DEPARTMENTAL WRITTEN EXAM

SPRING 2000 SOLUTIONS

SCORE: _____

DEPARTMENTAL EXAMINATION – SPRING 2000

DEPARTMENT OF PHYSICS

DEPARTMENTAL EXAMINATION – SPRING 2000

PART I

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (9) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. Please indicate with a "check" which of the (7) questions you wish to be graded below:

Section 1:	**Problem 1**_____	**Problem 2** _____
Section 2:	**Problem 3**_____	**Problem 4** _____
Section 3:	**Problem 5**_____	**Problem 6**_____
Section 4:	**Problem 7** _____	**Problem 8**_____
Section 5:	**Problem 9**_____	

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SECTION I.

Problem 1.

The internal energy and equation of state of one mole of a van der Waals gas are given by $U = \frac{f}{2}RT - \frac{a}{V}$ *and* $\left(p + \frac{a}{V^2}\right)(V - b) = RT.$

a) What are the slope and curvature of the isotherm at the critical point? Determine the volume and temperature at the critical point from a and b.

b) A van der Waals gas is adiabatically expanded. Compute the inversion temperature below which the expansion leads to cooling, from a and b. (Hints: Which thermodynamic potential is constant, so that its total differential, expressed as a function of T and V is 0? In the last step, make the approximation b << V.)

Solution

a) The slope and curvature are both 0.

$$\left(p + \frac{a}{V^2}\right)(V-b)RT \Rightarrow p = \frac{RT}{V-b} - \frac{a}{V^2}$$

$$\frac{\partial p}{\partial V} = 0, \quad \frac{\partial^2 p}{\partial V^2} = 0 \Rightarrow$$

$$-\frac{RT_c}{(V_c-b)^2} + \frac{2a}{V_c^3} = 0, \quad \frac{2RT_c}{(V_c-b)^3} - \frac{6a}{V_c^4} = 0 \Rightarrow$$

$$\underline{\underline{V_c = 3b}}, \quad \underline{\underline{T_c = \frac{8}{27}\frac{a}{Rb}}}$$

b) The enthalpy is constant, $dH = \frac{\partial H}{\partial V} dV + \frac{\partial H}{\partial T} dT = 0$.

$$H = U + pV = \frac{f}{2}RT - \frac{a}{V} + \frac{RTV}{V-b} - \frac{a}{V} = RT\left(\frac{f}{2} + \frac{V}{V-b}\right) - \frac{2a}{V}$$

$$\frac{dT}{dV} = -\frac{\partial H/\partial V}{\partial H/\partial T} = \frac{\frac{bT}{(V-b)^2} - \frac{2a}{RV^2}}{\frac{f}{2} + \frac{V}{V-b}} \approx \frac{RTb - 2a}{\left(\frac{f}{2}+1\right)RV^2}$$

The numerator is negative for $T < \underline{\underline{T_i = \frac{2a}{Rb}}}$

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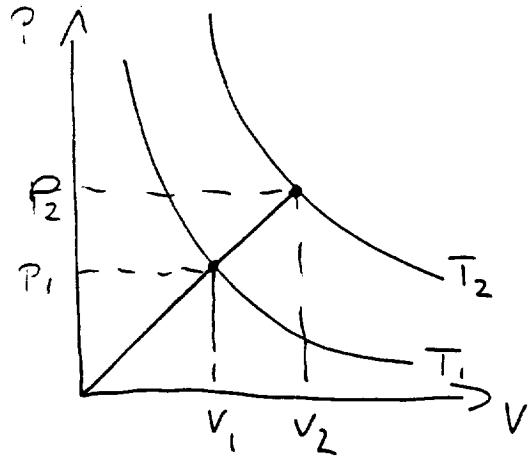
SECTION 1.

Problem 2.

An ideal gas is heated in such a way that the pressure is proportional to the volume. What is the effective specific heat for this process?

Solution:



initial state: P_1, V_1, T_1

final state: $P_2 = kP_1$, $V_2 = kV_1$, $T_2 = k^2 T_1$

$$\Delta Q = C_V (T_2 - T_1) = C_V T_1 (k^2 - 1)$$

$$\Delta W = \frac{1}{2} V_2 P_2 - \frac{1}{2} V_1 P_1 \qquad \text{(area of trapezium)}$$

$$= \frac{1}{2} V_1 P_1 (k^2 - 1)$$

$$= \frac{1}{2} R T_1 (k^2 - 1) = \frac{1}{2} R (T_2 - T_1)$$

$$C = \frac{\Delta Q + \Delta W}{T_2 - T_1} \Rightarrow$$

$$\underline{\underline{C = C_V + \frac{1}{2} R = \frac{1}{2}(C_V + C_P)}}$$

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SECTION 2.

Problem 3.

In 1995, M. Mayor and D. Queloz published precise measurements of the radial velocity of 51 Peg, a star that is similar to the Sun. They found near-sinusoidal variations of the radial velocity, which they interpreted as the reflex motion of the star due to a planetary companion. The period of the variations is 4.23 days, the amplitude 59 m/s.

a) What is the orbital radius of the planet?

b) Under the assumption that the mass of 51 Peg is one solar mass, derive a lower limit for the mass of the planet. (Note that we do not know the inclination of the orbit.)

c) If our mass estimate for 51 Peg is off by 20% (i.e. $M = (1 \pm 0.2)M_{sun}$), how does this affect the result in b)?

d) Under the assumption that binary star orbits are oriented randomly in space, what is the probability that the companion of 51 Peg is not a planet but a star ($m \geq 0.08 M_{sun}$)? (The orbit would have to be nearly face-on.)

e) What temperature do we expect for the planet?

$M_{sun} = 2 \cdot 10^{30} kg, \quad M_{Jup} = 2 \cdot 10^{27} kg, \quad 1AU = 1.5 \cdot 10^{8} km, \quad \gamma = 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$

solution:

M = mass of star
m = mass of planet
R = orbital radius of star
r = orbital radius of planet ~ distance star-planet
i = inclination
P = orbital period
v = velocity amplitude
T = temperature

a) $\frac{\gamma m M}{r^2} = m r \omega^2 = m r \left(\frac{2\pi}{P}\right)^2 \Rightarrow r = (\gamma M)^{1/3} \cdot \left(\frac{P}{2\pi}\right)^{2/3} = \frac{4.23}{365.25}\,AU = \underline{\underline{0.05\,AU}}$

b) $v = \frac{2\pi R}{T} \sin i \quad , \quad \frac{m}{M} = \frac{R}{r} \Rightarrow$

$$m = \frac{M v P}{2\pi r \sin i} = \frac{M^{2/3} v P^{1/3}}{(2\pi)^{1/3} \gamma^{1/3} \sin i} \Rightarrow$$

$$m \sin i = 9 \cdot 10^{26}\,kg = \underline{\underline{0.47\,M_{Jup}}} = 4.5 \cdot 10^{-4}\,M_\odot$$

c) We see that M enters with the $2/3$ power in the result of b), therefore a 20% error in M gives a $\frac{2}{3} \cdot 20\% = \underline{\underline{13\%}}$ error in $m \sin i$.

