sim 2\,\text{Å} \simeq 2\times10^{-10}\,m$

$e \simeq 1.6\times10^{-19}$ Coul

$4\pi\epsilon_0 \simeq 4\pi\, 8.5\, 10^{-12} \sim 10^{-10}\,F/m$

(b)

$$\nu = n\bar{v}\sigma = n\sqrt{\frac{kT}{m}}\,\sigma$$

$$n \sim \frac{6\times10^{23}\ \text{atoms}}{20\ \text{liters}} = 3\times10^{25}\ \frac{\text{atoms}}{m^3}$$

$$\sqrt{\frac{kT}{m}} \sim \sqrt{\frac{(1.4\times10^{-23}\ J/K)\,300K}{(20)\,(1.7\ 10^{-27})}} \sim 400\ \frac{m}{sec}$$

$$\sigma \sim (5\,\text{Å})^2 \simeq 2.5\times10^{-19}\ m^2$$

$$\nu \sim (3\times10^{25})(4\times10^{2})(2.5\times10^{-19})\ sec^{-1}$$

$$\nu \sim 3\times10^{9}\ sec^{-1}$$

(e) Assume there is one neutrino for every MeV of solar energy

$$N \sim \left(\frac{1\,kW}{m^2}\right)\left(\frac{3{,}3600\,sec}{MeV}\right) m^2$$

$$\sim \frac{10^4\,J}{10^{-19}\,J} \sim 10^{23}$$

Solution for 10

(a)

$$\cancel{Vol}\,\rho g \sim \cancel{Vol}\,\underline{M}\,\nabla B \sim \cancel{Vol}\,\chi_m \underset{\sim B/\mu_0}{H}\,\nabla B$$

$$\chi_m \sim \frac{\rho g \mu_0}{\nabla B^2} \sim \frac{\rho g \mu_0 l}{B^2}$$

$$B = 10\,T \quad , \quad l \sim .1\,m$$

$$g = 10\,\frac{m}{sec^2} \quad , \quad \rho \sim \rho_{water} = 10^3\,\frac{kg}{m^3}$$

$$\mu_0 = 4\pi \times 10^{-7}\,N/A^2$$

$$\chi_m \sim 10$$

$$\chi_m \sim \frac{10^3\,10\,10^{-6}\,10^{-1}}{10^2} \sim 10^{-5}$$

INSTRUCTIONS
PART II
PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (*E.g.* Section 1: problem 1 or problem 2.) Partial credit will be given for partial solutions for seven (7) questions only.

Circle the seven problems you wish to be graded:

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	**11 12**	**13 14**	**15 16**	**17 18**	**19 20**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, etc.) in the corner of the exam room.

2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

*Colored scratch paper is provided and may be discarded when the examination is over.

At the conclusion of the examination period, please staple each problem together and place completed exams on the table, separated in stacks according to problem numbers.

#11 (SECTION 1)

PROBLEM: A particle of mass m moves in the central potential

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} \ .$$

In this problem you may assume $C > 0$.

(a) Find and sketch the effective potential $U_{\text{eff}}(r)$.

(b) Find the equation for the shape of bound orbits, $r(\phi)$, in the orbital plane. *Hint*: You might consider deriving a differential equation for the quantity $s = 1/r$.

(c) Find the precession $\Delta\phi$ in the angle of periapsis (*i.e.* the distance of closest approach to the force center) over one radial cycle.

GRAD 11 SOC pg 1

GRADUATE MECHANICS

A particle of mass m moves in the central potential

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} \ .$$

In this problem you may assume $C > 0$.

3 (a) Find and sketch the effective potential $U_{\text{eff}}(r)$.

5 (b) Find the equation for the shape of bound orbits, $r(\phi)$, in the orbital plane. *Hint*: You might consider deriving a differential equation for the quantity $s = 1/r$.

2 (c) Find the precession $\Delta\phi$ in the angle of periapsis (*i.e.* the distance of closest approach to the force center) over one radial cycle.

SOLUTION

3 (a) The effective potential is

$$\begin{aligned} U_{\text{eff}}(r) &= U(r) + \frac{\ell^2}{2mr^2} \\ &= -\frac{k}{r} + \frac{\ell^2 + mC}{2mr^2} \ , \end{aligned}$$

where $\ell = mr^2\dot{\phi}$ is the angular momentum. The sketch follows.

5 (b) We know that total energy is conserved, with

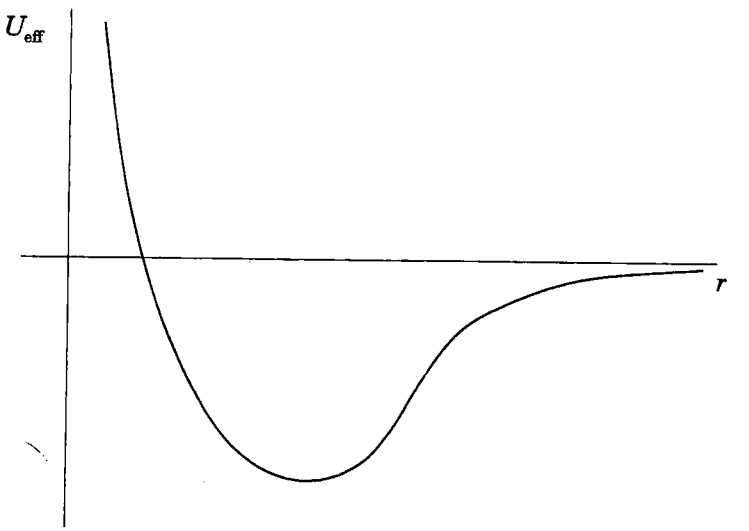
$$\begin{aligned} E &= \tfrac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) \\ &= \frac{m\dot{s}^2}{2s^4} - ks + \frac{1}{2m}(\ell^2 + mC)s^2 \ , \end{aligned}$$

where $s = 1/r$. Conservation of angular momentum allows us to write

$$\frac{1}{s^2}\frac{d}{dt} = \frac{\ell}{m}\frac{d}{d\phi} \ ,$$

hence

$$E + \frac{\ell^2}{2m}\left(\frac{ds}{d\phi}\right)^2 - ks + \frac{1}{2m}(\ell^2 + mC)s^2 \ .$$



