INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	**1 2**	**3 4**	**5 6**	**7 8**	**9 10**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

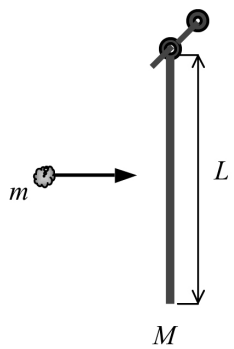
Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

PHYSICAL CONSTANTS

Gas constant	$R = 8.314 \times 10^3$ J kilomole^{-1} K^{-1} ($R = N_A k$)
Boltzmann's constant	$k = 1.381 \times 10^{-23}$ JK^{-1}
	$= 8.617 \times 10^{-5}$ eVK^{-1}
Avogadro's number	$N_A = 6.022 \times 10^{26}$ kilomole^{-1}
Volume of one kilomole of gas at STP	22.42 m^3 (STP = 0°C, 1atm)
Standard atmosphere	1.013×10^5 Pa
Mechanical equivalent of heat	4184 J kilocalorie^{-1}
Temperature of triple point of H_2O	273.16 K = 0.01°C
Atomic mass unit	1 amu = 1.661×10^{-27} kg
Planck's constant	$h = 6.626 \times 10^{-34}$ Js
	$\hbar = h/2\pi = 1.054 \times 10^{-34}$ Js
Bohr magneton	$\mu_B = 9.274 \times 10^{-24}$ JT^{-1}
Bohr radius	$a_0 = 5.292 \times 10^{-11}$ m
Electron charge	$e = 1.602 \times 10^{-19}$ C
Electron mass	$m_e = 9.109 \times 10^{-31}$ kg
Proton mass	$m_p = 1.673 \times 10^{-27}$ kg
Gravitational constant	$G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Standard acceleration due to gravity	$g = 9.807$ ms^{-2}
Speed of light	$c = 2.998 \times 10^8$ ms^{-1}
Permittivity of free space	$\varepsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}
Stefan constant	$\sigma = 5.670 \times 10^{-8}$ Wm^{-2} K^{-4}

#1 : UNDERGRADUATE CLASSICAL MECHANICS

`PROBLEM:` A rod of length L and mass M is suspended from a pivot. The rod is struck midway along its length by a lump of clay of mass m moving horizontally at speed v. The clay sticks to the rod. Find an expression for the minimum speed v that will result in the rod's making a complete circle rather than swinging like a pendulum.



SOLUTION:

Assume that during the collision of the clay with the rod, the angular momentum about the pivot is conserved. Before the collision (when only the clay is moving) the angular momentum is $mv(L/2)$ and just afterwards it is $I\omega_0$. Here,

$$I = ML^2/3 + m(L/2)^2$$

is the rotational inertia of the system, after the collision, about the pivot and ω_0 is the angular speed when the system is at the lowest point of its subsequent swing. If the pivot is frictionless, the mechanical energy will be conserved during this swing, or

$$K_0 + U_0 = K + U.$$

The center of mass is at the center of the rod (since the clay is stuck there) and so the maximum change in potential energy, when the rod is at the highest point of its swing, is $(m + M)gL$. If there is also kinetic energy at this point, the rod will make a complete circle, i.e., $K_{top} \geq 0$ implies

$$K_0 \geq U_{top} - U_0$$

or

$$I\omega_0^2/2 \geq (m + M)gL.$$

Since $\omega_0 = mvL/(2I)$, from above, this condition becomes

$$\omega_0^2 = [mvL/(2I)]^2 \geq 2(m + M)gL/I,$$

or

$$v^2 \geq (8I/L^2)(m + M)gL/m^2 = (m/4 + M/3)8(m + M)gL/m^2.$$

#2 : UNDERGRADUATE CLASSICAL MECHANICS

`PROBLEM:` A meter stick that lies parallel to the x axis moves in the y direction of the lab frame with relativistic speed, v_y. A rocket moves along the x axis of the lab frame with relativistic speed, v_x. In the rocket frame the meter stick appears tilted at an angle θ with respect to the x' axis. Find θ by the following steps.

`(a)` Let the center of the meter stick pass point $x = y = x' = y' = 0$ at time $t = t' = 0$.
`(b)` Where and when does the right end of the meter stick cross the x axis as observed in the lab frame?

`(c)` Where and when does this event occur in the rocket frame?

`(d)` To find the tilt angle in the rocket frame you need x' and y' at $t' = 0$. To find these calculate the x and y components of the stick speed in the rocket frame.