d) $m \geq 0.08\,M_\odot \Leftrightarrow \sin i \leq \frac{4.5 \cdot 10^{-4}}{0.08} = 0.0056$

$$p(\sin i \leq \vartheta) = \int_0^\vartheta \sin\vartheta \, d\vartheta = -\cos\vartheta \Big|_0^\vartheta = 1 - \cos\vartheta = \underline{\underline{1.6 \cdot 10^{-5}}}$$

e) $T^4 \sim r^{-2} \Rightarrow T = T_{Earth} \cdot r^{-0.5} = \sqrt{20} \cdot 300\,K = \underline{\underline{1340\,K}}$

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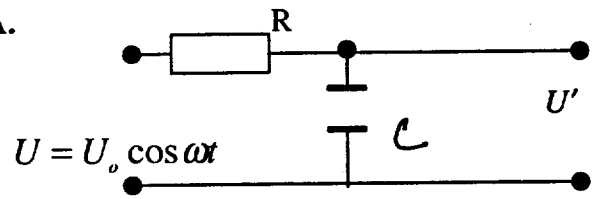
SECTION 2.

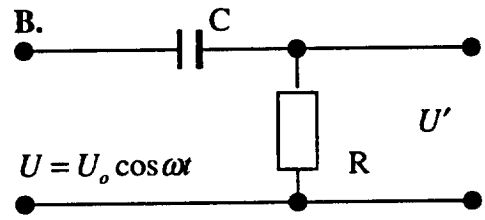
Problem 4.

The following three electronic circuits

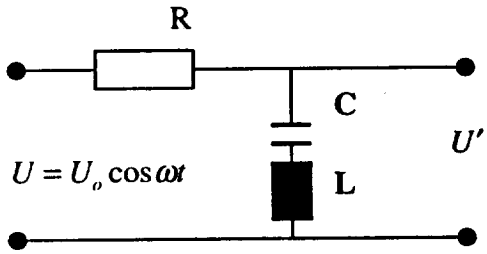
A.



B.



C.



are connected to an AC voltage $U = U_o \cos \omega t$.

a) Compute the amplitude of the output voltage $U_o' = |U'|$ for circuit A, and give the critical frequency ω_c for which $U_o' = \frac{1}{\sqrt{2}} U_o$

b) Compute U_o' for circuit B, and give ω_c for which $U_o' = \frac{1}{\sqrt{2}} U_o$

c) Compute U_o' for circuit C, and give ω_c for which U_o' is minimal.

d) Assign one of the terms "high pass filter", "low pass filter", "band pass filter", and, " band rejection filter" to each of the circuits A, B, and C, and sketch a circuit to which the remaining term applies.

Solution:

a) $$\frac{U'}{U} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{1 + i\omega RC} \Rightarrow \underline{\underline{U_0' = \frac{1}{\sqrt{1+(\omega RC)^2}} U_0}}$$

$$\omega_c RC = 1 \Rightarrow \underline{\underline{\omega_c = \frac{1}{RC}}}$$

b) $$\frac{U'}{U} = \frac{R}{R + \frac{1}{i\omega C}} = \frac{1}{1 - \frac{i}{\omega RC}} \Rightarrow \underline{\underline{U_0' = \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}} U_0}}$$

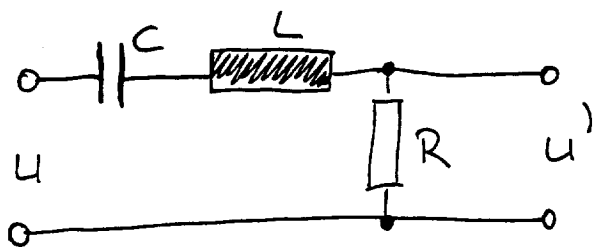
$$\omega_c RC = 1 \Rightarrow \underline{\underline{\omega_c = \frac{1}{RC}}}$$

c) $$\frac{U'}{U} = \frac{i\omega L + \frac{1}{i\omega C}}{R + i\omega L + \frac{1}{i\omega C}} = \frac{1}{1 + \frac{iR}{\frac{1}{\omega C} - \omega L}} = \frac{1}{1 + \frac{i\omega RC}{1 - \omega^2 LC}} \Rightarrow$$

$$\underline{\underline{U_0' = \frac{1}{\sqrt{1+\left(\frac{\omega RC}{1-\omega^2 LC}\right)^2}} U_0}}$$

$$U_0' = 0 \text{ for } \underline{\underline{\omega_c = \frac{1}{\sqrt{LC}}}}$$

d) (A) low pass, (B) high pass, (C) band rejection



band pass

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SECTION 3

Problem 5.

A train has length 120 ft. in its rest frame, and it is traveling at $4c/5$ with respect to the platform. The conductor walks back and forth in the train taking tickets at $4c/5$ with respect to the train. How long does it take him to get from the rear to the front, and how long does it take him to get from the front to the rear? (Recall $c = 1$ ft/nsec.)

(a) according to observers on the train

(b) according to the conductor's own clock

(c) according to observers on the platform

2n 5 (a) According to observers on the train, the train has length 120 ft and the conductor is moving with relative velocity either +4/5c or −4/5c, so

$$\Delta t = \frac{L}{v} = \frac{120 \text{ ft}}{\frac{4}{5} \text{ ft/nsec}} = 150 \text{ nsec}$$

for rear-to-front and for front-to-rear.

(b) The proper time of the conductor is related to Δt and Δx in the train frame by

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2 = (150)^2 - (120)^2 = (90)^2$$

$$\Delta\tau = 90 \text{ nsec}$$

in both directions. Equivalently,

$$\Delta\tau \doteq \frac{\Delta t}{\gamma} = (150)\sqrt{1 - \frac{16}{25}} = \frac{3}{5} \cdot (150) = 90 \text{ nsec}$$

(c) Define events A: Conductor at back of train; B: Conductor reaches front of train; and C: Conductor returns to back of train.
In train frame, $\Delta t_{AB} = 150$ and $\Delta x_{AB} = 120$, while $\Delta t_{BC} = 150$ and $\Delta x_{BC} = -120$.
Boost to platform frame with $v = 4/5$.

$$\Delta t'_{AB} = \gamma(\Delta t_{AB} - v\Delta x_{AB}) = \frac{5}{3}(150 - \frac{4}{5} \cdot 120) = 90 \text{ nsec}$$

$$\Delta t'_{BC} = \gamma(\Delta t_{BC} - v\Delta x_{BC}) = \frac{5}{3}(150 - \frac{4}{5} \cdot (-120)) = 410 \text{ nsec}$$

SECTION 3
Problem 6.

Consider two objects of equal mass M in a circular Keplerian orbit (due to their mutual attraction). Assume their separation is R.

a) Where should an infinitesimally small mass particle be placed such that it will also rotate about the overall center-of-mass at the same frequency as that of the original objects. Look for a solution where the three masses are <u>not</u> co-linear.