We can integrate this directly, or, what is somewhat simpler, differentiate to obtain

$$\frac{d^2s}{d\phi^2} + \left(1 + \frac{mC}{\ell^2}\right)s = \frac{mk}{\ell^2} \ .$$

This is instantly recognized as the equation of a harmonic oscillator under a constant force, with ϕ playing the role of time. The solution is

$$s(\phi) = s_0\big(1 - \epsilon\cos\beta\phi\big) \ ,$$

where $s_0 = mk/(\ell^2 + mC)$ and $\beta = \sqrt{1 + \frac{mC}{\ell^2}}$. Thus,

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos\beta\phi} \ ,$$

with $r_0 = s_0^{-1} = (\ell^2 + mC)/mk$. For the orbit to be bound, $|\epsilon| < 1$.

2

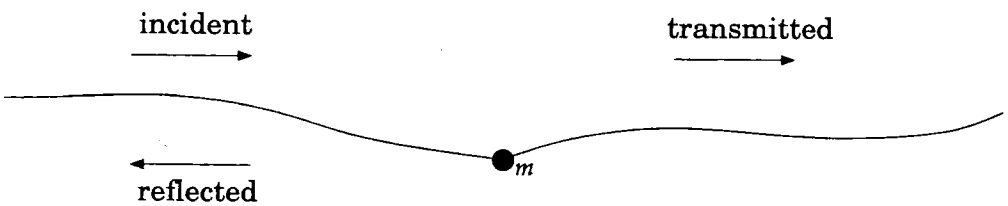
(c) We have chosen the zero of ϕ to correspond to apoapsis, In fact, apoapsis occurs for $\beta\phi = 2\pi n$ and periapsis for $\beta\phi = 2\pi(n + \frac{1}{2})$. The difference between angles at successive periapses is

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi\big(\beta^{-1} - 1\big) \ .$$

If $C > 0$ then $\beta > 1$ and $\Delta\phi < 0$, *i.e.* periapsis lags from one orbit to the next.

#12 (SECTION 1)

A mass m is affixed at position $x = 0$ to a string, as shown in the figure below. The tension in the string is τ and its linear mass density is μ. As the mass moves, it scrapes along a surface parallel to the string, hence it experiences a frictional force $-\gamma m v$, where v is the (vertical) velocity of the mass, in addition to the forces it is subjected to from the string. A pulse of shape $f(x - ct)$ is incident from the left at large negative values of the time t, where c is the speed of wave propagation in the string.



(a) Find the transmission coefficient $\mathtt{t}(k)$ as a function of wavevector. You may find it convenient to define $Q = \gamma/c$ and $\zeta = m/2\mu$.

(b) An incident pulse (*i.e.* a wavepacket) is made up of Fourier components with wavevectors $k \ll Q$. If the incident pulse shape is described by $y_{\text{in}}(x,t) = f(x - ct)$, what is the reflected pulse $y_{\text{ref}}(x,t)$?

GRADUATE MECHANICS

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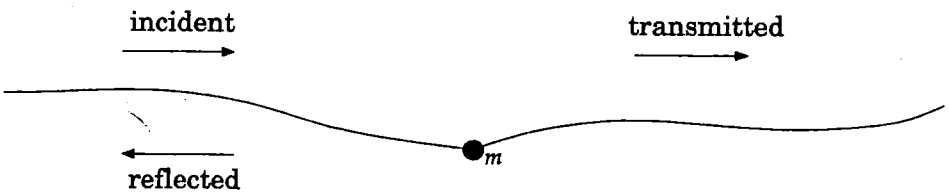

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SOLUTION

(a) Let $y(x,t)$ be the vertical displacement of the string at position x and time t. For $x \neq 0$, we have that the string obeys the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \ ,$$

where $c^2 = \tau/\mu$. At $x = 0$, Newton's second law gives

$$m\ddot{y}(0,t) + \gamma m \dot{y}(0,t) = \tau y'(0^+,t) - \tau y'(0^-,t),$$

where prime denotes derivative with respect to x and dot denotes derivative with respect to t. The equations of motion are linear, hence the solution on either side of $x = 0$ may be written as a Fourier integral. We therefore may write a general solution as a linear combination of functions of the following

GRAD 12 SOL PG 2

form:

$$y(x,t) = \begin{cases} e^{ik(x-ct)} + \mathtt{r}(k)\, e^{-ik(x+ct)} & x < 0 \\ \\ \mathtt{t}(k)\, e^{ik(x-ct)} & x > 0\ . \end{cases}$$

Continuity at $x = 0$ requires

$$1 + \mathtt{r}(k) = \mathtt{t}(k)\ .$$

The second equation comes from Mr. Newton:

$$-(mc^2k^2 + i\gamma mck)\,\mathtt{t}(\mathtt{k}) = ik\tau\big[\mathtt{t}(k) - 1 + \mathtt{r}(k)\big]\ .$$

Substituting $\mathtt{r} = \mathtt{t} - 1$ into the second equation and solving for $\mathtt{t}(k)$ yields

$$\mathtt{t}(k) = \left[1 + \frac{\gamma mc}{2\tau} - i\frac{mk}{2\mu}\right]^{-1} = \frac{1}{1 + Q\zeta - ik\zeta}\ ,$$

where $Q = \gamma/c$ and $\zeta = m/2\mu$.