Solution

Note I put c=1 through out)

Lorentz transformation

$t' = \gamma(t - v_x x)$ $\quad x' = \gamma(x - v_x t)$, $\gamma = \frac{1}{\sqrt{1 - v_x^2/c^2}}$

In lab frame center of stick and right end cross at $t = 0$

In rocket frame positive x' axis time is less than 0 =>

at $t' = 0$ right end has already passed x' axis. =>

stick tilted upward to right in rocket frame

right end cross x axis. $x = 1/2$, $t = 0$

$t' = -\frac{\gamma v_x}{2}$, $x' = \frac{\gamma}{2}$

We want right end put at $t' = -\gamma v_x/2$ but at time $t' = 0$

so wait $\frac{\gamma v_x}{2}$ seconds in rocket frame

where has right end moved? Need speed of stick in rocket frame. In lab frame ; Δx of right end is $\Delta x = 0$

$\Delta y' = \Delta y$

$\Delta x' = \gamma(\cancelto{0}{\Delta x} - v_x \Delta t) = -\gamma v_x \Delta t$

$\Delta t' = \gamma(\Delta t - v_x \cancelto{0}{\Delta x}) = \gamma \Delta t$

=> $v_x' = \frac{\Delta x'}{\Delta t'} = -v_x$ and $v_y' = \frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma \Delta t} = v_y/\gamma$

so in time $\Delta t' = -\frac{\gamma v_x}{2}$ right end of stick has moved by $v_x' \Delta t'$ in x' direction and $v_y' \Delta t'$ in y' direction

$x' \to \frac{\gamma}{2} - (v_x)\left(\frac{\gamma v_x}{2}\right) = \frac{\gamma}{2}(1 - v_x^2) = \frac{1}{2\gamma}$

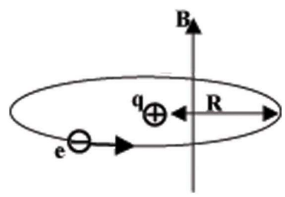
$y' \to 0 + \frac{v_y}{\gamma}\left(\frac{\gamma v_x}{2}\right) = \frac{v_x v_y}{2}$

So angle $\tan\theta = \frac{y'}{x'} = \frac{2 v_x v_y \gamma}{2} = v_x v_y \gamma$

$\theta = \tan^{-1}(v_x v_y \gamma)$

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM:



Consider a classical electron of charge e and mass m_e (and no spin) moving with speed v in a circular orbit of radius R around a positive charge q, as shown in the figure. A uniform magnetic field B in direction perpendicular to the plane of the orbit is then turned on.

(a) Find the change in the speed of this electron, Δv, when the magnetic field is turned on. Assume R is unchanged. Does this electron speed up or slow down?

(b) Assuming there are n such electrons per unit volume, give an expression for the magnetic susceptibility per unit volume. Assume all orbits are perpendicular to the magnetic field, and ignore the difference between B and H.

(c) Give an argument for why it is reasonable to ignore any change in R when the magnetic field is applied as long as $\Delta v/v << 1$.

`SOLUTION:`

`(a)` As B is turned on, an azimuthal electric field is generated by Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S}$$
$$\therefore E = -\frac{R}{2c}\frac{\partial B}{\partial t}$$

For azimuthal speed of electron

$$m_e \frac{\partial v}{\partial t} = eE = -\frac{eR}{2c}\frac{\partial B}{\partial t} ==> \Delta v = -\frac{eR}{2m_e c}B$$

The electron speeds up for the motion shown in the figure (Lenz's law).

`(b)` Kinetic energy of electron changes by

$$\Delta E = \frac{1}{2}m_e[(v+\Delta v)^2 - v^2] = m_e \Delta v + \frac{1}{2}m_e(\Delta v)^2$$

and magnetic susceptibility is

$$\chi = -n\frac{\partial^2 \Delta E}{\partial B^2} = -\frac{ne^2}{4m_e c^2}R^2$$

`(c)` In the absence of applied magnetic field the force balance is

$$\frac{m_e v^2}{R} = \frac{qe}{R^2}$$

and with magnetic field

$$\frac{m_e(v+\Delta v)^2}{R} = \frac{qe}{R^2} - \frac{e}{c}(v+\Delta v)B$$

Since

$$\frac{2m_e \Delta v}{R} = -\frac{e}{c}vB$$

the force balance equation is satisfied to linear order in B with no change in R.

#4 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` Consider a lossless metallic waveguide with dimensions $1\,\mathrm{cm}\,\hat{\mathbf{x}} \times 2\,\mathrm{cm}\,\hat{\mathbf{y}} \times 5\,\mathrm{cm}\,\hat{\mathbf{z}}$. Electromagnetic waves propagating along the $\hat{\mathbf{z}}$ axis of the waveguide with only $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ electric field components are called the TE modes. The lowest frequency TE mode which can propagate in a waveguide is the TE_{10} mode. It has the form $E_y \propto e^{i\beta z} \sin k_y y$ where $i = \sqrt{-1}$, $\beta = \sqrt{k^2 - k_y^2}$ is the propagation constant, and k is the wavenumber.

(a) Determine the cutoff frequency f_c in GHz, the lowest frequency for which the TE_{10} mode will propagate without loss along the $\hat{\mathbf{z}}$-axis. (Use the boundary conditions on E_y to determine the allowable k_y.)

