FALL 2002

DEPARTMENTAL EXAM

&

SOLUTIONS

NUMBER_____ SCORE _____

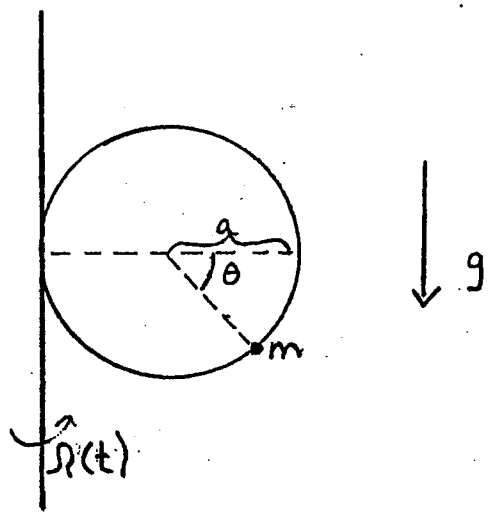
PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 1

Problem 1. Mechanics

A circular wire hoop is rigidly attached on one side to a vertical axle (see figure). The axle and hoop rotate about the vertical with angular velocity $\Omega(t) = \Omega_o \sin \omega t$. A bead of mass m slides without friction on the hoop. Gravity is directed vertically down, the radius of the hoop is a, and the generalized coordinate θ locates the position of the bead on the hoop.



a) Obtain a Lagrangian governing the motion of the bead.

b) Obtain the Hamiltonian.

c) Is the Hamiltonian equal to the energy of the bead?

d) Is the Hamiltonian conserved?

e) Is the energy conserved?

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A B C

solution (1)

UG mech (8)

DEPARTMENT OF PHYSICS

MID-TERM

Physics 200A
October 20, 1987

T.M. O'Neil
Your Name

(10 points)

1. A circular wire hoop is rigidly attached on one side to a vertical axle (see figure). The axle and hoop rotate about the vertical with angular velocity $\Omega(t) = \Omega_0 \sin \omega t$. A bead of mass m slides without friction on the hoop. Gravity is directed vertically down, the radius of the hoop is a, and the generalized coordinate θ locates the position of the bead on the hoop.

(a) Obtain a Lagrangian governing the motion of the bead.

$$L = T - V$$

$$T = \frac{m}{2}\left[a^2\dot{\theta}^2 + \Omega^2(t)\,a^2(1+\cos\theta)^2\right]$$

$$V = -mga\sin\theta$$

(b) Obtain the Hamiltonian.

$$H = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \frac{ma^2\dot{\theta}^2}{2} - \frac{m}{2}\Omega^2(t)\,a^2(1+\cos\theta)^2 - mga\sin\theta$$

$$\frac{ma^2}{2}\left(\frac{P_\theta}{ma^2}\right)^2$$

(c) Is the Hamiltonian equal to the energy of the bead?

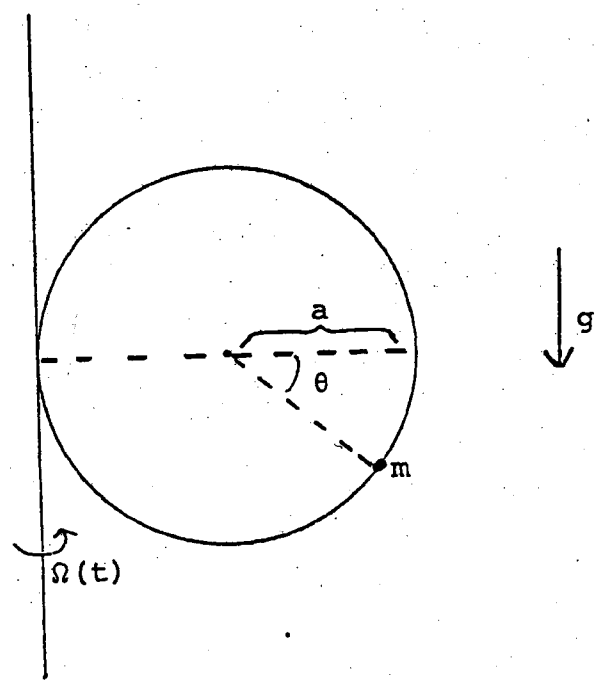
no $\quad H \neq T + V$

(d) Is the Hamiltonian conserved?

no, since $\frac{\partial L}{\partial t} \neq 0$

(e) Is the energy conserved?

no, since forces of constraint can do work on the bead



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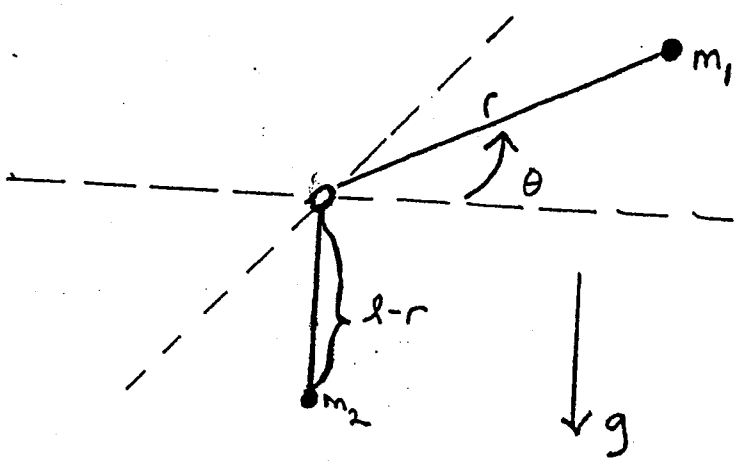
PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 1

PROBLEM 2. MECHANICS

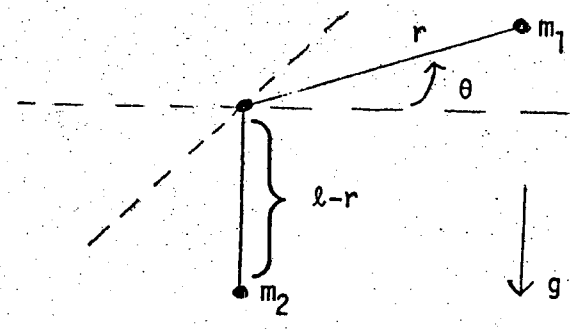
Two point masses are connected by a massless string that passes through a hole in a smooth table (see figure). The mass m_1 rests on the table and the mass m_2 hangs below. Gravity is directed vertically down, and the string is of length ℓ. You may assume that the masses move in such a way that the string remains taut and that neither mass comes near the hole.



Write down the Lagrangian for this system and determine two constants of the motion.

Solution 2

(2) Two point masses are connected by a massless string that passes through a hole in a smooth table (see figure). The mass m_1 rests on the table and the mass m_2 hangs below. Gravity is directed vertically down, and the string is of length ℓ. You may assume that the masses move in such a way that the string remains taut and that neither mass comes near the hole.



(a) Write down the Lagrangian for this system and determine two constants of the motion.

$$L = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2 - m_2 g r$$

$$\frac{\partial L}{\partial t} = 0 \quad \therefore \quad \text{const} = H = \frac{m_1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m_2}{2}\dot{r}^2 + m_2 g r$$

$$\frac{\partial L}{\partial \theta} = 0 \quad \therefore \quad \text{const} = P_\theta = m_1 r^2 \dot{\theta}$$

perhaps just partial

(b) For the initial conditions $(r,\theta) = (R,0)$ and $(\dot{r},\dot{\theta}) = (0,\omega)$, where $m_1\omega^2 R > m_2 g$, what is the maximum value of r reached by m_1?

$$m_2 g R + \frac{m_1}{2} R^2 \omega^2 = \frac{m_1}{2} r^2 \underbrace{\dot{\theta}^2}_{\left(\frac{R^2\omega}{r^2}\right)^2} + m_2 g r \quad , \text{ since } \dot{r} = 0 \text{ at } r = r_{max}$$

$$m_2 g (R - r) + \frac{m_1}{2}\omega^2\left(R^2 - \frac{R^4}{r^2}\right) = 0$$

$$m_2 g(R-r) + \frac{m_1}{2}\omega^2 R^2\left(\frac{r^2-R^2}{r^2}\right) = 0$$

$$\cancel{(R-r)}\left[m_2 g - \frac{m_1\omega^2}{2r^2}R^2(r+R)\right] = 0$$

$$r^2 - \frac{m_1\omega^2R^2}{2m_2g}r - \frac{m_1\omega^2R^3}{2m_2g} = 0$$

$$x = \frac{r}{R}$$

$$x^2 - \frac{m_1\omega^2R}{2m_2g}x - \underbrace{\frac{m_1\omega^2R}{2m_2g}}_{\alpha} = 0$$

correct sign (check for $\alpha = 1/2$)

$$x = \frac{+\alpha \pm \sqrt{\alpha^2+4\alpha}}{2}$$

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PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 2

PROBLEM 3. E & M

A perfectly conducting sphere of radius a is uniformly magnetized with a magnetization of magnitude M. It also has a net charge Q. Find the total angular momentum of the electromagnetic field of this system.