(b) The reflection coefficient is

$$\mathtt{r}(k) = \mathtt{t}(k) - 1 = -\frac{Q\zeta - ik\zeta}{1 + Q\zeta - ik\zeta}\ .$$

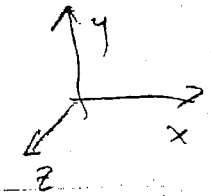
Assuming $k \ll Q$, we have $\mathtt{r}(k) \simeq -\frac{Q\zeta}{1+Q\zeta}$, so the reflected pulse is

$$y_{\text{ref}}(x,t) \simeq -\frac{Q\zeta}{1 + Q\zeta}\, f(x + ct)\ .$$

Thus, the reflected pulse is an inverted, attenuated version of the incident pulse.

#13 (SECTION 2)

PROBLEM: An infinite slab of copper occupies the region $x = -d$ to $x = +d$, where (x, y, z) is a rectangular coordinate system. The slab carries the oscillating current/length $\mathbf{K}(t) = \mathrm{Re}\, \hat{y} K_0\, e^{-i\omega t}$, where K_0 is a constant. Note that the current density in the slab, $J_y(x,t) = \mathrm{Re}\, J_y(x)\, e^{-i\omega t}$, is not uniform and not specified; only the integral $\int_{-d}^{+d} dx\, J_y(x) = K_0$ is specified. Determine the magnetic field in all of space and determine the current density $J_y(x)$ in the slab. You may assume that the conductivity of copper is much larger than the oscillation frequency (i.e., $\sigma \gg \omega$).



GRAD 13 SOC

Soln

The magnetic field is in the z-direction

$$\underset{\sim}{B}(\underset{\sim}{r},t) = \hat{z}\,\mathrm{Re}\,B_z(x)\,e^{-i\omega t}$$

for $x \leq -d$ $\quad B_z(x) = \dfrac{2\pi K_0}{c}$

$x \geq d$ $\quad B_z(x) = -\dfrac{2\pi K_0}{c}$

for $-d < x < d$

$$\underbrace{\nabla^2}_{\frac{\partial^2}{\partial x^2}} B_z(x) = \frac{4\pi\sigma}{c^2}(-i\omega)B_z$$

$$B_z(x) = -\frac{2\pi K_0}{c}\,\frac{\sin kx}{\sin kd}$$

$$k = \sqrt{\frac{4\pi\sigma i\omega}{c^2}} = \sqrt{\frac{4\pi\sigma\omega}{c^2}}\left(\frac{1-i}{\sqrt{2}}\right)$$

$$\frac{4\pi}{c}J_y = -\frac{\partial B_z}{\partial x} = \frac{2\pi K_0}{c}\,k\,\frac{\cos kx}{\sin kd}$$

#14 (SECTION 2)

PROBLEM: In a particular inertial frame there is a uniform electric field $\mathbf{E} = E_0(\hat{x} + \hat{y})$ and a uniform magnetic field $\mathbf{B} = E_0\hat{y}$.

(a) Is there an inertial frame where either the electric field or the magnetic field vanishes? If so, find that frame. If not, state why not.

(b) Is there a frame where the electric and magnetic field are parallel? If so, find that frame. It not, state why not. You may need the equations for transformation of the fields:

$$\bar{E}_{\|} = E_{\|}\,, \bar{B}_{\|} = B_{\|}\,, \bar{\mathbf{E}}_{\perp} = \gamma\left[\mathbf{E}_{\perp} + \frac{\mathbf{V}\times\mathbf{B}_{\perp}}{c}\right]\,, \bar{\mathbf{B}}_{\perp} = \gamma\left[\mathbf{B}_{\perp} - \frac{\mathbf{V}\times\mathbf{E}_{\perp}}{c}\right]$$

where $\|$ and $\perp$ to the direction of the relative velocity between the frames, $\mathbf{V}$.

GRAD 14 SOC

4. In a particular inertial frame there is a uniform electric field $\mathbf{E} = E_0(\hat{x} + \hat{y})$ and a uniform magnetic field $\mathbf{B} = E_0\hat{y}$.

a. Is there an inertial frame where either the electric field or the magnetic field vanishes? If so, find that frame. If not, state why not.

no because $\underset{\sim}{E} \cdot \underset{\sim}{B} = E_0^2 \neq 0$

and $\bar{\underset{\sim}{E}} \cdot \bar{\underset{\sim}{B}} = \underset{\sim}{E} \cdot \underset{\sim}{B}$.

b. Is there a frame where the electric and magnetic field are parallel? If so, find that frame. If not, state why not.

yes, look for frame traveling with velocity $\underset{\sim}{v} = \hat{z} v$

$E_{\parallel} = B_{\parallel} = 0 \qquad \bar{\underset{\sim}{E}} = \gamma\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right), \quad \bar{\underset{\sim}{B}} = \gamma\left(\underset{\sim}{B} - \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c}\right)$

require

$$0 = \bar{\underset{\sim}{E}} \times \bar{\underset{\sim}{B}} = \gamma^2\left[\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right] \otimes \left[\underset{\sim}{B} - \frac{\underset{\sim}{v} \times \underset{\sim}{E}}{c}\right]$$

$$0 = \gamma^2 E_0^2\left[(\hat{x} + \hat{y}) - \frac{v_z}{c}\hat{x}\right] \otimes \left[\hat{y} - \frac{v_z}{c}(\hat{y} - \hat{x})\right]$$

$$1 - 3\frac{v_z}{c} + \left(\frac{v_z}{c}\right)^2 = 0 \qquad \left(\frac{\hat{z}\, v_z/c}{1 + (v_z/c)^2} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{E^2 + B^2} = \frac{\hat{z}}{3}\right)$$

$$\frac{v_z}{c} = \frac{+3 \pm \sqrt{9 - 4}}{2} = \frac{+3 \pm \sqrt{5}}{2}$$

#15 (SECTION 3)

PROBLEM : An isotropic ferromagnet may be described by the Heisenberg model

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j - \vec{H} \cdot \sum_i \vec{\sigma}_i$$

where $\vec{\sigma}_i$ is a 3d unit vector describing the direction of spin on site i of a simple cubic lattice, $J > 0$ is the strength of the ferromagnetic interaction, $\vec{H}$ is the external magnetic field, and $\langle i, j \rangle$ denotes nearest neighbor sites.