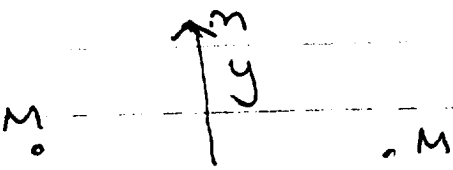
b) Now imagine making the two large masses different, say M_1 and M_2. Show that the same configuration as found in part a) will still satisfy the equation of motion, albeit with the rotation around the (now altered) center-of-mass position.

1) For the two original masses

$$M\omega^2 \frac{R}{2} = \frac{GM^2}{R^2} \qquad \omega^2 = \frac{2GM}{R^3}$$

a) By symmetry, the third particle should be placed on a vertical line through the center-of-mass. Then, for it to rotate at the same rate, we need

$$\frac{2MGy}{\left(\left(\frac{R}{2}\right)^2 + y^2\right)^{3/2}} = \omega^2 y$$



or, using result for ω^2

$$\left(\frac{R}{2}\right)^2 + y^2 = R^2 \qquad y = \frac{\sqrt{3}}{2} R$$

$\Rightarrow$ three particles are at vertices of an equilateral triangle

b) The new center of mass is at $X = \dfrac{M_1 X_1 + M_2 X_2}{M_1 + M_2}$

$$= \frac{R}{2}\left(\frac{M_1 - M_2}{M_1 + M_2}\right)$$

Let us take this to be at the origin and defn $\vec{d}$ to be the position of the mass m particle.

The force on this particle is

$$-\frac{G}{R^3}\left(M_1(\vec{d}-\vec{r}_1) + M_2(\vec{d}-\vec{r}_2)\right)$$

where $\vec{r}_1$ and $\vec{r}_2$ are the c.o.m. coordinates of M_1 and M_2. But

$M_1\vec{r}_1 + M_2\vec{r}_2 = 0$, so we get

$$-\frac{G(M_1+M_2)}{R^3}\vec{d}$$

this is a radially inward force which exactly provides the needed centripetal acceleration $\omega^2 d$ for the Keplerian frequency $\omega^2 = \frac{G(M_1+M_2)}{R^3}$

SECTION 4

Problem 7.

A beam of particles of energy E impinges on a barrier modeled by repulsive one-dimensional δ function $V(x) = V_o\delta(x)$.

a) Find the reflection and transmission coefficients for a beam arriving from $x = -\infty$.

b) Imagine placing a second, identical barrier at x = L such that $\sqrt{\frac{2mE}{\hbar^2}}L = \pi/2$.

Recompute the transmission coefficient. Is it larger or smaller than that for the single barrier problem, if V_o is small?

undergrad QM

a) $\psi(x<0) = e^{ikx} + re^{-ikx}$

$\psi(x>0) = te^{ikx}$ $\qquad k = \sqrt{\frac{2mE}{\hbar}}$

Schroedinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi$$

integrating across $x=0$ gives

$$-\frac{\hbar^2}{2m}\left(\psi'(0^+) - \psi'(0^-)\right) + V_0\psi(0) = 0.$$

$$\Rightarrow \psi'(0^+) - \psi'(0^-) = \frac{2mV_0}{\hbar^2}\psi(0)$$

So, we obtain

$$1 + r = t$$

$$itk - ik(1-r) = \frac{2mV_0}{\hbar^2}t$$

solving, we get

$$(1+r) - (1-r) = \frac{2mV_0}{i\hbar^2 k}(1+r)$$

$$\Rightarrow r = \frac{mV_0/i\hbar^2 k}{1 - mV_0/i\hbar^2 k}$$

$$t = \frac{1}{1 - mV_0/i\hbar^2 k}$$

The equations are now $\left(\psi_{\mp} \sim a e^{ikx} + b e^{-ikx}\right)$

$$1 + r = a + b$$

$$(a-b) - (1-r) = \frac{2mV_0}{i\hbar^2 K}(1+r)$$

$$i(a-b) = it$$

$$it - i(a+b) = \frac{2mV_0}{i\hbar^2 K}(it)$$

eliminating $a+b$ and $a-b$, we get

$$(1+r) = t\left(1 - \frac{2mV_0}{i\hbar^2 K}\right)$$

$$t - (1-r) = \frac{2mV_0}{i\hbar^2 K}(1+r)$$

thus

$$(1+r) = \left(1 - \frac{2mV_0}{i\hbar^2 K}\right)\left((1+r)\frac{2mV_0}{i\hbar^2 K} + (1-r)\right)$$

$$1 + r = -\left(1 - \frac{2mV_0}{i\hbar^2 K}\right)^2 r + 1 + \left(\frac{2mV_0}{\hbar^2 K}\right)^2$$

$$r = \boxed{\frac{\left(\frac{2mV_0}{\hbar^2 K}\right)^2}{1 + \left(1 - \frac{2mV_0}{i\hbar^2 K}\right)^2}} \Rightarrow$$

$$t = \frac{1+r}{1 - \frac{2mV_0}{i\hbar^2 K}} = \frac{1 + \left(1 - \frac{2mV_0}{i\hbar^2 K}\right)^2 + \left(\frac{2mV_0}{\hbar^2 K}\right)^2}{\left(1 + \left(1 - 2mV_0/i\hbar^2 K\right)^2\right)\left(1 - \frac{2mV_0}{i\hbar^2 K}\right)}$$

$$= \frac{2}{1 + \left(1 - \frac{2mV_0}{i\hbar^2 K}\right)^2}$$

Note that as a check, $t = 1$ and $r = 0$ if $V_0 = 0$ and

$$|r|^2 + |t|^2 =$$

$$\frac{\left(\frac{2mV_0}{\hbar^2 K}\right)^4 + 4}{\left(2 - \left(\frac{2mV_0}{\hbar^2 K}\right)^2\right)^2 + \left(\frac{4mV_0}{\hbar^2 K}\right)^2} = \underline{1} \checkmark$$

<u>Finally</u>, comparing case (a) with (b)

(a) $|t|^2 \sim \frac{1}{1+\delta^2}$ $\qquad \delta = \frac{2mV_0}{\hbar^2 K}$

(b) $|t|^2 \sim \frac{4}{4+\delta^4}$ which is larger than that of (a)

So, adding a barrier increases the amount of transmission!

SECTION 4

Problem 8.

A three-dimensional quantum system has potential $V(r) = r^4$. Use the trial wavefunction $\psi = e^{-ar^2}$ and get a variational estimate for the ground state energy.

$$H = \frac{p^2}{2m} + r^4$$

$$\langle \psi | H | \psi \rangle = \int \frac{1}{2m} (\nabla \psi)^2 + r^4 |\psi|^2$$

$$= \int_0^\infty (4\pi r^2 dr) \left[\frac{1}{2m} \cdot \left(e^{-ar^2} \cdot (-2ar) \right)^2 + r^4 e^{-2ar^2} \right]$$

$$= \frac{3(8a^3 + 5m)\pi^{3/2}}{32\sqrt{2}\, a^{7/2} m}$$

$$\langle \psi | \psi \rangle = \int (4\pi r^2 dr)\, e^{-2ar^2} = \frac{\pi^{3/2}}{2\sqrt{2}\, a^{3/2}}$$

$$\langle E \rangle = \frac{3(8a^3 + 5m)}{16 a^2 m}$$

$$\frac{\partial E}{\partial a} = \frac{3}{8}\left(\frac{4}{m} - \frac{5}{a^3} \right) \Rightarrow a = \left(\frac{5m}{4} \right)^{1/3}$$

$$\Rightarrow E = \frac{9}{4} \left(\frac{5}{2m^2} \right)^{1/3}$$

Section 5

Problem 9.