Solution: Problem #3

E-Field: Gauss Law $\rightarrow$ $\vec{E} = \begin{cases} 0 & r<a \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r>a \end{cases}$

B-Field: pure dipole source, $\vec{B} = \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

for dipole $\vec{m} = m\hat{z} = \dfrac{4}{3}\pi a^3 M\hat{z}$ along z-axis.

$\therefore$ momentum density in EM fields $\quad \vec{\mathcal{P}}_{EM} = \epsilon_0 \vec{E}\times\vec{B}$

$$= \begin{cases} 0 & r<a \\ \dfrac{QM}{4\pi r^2}\dfrac{\mu_0 a^3 \sin\theta}{3r^3}\hat{\varphi} & r>a \end{cases}$$

angular momentum density $\quad \vec{\mathcal{L}}_{EM} = \vec{r}\times\vec{\mathcal{P}}_{EM} = \begin{cases} 0 & r<a \\ -\dfrac{QM}{4\pi}\dfrac{\mu_0 a^3 \sin\theta}{3r^4}\hat{\theta} & r>a \end{cases}$

Integrate over all space; φ symmetry projects component along z-axis, with $-\hat{\theta}\cdot\hat{z} = +\sin\theta$

$$\vec{\mathcal{L}}_{TOTAL} = \hat{z}\,\mu_0\frac{QMa^3}{12\pi}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\underbrace{\int_0^{\pi}\sin^2\theta\sin\theta\,d\theta}_{4/3}\underbrace{\int_a^{\infty}\frac{r^2\,dr}{r^4}}_{1/a}$$

$$= \hat{z}\,\frac{2}{9}\mu_0 MQa^2$$

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PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 2

PROBLEM 4. E&M

A long cylindrical nonmagnetic conductor of radius b has a coaxial cylindrical hole of radius a drilled along it. It carries a current I distributed uniformly over the cross section. Find the magnetic energy associated with the magnetic field in a length L of the conductor. From this determine the self-inductance of the same length L.

Solution: Problem #4

Ampère's Law $\longrightarrow$ $\vec{B} = \hat{\varphi}\,\mu_0 I \dfrac{(\rho^2 - a^2)}{(b^2 - a^2)} \cdot \dfrac{1}{2\pi\rho}$ for $a < \rho < b$.

$$U_{Em} = \int \frac{1}{2\mu_0} B^2 d\tau = \int_0^l dz \int_0^{2\pi} d\varphi \int_a^b \frac{B^2}{2\mu_0} \rho\, d\rho$$

$$= \frac{\mu_0 l I^2}{4\pi} \int_a^b \left(\frac{\rho^2 - a^2}{b^2 - a^2} \right)^2 \frac{1}{\rho^2} \rho\, d\rho$$

$$= \frac{\mu_0 l I^2}{4\pi (b^2 - a^2)^2} \left\{ \frac{1}{4}\rho^4 - \frac{2a^2\rho^2}{2} + a^4 \ln \rho \right\} \Bigg|_a^b$$

$$= \frac{\mu_0 l I^2}{16\pi (b^2 - a^2)^2} \left\{ b^4 - a^4 - 4a^2(b^2 - a^2) + 4a^4 \ln b/a \right\}$$

$$= \frac{\mu_0 l I^2}{16\pi (b^2 - a^2)^2} \left\{ (b^2 - a^2)(b^2 - 3a^2) + 4a^4 \ln b/a \right\}$$

But $U_{Em} = \frac{1}{2} L I^2 \quad \therefore \quad L = \dfrac{\mu_0 l}{8\pi (b^2 - a^2)} \left\{ (b^2 - a^2)(b^2 - 3a^2) + 4a^4 \ln b/a \right\}$.

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PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 3

PROBLEM 5. STAT MECH/THERMAL

The pressure p in a spherical bubble of radius r is given by $p = \sigma/2r$, where σ is a constant. A bubble is in a vacuum and filled with ν moles of ideal gas with heat capacity C_V at temperature T_0. (Assume the heat capacity of the bubble wall is negligible.) Then a small metal particle with heat capacity C at temperature T_m comes in contact with the bubble. The bubble and solid radius has increased by a factor of two.

a) What is the final temperature T_f in terms of T_0?

b) Calculate the total change in entropy of the gas bubble and the solid as a result of this process.

c) Calculate the initial temperature of the metal particle T_m in terms of the parameters given, *i.e.*, T_0, C_V, C, ν, and the gas constant R.

Solution : Problem #5

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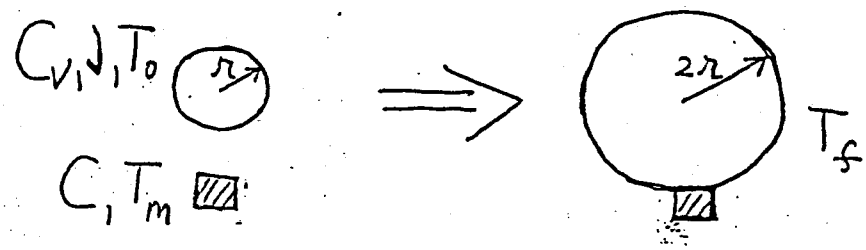
Midterm #2

11/16/92



3. (36 points) The pressure, p, in a spherical bubble of radius r is given by $p = \frac{\sigma}{2r}$, where σ is a constant. A bubble is in a vacuum and filled with ν moles of ideal gas with heat capacity C_V at temperature T_0. [Assume the heat capacity of the bubble wall is negligible.] Then a small metal particle with heat capacity C at temperature T_m comes in contact with the bubble. The bubble and solid reach a new equilibrium at temperature T_f, where the bubble radius has increased by a factor of two.

a) What is the final temperature, T_f, in terms of T_0?

$$pV = \nu RT \qquad V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{\sigma}{2r}\frac{4}{3}\pi r^3 = \nu RT = \frac{2}{3}\pi\sigma r^2$$

$$\frac{T_f}{T_0} = \frac{r_f^2}{r_i^2} = 4 \qquad \underline{\underline{T_f = 4T_0}}$$

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3. (cont'd)

b) Calculate the total change in entropy of the gas bubble and the solid as a result of this process:

$$\Delta S = \int_{metal} \frac{dQ}{T} + \int_{bubble} \frac{dQ}{T} = C\int_{T_m}^{T_f} \frac{dT}{T} + C_v \int_{T_0}^{T_f} \frac{dT}{T} + \int_{V_i}^{V_f} \frac{p\,dV}{T}$$

$$\int \frac{p\,dV}{T} = \int \frac{p\,dV}{pV/JR} = JR\int \frac{dV}{V} = JR \ln \frac{V_f}{V_i} = JR \ln 8$$

$V \propto r^3$

$$\Delta S = C \ln\left(\frac{T_f}{T_m}\right) + C_v \ln 4 + JR \ln 8$$

{ As shown below, $T_f/T_m = 4T_0/T_m$ is known in terms of $C, C_v + J$ }

c) Calculate the initial temperature of the metal particle, T_m, in terms of the parameters given (i.e., T_0, C_v, C, v, and the gas constant R).

$$C(T_m - T_f) = C_v(T_f - T_0) + \int p\,dV$$

$$\int p\,dV = \int \frac{\sigma}{2r} 4\pi r^2 dr = \pi\sigma r^2 \Big|_{r_i}^{r_f} = 3\pi\sigma r_i^2$$

but $JRT_0 = \frac{2}{3}\pi\sigma r_i^2$, so

$$C(T_m - T_f) = C_v(T_f - T_0) + \frac{9}{2} JRT_0$$

$$T_m = \left[4C + 3C_v + \frac{9}{2}JR\right] T_0/C$$

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SECTION 3

PROBLEM 6. STAT MECH/THERM

A chain has N frictionless links of length L that can be oriented in four directions, up, down, left and right, as shown. The top is tied to a fixed point and a weight of mass M is attached to the bottom. The system is at temperature T in a gravitational field of strength g. (This is an ideal chain, so it can cross itself.)

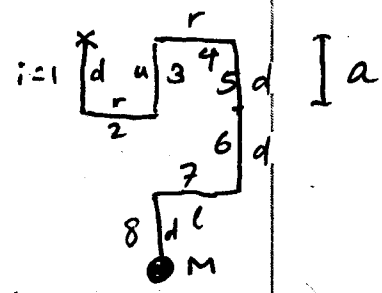
a) What are the relative probabilities of the four different orientations of the links P_r, P_l, P_u, and P_d?

b) Obtain an expression for the mean length $\bar{L}$ of the system at temperature T. Evaluate this expression in the limit of high and low temperatures, stating clearly which is which.

c) Write down the partition function for this system.

d) Calculate the entropy of the system and evaluate this expression in the limit of high and low temperatures.