(a) Apply the Weiss mean-field approximation, i.e., replace the spin variable for each of the 6 nearest neighbors of a site by the thermodynamic average, $\vec{m} \equiv \langle \vec{\sigma}_i \rangle$, and write down the mean-field Hamiltonian $\mathcal{H}_{\mathrm{MF}}$ in terms of $\vec{m}$.

(b) Using the mean-field Hamiltonian, derive a self-consistent equation of state describing how the average magnetization $m \equiv |\vec{m}|$ depends on $\vec{H}$ at temperature T. [Hint: You may assume that $\vec{m}$ is in the direction of $\vec{H}$.]

(c) Find the critical temperature T_c in the absence of external field, i.e., for $\vec{H} = 0$.

Grad 15 SOL

Solution:

(a) The average magnetization per site $\vec{m} \equiv \langle \vec{\sigma}_i \rangle$ is clearly independent of the location of the site i. The mean-field approximation amounts to replacing the spin $\vec{\sigma}_j$ neighboring site i by the average magnetization. Summing over the 6 nearest neighbors of each site in a simple cubic lattice, we have

2

$$H_{\mathrm{MF}} = -6J \sum_i \vec{\sigma}_i \cdot \vec{m} - \vec{H} \cdot \sum_i \vec{\sigma}_i .$$

(b) Assuming that $\vec{m}$ points in the direction of the external field $\vec{H}$, then the spin variable $\vec{\sigma}_j$ can be represented simply by the angle θ it makes with $\vec{H}$. Denoting the magnitude of $\vec{m}$ and $\vec{H}$ by m and H respectively, the mean-field Hamiltonian becomes

2

$$H_{\mathrm{MF}} = -6J \sum_i m \cos\theta_i - H \sum_i \cos\theta_i .$$

The mean-field partition function is then $Z_{MF} \equiv \int d\vec{\sigma}_1 .. d\vec{\sigma}_N \; e^{-\beta H_{\mathrm{MF}}} = Z_1^N$ where N is the number of sites in the system, and

2

$$\begin{aligned} Z_1 &= \int d\theta \, 2\pi \sin\theta \; e^{(\mathrm{J}m+h)\cos\theta} \\ &= \frac{1}{\mathrm{J}m+h}\left(e^{\mathrm{J}m+h} - e^{-(\mathrm{J}m+h)}\right) \end{aligned}$$

where $\mathrm{J} \equiv 6\beta J$ and $h \equiv \beta H$.

The mean-magnetization is then

2

$$m = \frac{\partial}{\partial h} \ln Z_1 = -\frac{1}{\mathrm{J}m+h} + \frac{e^{\mathrm{J}m+h} + e^{-(\mathrm{J}m+h)}}{e^{\mathrm{J}m+h} - e^{-(\mathrm{J}m+h)}} .$$

(c) In the vicinity of T_c (and with $h = 0$), we expect the magnetization to be small. Expanding the self-consistent equation above for m, we find

2

$$m \approx -\frac{1}{\mathrm{J}m} + \frac{1 + \frac{(\mathrm{J}m)^2}{2!} + \frac{(\mathrm{J}m)^4}{4!}}{\mathrm{J}m + \frac{(\mathrm{J}m)^3}{3!} + \frac{(\mathrm{J}m)^5}{5!}} = \frac{\mathrm{J}m}{3} + O(\mathrm{J}m)^3$$

Critical point occurs when the self-consistent equation begins to admit multiple solutions, i.e., when spontaneous symmetry breaking appears. This occurs at $\mathrm{J} = 3$ or $T_c = 2J / k_B$.

#16 (SECTION 3)

PROBLEM : A three-dimensional system of spin-0 bosonic particles obeys the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \Delta + \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ .$$

The quantity Δ is the formation energy and m the mass of each particle. These particles are not conserved – they may be created and destroyed at the boundaries of their environment. (A possible example: vacancies in a crystalline lattice.) The Hamiltonian for these particles is

$$\mathcal{H} = \sum_{\boldsymbol{k}} \varepsilon(\boldsymbol{k})\, \hat{n}_{\boldsymbol{k}} + \frac{U}{2V}\, \hat{N}^2 \ ,$$

where $\hat{n}_{\boldsymbol{k}}$ is the number operator for particles with wavevector $\boldsymbol{k}$, $\hat{N} = \sum_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}}$ is the total number of particles, V is the volume of the system, and U is an interaction potential.

(a) Treat the interaction term within mean field theory. That is, define $\hat{N} = \langle \hat{N} \rangle + \delta \hat{N}$, where $\langle \hat{N} \rangle$ is the thermodynamic average of $\hat{N}$, and derive the mean field self-consistency equation for the number density $\rho = \langle \hat{N} \rangle / V$ by neglecting terms quadratic in the fluctuations $\delta \hat{N}$. Show that the mean field Hamiltonian is

$$\mathcal{H}_{\rm MF} = -\tfrac{1}{2} V U \rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho \Big]\, \hat{n}_{\boldsymbol{k}} \ ,$$

(b) Derive the criterion for Bose condensation. Show that this requires $\Delta < 0$. For $\Delta = -|\Delta_0|$, find an equation relating $T_{\rm c}$, U, and Δ_0.