Imagine that a charged particle of charge Q is placed in a gas of electrons at temperature T, with initial density N_o.

a) What is the equilibrium electron density in terms of the electrostatic potential $\Phi(r)$.

b) Drive an equation for the potential $\Phi(r)$ (due both to the added charge plus the re-distribution of the electrons).

c) Find $\Phi(r)$, in the high temperature limit.

Solution

a) The energy of the electrons is changed by the term $-e\Phi$. Hence, the density is given by the Boltzmann formula

$$\boxed{N(r) = N_0 \exp e\Phi/KT}$$

b) $$\boxed{\nabla^2 \Phi = -4\pi \left(Q\,\delta^3(x) - eN_0\left(\exp \frac{e\Phi}{KT} - 1\right)\right)}$$

c) if T is large, we can expand the exponential to get

$$\nabla^2 \Phi + \frac{4\pi N_0 e^2}{KT}\Phi = -4\pi Q\,\delta^3(x)$$

This can be solved by going to Fourier space

$$\Phi(\vec{x}) = \int \frac{d^3K}{(2\pi)^3}\,\Phi(k)\,e^{i\vec{K}\cdot\vec{x}}$$

$$\Phi(k) = \frac{4\pi Q}{K^2 + K_0^2} \qquad K_0 = \sqrt{\frac{4\pi Ne^2}{KT}}$$

$$\Rightarrow 4\pi Q \int_0^\infty \frac{K^2\,d^2K}{2\pi^2(K^2+K_0^2)}\,\frac{\sin Kr}{Kr} = \boxed{\frac{Q}{r}\,e^{-K_0 r}}$$

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PART II

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Section 1:	**Problem 11**_____	**Problem 12** _____
Section 2:	**Problem 13**_____	**Problem 14** _____
Section 3:	**Problem 15**_____	**Problem 16**_____
Section 4:	**Problem 17** _____	
Section 5	**Problem 18**_____	**Problem 19**_____

SCORE _____

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SECTION I

Problem 11.

A Z_2 (Ising) gauge theory on the square lattice has the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ijkl \rangle} \sigma_{ij}\sigma_{jk}\sigma_{kl}\sigma_{li} \ ,$$

where the sum is over all *plaquettes* (*i.e.* squares). The sites i, j, k, and l are vertices, and ij, jk, kl, and li links along the plaquette $ijkl$. On each link, the "gauge field" σ may take values ± 1.

Solve for the thermodynamics of this model in the mean field approximation. Use the trial density matrix

$$\varrho(\{\sigma_{ij}\}) = \prod_{\langle ij \rangle} P(\sigma_{ij})$$

$$P(\sigma) = \tfrac{1}{2}(1+x)\delta_{\sigma,1} + \tfrac{1}{2}(1-x)\delta_{\sigma,-1} \ ,$$

where the product is over all links of the lattice.

(a) What is the variational free energy per site, $f(x,T)$?

(b) What is the mean field equation which determines $x(T)$?

(c) Is the mean field transition first or second order? If first order, derive an equation for the discontinuity in x at T_c. If second order, derive an equation for the discontinuity in c, the specific heat per site.

Solution

The energy is $E = \mathrm{Tr}(\varrho\mathcal{H})$. The variational density matrix factorizes into a product, hence $\langle\sigma_{ij}\rangle = x$ for each link, and $E = -NJx^4$. Note that the number of squares is the same as the number of sites, N.

The entropy is given by $S = -k_B\,\mathrm{Tr}(\varrho\ln\varrho)$, hence

$$S = -2Nk_B\left[\frac{1+x}{2}\ln\left(\frac{1+x}{2}\right) + \frac{1-x}{2}\ln\left(\frac{1-x}{2}\right)\right] ,$$

since each *link* contributes to the entropy, and there are $2N$ links. (In general, a lattice of coordination number z has $\frac{1}{2}z$ links per site, which is made clear by the observation that each link connects two sites. For the square lattice, $z = 4$.)

(a) The variational free energy per site, $f = (E - TS)/N$, is

$$f(x,T) = -Jx^4 + k_BT\left[\ln\left(\frac{1-x^2}{4}\right) + x\ln\left(\frac{1+x}{1-x}\right)\right] .$$

(b) The mean field equation which determines $x(T)$ is $\partial f/\partial x = 0$, which gives

$$\frac{\partial f}{\partial x} = -4Jx^3 + k_BT\ln\left(\frac{1+x}{1-x}\right) = 0 .$$

(c) The free energy in the vicinity of $x = 0$ is $f(x,T) = -k_BT\ln 4 + k_BTx^2 + \mathcal{O}(x^4)$. It always has a positive curvature. However, $f(x=1,T) = -J$, so for $T < J/\ln 4$, $f(x=1,T) < f(x=0,T)$. This tells us that there is a phase transition, and that it must be a first order one. To determine T_c and $x_c \equiv x(T_c^-)$, set $f(x,T) = f(0,T)$, which, in addition to $\partial f/\partial x = 0$, gives us two equations in the two unknowns T_c and x_c. Eliminating T from the equations gives us a transcendental equation for x_c:

$$-\tfrac{4}{3}\ln(1-x_c^2) = x_c\ln\left(\frac{1+x_c}{1-x_c}\right) .$$

This equation has a unique nontrivial ($x_c \neq 0$) solution, numerically found to be $x_c \simeq 0.990611$. Once x_c is found, we may invoke $\partial f/\partial x = 0$ to find $T_c \simeq 0.725899\,J/k_B$.

SECTION 1

Problem 12.

Consider as a model for a white dwarf star a degenerate gas of electrons in the limit where the Fermi momentum is much bigger than $M_e c$.

a) Find the ground state energy in terms of the electron density and the total particle number N (keep the leading two terms)

b) Find the pressure.

c) Assume that the outward force exerted by the pressure of the electron gas is balanced by gravitational self-attraction of the protons (of which there are also N). Derive an estimate, in terms of fundamental physical constants, for the largest possible mass for which this balance is possible.