SOLUTION #6



$i = 1, \ldots, N$ labels each link

$\sigma_i \in \{u, d, l, r\}$ – 4 states per link

Hamiltonian : $H = -MgL$

Kronecker-δ

Length : $L = a \sum_{i=1}^{N} (\delta_{\sigma_i,d} - \delta_{\sigma_i,u})$

We can write $H = -Mg \sum_{i=1}^{N} L_i$; $L_i = a(\delta_{\sigma_i,d} - \delta_{\sigma_i,u})$

$$= \begin{cases} +a & \text{if } \sigma_i = d \\ -a & = \sigma_i = u \\ 0 & \text{otherwise} \end{cases}$$

Hence, $Z_1 = z^N$, with

$$z = \mathrm{Tr}_{\sigma}\, e^{\beta Mga(\delta_{\sigma,d} - \delta_{\sigma,u})}$$

$$= 2 + 2\cosh(\beta Mga) = 4\cosh^2(\tfrac{1}{2}\beta Mga)$$

2 (a) $$P_l = P_r = \frac{1}{4\cosh^2(\frac{1}{2}\beta Mga)} = \frac{1}{2(1+\cosh\beta Mga)}$$

$$P_{d,u} = \frac{e^{\pm\beta Mga}}{4\cosh^2(\frac{1}{2}\beta Mga)} = \frac{e^{\pm\beta Mga}}{2(1+\cosh\beta Mga)}$$

3 (b) $$Z_1 = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{\beta MgL} = z^N$$

$$\Rightarrow \langle L \rangle = \frac{1}{Mg}\frac{\partial}{\partial\beta}\ln Z_1 = \frac{N}{Mg}\frac{\partial}{\partial\beta}\ln z$$

$$= \frac{Na\sinh\beta Mga}{1+\cosh\beta Mga} = Na\tanh(\tfrac{1}{2}\beta Mga)$$

(b) cont'd low $T \Rightarrow \beta \to \infty \Rightarrow \langle L \rangle = Na$ ✓

high $T \Rightarrow \beta \to 0 \Rightarrow \langle L \rangle = \dfrac{NMga^2}{2k_BT}$

3 (c) $Z_1 = z^N = 2^N (1+\cosh \beta Mga)^N = 4^N \cosh^{2N}(\frac{1}{2}\beta Mga)$

2 (d) $dF = -SdT + \ldots \Rightarrow S = -\dfrac{\partial F}{\partial T} = -\dfrac{\partial \beta}{\partial T}\dfrac{\partial F}{\partial \beta} = k_B\beta^2 \dfrac{\partial F}{\partial \beta}$

$$F = -\frac{1}{\beta}\ln Z_1 = -\frac{1}{\beta}\cdot\left\{N\ln 4 + 2N\ln\cosh(\tfrac{1}{2}\beta Mga)\right\}$$

$$\frac{\partial F}{\partial \beta} = \frac{1}{\beta^2}\left[N\ln 4 + 2N\ln\cosh(\tfrac{1}{2}\beta Mga)\right] - \frac{N}{\beta}Mga\tanh(\tfrac{1}{2}\beta Mga)$$

$$S = k_B\beta^2\frac{\partial F}{\partial \beta}$$

$$= Nk_B\cdot\left\{\ln 4 + 2\ln\cosh(\tfrac{1}{2}\beta Mga) - \beta Mga\tanh(\tfrac{1}{2}\beta Mga)\right\}$$

$\beta \to \infty$ $(T \to 0)$: $S = Nk_B\left\{\ln(2+e^{\beta Mga}+e^{-\beta Mga}) - \beta Mga\dfrac{1-e^{-\beta Mga}}{1+e^{-\beta Mga}}\right\}$

$$= Nk_B\left\{\beta Mga + \ln(1+2e^{-\beta Mga}+e^{-2\beta Mga}) - \beta Mga + 2\beta Mga\, e^{-\beta Mga} + \ldots\right\}$$

$$= 2(1+\beta Mga)e^{-\beta Mga} + \ldots$$

$\beta \to 0$ $(T \to \infty)$: $S = Nk_B\left\{\ln 4 + \frac{1}{4}(\beta Mga)^2 - \frac{1}{2}(\beta Mga)^2 + \ldots\right\}$

$$= Nk_B\left\{\ln 4 - (\tfrac{1}{2}\beta Mga)^2 + \ldots\right\}$$

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PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 4

PROBLEM 7. QUANTUM MECHANICS

An excited nucleus under study decays by the process of internal conversion , *i.e.*, instead of emitting a photon, it ejects one of the atomic electrons $N^* \rightarrow N + e$. Let E be the kinetic energy of the electron and E_0 the energy liberated in the decay. Compute the density of final states $\rho_f(E)$ in a volume V as needed for Fermi's golden rule. You should assume that the nucleus is massive so that recoil can be neglected. What is the density of final states if the electron is non-relativistic? What is the density of final states if the electron is highly relativistic?

Prob. 7 Solution

Using PBC in a volume $V = L^3$, we have

$$dk_x L = 2\pi n_x \quad \ldots$$

$$dn_x = \frac{L}{2\pi} dk_x$$

$$d^3 n = \frac{L^3}{(2\pi)^3} d^3k = \frac{V d^3k}{(2\pi)^3}$$

$$\rho_f(E) = \int \frac{V d^3k}{(2\pi)^3} \delta(E - E_0) \, d$$

$$= \frac{V}{(2\pi\hbar)^3} \int d^3p \; \delta(E - E_0)$$

$$= \frac{V}{2\pi^2\hbar^3} \int p^2 dp \; \delta(E - E_0)$$

Non-relativistic limit: $E = \frac{p^2}{2m}$ $\quad dE = \frac{p}{m} dp$

$p = \sqrt{2mE}$ $\quad dp = \frac{m}{p} dE$

$$\rho_f(E) = \frac{V}{2\pi^2\hbar^3} \int \sqrt{2mE} \, m \, dE \, \delta(E - E_0)$$

$$= \frac{V}{2\pi^2\hbar^3} \sqrt{2m^3 E_0} = \frac{V m^{3/2} \sqrt{E_0}}{\sqrt{2}\, \pi^2 \hbar^3}$$

Relativistic limit: $E = pc$ $\quad dE = c\,dp$

$p = E/c$ $\quad dp = \frac{1}{c} dE$

$$\rho_f = \frac{V}{2\pi^2\hbar^3} \int \frac{E^2}{c^3} dE \, \delta(E - E_0) = \frac{V E_0^2}{2\pi^2 \hbar^3 c^3}$$

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PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 4

PROBLEM 8. QUANTUM MECHANICS

A rotating system is described by the Hamiltonian

$$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_3}$$

with **L** the orbital angular momentum and I_1 and I_3 inertial moments.

a) Comment on the symmetry of the system;

b) Write the Hamiltonian in a form that only involves compatible operators and find its eigenvalues and eigenstates;

c) What are the degeneracies of the system? Is there a relation with your answer in a)?

Problem 8 | Solution

8. $$H = \frac{L_x^2 + L_y^2}{2I_1} + \frac{L_z^2}{2I_3}$$

(a) Hamiltonian has cylindrical symmetry about $\hat{z}$-axis since $I_x = I_y = I_1$

Rotations about $\hat{z}$-axis do not change energy
Reflections $z \to -z$ about (xy) plane do not change energy.

(b) $$H = \frac{L^2}{2I_1} + L_z^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)$$

eigenstates $|L\, L_z\rangle = |l\, m\rangle$

$$L^2\, |l\, m\rangle = l(l+1)\hbar^2\, |l\, m\rangle$$
$$L_z\, |l\, m\rangle = m\hbar\, |l\, m\rangle$$

$$H\, |l\, m\rangle = \frac{l(l+1)\hbar^2}{2I_1} + m^2\hbar^2\left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)$$

for $m = |l|, |l-1|, \ldots, -|l|$

(c) $|l\, m\rangle$ and $|l\, -m\rangle$, $m \neq 0$ are degenerate energy eigenstates.

2-fold degeneracy expected from $\hat{z} \to -\hat{z}$ symmetry.

Section 4

Problem 8 QM

a) The system is symmetric around the z-axis because the inertial moment is the same for L_x and L_y

b) L_x, L_y, L_z do not commute with each other. On the other hand L_z and L^2 do commute. So L_x, L_y, L_z are incompatible operators and L_z an L^2 are compatible operators

Change expression for Hamiltonian H using $L_x^2 + L_y^2 = L^2 - L_z^2$:

$$H = \frac{L^2 - L_z^2}{2I_1} + \frac{L_z^2}{2I_3} = \frac{L^2}{2I_1} + \frac{L_z^2}{2} \frac{I_1 - I_3}{I_1 I_3}$$

Eigenvalues and eigenstates are the usual ones for $Y_l^m(\cos\theta, \varphi)$ or $|l, m\rangle$.