(b) Suppose a microwave generator that produces $P_{in} = 1\,\mathrm{kW}$ at $f = 1\,\mathrm{GHz}$ is attached to one end of the waveguide. Examine whether or not it can cook an egg positioned at the other end. Assume that you need $100\,\mathrm{W}$ of power to cook the egg and that the power absorbed by the egg is proportional to the square of the electric field: $P = \kappa E_y^2$. Here κ is a proportionality constant and E_y is the average electric field within this egg. Assume that $\kappa = 10^{-6}\,\mathrm{W} \times \mathrm{m}^2/\mathrm{V}^2$. What is the total microwave power P_{out} leaving the waveguide at the other end?

SOLUTION:

(a) The cutoff frequency f_c corresponds to $\beta = 0$, i.e., $k_c = k_y$. The allowed k_y is determined by the boundary condition $E_y = 0$ at $y = 0$ and $y = a \equiv 2\,\mathrm{cm}$. Therefore, $k_y = \pi/a$ and the cutoff frequency is

$$f_c = k_c c/(2\pi) = c/(2a) = 7.5\,\mathrm{GHz}\,.$$

(b) The frequency of 1 GHz is below the cutoff. The wave's amplitude/electric field decays as $\sim \exp(i\beta L)$ with:

$$\begin{aligned}\beta &= \sqrt{(2\pi f/c)^2 - (\pi/a)^2} \\ &= \sqrt{[(2\pi \times 10^9\,\mathrm{s}^{-1}/(3 \times 10^8\,\mathrm{m/s})]^2 - (\pi/0.02\,\mathrm{m})^2} \\ &= 155.7i\,\mathrm{m}^{-1}.\end{aligned}$$

At the output the field will be attenuated by the factor of

$$e^{-L\mathrm{Im}\beta} = 4.1 \times 10^{-4}.$$

The power goes as the square of the field, and so

$$P_{out} = P_{in}\exp(-2L\mathrm{Im}\beta) = 1.7 \times 10^{-4}\,\mathrm{W}\,.$$

Since P_{out} is far below the power needed to cook the egg, we do not need to calculate the power absorption by the egg explicitly. It is clear that the answer should be negative.

#5 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` A long molecule is composed of N chemical units ("monomers"), each of which can be in one of two states of different lengths a and b, were $b > a$. The whole molecule therefore can be between Na and Nb in length. The energy of a monomer in the longer state is ϵ larger than the energy of a monomer in the shorter state. You may consider the thermodynamic limit $N \gg 1$ to simplify the calculations. The following mathematical result may be useful: $\ln N! \approx N \ln N - N$ for $N \gg 1$.

(a) At a given temperature T, find the average number of monomers in each state, and hence the equilibrium length of the entire molecule.

(b) Now, suppose that the molecule is forced to be a *fixed* length L, where $Na < L < Nb$, so that $(L - Na)/(b - a)$ of its monomers are in the stretched (length b) state. Find the internal energy $E(N, L)$ and the entropy $S(N, T, L)$.

(c) From (b) calculate the Helmholtz free energy $F(N, T, L)$, and finally the force needed to extend the molecule to length L at fixed temperature T.

SOLUTION:

(a) Since monomers are independent, we can compute the partition function for one monomer, $Z_1 = 1 + e^{-\beta\epsilon}$, and then compute

$$Z_N = (1 + e^{-\beta\epsilon})^N.$$

Since the energy is just proportional to the number of elements in state b, we can calculate

$$\langle N_B \rangle = -\frac{\partial \ln Z}{\partial \beta\epsilon} = \frac{N}{e^{\beta\epsilon}+1}$$

Note that $\langle N_A \rangle = N - \langle N_B \rangle$, so

$$\langle L \rangle = a\langle N_A \rangle + b\langle N_B \rangle = Na + \frac{N(b-a)}{e^{\beta\epsilon}+1}$$

(b) If we force length to be L, this means we force $N_B = (L - Na)/(b - a)$. The energy is $E = N_B\epsilon$, and the entropy is just the log of the number of ways to choose N_B of the monomers to be in state b, $S = k_B \ln \begin{pmatrix} N \\ N_B \end{pmatrix}$. So, $S = k_B \ln \frac{N!}{N_B!(N-N_B)!}$, which can be written in the large N and N_B limit as

$$S = k_B \left[N_B \ln \frac{N}{N_B} + (N - N_B) \ln \frac{N}{N-N_B} \right]$$

so

$$F = E - TS = N_B\epsilon + k_B T \left[N_B \ln \frac{N_B}{N} + (N - N_B) \ln \frac{N-N_B}{N} \right]$$

where $N_B/N = (L - Na)/[N(b - a)]$ and where $1 - N_B/N = (Nb - L)/[N(b - a)]$.