Solution

The usual relationship between Fermi momentum and density is $\left(n = N/L^3 \right)$

$$2\int^{p_f} \frac{d^3p}{(2\pi\hbar)^3} = n \quad , \quad p_f = \hbar\left(3\pi^2 n\right)^{1/3}$$

a) The energy is

$$E = 2L^3 \int^{p_f} \frac{d^3p}{(2\pi\hbar)^3} \sqrt{(pc)^2 + (mc^2)^2}$$

$$= \frac{m^4c^5}{\pi^2\hbar^3} L^3 \int_0^{x_f} dx\, x^2 \sqrt{1+x^2}$$

with $x_f = \dfrac{p_F}{mc}$

$$\boxed{\frac{E}{N} = \frac{m^4c^5}{\pi^2\hbar^3 n}\left(\frac{1}{4}x_f^4 + \frac{1}{4}x_f^2\right)}$$

b) The pressure is just $-\dfrac{\partial E}{\partial V} =$

$$-\frac{1}{N}\frac{\partial E}{\partial(1/n)} =$$

$$= \frac{m^4c^5}{12\pi^2\hbar^3}\left(x_f^4 - x_f^2\right)$$

c) Outward force $\sim 4\pi R^2 p(R)$

inward force $\sim \alpha \frac{GM^2}{R^2}$ $\quad \alpha$ is $O(1)$

to balance

$$p(R) = \frac{GM^2}{4\pi R^4}$$

using $n = N/\frac{4\pi}{3}R^3$, we get

$$\frac{m^4c^5}{12\pi^2\hbar^3}\left(\frac{\hbar^4\left(\frac{9\pi}{8}N\right)^{4/3}}{m^4c^4R^4} - \frac{\hbar^2\left(\frac{9\pi}{8}N\right)^{2/3}}{m^2c^2R^2}\right) = \frac{\alpha GM^2}{4\pi R^4}$$

We assume $Nm_p = M$ $\quad m_p$ = proton mass.
After some algebra, we require

$$0 > \left(\frac{9\pi}{8}\right)^{2/3}\frac{R^2}{m_p^{2/3}m^2} = \frac{1}{m^2c^2}\left(\frac{9\pi}{8}\right)^{4/3}\left(\frac{M}{m_p}\right)^{4/3} - \frac{3\pi\alpha GM^2\hbar^3}{m^4c^5}$$

or $$\boxed{M^{2/3} < \left(\left(\frac{8}{9\pi}\right)^{3/4}\frac{3\pi\alpha G\hbar^3}{m^4c^5}m_p^{4/3}\left(\frac{\hbar}{mc}\right)^2\right)^{-1}}$$

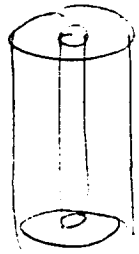
SCORE _____

DEPARTMENTAL EXAMINATION – SPRING 2000

SECTION 2

Problem 13.

A particle of mass M is confined to a cylinder of height L, inner radius a and outer radius b, with boundary conditions that ψ vanishes on the boundary. A magnetic field with total flux Φ_B passes through the central core of the cylinder. The magnetic field vanishes in the region where the particle is present. Find the allowed wavefunctions for the system. You can leave the energy quantization conditions as a relation on Bessel functions. Find the condition on the total flux Φ_B so that the energies are the same as those for the system with no magnetic field.



$$H = \frac{1}{2M}\left(p - \frac{e}{c}A\right)^2$$

$$A \text{ in cavity} = \frac{\Phi_B}{2\pi\rho}\hat{\theta}$$

$$H = \frac{1}{2M}\left(p^2 - \frac{e}{c}(p\cdot A + A\cdot p) + \frac{e^2}{c^2}A^2\right)$$

$$= \frac{1}{2M}\left[p^2 - \frac{e}{c}\frac{\Phi_B}{\pi\rho}p_\theta + \frac{e^2}{c^2}\frac{\Phi_B^2}{4\pi^2\rho^2}\right]$$

But $p_\theta = \frac{L_z}{r}$

$$H = \frac{1}{2M}\left[p^2 - \frac{e\Phi_B}{\pi c\rho^2}L_z + \frac{e^2\Phi_B^2}{4\pi^2c^2\rho^2}\right]$$

$$\psi = \psi(\rho)\, e^{im\theta}\sin(kz) \qquad kL = n\pi \qquad \text{in cylindrical coordinates}$$

$$H\psi = \frac{1}{2M}\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{m^2}{\rho^2} + \frac{n^2\pi^2}{L^2} - \frac{e\Phi_B m}{\pi c\rho^2} + \frac{e^2\Phi_B^2}{4\pi^2c^2\rho^2}\right]$$

$$= E\psi$$

$$\Rightarrow \quad \rho^2\frac{\partial^2\psi}{\partial\rho^2} + \rho\frac{\partial\psi}{\partial\rho} - m^2 + \frac{e\Phi_B m}{\pi c} - \frac{e^2\Phi_B^2}{4\pi^2c^2}$$

$$= -2ME'\rho^2\psi \qquad E' = E - \frac{n^2\pi^2}{L^2}$$

$$\Rightarrow \quad \rho\frac{\partial^2\psi}{\partial\rho^2} + \rho\frac{\partial\psi}{\partial\rho} + 2ME'\rho^2\psi - \left(m - \frac{e\Phi_B}{2\pi c}\right)^2\psi = 0$$

$$\Rightarrow \quad \psi = \alpha J_\nu\left[\sqrt{2ME'}\,\rho\right] + \beta N_\nu\left[\sqrt{2ME'}\,\rho\right]$$

α, β chosen so that ψ vanishes at a, b

Solution

13.1

$$\psi = N_\nu\left[\sqrt{2ME'}\,a\right] J_\nu\left[\sqrt{2ME'}\,\rho\right] - J_\nu\left[\sqrt{2ME'}\,a\right] N_\nu\left[\sqrt{2ME'}\,\rho\right]$$

vanishes at $\rho = a$

to vanish at $\rho = b$ need

$$N_\nu\left[\sqrt{2ME'}\,a\right] J_\nu\left(\sqrt{2ME'}\,b\right) - J_\nu\left(\sqrt{2ME'}\,a\right) N_\nu\left(\sqrt{2ME'}\,b\right) = 0$$

To have energy remain the same as $\Phi_B = 0$:
~~and~~ need ν's to be the same.

$$\Rightarrow \quad m\nu_E \quad \frac{e\,\Phi_B}{2\pi c} = \text{integer}$$

SECTION 2

Problem 14.