Use $L^2 |l, m\rangle = l(l+1)\hbar^2 |l, m\rangle$

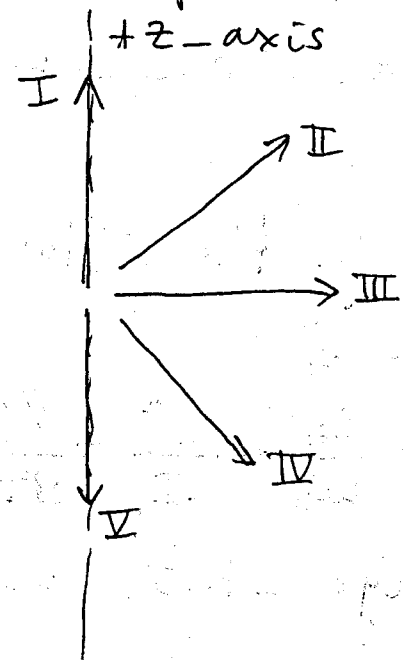
$$L_z |l, m\rangle = m\hbar |l, m\rangle$$

to get $E_{l,m} = \frac{l(l+1)}{2I_1}\hbar^2 + \frac{m^2}{2}\frac{I_1 - I_3}{I_1 I_3}\hbar^2$ with $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, +l$

c) The energies depend upon l and m^2 so l, m and $l, -m$ have the same energy and thus we have two-fold degeneracy for $l \geq 1$ and $m^2 \geq 1$

Thus ~~keeping~~ flipping the z-component of L from positive to negative, while keeping its magnitude the same, does not change the energy. [Compare with situation where $I_1 = I_3$]

Example: $l = 2$



I & V are degenerate

II & IV are degenerate

III is not degenerate

Flipping the sign of the z-component leaves the component in the x,y plane the same in magnitude. The direction in the x-y plane does not matter (classically speaking) because the inertial moment is the same in the x- and y-direction

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PART I

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SECTION 5

PROBLEM 9. EXPERIMENTAL/QUANTUM

A surprising and as yet not well understood feature of the strong interaction is that the nucleons inside nuclei can be taken to be almost non-interacting particles moving in a collective potential, the so-called single particle approximation. (The protons still experience the Coulomb interaction with other protons, but this is weak compared to the strong interaction.) In fact, an excellent first approximation is to take the strongly interacting system of neutrons and protons to be separately represented by gases of non-interacting spin-1/2 fermion quasi-particles in an overall spherical square well potential of depth $V_0 \approx 50\,\text{MeV}$. The radius of this potential is the radius of the nucleus, empirically $r \approx r_0 A^{1/3}$, where $r_0 \approx 1.2\,\text{fm}$ and where one Fermi, 1fm, is $10^{-13}\,\text{cm}$, roughly the size of a bare nucleon. Here A is the nuclear mass number, or total number of nucleons in the nucleus. The nucleon quasi particles can be taken to have the quantum numbers and masses of bare nucleons. Crudely, large nuclei have equal numbers of neutrons and protons. (Actually, large nuclei tend to have an excess of neutrons over protons, but ignore this.)

a) Give an estimate of the minimum amount of energy required to eject a nucleon from a nucleus. In other words, what is the binding energy of the least tightly bound nucleon in the nucleus? Ignore the Coulomb interaction.

b) What is the average kinetic energy of a nucleon (either a proton or a neutron) in a nucleus?

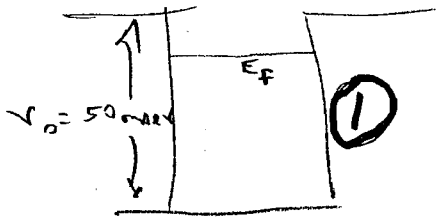
c) What is the average speed of a nucleon in the nucleus? Express your result as a fraction of the speed of light.

Quantum / stat mech problem 2

PART I: PROBLEM #9

6 pts

a.) Binding energy? ⇒ get fermi level



1

$$dn_{n\,or\,p} = \frac{g\, d^3k}{(2\pi)^3} \frac{1}{e^{\frac{E-\mu}{T}}+1}$$ 1

take $T \to 0$, $\mu \to E_f$

$$n_{n\,or\,p} = g\int_0^{p_f} \frac{c^3 d^3p}{(2\pi\hbar c)^3} \qquad \left\{ g = 2 \cdot 2 \cdot 2 \right.$$

$$= \frac{4\pi}{8\pi^3(\hbar c)^3}\int_0^{p_f} p^2\, dp$$

$$= \frac{1}{3\pi^2}\frac{1}{(\hbar c)^3} p_f^3$$ 1

so $$E_f = \frac{p_f^2}{2m} = \frac{[3\pi^2]^{2/3}(n_{n,p})^{2/3}(\hbar c)^2}{2m}$$

since $$p_f = \left[3\pi^2 n_{n,p}(\hbar c)^3\right]^{1/3}$$

$$n_{n,p} = \frac{A/2}{(r_0 A^{1/3})^3} \simeq 0.29 \simeq 0.3\ fm^{-3}$$ 1

$$E_f = \frac{(3\pi^2)^{2/3}}{2}\frac{(0.3\ fm^{-3})^{2/3}(197\ MeV\text{-}fm)^2}{931.494\ MeV}$$

$m_{nuc} \sim 1$ GeV

1 or $\frac{1}{\text{Avogadro's}}$ #1

(estimation)

~~≃ 44.8 MeV ≃ 44 MeV~~

≃ 42 MeV

40 - 45 MeV 1

Binding energy $= V_0 - E_f \approx 50 - 42$ MeV ~ 8 MeV

~ 10 MeV

2 pts

b.) average K.E. ⇒ average momentum squared in the Fermi gas

$$\langle E\rangle \simeq \left\langle \frac{p^2}{2m}\right\rangle$$ 1

$$\simeq \frac{1}{2m}\langle p^2\rangle$$

$$\simeq \frac{3}{5}E_f \simeq 25\ MeV$$

$$\langle p^2\rangle = \frac{\int_0^{p_f} p^2 \cdot p^2\, dp}{\int_0^{p_f} p^2\, dp} = \frac{\frac{1}{5}p_f^5}{\frac{1}{3}p_f^3}$$ 1

$$\simeq \frac{3}{5}p_f^2$$

c.) $\boxed{2\text{ pts}}$ $\langle E \rangle = \dfrac{\langle p^2 \rangle}{2m} = 25 \text{ meV} = \frac{1}{2} m v^2$

9

pag 2

① $v^2 = \dfrac{2 \cdot \langle E \rangle}{m}$ or $v = \dfrac{\sqrt{2}\,(\langle E \rangle)^{1/2}}{m^{1/2}}$

$v^2/c^2 \simeq \dfrac{2 \cdot 25 \text{ meV}}{939{,}494 \text{ meV}} \simeq \dfrac{50 \text{ meV}}{1000 \text{ meV}} = 5 \times 10^{-2}$ ①

$\Rightarrow v/c \simeq 0.22 \sim 1/4$

NUMBER_____ SCORE _____

PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 5

PROBLEM 10. EXPERIMENTAL/QUANTUM

a) Riff Raff measures the brain waves of his sister, Magenta. He initially takes data as a sampling rate of 200 Hz (200 samples per second) for periods of 100 s. What is the highest spectral frequency that he can resolve?

b) What is the lowest spectral frequency that Riff Raff can resolve?

c) Magenta has a rare disease that leads to brain oscillations at 130 Hz. How will these appear in Riff Raff s data?

d) Dr. Everet Scott starts an NMR experiment that requires that he sweep the strength of a magnetic field and measure the change in magnetization of the sample as a function of the field. Dr. Scott ponders whether, all other things being equal:

   i) Is it best to sweep slowly, and use a low pass filter with a small bandwidth?

   ii) Is it best to sweep fast, use a filter with a larger bandwidth and average together many traces?

Which of these strategies are best (i) or (ii), or are they equal?

e) Just as Dr. Scott gets down to work, Columbia starts to dance in the laboratory. This leads to 1/f noise, *i.e.*, noise with a spectrum that increases with *decreasing* frequency, in the measurement process. Under this new condition, which of the above measurement strategies is best? Why (be brief)?