(c) Force is given by L derivative of free energy at fixed temperature:

$$f = \left(\frac{\partial F}{\partial L}\right)_T = \left(\frac{\partial N_B}{\partial L}\right)\left(\frac{\partial F}{\partial N_B}\right)$$

$$f = \frac{\epsilon + k_B T[\ln N_B/N - \ln(1 - N_B/N)]}{b-a} = \frac{\epsilon + k_B T \ln \frac{L-Na}{Nb-L}}{b-a}$$

Note that force diverges to infinite tension if $L \longrightarrow Nb$. Force also diverges to infinite compression if $L \longrightarrow Na$.

#6 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:

A container is divided into two equal chambers, each of the same volume V. One chamber contains N_0 molecules of an ideal gas at temperature T, and the other chamber is completely empty, a perfect vacuum. A small hole, of area A, is punched in the wall separating the two chambers, and gas begins to leak into the empty chamber. The temperature of the gas is kept constant.

(a) Given that the flux of molecules moving in the $+x$ direction can be written as $\Phi = n\overline{v}/4$, where n is the number of molecules per volume. Write down an expression for the rate at which molecules leave the filled chamber at the instant the hole is punched ($t = 0$). Next apply the same consideration to the other side of the chamber and obtain an equation describing the number of molecules $N(t)$ in the initially filled chamber, for all time $t > 0$ after the hole is punched. Solve the equation and express your result in terms of A, V, N_0, and $\overline{v}$.

(b) As the gas redistributes itself between the two chambers, the entropy of the system increases. Calculate the difference between the entropy of the final state with both chambers holding $N_0/2$ molecules, and the initial state when all the molecules were in one chamber.

[Note: The following approximation may be used for simplification: $\ln N! \approx N \ln N - N$ for $N \gg 1$.]

SOLUTION:

(a) At the instant the hole is punched, the density of molecule is $n = N_0/V$ and the rate of molecules exiting the chamber is $\Phi \cdot A = \frac{1}{4}\overline{v}AN_0/V$.

At a later time $t > 0$, if there are N molecules in the first chamber, there are $N_0 - N$ molecules in the other. Then, the outflux from the first chamber is $\Phi_{\text{out}} = \frac{1}{4}\overline{v}N(t)/V$ while the influx from the other chamber is $\Phi_{\text{in}} = \frac{1}{4}\overline{v}(N_0 - N(t))/V$. We thus arrive at the following differential equation for $N(t)$:

$$\frac{dN}{dt} = [\Phi_{\text{in}} - \Phi_{\text{out}}] \cdot A = \frac{\overline{v}A}{2V}\left(\frac{N_0}{2} - N\right).$$

This equation can be solved by direct integration:

$$\frac{dN}{N - \frac{N_0}{2}} = -\frac{\overline{v}A}{2V}dt.$$

After integrating, $\ln(N - \frac{N_0}{2}) = -\overline{v}At/(2V) + \text{const}$.

Including the initial condition $N(t = 0) = N_0$, we finally obtain

$$N(t) = \frac{N_0}{2}\left(1 + e^{-\overline{v}At/(2V)}\right).$$

(b) $\Delta S = S_f - S_i = k \ln w_f - k \ln w_i = k \ln(w_f/w_i)$, where w is the number of ways to put N particles in one box and $N_0 - N$ particles in the other box. Clearly, $w_i = 1$. $w_f = N_0!/[(N_0/2)!]^2$. Then

$$\Delta S = k \ln\left(N_0!/[(N_0/2)!]^2\right) = k[\ln N_0! - 2\ln(N_0/2)!].$$

Applying Stirling's formula, $\ln N! \approx N \ln N - N$, we obtain,

$$\Delta S \approx k[N_0 \ln N_0 - N_0 \ln(N_0/2)] = N_0 k \ln 2.$$

#7 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM:

Consider an electron in an hydrogen atom in a state with quantum numbers n, l, m. Ignore electron spin for this problem. Assume a uniform magnetic field $\vec{B}$ in the z direction is slowly turned on.
(a) Show that a possible magnetic vector potential is $\vec{A} = (\vec{B} \times \vec{r})/2$.
(b) Show that the wavefunction for this electron in the presence of the magnetic field is the same as in the absence of the magnetic field to lowest order in $\vec{A}$. Use that $\vec{p} \to \vec{p} - (e/c)\vec{A}$.
(c) Estimate the order of magnitude of B for which the change in the wavefunction is no longer negligible.
(d) Assuming there are n_a atoms per unit volume in the same state, find an expression for the magnetic susceptibility per unit volume χ of this system. Ignore the difference between B and H.
Hint: $\chi = -\partial^2 E/\partial B^2$, with E the energy per unit volume.