Consider the Hamiltonian

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right)$$

for a spin 3/2 particle.

a) Find the energy eigenstates of the system

b) Assume that at t = 0, a perturbation of the form CS_x is turned on. If the system at t=0 is in the ground state with the highest value of $\langle S_z \rangle$, find the probability that it will still be in this state at t=T. Assume C <<1, and assume for simplicity that B=A (in this part only).

a) $S_x^2 - S_y^2 = \frac{1}{2}(S_+^2 + S_-^2)$

for spin 3/2

$$S_+|m\rangle = \sqrt{\left(\tfrac{3}{2}-m\right)\left(\tfrac{5}{2}+m\right)}\;|m+1\rangle$$

$$S_-|m\rangle = \sqrt{\left(\tfrac{3}{2}+m\right)\left(\tfrac{5}{2}-m\right)}\;|m-1\rangle$$

giving two decoupled 2x2 systems $|\tfrac{3}{2}\rangle$, $|-\tfrac{1}{2}\rangle$

$$\hbar^2 \begin{pmatrix} \tfrac{9}{4}A & \sqrt{3}B \\ \sqrt{3}B & \tfrac{1}{4}A \end{pmatrix}$$

eigenvalue equation

$$\left(\tfrac{9}{4}A - \lambda\right)\left(\tfrac{1}{4}A - \lambda\right) = 3B^2$$

$$\frac{\lambda}{\hbar^2} = \frac{5}{4}A \pm \sqrt{A^2 + 3B^2}$$

with corresponding eigenvectors

b) We now specialize to $A = B$, $\frac{\lambda}{\hbar^2} = \frac{13}{4}A, -\frac{3}{4}A$

states are

$$\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \; ; \; \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$$

Solution
14.1

Ground state is the second one. Now, a perturbation $C S_x$ can cause transitions from

$$\frac{1}{2}\left(- |3/2\rangle + \sqrt{3}\,|-1/2\rangle \right) \equiv |0\rangle$$

to the state $\frac{1}{2}\left(- |-3/2\rangle + \sqrt{3}\,|1/2\rangle \right) \equiv |\phi\rangle$
(at the same energy)

and to the state $\frac{1}{2}\left(\sqrt{3}\,|-3/2\rangle + |1/2\rangle \right) \equiv |\psi\rangle$

at an energy change of $4A\hbar^2$

Standard time-dependent perturbation theory gives these transition amplitudes as

$$-\frac{iC\hbar}{\hbar}\int_0^T dt \left(\langle \phi | S_x | 0 \rangle \right)$$

$$-iC \int_0^T dt\, e^{4iA\hbar t} \langle \psi | S_x | 0 \rangle$$

The loss of probability is then given by

$$\cancel{1 - A^2 T^2 \left(\frac{3}{4}\right) - C^2 \frac{\sin^2(2A\hbar T)}{(2A\hbar)^2} \frac{\sqrt{3}}{4}}$$

Solution 14.2

$$c^2 T^2 \left|\langle \phi | S_x | 0 \rangle\right|^2$$

$$+ \quad c^2 \frac{\sin^2 2A\hbar T}{(2A\hbar)^2} \left|\langle \psi | S_x | 0 \rangle\right|^2$$

Finally, the matrix elements are found via

$$S_x |0\rangle =$$

$$\frac{1}{2}(S_+ + S_-)|0\rangle =$$

$$\frac{1}{4}\left(-\sqrt{3}\,|1/2\rangle + \sqrt{3}\left(2\,|1/2\rangle + \sqrt{3}\,|-3/2\rangle\right)\right)$$

$$=$$

$$\frac{1}{4}\left(\sqrt{3}\,|1/2\rangle + 3\,|-3/2\rangle\right)$$

overlap with other states are thus

$$\langle \phi | S_x | 0 \rangle = \frac{1}{8}(3 - 3) = 0$$

$$\langle \psi | S_x | 0 \rangle = \frac{1}{8}(2 \cdot 3\sqrt{3}) = \frac{3\sqrt{3}}{4}$$

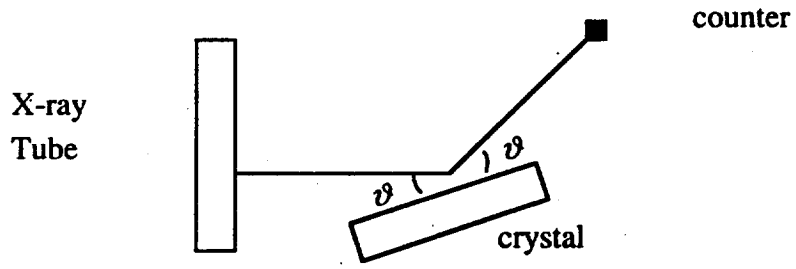
SCORE _____

DEPARTMENTAL EXAMINATION – SPRING 2000

SECTION 3

Problem 15

The attached X-ray spectra of Tungsten was recorded with a Bragg apparatus by varying the angle ϑ.



For one of the spectra, the crystal was LiF (lattice constant d= 2.01 Å), for the other spectrum NaCl. Both spectra were recorded in first order.

a) Determine the transition wavelength and transition energy of the line marked with an arrow, and identify the corresponding upper and lower levels in the attached table.

b) Determine the Lattice constant of NaCl. Average over 7 lines to reduce the statistical error.

(three pages attached)

Solution:

Spectroscopy 15

a) $n\lambda = 2d\sin\vartheta$, $n=1$, $d = 2.01\,\text{Å}$

$\vartheta = 24.73^\circ \Rightarrow \underline{\underline{\lambda = 1.682\,\text{Å}}} \Rightarrow \underline{\underline{\Delta E = 7.38\,\text{keV}}}$