NUMBER_____ SCORE Brief Answers

PART I

PHYSICS DEPARTMENTAL EXAM — FALL 2002

SECTION 5

PROBLEM 10. EXPERIMENTAL/QUANTUM

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+10

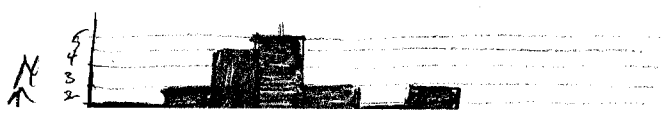
a) $f_{max} = f_{sample}/2 = 100\,Hz$

b) $f_{min} = 1/T = 0.01\,Hz$

c) Aliasing will move oscillation from 100+30 to 100−30 = 70 Hz

d) Equal for white noise

e) Many fast sweeps to avoid 1/f noise.



mode = 4
mean = 3.7

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 11. MATH

Find the eigenvalues of the integral operator

$$Ku(x) = \int_0^\infty e^{-|x-y|} u(y)\, dy$$

If you cannot find a closed form for the eigenvalues, come as close as you can.

FREDKIN
PROBLEM #11

Problem: Find the eigenvalues of the integral operator

$$Ku(x) = \int_0^\infty e^{-|x-y|} u(y)\, dy.$$

If you cannot find a closed form for the eigenvalues, come as close as you can.
Solution: The kernel is the Green's function for the differential operator

$$\frac{d^2}{dx^2} - 1$$

with the boundary conditions $u'(0) = u(0)$, $u'(1) = -u(1)$. Namely,

$$k(x, y) = e^{-|x-y|}$$

satisfies

$$\frac{\partial^2 k(x,y)}{\partial x^2} - k(x,y) = -2\delta(x-y)$$

and the boundary conditions (in x for fixed y). Now, if $Ku = \lambda u$, u must satisfy the boundary conditions and

$$\lambda(u'' - u) = -2u$$

or

$$u'' - \mu^2 u = 0$$

with

$$\mu^2 = 1 - \frac{2}{\lambda}$$

or

$$\lambda = \frac{2}{1-\mu^2}.$$

We must have

$$u(x) = Ae^{\mu x} + Be^{-\mu x}$$

and the boundary conditions are

$$\mu(A - B) = (A + B)$$
$$\mu\left(Ae^{\mu} - Be^{-\mu}\right) = -\left(Ae^{\mu} + Be^{-\mu}\right)$$

which is consistent with nontrivial u only if

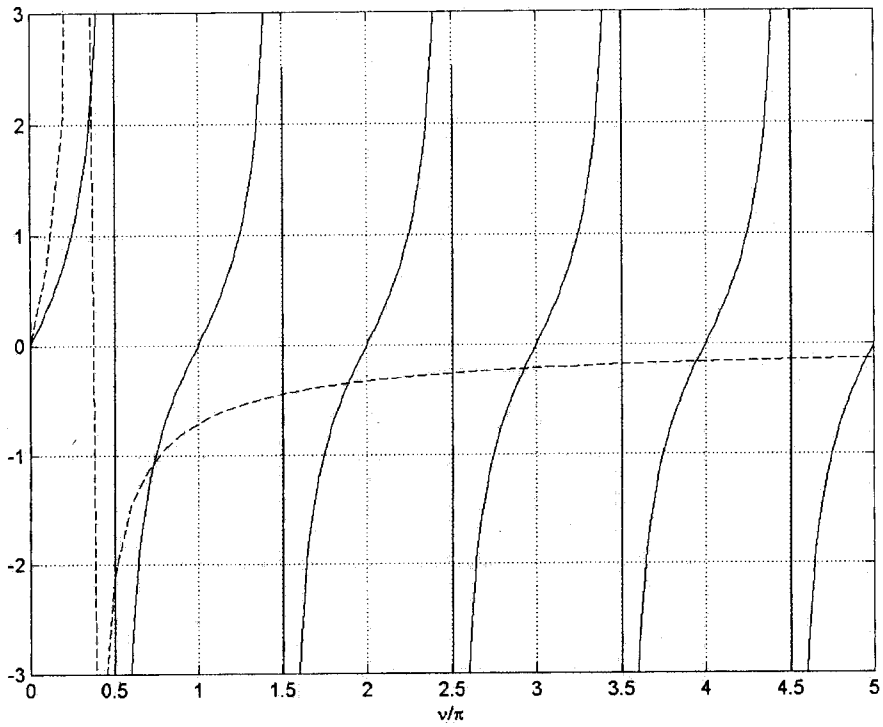
$$(\mu+1)^2 e^{\mu} = (1-\mu)^2 e^{-\mu}.$$

This equation cannot be satisfied for any real μ other than $\mu = 0$, for which $u(x)$ is identically zero. But for $\mu = -i\nu$, we have the equation

$$e^{i\nu} = \frac{1+i\nu}{1-i\nu}$$

or

$$\tan\nu = \frac{2\nu}{1-\nu^2}.$$



The figure shows that there is a sequence of solutions ν_n, $n = 1\ldots$, with $\nu_n \approx n\pi$ for all but the first few values of n. $\nu_1 \approx 0.742\pi$. The eigenvalues are

$$\lambda_n = \frac{2}{1+\nu_n^2}$$

and the lowest eigenvalue is $\lambda_1 \approx 0.3108$.

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 12. E & M

A sphere of magnetic material has permanent magnetization $\mathbf{M} = M(r)\hat{\mathbf{r}}$, where (r,θ,φ) are spherical coordinates. Determine $\mathbf{B}(r,\theta,\varphi)$ and $\mathbf{H}(r,\theta,\varphi)$.

O'NEIL
PROBLEM #12

10 points

5. A sphere of magnetic material has the permanent magnetization $\mathbf{M} = M(r)\hat{r}$, where (r,θ,ϕ) are spherical coordinates. Determine $\mathbf{B}(r,\theta,\phi)$ and $\mathbf{H}(r,\theta,\phi)$.

$$4\pi r^2 B_r = \oiint \underline{B} \cdot d\underline{S} = 0 \quad \therefore B_r = 0$$

$$0 = B_r = H_r(r) + 4\pi M_r(r)$$

$$\therefore H_r(r) = -4\pi M_r(r)$$

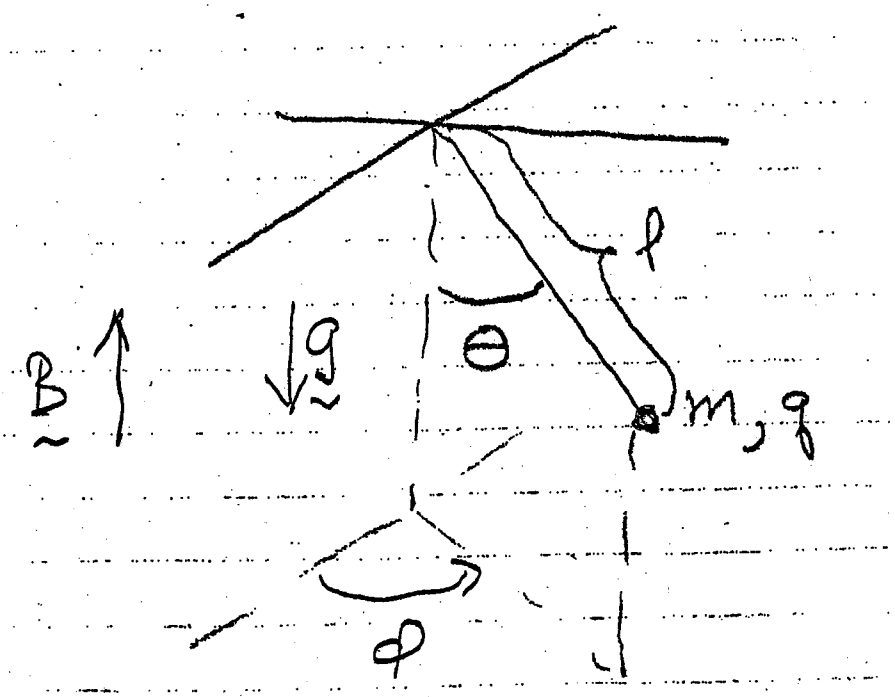
NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

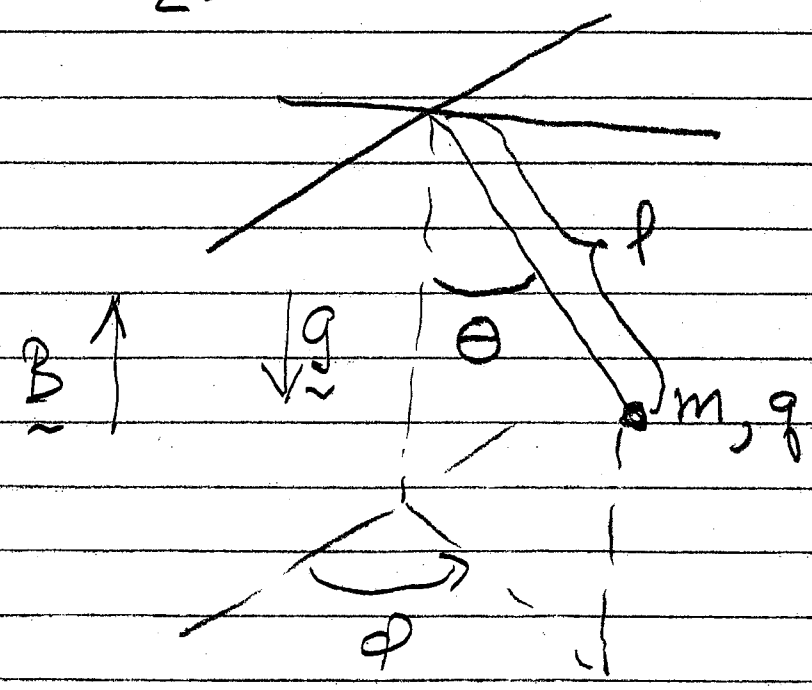
PROBLEM 13. MECHANICS/ E& M

The figure below shows a pendulum that consists of a mass m attached to a string of length ℓ. The mass carries charge q. Gravity points vertically down and a uniform magnetic field, **B**, points vertically up. What is the minimum value of the angle θ reached by the mass when it is released from rest at $\theta = \pi/2$. Here, $(\theta, \vartheta, \ell)$ are spherical coordinates defined with the origin at the point on the ceiling where the upper end of the string is attached.