`SOLUTION:`

`(a)` For $\vec{A} = (\vec{B} \times \vec{r})/2$ and $\vec{B} = B\hat{z}$,

$$\vec{A} = \frac{B}{2}(-y\hat{x} + x\hat{y})$$

$$\vec{\nabla} \times \vec{A} = \hat{z}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) = B\hat{z}$$

`(b)` In the Hamiltonian $\vec{p} \to \vec{p} - (e/c)\vec{A}$, hence

$$\frac{p^2}{2m_e} \to \frac{1}{2m_e}(\vec{p} - \frac{e}{c}\vec{A})^2 = \frac{p^2}{2m_e} - \frac{e}{m_e c}\vec{A} \cdot \vec{p} + \frac{e^2}{2m_e c^2}A^2$$

where we used that $\vec{\nabla} \cdot \vec{A} = 0$ to commute $\vec{A}$ and $\vec{p}$. Now

$$\vec{A} \cdot \vec{p} = \frac{1}{2}(\vec{B} \times \vec{r}) \cdot \vec{p} = \frac{1}{2}(\vec{r} \times \vec{p}) \cdot \vec{B} = \frac{1}{2}\vec{L} \cdot \vec{B} = \frac{1}{2}BL_z$$

The wavefunction with magnetic quantum number m is an eigenfunction of L_z, hence it is an eigenfunction of $\vec{A} \cdot \vec{p}$. Therefore, the wavefunction doesn't change provided we ignore the term proportional to A^2.

`(c)` An estimate of the error can be obtained by taking the ratio of the A^2 term to the A term in the above equation, yielding

$$\frac{eA}{pc} = \frac{eBr}{2pc} \sim \frac{eBr^2}{2\hbar c}$$

taking $p \sim \hbar/r$. For $r \sim 10^{-10}m$ this yields $eA/pc \sim 0.76 \times 10^{-5}B/Tesla$ Therefore, this term is negligible unless $B \sim 10^5 T$. Laboratory fields are typically a few T.

`(d)` In lowest order perturbation theory, we take the expectation value of the Hamiltonian with the unperturbed wave function $|\Psi>$ to compute the energy E. Only the term proportional to A^2 contributes to the second derivative of E with respect to B, hence

$$\chi = - < \Psi | n_a \frac{\partial^2}{\partial B^2}(\frac{e^2}{2m_e c^2}A^2)|\Psi >= -\frac{n_a e^2}{4m_e c^2} < \Psi | r^2 | \Psi >$$

#8 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Use the variational method with a Gaussian trial function, $\Psi(x) = \exp\left(-ax^2/2\right)$ to estimate the ground state energy of a particle in the potential $V(x) = \lambda x^6$.

Hint: the integral $\int_{-\infty}^{\infty} dx x^n \exp\left(-ax^2\right) = \Gamma\left(\frac{n+1}{2}\right)/a^{\frac{n+1}{2}}$, where Γ is the Gamma function, $\Gamma(1/2) = \sqrt{\pi}$.

$\boxed{\text{Solution:}}$ $E_0 \le E(a) = \langle\psi|H|\psi\rangle / \langle\psi|\psi\rangle$

So calculat $E(a)$ and find a that minimizes

$$\frac{dE(a)}{da} = 0 \Rightarrow a \to a_0 \quad , \quad E(a_0) \text{ is estimate}$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} e^{-\frac{ax^2}{2}} dx = \int e^{-ax^2} dx = \frac{\Gamma(\frac{1}{2})}{a^{1/2}} = \sqrt{\frac{\pi}{a}}$$

$$H|\psi\rangle = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \lambda x^6 \psi$$

$$d\psi/dx = -ax\, e^{-ax^2/2} \qquad d^2\psi/dx^2 = (-a + a^2x^2)\, e^{-ax^2/2}$$

$$\langle\psi|H|\psi\rangle = \int_{-\infty}^{\infty} dx\, e^{-ax^2/2}\left\{\left(-\frac{\hbar^2}{2m}\right)(-a + a^2x^2) + \lambda x^6\right\} e^{-ax^2/2}$$

$$= \frac{\hbar^2}{2m} a \int dx\, e^{-ax^2} - \frac{\hbar^2 a^2}{2m}\int dx\, x^2 e^{-ax^2} + \lambda \int dx\, x^6 e^{-ax^2}$$

$$= \frac{\hbar^2 a}{2m}\sqrt{\frac{\pi}{a}} - \frac{\hbar^2 a^2}{2m}\frac{\Gamma(3/2)}{a^{3/2}} + \frac{\lambda\,\Gamma(7/2)}{a^{7/2}} = \sqrt{\frac{\pi}{2}}\left\{\frac{\hbar^2 a}{2m} - \frac{\hbar^2 a}{4m} + \frac{\lambda 15}{8a^3}\right\}$$

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

$$\Gamma(7/2) = 5/2\,\Gamma(5/2) = \tfrac{5}{2}\tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{15}{8}\sqrt{\pi}.$$