GRAD 16 SOL pg 1

GRADUATE STATISTICAL MECHANICS

A three-dimensional system of spin-0 bosonic particles obeys the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \Delta + \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ .$$

The quantity Δ is the formation energy and m the mass of each particle. These particles are not conserved – they may be created and destroyed at the boundaries of their environment. (A possible example: vacancies in a crystalline lattice.) The Hamiltonian for these particles is

$$\mathcal{H} = \sum_{\boldsymbol{k}} \varepsilon(\boldsymbol{k})\, \hat{n}_{\boldsymbol{k}} + \frac{U}{2V}\, \hat{N}^2 \ ,$$

where $\hat{n}_{\boldsymbol{k}}$ is the number operator for particles with wavevector $\boldsymbol{k}$, $\hat{N} = \sum_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}}$ is the total number of particles, V is the volume of the system, and U is an interaction potential.

4 (a) Treat the interaction term within mean field theory. That is, define $\hat{N} = \langle \hat{N} \rangle + \delta\hat{N}$, where $\langle \hat{N} \rangle$ is the thermodynamic average of $\hat{N}$, and derive the mean field self-consistency equation for the number density $\rho = \langle \hat{N} \rangle / V$ by neglecting terms quadratic in the fluctuations $\delta\hat{N}$. Show that the mean field Hamiltonian is

$$\mathcal{H}_{\rm MF} = -\tfrac{1}{2} V U \rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho \Big]\, \hat{n}_{\boldsymbol{k}} \ ,$$

6 (b) Derive the criterion for Bose condensation. Show that this requires $\Delta < 0$. For $\Delta = -|\Delta_0|$, find an equation relating $T_{\rm c}$, U, and Δ_0.

SOLUTION

(a) We write

$$\begin{aligned} \hat{N}^2 &= \big(\langle \hat{N} \rangle + \delta\hat{N} \big)^2 \\ &= \langle \hat{N} \rangle^2 + 2 \langle \hat{N} \rangle\, \delta\hat{N} + (\delta\hat{N})^2 \\ &= -\langle \hat{N} \rangle^2 + 2 \langle \hat{N} \rangle\, \hat{N} + (\delta\hat{N})^2 \ . \end{aligned}$$

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We drop the last term, $(\delta\hat{N})^2$, because it is quadratic in the fluctuations. This is the mean field assumption. The Hamiltonian now becomes

$$\mathcal{H}_{\rm MF} = -\tfrac{1}{2}VU\rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho\Big]\,\hat{n}_{\boldsymbol{k}}\ ,$$

where $\rho = \langle\hat{N}\rangle/V$ is the number density. This, the dispersion is effectively changed, to

$$\tilde{\varepsilon}(\boldsymbol{k}) = \frac{\hbar^2\boldsymbol{k}^2}{2m} + \Delta + U\rho\ .$$

The average number of particles in state $\big|\boldsymbol{k}\big\rangle$ is given by the Bose function,

$$\langle\hat{n}_{\boldsymbol{k}}\rangle = \frac{1}{\exp\big[\tilde{\varepsilon}(\boldsymbol{k})/k_{\rm B}T\big] - 1}\ .$$

Summing over all $\boldsymbol{k}$ states, and using

$$\frac{1}{V}\sum_{\boldsymbol{k}} \longrightarrow \int\!\frac{d^3k}{(2\pi)^3}\ ,$$

we obtain

$$\begin{aligned}
\rho &= \frac{1}{V}\sum_{\boldsymbol{k}}\langle\hat{n}_{\boldsymbol{k}}\rangle \\
&= \rho_0 + \int\!\frac{d^3k}{(2\pi)^3}\,\frac{1}{e^{\hbar^2k^2/2mk_{\rm B}T}\,e^{(\Delta+U\rho)/k_{\rm B}T} - 1} \\
&= \rho_0 + \int\limits_0^\infty d\omega\,\frac{D(\omega)}{e^{(\omega+\Delta+U\rho)/k_{\rm B}T} - 1}
\end{aligned}$$

where $\rho_0 = \langle\hat{n}_{\boldsymbol{k}=0}\rangle/V$ is the number density of the $\boldsymbol{k} = 0$ state alone, *i.e.*the condensate density. When there is no condensate, $\rho_0 = 0$. The above equation is the mean field equation. It is equivalent to demanding $\partial F/\partial\rho = 0$, *i.e.* to extremizing the free energy with respect to the mean field parameter ρ. Though it is not a required part of the solution, we have here written this relation in terms of the density of states $D(\omega)$, defined according to

$$D(\omega) \equiv \int\!\frac{d^3k}{(2\pi)^3}\,\delta\!\left(\omega - \frac{\hbar^2\boldsymbol{k}^2}{2m}\right) = \frac{m^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\,\sqrt{\omega}\ .$$

(b) Bose condensation requires

$$\Delta + U\rho = 0 \ ,$$

which clearly requires $\Delta < 0$. Writing $\Delta = -|\Delta_0|$, we have, just at $T = T_c$,

$$\rho(T_c) = \frac{|\Delta_0|}{U} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\hbar^2 k^2/2mk_B T_c} - 1} \ ,$$

since $\rho_0(T_c) = 0$. This relation determines T_c. Although this is as far as you were asked to go, it is useful to note that the integral can be done and an explicit expression obtained for T_c:

$$\begin{aligned} \frac{|\Delta_0|}{U} &= \int_0^\infty d\omega \, D(\omega) \sum_{j=1}^\infty e^{-j\omega/k_B T_c} \\ &= \left(\frac{mk_B T_c}{2\pi\hbar^2}\right)^{3/2} \zeta(\tfrac{3}{2}) \ , \end{aligned}$$

where $\zeta(\ell) = \sum_{n=1}^\infty n^{-\ell}$ is the Riemann zeta function. Thus,

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left(\frac{\Delta_0}{\zeta(\frac{3}{2})\, U}\right)^{2/3} .$$

#17 (SECTION 4)

PROBLEM: An electron is initially spin up along the $\hat{z}$-axis. This state is then rotated by an angle α about an axis $\hat{\alpha}$, where $\vec{\alpha} = \alpha\hat{\alpha} = \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z}$ and $\alpha = (\vec{\alpha} \cdot \vec{\alpha})^{1/2}$.