Prob 13 Solution

(1) The figure below shows a pendulum that consists of a mass m attached to a string of length l. The mass carries charge q. Gravity points vertically down and a uniform magnetic field, $\underline{B}$, points vertically up. What is the minimum value of the angle θ reached if the mass is released from rest at $\theta = \pi/2$. Here, (θ, ϕ, r) are spherical coordinates defined with the origin at the point on the ceiling where the upper end of string is attached.



Soln

$$L = \frac{m}{2}\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2\right) + mgl\cos\theta$$

$$+ \frac{q}{c} A_\varphi\, l\sin\theta\,\dot{\varphi}$$

To determine A_φ use

$$A_\varphi\, 2\pi \overset{l\sin\theta}{r} \equiv \oint \underset{\sim}{A}\cdot dl = \iint \underset{\sim}{B}\cdot ds = B\pi r^2$$

$$A_\varphi = \frac{Br}{2} = \frac{Bl\sin\theta}{2}$$

$$\therefore L = \frac{m}{2}\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2\right) + mgl\cos\theta$$

$$+ \frac{qB}{c}\frac{l^2\sin^2\theta}{2}\dot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = 0 \implies \text{const} = \frac{\partial L}{\partial \dot{\varphi}}$$

$$\underset{\frac{qBl^2}{2c}}{\underset{\parallel}{\text{const}}} = m\cancel{l^2}\sin^2\theta\,\dot{\varphi} + \frac{q}{c}\frac{B\cancel{l^2}\sin^2\theta}{2}$$

$$\frac{\partial L}{\partial t} = 0 \implies H = \text{const.}$$

$$\underset{0}{\underset{\parallel}{H}} = \frac{m}{2}\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2\right) - mgl\cos\theta$$

At the minimum value of θ, $\dot{\theta}$ is zero.

$$\therefore \quad 0 = \frac{m}{2} l^2 \sin^2\theta \, \dot{\varphi}^2 - mgl\cos\theta$$

$$0 = \frac{\cancel{m} l^{\cancel{2}} \sin^2\theta}{2} \left(\frac{qB}{2cm}\right)^2 \frac{\overbrace{(1-\sin^2\theta)^2}^{\cos^4\theta}}{\sin^4\theta} - \cancel{m}g\cancel{l}\cos\theta$$

$$\therefore \quad \frac{l}{8g}\left(\frac{qB}{mc}\right)^2 = \cos\theta \frac{\sin^2\theta}{\cos^{\cancel{2}4}\theta} = \frac{\sin^2\theta}{\cos^3\theta}$$

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 14. MECHANICS/E&M

An electron is incident with impact parameter ρ and speed v_o on a proton at rest. Calculate the energy radiated during the collision assuming the ordering $e^2/\rho << mv_0^2 << mc^2$. (Hint: for this ordering there is a simple approximation for the orbit.)

Solution
Problem 14

T. M. O'Neil
Phys. 203B

Your Name

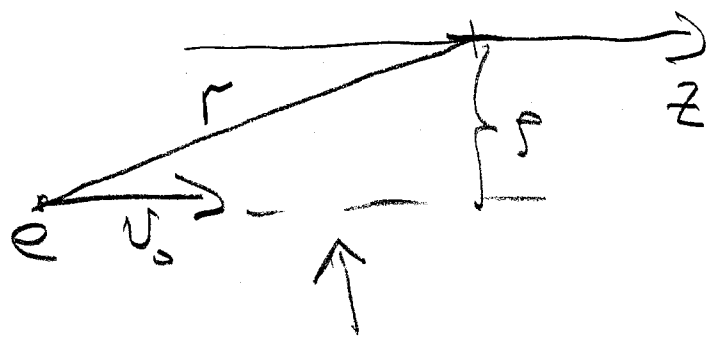
FINAL EXAM
June 10, 1994

Each problem is worth 10 points.

1. An electron is incident with impact parameter ρ and speed v_0 on a proton at rest. Calculate the energy radiated during the collision assuming the ordering $e^2/\rho \ll m v_0^2 \ll mc^2$. (Hint: for this ordering there is a simple approximation for the orbit.)

$$\text{Power} = \frac{2}{3}\frac{e^2 a^2}{c^3}$$

$$|a| = \frac{e^2}{m r^2}$$



Approximate orbit by straight line

$$\text{Energy radiated} = \int \text{Power}\, dt$$

$$= \frac{2}{3}\frac{e^2}{c^3}\int_{-\infty}^{+\infty} dt \left(\frac{e^2}{m}\right)^2 \frac{1}{\left[\rho^2 + (v_0 t)^2\right]^2}$$

$$= \frac{2 e^6}{3 m^2 c^3}\frac{1}{v_0}\frac{1}{\rho^4}\overbrace{\int_{-\infty}^{+\infty} dx \frac{1}{(1+x^2)^2}}^{\pi/2}$$

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 15. E & M

An electron resides in a large evacuated region of space that is permeated by the uniform magnetic field $\mathbf{B}_0$. Initially the electron velocity $\mathbf{v}_0$ is perpendicular to $\mathbf{B}_0$. Taking into account radiation, calculate the electron kinetic energy as a function of time. You may assume that $|\mathbf{v}_0| << c$ and that the energy radiated in one cyclotron period is small compared to the instantaneous kinetic energy. Is there some small value of the kinetic energy where you expect your result to fail?

40

PROBLEM #15

2. An electron resides in a large evacuated region of space that is permeated by the uniform magnetic field $\mathbf{B}_0$. Initially the electron velocity $\mathbf{v}_0$ is perpendicular to $\mathbf{B}_0$. Taking into account radiation, calculate the electron kinetic energy as a function of time. You may assume that $v_0 \ll c$ and that the energy radiated in one cyclotron period is small compared to the instantaneous kinetic energy. Is there some small value of the kinetic energy where you expect your result to fail?

$$m\frac{d\underset{\sim}{v}}{dt} = -e\frac{\underset{\sim}{v}}{c}\times\underset{\sim}{B}_0$$

Power radiated is given by

$$P = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\underset{\sim}{v}}{dt}\right)^2 = \frac{2}{3}\frac{e^2}{c^3}\frac{e^2B_0^2v_\perp^2}{c^2m^2}$$

$$\frac{d}{dt}\frac{mv_\perp^2}{2} = -P = -\frac{4}{3}\frac{e^4B_0^2}{c^5m^3}\frac{mv_\perp^2}{2}$$

$$\frac{mv_\perp^2(t)}{2} = \frac{mv_0^2}{2}\exp\left[-\frac{4}{3}\frac{e^4B_0^2}{c^5m^3}t\right]$$

expect to fail when

$$mv_\perp^2 \sim \hbar\Omega_c = \hbar\frac{eB_0}{mc}$$

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PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 16. MECHANICS/E&M

A non-relativistic electron is ejected from the origin with velocity $\mathbf{V} = v_o(\hat{z} + \hat{y})$, where $v_o(eB/mc) << L$. The magnetic field in a region of space is given by $\mathbf{B} = B_o[\hat{z}(1 + z/L) - \hat{x}x/L]$. Determine the maximum value of z reached by the electron.

Solution 16

7. The magnetic field in a region of space is given by $\mathbf{B} = B_0[\hat{z}(1+z/L) - \hat{x}\,x/L]$. A non-relativistic electron is ejected from the origin with velocity $\mathbf{v} \simeq v_0(\hat{z}+\hat{y})$, where $v_0/(eB/mc) \ll L$. Determine the maximum value of z reached by the electron.