So

$$\boxed{E(a) = \frac{\hbar^2 a}{4m} + \frac{\lambda 15}{8a^3}}$$

Minimize $\frac{dE}{da} = 0 = \frac{\hbar^2}{4m} - \frac{45\lambda}{8a^4}$ or $a_0^4 = \frac{45}{2}\frac{\lambda m}{\hbar^2}$

$$a_0 = \left(\frac{45}{2}\frac{\lambda m}{\hbar^2}\right)^{1/4}$$

so our energy estimate is

$$E(a_0) = \frac{\hbar^2}{4m}\left(\frac{45}{2}\frac{\lambda m}{\hbar^2}\right)^{1/4} + \frac{\lambda 15}{8\left(\frac{45}{2}\frac{\lambda m}{\hbar^2}\right)^{3/4}}$$

$$E(a_0) = \frac{\hbar^{3/2}\lambda^{1/4}}{m^{3/4}}\left(\frac{1}{4}\left(\frac{45}{2}\right)^{1/4} + \frac{15}{8}\left(\frac{2}{45}\right)^{3/4}\right)$$

$$E(a_0) = \frac{1}{3}\left(\frac{45\lambda\hbar^3}{2m^3}\right)^{1/4}$$

#9 : UNDERGRADUATE MATH METHODS

`PROBLEM:` Consider the transcendental equation

$$Aue^{-\varepsilon^2 u^2} = 1\,,$$

where $A \gg 1$ and $\varepsilon \ll 1$ are known. Find all real solution(s) for u to the leading order in small parameters ε and $1/A$.

SOLUTION:

First of all, let us get rid of one of the parameters by rescaling. Define $v = \varepsilon u$, then

$$ve^{-v^2} = \delta \equiv \varepsilon/A\,.$$

We need to find v to the leading-order in $\delta \ll 1$. Clearly, v must be positive. The expression of the left-hand side of the last equation is zero both at $v = 0$ and at $v = \infty$ and has a single maximum in between, at $v_{\max} = \sqrt{2}/2$. The magnitude of the maximum ~ 1 is much larger than δ; hence, there are exactly two solutions, $0 < v_1 < v_{\max}$ and $v_2 > v_{\max}$. To find $v_1 \ll 1$, it is permissible to use Taylor expansion

$$v_1 = \delta e^{v_1^2} = \delta(1 + v_1^2 + \ldots)\,.$$

To the leading order, $v_1 = \delta$, and so

$$u_1 \simeq 1/A\,.$$

Next, to get v_2, we write $v_2 = \sqrt{\ln(v_2/\delta)}\,.$ Since v_2 is large, we cannot expand in v_2. However, we can separate the large logarithm:

$$v_2 = \left(\ln\frac{1}{\delta} + \ln v_2\right)^{1/2}.$$

For large v_2 we have $v_2 \gg \sqrt{\ln v_2}$. Therefore, from the two terms in the parenthesis it is the first one that is dominant. This yields

$$v_2 \simeq \left(\ln\frac{1}{\delta}\right)^{1/2},$$

and so

$$u_2 \simeq \frac{1}{\varepsilon}\left(\ln\frac{A}{\varepsilon}\right)^{1/2}.$$

#10 : UNDERGRADUATE GENERAL PHYSICS

`PROBLEM:` Carbon-14 is produced in the atmosphere when cosmic ray neutrons react with nitrogen atoms in air: $^{14}\mathrm{N} + {}^{1}\mathrm{n} \rightarrow {}^{14}\mathrm{C} + {}^{1}\mathrm{H}$. The carbon-14 produced in this reaction forms carbon dioxide, CO_2, a component of air. Atmospheric CO_2, has approximately one atom of carbon-14 per every 10^{12} atoms of carbon-12. A living tree takes in carbon dioxide with the same $^{14}C/^{12}\mathrm{C}$ ratio as the atmosphere. When a tree dies, it stops intaking all carbon. The radioactive decay of the carbon in the tree starts to change the ratio of $^{14}C/^{12}\mathrm{C}$ ratio. By measuring how much the ratio is lowered, it is possible to estimate the elapsed time since the tree died, or the age of the tree. The decay of carbon-14 has a half-life of 5720 years and proceeds as $^{14}C \rightarrow ^{14}N + e^{-}$.

(a) Express the $^{14}C/^{12}\mathrm{C}$ ratio as a function of time.

(b) UCSD operates a high altitude research station in the White Mountains of East-Central California at an altitude of 14,000 feet. Some of the oldest living organisms on earth, called Bristlecone Pine Trees, live there. A dead Bristlecone Pine Tree sample there was found to have a $^{14}C/^{12}C$ ratio of $r = 0.795$ times that found in plants living today. How long ago did the tree die?

(c) What if you find out that your estimate of the half-life of $^{14}\mathrm{C}$ is uncertain by X years and that the ratio r is uncertain by $Y\%$. Using the laws of error propagation, how should you express the age of the dead Bristlecone Pine Tree now?