the transition is $\underline{\underline{M_{I} \rightarrow L_{III}}}$

b) $$d_{NaCL} = \frac{\sin\vartheta_{LiF}}{\sin\vartheta_{NaCL}} \cdot d_{LiF}$$

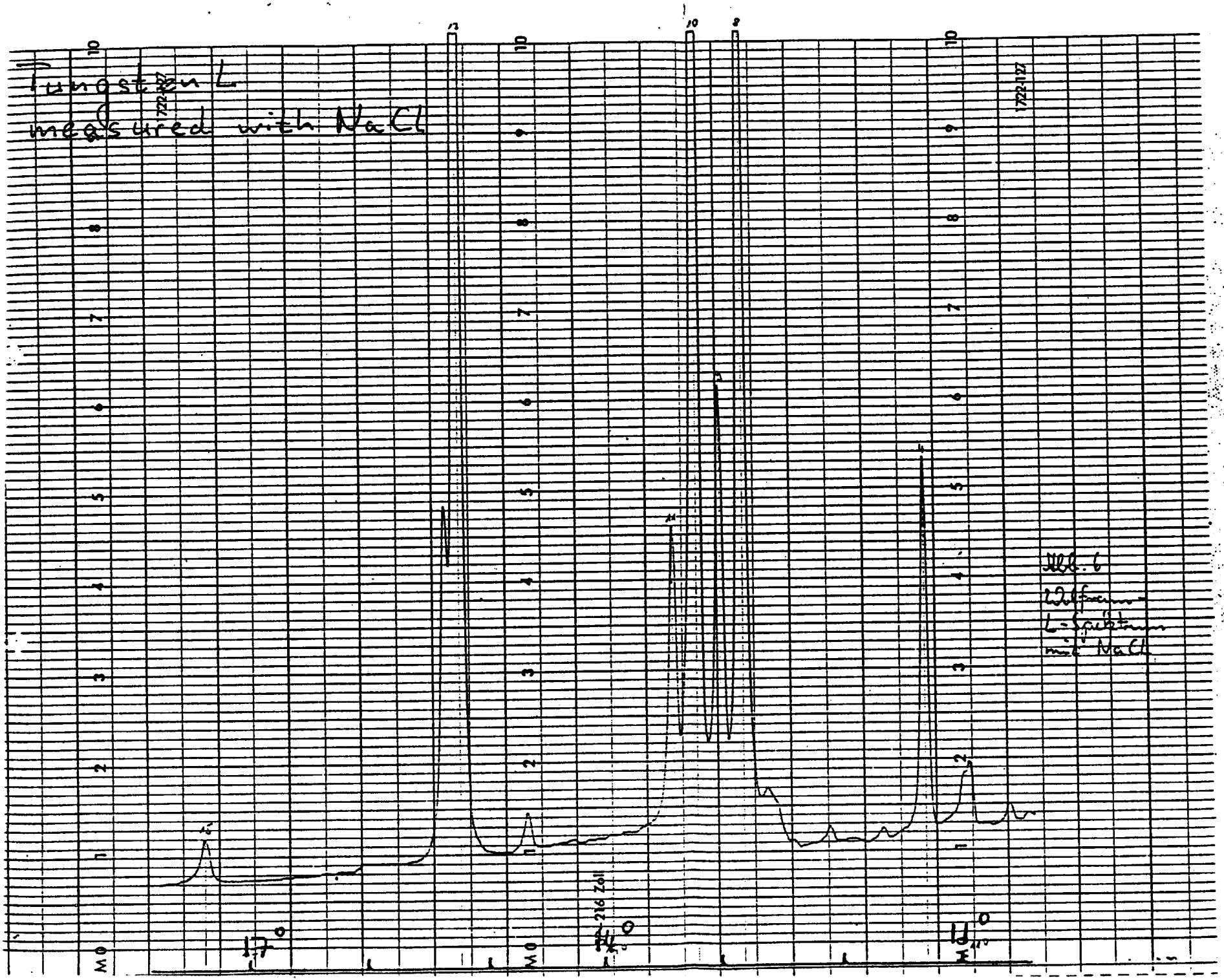
Line #	ϑ_{LiF}	ϑ_{NaCL}	d_{NaCL} [Å]
4	15.91	11.30	2.812
8	18.09	12.85	2.806
9	18.36	13.00	2.814
10	18.64	13.21	2.811
11	18.94	13.41	2.813
13	21.58	15.26	2.809
15	24.73	17.41	2.810

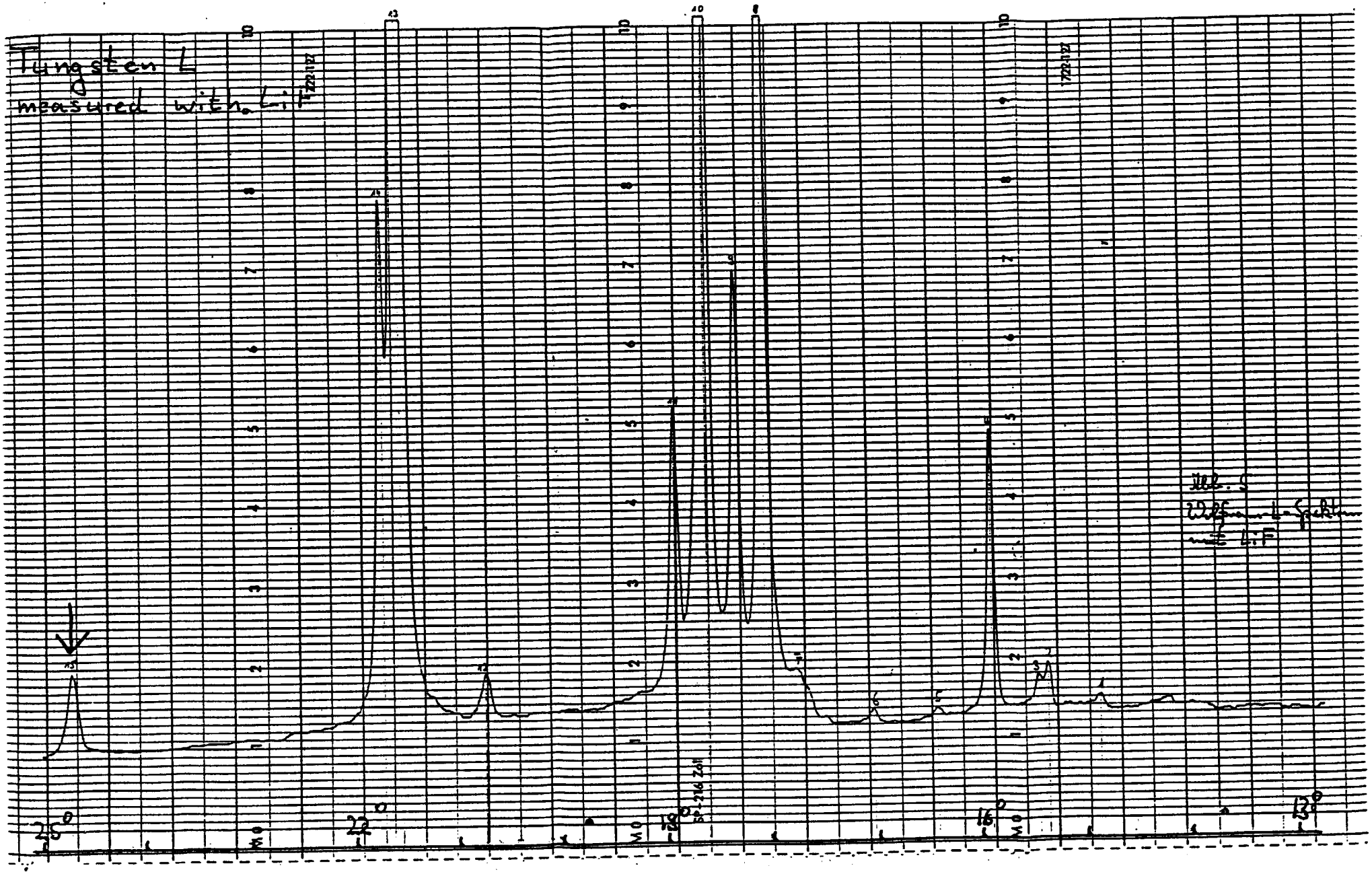
$\underline{\underline{d_{NaCL} = 2.81\,\text{Å}}}$

Tungsten

Aus: Handbuch der Physik, Bd XXX (227)

L_I	-12098	eV
L_{II}	-11540	"
L_{III}	-10202	"
M_I	-2816	"
M_{II}	-2570	"
M_{III}	-2276	"
M_{IV}	-1867	"
M_V	-1804	"
N_I	-589	"
N_{II}	-488	"
N_{III}	-421	"
N_{IV}	-254	"
N_V	-240	"
N_{VI}	-31	"
N_{VII}	-28	"
O_I	-73	"
O_{II}	-43	"
O_{III}	-34	"
O_{IV}	0	"
O_V	0	"


Tungsten L
measured with NaCl
Abb. 6
Wolfram-L-Spektrum mit NaCl
17°
14°
11°
216 Z01
7722-127


Tungsten L
measured with LiF
25°
22°
20°
16°
13°

SECTION 3

Problem 16:

Small Latex spheres with 0.6 μm diameter and $\rho = 1200 kgm^{-3}$ are suspended in a sugar solution with $\rho = 1190 kgm^{-3}$. After the suspension settles down, samples are taken at heights of 2, 4, 6, 8 mm above the bottom. The samples contain 1.16 x 10^6, 13500, 95, and 2 spheres, respectively. Use these numbers to determine Boltzmann's constant, the mass of an average air molecule, and Avogadro's constant! (The scale height of air is 8km.)