The electron conserves the adiabatic invariant $\frac{mv_\perp^2}{2|\underset{\sim}{B}|}$ as it moves along the field line. The electron kinetic energy also is conserved.

$$\therefore \frac{mv_0^2}{2}2 = \frac{mv_z^2}{2} + \frac{mv_\perp^2}{2}$$

$$\parallel \quad = \frac{mv_z^2}{2} + \frac{mv_0^2}{2}\frac{\cancel{B_0}}{\cancel{B_0}}\left[(1+z/L)^2 + \cancel{(x/L)^2}\right]^{1/2}$$

$O\left(\frac{r_c}{L}\right)^2 \ll 1$

The electron reaches the max value of z when $v_z = 0$.

$$\therefore \quad mv_0^2 = \frac{mv_0^2}{2}\left[1 + \frac{z_{max}}{L}\right]$$

$$z_{max} = L$$

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 17. QUANTUM MECHANICS

Protons (contained in Hydrogen atoms) are in a strong, constant magnetic field $B\hat{\mathbf{z}}$ for enough time to relax to the ground state. At $t = 0$ a radio-frequency electromagnetic wave is applied,

$$\mathbf{B}(t) = B_1 f(\omega)\cos(\omega t)\hat{\mathbf{x}}$$

where $f(\omega)$ is a frequency distribution peaked at ω_1:

$$f(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\omega-\omega_1)^2/2\sigma^2}$$

What is the maximum spin flip transition rate that can be achieved as ω_1 is tuned?

Problem 17 SOLUTIONS

If transients due to turning on $\cos\omega t$ at time $t=0$ are neglected, then the flip rate at frequency ω is proportional to

$$(f(\omega))^2\,\delta(\omega-\omega_0)$$

where

$$\omega_0 = \gamma\hbar B \qquad \gamma = e/2mc$$

Presumably the total rate is prop to

$$\int (f(\omega))^2\,\delta(\omega-\omega_0)\,d\omega$$

$$= f(\omega_0)^2 \propto e^{-(\omega_0-\omega_1)^2/2\sigma^2}$$

and this maximizes at

$$\omega_1 = \omega_0$$

If the relaxation rate $\frac{1}{\tau}$ of the spin is included, the δ-function is replaced by something like $\frac{1}{(\omega-\omega_0)^2+\frac{1}{\tau^2}}$. Then one must maximize

$$\int \frac{e^{-(\omega-\omega_1)^2/2\sigma^2}}{(\omega-\omega_0)^2+\frac{1}{\tau^2}}\,d\omega$$

$$= \int \frac{e^{-(\nu+\omega_0-\omega_1)^2/2\sigma^2}}{\nu^2+\frac{1}{\tau^2}}\,d\nu$$

which also maximizes at $\omega_1 = \omega_0$

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 18. QUANTUM MECHANICS

A spin-less charged particle (charge e) in free space moves under the influence of a *uniform* magnetic field **B**. Find and discuss the energy levels for this particle. Show explicitly how you obtain your result.

SOLUTION

Problem 18 Solution

For a magnetic field $\vec{B} = B\hat{z}$

the corresponding vector potential

can be $A_x = -By/2$, $A_y = Bx/2$, $A_z = 0$

momentum operator (generalized):

$$\Pi_i = p_i - \frac{e}{c} A_i$$

$$\Rightarrow \left[\Pi_x, \Pi_y\right] = \left(p_x - \frac{e}{c}A_x\right)\left(p_y - \frac{e}{c}A_y\right) - \left(p_y - \frac{e}{c}A_y\right)\left(p_x - \frac{e}{c}A_x\right)$$

$$= i\frac{e}{c}\hbar B$$

This looks like the commutator for angular momentum components which suggests that we may be able to exploit that simple algebra by defining "raising" & "lowering" operators:

$$a = \sqrt{\frac{c}{2\hbar e B}}\left(\Pi_x + i\Pi_y\right)$$

$$a^+ = \sqrt{\frac{c}{2\hbar e B}}\left(\Pi_x - i\Pi_y\right) \quad \Rightarrow \quad \left[a, a^+\right] = 1$$

$$\Rightarrow a^+ a = \frac{c}{\ } \left\{p_x^2 + p_y^2 + \frac{e^2}{c^2}\left(A_x^2 + A_y^2\right) - \frac{e}{c}\left([A_x, p_x] + [A_y, p_y]\right) + i[\Pi_x, \Pi_y]\right.$$

Problem 17 Solution

The appropriate Hamiltonian for a particle in a magnetic field is

$$H = \frac{(\mathbb{P} - e/c\, \mathbb{A})^2}{2m} + e\varphi$$

or $$= \frac{\Pi^2}{2m} + e\varphi$$

or $$\hat{H} = \frac{p_z^2}{2m} + \frac{\hbar e B}{mc} a^+ a - \frac{i}{2m}[\Pi_x, \Pi_y]$$

$$= \frac{p_z^2}{2m} + \frac{\hbar e B}{mc}\left(a^+ a + \tfrac{1}{2}\right)$$

define $N = a^+ a$ as in S.H.O.

p_z commutes with $a^+ a$ $\Rightarrow$ we can construct simultaneous eigenkets

$$\omega \equiv \frac{eB}{mc} \qquad \hat{N} = \Pi_- \Pi_+$$

$$\hat{H} = \hbar\omega\left(N + \tfrac{1}{2}\right) + \frac{p_z^2}{2m}$$

$\Downarrow$

$$\left\{ \begin{array}{l} \hat{N}\,|n, \hbar k\rangle = n\,|n, \hbar k\rangle \\ \& \\ \hat{p}_z\,|n, \hbar k\rangle = \hbar k\,|n, \hbar k\rangle \end{array} \right.$$

so energy eigenvalues are

$$E_{k} = \frac{\hbar^2 k^2}{2m} + \frac{\hbar e B}{mc}\left(n + \tfrac{1}{2}\right)$$

NUMBER_____ SCORE _____

PART II

PHYSICS DEPARTMENTAL EXAM — FALL 2002

PROBLEM 19. STATISTICAL MECHANICS

Consider a drunk performing a random walk on a square grid (of grid size α), starting from the origin.

d) Suppose at every step, the drunk takes a step to one of the four neighboring grids with equal probability. Derive the expression for the average position and root-mean-square distance from the origin after N steps. [You may choose to assume that N is large.]

e) Repeat (a) for random walk performed on a cubic lattice.

f) Now suppose the walker is not completely drunk: on a square grid, he either continues to advance in the same direction with probability 0.4, or picks one of the three remaining directions with equal probability (0.2 each). The first step was taken with equal probability in all four directions. Compute the average position and root-mean-square distance from the origin after this walker takes N steps.

[Hint: It will be useful to first show the *correlation* in the directions of successive steps , and then use it to compute recursively the correlation in the directions between two arbitrary steps.]

Written Exam — Fall 2002

Solution for Problem 19. Statistical Mechanics

(a) Let $\vec{a}_n$ describe the displacement vector associated with step n. The total displacement after N steps is

$$\vec{R}_N = \sum_{n=1}^{N} \vec{a}_n. \tag{1}$$

For a completely random walker, $\vec{a}_n \in \{\pm a\hat{x}, \pm a\hat{y}\}$ with equal probability, and is uncorrelated with $\vec{a}_m$ for all $m \neq n$. Hence

$$\langle \vec{R}_N \rangle = \sum_{n=1}^{N} \langle \vec{a}_n \rangle = 0 \tag{2}$$

$$\langle \vec{R}_N^2 \rangle = \sum_{m,n=1}^{N} \langle \vec{a}_m \cdot \vec{a}_n \rangle = \sum_{n=1}^{N} \langle \vec{a}_n^2 \rangle = Na^2. \tag{3}$$

(b) The above result is independent of the spatial dimensionality.

(c) The total displacement is still given by Eq. (1). Since there is no asymmetry in the four direction, the result (2) still holds.

There is however explicit correlation in the directions of successive steps taken by the walker, and the result (3) is no longer valid. Given that step n is in direction a_n, the average direction of step $n+1$ is $\langle \vec{a}_{n+1} \rangle = 0.2\, \vec{a}_n$. Similarly,

$$\langle \vec{a}_{n+2} \rangle = 0.2\, \langle \vec{a}_{n+1} \rangle = (0.2)^2\, \vec{a}_n,$$

and generally, $\langle \vec{a}_n \vec{a}_{n+\ell} \rangle = (0.2)^\ell\, a^2$.