SOLUTION:

(a) The number of carbon-14 nuclei remaining after time t in terms of the initial amount $N(0)$ is given by

$$N(t) = N(0)e^{-t/\tau}.$$

Therefore

$$r = N(t)/N(0) = e^{-t/\tau}.$$

(b) The expression for the age is

$$t = -\tau \ln r\,.$$

The time constant τ in these equations is related to the half-life as follows:

$$\tau_{1/2} \equiv t(r = 1/2) = \tau \ln 2\,.$$

In our case $\tau = 5720/\ln 2 = 8254$ years. Substituting this τ and $r = 0.795$ into the above equation for t, we get

$$t = \tau_{1/2} \left|\frac{\ln r}{\ln 2}\right| = 1893\,\mathrm{yr}.$$

(c) The error in t due to errors in r and τ is:

$$\begin{aligned}
\delta t &= \sqrt{\left(\delta\tau_{1/2}\frac{\partial t}{\partial \tau_{1/2}}\right)^2 + \left(\delta r\frac{\partial t}{\partial r}\right)^2} \\
&= \frac{1}{\ln 2}\sqrt{(X\ln r)^2 + \left(\frac{Y}{100}\tau_{1/2}\right)^2} \\
&= \sqrt{\Big(0.331\,X\Big)^2 + \Big(57.2\,Y\Big)^2}. \qquad (1)
\end{aligned}$$

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1		§2		§3		§4		§5	
Problems: (Circle your seven choices)	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

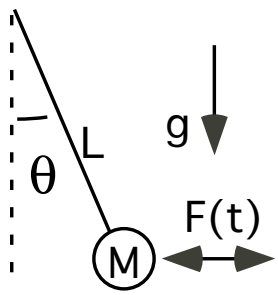
Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

PHYSICAL CONSTANTS

Gas constant	$R = 8.314 \times 10^3$ J kilomole^{-1} K^{-1} ($R = N_A k$)
Boltzmann's constant	$k = 1.381 \times 10^{-23}$ JK^{-1}
	$= 8.617 \times 10^{-5}$ eVK^{-1}
Avogadro's number	$N_A = 6.022 \times 10^{26}$ kilomole^{-1}
Volume of one kilomole of gas at STP	22.42 m^3 (STP = 0°C, 1atm)
Standard atmosphere	1.013×10^5 Pa
Mechanical equivalent of heat	4184 J kilocalorie^{-1}
Temperature of triple point of H_2O	273.16 K = 0.01°C
Atomic mass unit	1 amu = 1.661×10^{-27} kg
Planck's constant	$h = 6.626 \times 10^{-34}$ Js
	$\hbar = h/2\pi = 1.054 \times 10^{-34}$ Js
Bohr magneton	$\mu_B = 9.274 \times 10^{-24}$ JT^{-1}
Bohr radius	$a_0 = 5.292 \times 10^{-11}$ m
Electron charge	$e = 1.602 \times 10^{-19}$ C
Electron mass	$m_e = 9.109 \times 10^{-31}$ kg
Proton mass	$m_p = 1.673 \times 10^{-27}$ kg
Gravitational constant	$G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Standard acceleration due to gravity	$g = 9.807$ ms^{-2}
Speed of light	$c = 2.998 \times 10^8$ ms^{-1}
Permittivity of free space	$\varepsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}
Stefan constant	$\sigma = 5.670 \times 10^{-8}$ Wm^{-2} K^{-4}

#11 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` A simple pendulum of length L and mass M in gravity is subjected to a rapidly-oscillating horizontal force, $F(t) = F_0 \cos \Omega t$, where $\Omega^2 \gg g/L$. By averaging over the rapid oscillations find a condition on F_0 such that $\theta = 0$ becomes an unstable equilibrium, and obtain the new equilibrium angle θ.



`SOLUTION:`

The solution can be found using Hamiltonian averaging methods, or by averaging the equations of motion. We will use the second method.

θ satisfies

$$ML^2\ddot{\theta} = F_0 L \cos\theta \cos\Omega t - MgL \sin\theta.$$

Break θ into a slowly varying part $\overline{\theta}$ and a rapidly-varying part θ_1:

$$\theta = \overline{\theta} + \theta_1$$

$$ML^2\ddot{\overline{\theta}} + ML^2\ddot{\theta}_1 = F_0 L \cos(\overline{\theta} + \theta_1) \cos\Omega t - MgL \sin(\overline{\theta} + \theta_1)$$

$$\Longrightarrow \theta_1 \approx -\frac{F_0}{ML\Omega^2} \cos\overline{\theta} \cos\Omega t.$$

Use this to compute $\overline{\theta}$ by averaging the equation of motion, dropping small terms:

$$\begin{aligned} ML^2\ddot{\overline{\theta}} &\approx -F_0 L \sin\overline{\theta} \langle \theta_1 \cos\Omega t \rangle - MgL \sin\overline{\theta} \\ &= \frac{F_0^2}{2M\Omega^2} \sin\overline{\theta} \cos\overline{\theta} - MgL \sin\overline{\theta} \\ &= -\sin\overline{\theta} \left(MgL - \frac{F_0^2}{2M\Omega^2} \cos\overline{\theta} \right) \end{aligned}$$

$\Longrightarrow \theta = 0$ is unstable if $\dfrac{F_0^2}{2M\Omega^2} > MgL$

New equilibrium at $\cos\overline{\theta} = \dfrac{2M^2 gL\Omega^2}{F_0^2}$.