Solution:

$$n = n_0 e^{-\frac{mgh}{kT}} = n_0 e^{-\frac{h}{H}}, \quad \ln n = \ln n_0 - \frac{h}{H}$$

note: m is "effective" mass $m = (\rho - \rho') V$

h	2 mm	4 mm	6 mm	8 mm
n	$1.16 \cdot 10^6$	13500	95	2
$\ln n$	14	9.5	4.6	0.7

From the table, $-\frac{1}{H} = \frac{-4.5}{2\,mm} \Rightarrow H = 0.45\ mm$

$$H = \frac{kT}{mg} = \frac{kT}{(\rho-\rho')Vg} \Rightarrow k = \frac{4}{3}\pi R^3 g H \frac{\rho-\rho'}{T} = \underline{\underline{1.7 \cdot 10^{-23}\ J/K}}$$

$$\frac{m_{air}}{m} = \frac{H}{H_{air}} \Rightarrow m_{air} = \frac{mH}{H_{air}} = \underline{\underline{6 \cdot 10^{-26}\ kg}}.$$

$$m_{air} = (0.75 \cdot 28 + 0.25 \cdot 32)\, m_H \Rightarrow m_H = \frac{m_{air}}{29} = 2 \cdot 10^{-27}\ kg$$

$$N_A = \frac{1\ g}{m_H} = \underline{\underline{5 \cdot 10^{-23}}}$$

SCORE _____

DEPARTMENTAL EXAMINATION – SPRING 2000

SECTION 4.

Problem 17.

Prove the addition formula

$$J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y).$$

Hint: The integral representation of the Bessel function is

$$J_n(x) = \oint e^{x(t-1/t)/2} \frac{dt}{t^{n+1}}$$

where the integral is around a closed contour enclosing the origin.

Solution

17

Prove the addition formula

$$J_n(\alpha+\beta) = \sum_{m=-\infty}^{\infty} J_m(\alpha)\, J_{n-m}(\beta)$$

$\boxed{1/2t}$

inter integral

<u>Proof</u>: $\quad e^{\frac{1}{2}z(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n$ = generating function.

$$\therefore \sum_{n=-\infty}^{\infty} J_n(\alpha+\beta)\, t^n = e^{\frac{1}{2}(t-1/t)(\alpha+\beta)}$$

$$= \left[e^{\frac{1}{2}(t-1/t)\alpha}\right]\left[e^{\frac{1}{2}(t-1/t)\beta}\right]$$

$$= \sum_{r=-\infty}^{\infty} J_r(\alpha)\, t^r \sum_{s=-\infty}^{\infty} J_s(\beta)\, t^s$$

equating coefficients of t^n on both sides gives the desired result.

SECTION 5

Problem 18.

A particle of mass m and charge e moves in the xy plane in a potential $V(r)$, and a uniform magnetic field B in the z direction, with vector potential $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$. Find the Hamiltonian in polar coordinates, and find two constants of the motion.

$$L = \frac{1}{2} m (\dot{\vec{r}})^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - V(r)$$

$$= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{qB}{2c} (\hat{z} \times \vec{r}) \cdot \vec{v} - V(r)$$

$$\hat{z} \times \vec{r} = r \hat{\theta} \qquad (\hat{z} \times \vec{r}) \cdot \vec{v} = r v_\theta = r^2 \dot{\theta}$$

$$\therefore \quad L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{qB}{2c} r^2 \dot{\theta} - V(r)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} + \frac{qB}{2c} r^2$$

$$H = \dot{r} p_r + \dot{\theta} p_\theta - L = \frac{p_r^2}{2m} + \frac{1}{2m} \left(p_\theta - \frac{qB}{2c} r^2 \right)^2 + V(r)$$

θ is a cyclic coordinate, so

$$p_\theta = m r^2 \dot{\theta} + \frac{qB}{2c} r^2 = \text{constant}.$$

also $\quad H = \text{constant}$

SCORE _____

DEPARTMENTAL EXAMINATION – SPRING 2000

SECTION 5

Problem 19

Consider a gas of microscopic dielectric spheres of radius and a dielectric constant ε. A linearly polarized electromagnetic wave of frequency ω and $\omega a/c \ll 1$ is incident on the gas. Assume the particulate density to be N_o. Because of the scattering of light by these spheres, the coherent energy in the wave will decay exponentially over some length scale. Calculate this scattering extinction length.

Solution

Note that since $\omega a/c \ll 1$, a dipole approximation is valid. Thus, can write

$$\ell_{ext} = 1/n_0\sigma$$

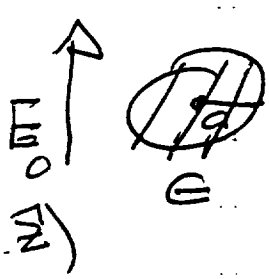
$\downarrow$
scattering extinction length

$$\sigma = P_{rad} \Big/ \frac{c}{4\pi}|E_0|^2$$

where, a la Larmor,

$$P_{rad} = \frac{ck^4}{3}|\underline{d}|^2 = \frac{c\,\omega^4}{3\,c^4}|\underline{d}|^2$$

Thus, it remains to calculate the dipole moment of a dielectric sphere in a uniform E field



$$r > a, \quad \phi = -E_0 r\cos\theta + \frac{\underline{d}\cdot\underline{r}}{r^3}$$

$$= -E_0 r\cos\theta + \frac{d\,r\cos\theta}{r^3}$$

$$r < a \quad \phi = c\,r\cos\theta$$

19.1
Solution

Demand $\epsilon \frac{\partial \phi_<}{\partial r}\Big|_a = \frac{\partial \phi_>}{\partial r}\Big|_a$

$$\frac{\partial \phi_<}{\partial \theta}\Big|_a = \frac{\partial \phi_>}{\partial \theta}\Big|_a$$

$$\Rightarrow \quad c = \frac{-3E_0}{\epsilon+2}, \quad d = \frac{\epsilon-1}{\epsilon+2} E_0 a^3$$

$$\underline{d} = d\hat{z}$$

$$\therefore \quad \sigma = \frac{\cancel{c}}{3}\frac{\omega^4}{c^4}\left(\frac{\epsilon-1}{\epsilon+2}\right)^3 \cancel{E_0^2} a^6 \Big/ \frac{\cancel{c}}{4\pi}\cancel{|E_0|^2}$$

$$= \frac{4\pi}{3}\left(\frac{\omega a}{c}\right)^4 a^2$$

$$l_{ext} = 1/n_0\sigma$$