(a) What is the **probability** that the electron is still spin up along the original $\hat{z}$-axis? Spin down along this axis?

(b) Now suppose that an electron initially spin down along the $\hat{z}$-axis is **sequentially** rotated first by α_z about $\hat{z}$, then by α_y about $\hat{y}$, and finally by α_x about $\hat{x}$. What is the probability that the electron is spin down along the original $\hat{z}$-axis?

(1.) <u>Quantum Problem</u>

An electron is initially spin up along the $\hat{z}$-axis. This state is then rotated by an angle θ about an axis $\hat{\theta}$, where $\vec{\theta} = \theta\hat{\theta} = \theta_x\hat{x} + \theta_y\hat{y} + \theta_z\hat{z}$ and $\theta = (\vec{\theta}\cdot\vec{\theta})^{1/2}$.

a.) What is the <u>probability</u> that the electron is still spin up along the original $\hat{z}$-axis? spin down along this axis?

b.) Now suppose that an electron initially spin down along the $\hat{z}$-axis is <u>sequentially</u> rotated first by θ_z about $\hat{z}$, then by θ_y about $\hat{y}$, and finally by θ_x about $\hat{x}$. What is the probability that the electron is spin down along the original $\hat{z}$-axis?

---

solution: a)

$$\begin{pmatrix} \langle +\hat{z}|\text{up on }\hat{\theta}\rangle \\ \langle -\hat{z}|\text{down on }\hat{\theta}\rangle \end{pmatrix} = e^{-i\vec{\sigma}\cdot\vec{\theta}/2}\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{set } \hbar = 1$$

$$= \left(\cos\tfrac{\theta}{2} - i\,\vec{\sigma}\cdot\hat{\theta}\sin\tfrac{\theta}{2}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 - i\,\hat{\theta}_z\sin\theta/2 \\ (-i\,\hat{\theta}_x + \hat{\theta}_y)\sin\theta/2 \end{pmatrix}$$

$\Rightarrow$ probability still spin up on $\hat{z}$

$\Rightarrow |\cos\theta/2 - i\,\hat{\theta}_z\sin\theta/2|^2$

$= \cos^2\theta/2 + \left(\dfrac{\theta_z^2}{\theta_x^2+\theta_y^2+\theta_z^2}\right)\sin^2\theta/2$

$\Rightarrow$ probability spin down on $\hat{z}$

$\Rightarrow |(-i\,\theta_x + \theta_y)\sin\theta/2|^2$

$= \sin^2\theta/2\,(\hat{\theta}_y^2 + \hat{\theta}_x^2) = \sin^2\theta/2\left(\dfrac{\theta_y^2+\theta_x^2}{\theta_x^2+\theta_y^2+\theta_z^2}\right)$

(over)

b.) now evaluate

$$e^{-i\sigma_x \theta_x/2}\, e^{-i\sigma_y \theta_y/2}\, e^{-i\sigma_3 \theta_3/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$e^{-i\sigma_3 \theta_3/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta_3/2 + i\sin\theta_3/2 \end{pmatrix} = e^{+i\theta_3/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

---

now, next rotation in sequence is $e^{-i\sigma_y \theta_y/2}\left\{ e^{+i\theta_3/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$= e^{i\theta_3/2} \left\{ e^{-i\sigma_y \theta_y/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$= e^{i\theta_3/2} \begin{pmatrix} -\sin\theta_y/2 \\ \cos\theta_y/2 \end{pmatrix}$$

---

now, $e^{-i\sigma_x \theta_x/2}$

$$= \cos\frac{\theta_x}{2} - i\sigma_x \sin\frac{\theta_x}{2} = \begin{bmatrix} \cos\theta_x/2 & 0 \\ 0 & \cos\theta_x/2 \end{bmatrix} - i\sin\frac{\theta_x}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} \cos\theta_x/2 & -i\sin\theta_x/2 \\ -i\sin\theta_x/2 & \cos\theta_x/2 \end{bmatrix}$$

---

so now full rotation sequence is $= e^{-i\theta_3/2} \begin{bmatrix} & \\ & \end{bmatrix} \begin{pmatrix} -\sin\theta_y/2 \\ \cos\theta_y/2 \end{pmatrix}$

$$= e^{-i\theta_3/2} \begin{bmatrix} -\sin\theta_y/2 \cos\theta_x/2 - i\sin\theta_x/2 \cos\theta_y/2 \\ \\ i\sin\theta_x/2 \sin\theta_y/2 + \cos\theta_x/2 \sin\theta_y/2 \end{bmatrix}$$

$\Rightarrow$ Probability that we are still spin down *

$$= \cos^2\frac{\theta_x}{2}\cos^2\frac{\theta_y}{2} + \sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}$$

---

we have used the identity

$$e^{-i\frac{\vec{\sigma}\cdot\hat{\theta}\theta}{2}} = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\,\vec{\sigma}\cdot\hat{\theta}\,\sin\left(\frac{\theta}{2}\right)$$

$$\begin{cases} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

#18 (SECTION 4)

PROBLEM: A spinless charged particle (charge e) in free space moves under the influence of a uniform magnetic field $\vec{B}$. Determine the energy levels for this particle.