Now, the rms distance is given by

$$\begin{aligned} \langle \vec{R}_N^2 \rangle &= \sum_{m,n=1}^{N} \langle \vec{a}_m \cdot \vec{a}_n \rangle \\ &= Na^2 + 2(N-1) \times \langle \vec{a}_n \vec{a}_{n+1} \rangle + 2(N-2) \times \langle \vec{a}_n \vec{a}_{n+2} \rangle + \ldots \\ &= Na^2 + 2a^2 \sum_{\ell=1}^{N-1} (N-\ell)(0.2)^\ell. \end{aligned} \tag{4}$$

Performing the sum for $N \gg 1$, with $S(\gamma) = \sum_{\ell=1}^{N-1} \gamma^\ell = \dfrac{\gamma}{1-\gamma}$,

$$\text{and} \quad \sum_{\ell=1}^{N-1} \ell\, \gamma^\ell = \gamma \frac{d}{d\gamma} S(\gamma) = \frac{1}{(1-\gamma)^2},$$

we find

$$\lim_{N\to\infty} \langle \vec{R}_N^2 \rangle = a^2 \left[N + 2NS(\gamma = 0.2) \right] = 1.5Na^2.$$

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PART II

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PROBLEM 20. RELATIVITY/MECHANICS/ESTIMATE

Suppose that we have a uniform (homogeneous) and isotropic distribution of mass-energy that fills all of space. (That is, at a given time t, the density of mass-energy is the same everywhere and there are no preferred directions.) This distribution of mass and energy could be uniformly contracting or expanding. Consider a test mass point of mass m that is on the surface of a co-moving spherical surface with radius $a(t)$ whose center is at an arbitrary point in space. By co-moving, we mean a spherical surface that rides along with the expansion/contraction. Take the test mass to be negligible compared to the total mass energy inside of radius $a(t)$ for any time we consider here. The physical density of mass-energy at any time t and at any point in space is $\rho(t)$. Take the gravitational constant to be $G = 1/m_{\rm pl}^2$, where the Planck mass is $m_{\rm pl} \approx 1.22 \times 10^{22}\,\text{MeV}$, where we have taken $\hbar = c = 1$.

a) By considering the kinetic and gravitational potential energies of the test mass find an equation for the time rate of change of $a(t)$. Use Newtonian expressions for kinetic energy and gravitational potential energy. (Call the time rate of change of the radius of the sphere $\dot{a} \equiv da/dt$.) Express your result in terms of G, $a(t)$, $\rho(t)$, and a constant $k = -2E_{\rm tot}/m$, where $E_{\rm tot}$ is the total mechanical energy which can be taken to be positive, negative or zero. Describe the evolution of the mass-energy distribution, *i.e.*, the evolution of $a(t)$ for the cases where k is positive, negative, and zero. The equation you derived turns out to be general relativistically correct. The technique for deriving it suggested here is correct so long as our imaginary comoving sphere is small compared to the causal horizon size (roughly the speed of light times the age of the universe).

b) Suppose that $\rho(t)$ comes almost entirely from electromagnetic radiation (photons) with a thermal energy spectrum. That is take $\rho(t) = AT^4$, where the temperature characterizing the Planck distribution of the photon energies is $T(t)$ and A is a constant. Since the number density of photons scales like $T^3(t)$ and the average energy of a photon scales like $T(t)$, we can show that $T(t)$ scales like $a^{-1}(t)$. If this is the case, how does $a(t)$ scale with time for the case where the total mechanical energy is zero?

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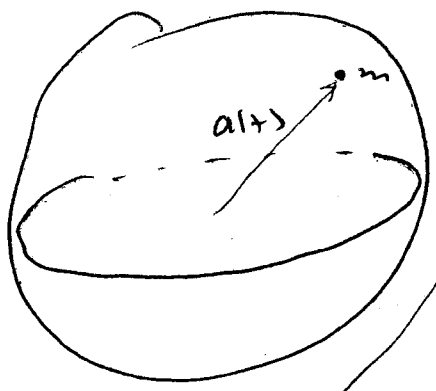
c) Now suppose that the mass-energy is dominated by a constant component. That is $\rho(t) = \rho_0$, independent of time or position. How does $a(t)$ depend on time in this case if the total mechanical energy of our commoving sphere is zero ($k = 0$)?

d) In fact, present astronomical observations indicate that your result from part c) is the best fit to the time dependence of the expansion rate of the universe. If the present expansion rate is measured to be $H_o = \dot{a}/a \approx 100\text{km-s}^{-1}\text{-Mpc}^{-1}$ (where $1\text{Mpc} \approx 3\times10^{24}$ cm), what is ρ_o? (ρ_o is the so-called vacuum energy density or dark energy density .)

(1.) Problem 20

Solution

a.)



total mechanical energy

K.E. + P.E. = const.

$$\frac{1}{2} m \dot{a}^2(t) + \left[- \frac{G\left(\frac{4}{3}\pi\rho(t)\, a^3(t)\right) m}{a(t)} \right] = \underset{\text{const}}{\overset{E_{tot}}{\shortparallel}}$$

$$\dot{a}(t) \equiv \frac{da(t)}{dt}$$

$$\left[\dot{a}^2 - \frac{8}{3}\pi G \rho\, a^2\right] m = \underset{2E_{tot}}{\overset{\text{const.}}{\shortparallel}} = -m k$$

define a constant $k = -\frac{\text{const}}{m} = \frac{-2E_{tot}}{m}$

$$\Rightarrow \boxed{\dot{a}^2 + k = \frac{8}{3}\pi G \rho\, a^2}$$

or $$\dot{a}^2 + k = \frac{8}{3}\pi \frac{\rho}{m_{pl}^2}\, a^2$$

This is the so-called Friedmann equation and its derivation is fully generally relativistically correct so long as $|a(t)|$ is small compared to the causal horizon size. We are able to use these simple Newtonian considerations because of the spherical symmetry here.

(2) Problem 20 Solution

b.) radiation $\Rightarrow \rho(t) = A T^4$

$$T(t) \propto \frac{1}{a(t)} \qquad T/T_0 = a_0/a$$

$$\frac{\dot{a}^2}{a^2} \simeq \frac{8}{3}\pi \frac{1}{m_{pl}^2} A T^4 = \frac{8}{3}\pi \frac{A}{m_{pl}^2} T_0^4 \frac{a_0^4}{a^4}$$

$$\dot{a}^2 a^2 \simeq \text{const}$$

$$\text{or } \dot{a}\, a = \text{const}$$

$$a\, da = \text{const} \cdot dt$$

$$\frac{1}{2} a^2 \propto t$$

$$\Rightarrow \boxed{a \propto t^{1/2}}$$

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c.) $\rho(t) = \rho_0$

$$\Rightarrow \dot{a}^2 = \frac{8}{3}\pi \frac{1}{m_{pl}^2} \rho_0 a^2$$

this is of the form

$\dot{a} = \text{const}\, a$

or $\frac{da}{a} = \text{const}\, dt$

or $\ln(a/a_0) \propto (t - t_0)$

or $a(t) = a_0 e^{H(t-t_0)}$

where H is a constant

$$\left.\frac{\dot{a}^2}{a^2}\right|_0 = H_0^2 = \frac{8}{3}\pi \frac{\rho_0}{m_{pl}^2}$$

more carefully $\frac{da}{a} = \left(\frac{8}{3}\pi\right)^{1/2} \frac{\rho_0^{1/2}}{m_{pl}} dt$

(3.) Problem 20 Solution

$$\ln\left(a/a_0\right) = \left(\tfrac{8}{3}\pi\right)^{1/2} \frac{\rho_0^{1/2}}{m_{pl}}\, dt = H_0\, dt$$

$$a = a_0\, e^{H_0 (t - t_0)}$$

$$H_0^2 = \tfrac{8}{3}\pi \frac{\rho_0}{m_{pl}^2} \quad \text{or} \quad \rho_0 = \frac{3}{8}\frac{m_{pl}^2 H_0^2}{\pi}$$

$$\boxed{\begin{aligned} \rho_0 &\simeq \frac{3}{8\pi}\left(1.22\times 10^{22}\ \text{MeV}\right)^2 \left(2\times 10^{-39}\ \text{MeV}\right)^2 \\ &\simeq 8.5\times 10^{-35}\ \text{MeV}^4 \simeq 10^{-34}\ \text{MeV}^4 = \frac{10^{-34}\ \text{MeV}}{(\hbar c)^3} \simeq \frac{10\ \text{keV}}{\text{cm}^3} \end{aligned}}$$

where

$$H_0 = \frac{100\ \text{km s}^{-1}}{\text{Mpc}} = \frac{10^7\ \text{cm}}{3\times 10^{24}\ \text{cm}}\ \frac{1}{\text{s}} \simeq 3\times 10^{-17}\ \frac{1}{\text{s}}$$

$$\simeq 3\times 10^{-17}\ \text{s}^{-1}\left(\frac{\hbar c}{c}\right) = 3\times 10^{-17}\ \text{s}^{-1}\ \frac{200\ \text{MeV-fm}}{3\times 10^{23}\ \text{fm s}^{-1}}$$

$$\simeq 2.2\times 10^{-39}\ \text{MeV}$$