(2.) Grad Quantum Problem

GRAD 18 SOL

A spin-less charged particle (charge e) in free space moves under the influence of a uniform magnetic field $\vec{B}$.

Find and discuss the energy levels for this particle. Hint: define new variables which are canonically conjugate and otherwise make the Hamiltonian look simple. Then write the answer by inspection.

Solution: Choose $\hat{z}$-axis to lie along $\vec{B}$,

i.e., $\vec{B} = B\hat{e}_z$

Then only non-vanishing components of vector potential are $A_x, A_y \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ or $B_z = \partial_x A_y - \partial_y A_x$

$\Rightarrow$ Hamiltonian is $\hat{H} = \frac{1}{2m}\left\{ (p_x - e/c\, A_x)^2 + (p_y - e/c\, A_y)^2 + p_z^2 \right)$

now define new variables $Q \equiv \left(\frac{c}{eB}\right)^{1/2} (p_x - e/c\, A_x)$

$P \equiv \left(\frac{c}{eB}\right)^{1/2} (p_y - e/c\, A_y)$

$\Rightarrow$ so, Hamiltonian in terms of these new variables is

$$\hat{H} = \frac{eB}{2mc}\left[P^2 + Q^2\right] + \frac{p_z^2}{2m}$$

note that $[Q, P] = \frac{-1}{B}\left\{ [A_x, p_y] + [p_x, A_y] \right\}$

and use identity $[p_i, G(\vec{x})] = -i\hbar \frac{\partial G}{\partial x_i}$

$$\Rightarrow [Q, P] = -\frac{1}{B}\underbrace{\left\{ i\hbar \frac{\partial A_x}{\partial y} - i\hbar \frac{\partial A_y}{\partial x} \right\}}_{i\hbar(-B)} = i\hbar$$

So Hamiltonian is of form of a harmonic oscillator of frequency $\hbar\omega = \frac{eB\hbar}{mc}$ and a free K.E. term $\Rightarrow$

energy eigenvalues are $E = \frac{p_z^2}{2m} + \frac{eB\hbar}{mc}\left(n + \frac{1}{2}\right)$ with $n = 0, 1, 2, \ldots$

#19 (SECTION 5)

PROBLEM: A nucleus has a ground state with angular momentum quantum number $J_0 = 0$ and energy E_0, a first excited state with excitation energy above the ground state $E_1 - E_0 = 0.1\,\mathrm{MeV}$ and angular momentum $J_1 = 1$, and a second excited state with excitation energy $E_2 - E_0 = 1.0\,\mathrm{MeV}$ and angular momentum $J_2 = 2$. In the laboratory at zero temperature this nucleus is known to have a mean lifetime (inverse decay rate) of $10^7\,\mathrm{s}$. The first excited state is known to decay 10^6 times as fast as the ground state, while the second excited state decays 10^{11} times faster than does the ground state.

What is the lifetime of this nucleus inside a massive star at temperature $kT = 0.1\,\mathrm{MeV}$? (Assume that thermal equilibrium obtains and that the populations of the states of the nucleus are thermal.)

Solution

CODE NUMBER: 50 SCORE: 10

#19 (SECTION 5)

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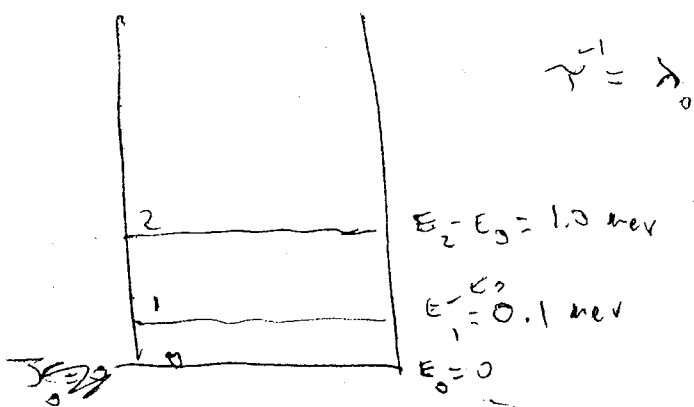
What is the lifetime of this nucleus inside a massive star at temperature $kT = 0.1\,\text{MeV}$? (Assume that thermal equilibrium obtains and that the populations of the states of the nucleus are thermal.)



$\tau^{-1} = \lambda_0 = 10^{-7}\,s^{-1}$

$\lambda_0 = 10^{-7}\,s^{-1}$

$\lambda_1 = 10^{-1}\,s^{-1} = 10^6 \lambda_0$

$\lambda_2 = 10^{11}\lambda_0 = 10^4\,s^{-1}$

$kT = 0.1$ MeV

partition function $Z = \sum_i (2J_i+1)\, e^{-E_i/T}$

$$\lambda_{TOT} = \sum_i P_i \lambda_i = \frac{\lambda_0 \cdot (2\cdot 0+1)\cdot e^{-0/T} + \lambda_1 \cdot (2\cdot 1+1)\cdot e^{-0.1/0.1} + \lambda_2 \cdot (2\cdot 2+1)\, e^{-\frac{1.0}{0.1}}}{1 + 3\cdot e^{-1} + 5 e^{-10}}$$

$$\boxed{P_i = \frac{(2J_i+1)\, e^{-E_i/T}}{Z}}$$

$$= \frac{\lambda_0 \left[1 + 3\cdot 10^6 e^{-1} + 5\cdot 10^{11} e^{-10}\right]}{1 + 3\cdot e^{-1} + 5\cdot e^{-10}}$$

$\lambda_0 \to 10^{-7}\,s^{-1}$

#19 (SECTION 5)

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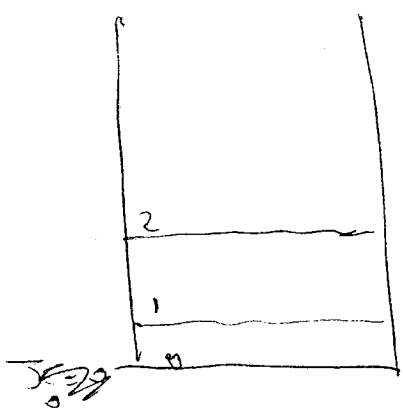

$\tau^{-1} = \lambda_0 = 10^{-7}\,s^{-1}$

$E_2 - E_0 = 1.0$ mev

$E_1 - E_0 = 0.1$ mev

$E_0 = 0$

$\lambda_0 = 10^{-7}\,s^{-1}$

$\lambda_1 = 10^{-1}\,s^{-1} = 10^6 \lambda_0$

$\lambda_2 = 10^{11} \lambda_0 = 10^4\,s^{-1}$

$kT = 0.1$ MeV

partition function $Z = \sum_i (2J_i+1)\, e^{-E_i/T}$

$$\lambda_{TOT} = \sum_i P_i \lambda_i = \frac{\lambda_0 \cdot (2\cdot 0+1)\cdot e^{-0/T} + \lambda_1 \cdot (2\cdot 1+1)\cdot e^{-0.1/0.1} + \lambda_2 \cdot (2\cdot 2+1)\, e^{-1.0/0.1}}{1 + 3\cdot e^{-1} + 5 e^{-10}}$$

$$P_i = \frac{(2J_i+1)\, e^{-E_i/T}}{Z}$$

$$= \frac{\lambda_0 \left[1 + 3\cdot 10^6 e^{-1} + 5\cdot 10^{11} e^{-10}\right]}{1 + 3\cdot e^{-1} + 5\cdot e^{-10}}$$

$10^{-7}\,s^{-1}$

#20 (SECTION 5)

PROBLEM: Evaluate the integral

$$I(x) = \int_{-1}^{1} \frac{dt}{x-t} \sqrt{1-t^2}$$

using contour integration methods for real x, $|x| > 1$. Make sure your result applies both for positive and for negative x.

GRAD 20 SOL

Problem: Evaluate the integral $I(x) = \int_{-1}^{1} \frac{dt}{x-t}\sqrt{1-t^2}$ using contour integration methods for real x, $|x|>1$. Make sure your result applies both for positive and for negative x.

Solution: Consider $x>1$ first. We have:

$$I(x) = \int_{-1}^{1} \frac{dt}{x-t}\sqrt{1-t^2}$$

$$= \frac{1}{2}\oint_C \frac{dt}{x-t}\sqrt{1-t^2},$$

where contour C goes around the branch cut $-1<t<1$ as shown in Fig. 1.

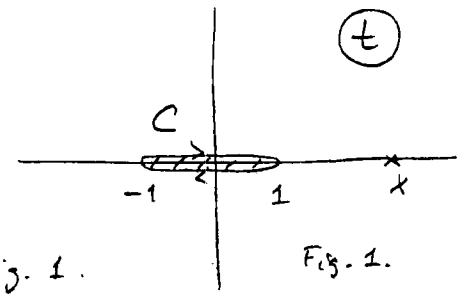

Fig. 1.

Deforming C into C_1 (see Fig. 2), we get

$$I(x) = \frac{1}{2}\oint_{C_1} \frac{dt}{x-t}\sqrt{1-t^2}$$

$$= \frac{1}{2}\int_{|t|=R} \frac{dt}{x-t}\sqrt{1-t^2} + \frac{1}{2}\cdot 2\pi i \operatorname*{res}_{t=x} \frac{\sqrt{1-t^2}}{x-t}.$$

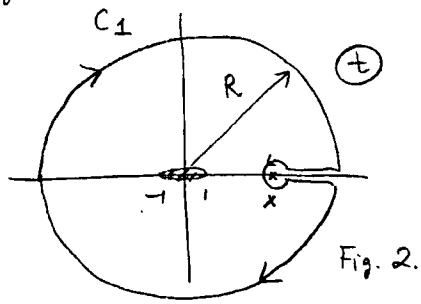

Fig. 2.

(1) On the large circle, $|t|=R$, we expand the integrand as follows:

$$\sqrt{1-t^2} = -it\left(1-\frac{1}{t^2}\right)^{1/2} = -it\left(1-\frac{1}{2}\frac{1}{t^2}+\dots\right),$$

$$\frac{1}{x-t} = -\frac{1}{t}\left(1+\frac{x}{t}+\dots\right),$$

$$\frac{\sqrt{1-t^2}}{x-t} \simeq i\left(1+\frac{x}{t}\right). \Rightarrow \frac{1}{2}\int_{|t|=R} \frac{dt}{x-t}\sqrt{1-t^2} = \frac{1}{2}\int dt\cdot\frac{ix}{t}$$

$$= \frac{1}{2}(-2\pi i)(ix) = \pi x.$$

(2) $$\operatorname*{res}_{t=x} \frac{\sqrt{1-t^2}}{x-t} = -\sqrt{1-t^2}\Big|_{t=x} = i\sqrt{x^2-1}.$$

Thus, $I(x) = \pi\left(x-\sqrt{x^2-1}\right)$. For negative x, we can use the easy to prove relation $I(-x) = -I(x) \Rightarrow I(x) = \pi\left[x - \operatorname{sign}(x)\sqrt{1-x^2}\right]$ for all $|x|\geq 1$.