#12 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` (a) Consider a particle in potential $V(x)$. Derive the eikonal equation for the Schrodinger equation for this problem. What is the physical significance of the eikonal phase?

(b) Derive the Lagrange equations for a ray trajectory for sound waves which obey Fermat's Principle. Note that Fermat's Principle states that a ray will traverse a path of minimum *time* of propagation. Assume propagation is in 2D, with speed $C_s = C_s(y)$ (i.e. y is depth).

(c) Derive the Hamiltonian for this problem and write the Hamilton equations of motion.

`SOLUTION:`

`(a.)` $-\hbar\frac{\partial\psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi$

$$\psi = \psi_\partial e^{i\phi/\hbar}$$
$$\uparrow$$
$$eikonal\,phase \rightarrow fast\,variation$$

$\Rightarrow \frac{\partial\phi}{\partial t} = \frac{1}{2m}(\underline{\nabla}\phi)^2 + V(\underline{x})$

but if re-write $\phi = S$

$\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V(x)$

then recognize Hamilton-Jacobi equation for classical problem!

`(b.)` $T = \int\limits_{\underline{x}_1}^{\underline{x}_2} \frac{d\ell}{C_s(\underline{x})} n(y)$

$C_s(\underline{x}) = \frac{c_0}{n(\underline{x})} \begin{matrix} \rightarrow ref.\ speed \\ \rightarrow index\ of\ refraction \end{matrix}$

$\underline{x}_{1,2} = (x_{\substack{1\\2}}, y_{\substack{1\\2}})$

$\underline{\text{so}}\ T = \int\limits_{\underline{x}_1}^{\underline{x}_2} d\ell\ n(\underline{x}) = \int\limits_{\underline{x}_1}^{\underline{x}_2} dx(1 + (\frac{dy}{dx})^2)^{1/2}\ n(y)$

$\delta T = \int\limits_{\underline{x}_1}^{\underline{x}_2} dx\{\frac{\partial n}{\partial y}(1 + y'^2)^{1/2}\delta y + \frac{n(y)y'}{(1+y'^2)^{1/2}}\delta y\prime\}$

$= \int\limits_{\underline{x}_1}^{\underline{x}_2} dx[(1 + y'^2)^{1/2}\frac{\partial n}{\partial y} - \frac{d}{dx}(\frac{n(y)y\prime}{(1+y'^2)^{1/2}})]\delta y$

$+(\frac{n(y)y'}{(1+y'^2)'^2})\delta y\Big|_{\underline{x}_1}^{\underline{x}_2}$

$\delta y = 0$ on end points

<u>so</u> $\delta T = 0 \Rightarrow$

$\frac{d}{dx}(\frac{n(y)y\prime}{(1+y'^2)^{1/2}}) - (1+y'^2)^{1/2}\frac{\partial n}{\partial y} = 0$

is Lagrange equation for ray.

`(c.)` $p = \frac{\partial L}{\partial \dot{q}}$, $H = p\dot{q}(q,p) - L$

here

$\Rightarrow p = \frac{\partial L}{\partial y\prime} = \frac{n(y)y\prime}{(1+y'^2)^{1/2}}$

$p^2(1+y'^2) = n(y)^2 y'^2$

$[p^2 - n(y)^2]y'^2 = -p^2$

$y\prime = [n(y)^2 - p^2]^{-1/2}p$

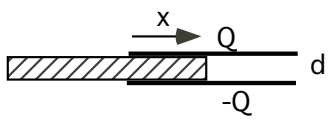
$H = p^2[n(y)^2 - p^2]^{-1/2} - n(y)(\frac{1+p^2}{n(y)^2-p^2})^{1/2}$

EOM: $\frac{dp}{dt} = \frac{-\partial H}{\partial y}$

$\frac{dy}{dt} = \frac{\partial H}{\partial p}$

#13 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` A thin dielectric slab, with dielectric constant ε and thickness d, is inserted a distance $x \gg d$ between two parallel square *isolated* conducting plates of side $L \gg d$, separated by d. The plates have equal and opposite charge $\pm Q$. Find the electrostatic force on the slab as a function of x.



SOLUTION

The free charge on the plates arranges itself so that the charge per unit area in the region adjacent to the slab (region 1) is σ_1, and the charge per unit area in the region far from the slab (region 2) is σ_2 (far being a distance $\gg d$). Since the plates are isolated and $L \gg d$, σ_1 and σ_2 satisfy

$$\sigma_1 Lx + \sigma_2 L(L-x) = Q.$$

In region 2, the electric field between the plates is given by the vacuum relation, $E_2 = 4\pi\sigma_2$. In region 1, the electric field is reduced by the dielectric $E_1 = 4\pi\sigma_1/\varepsilon$. However, the *potential* between the plates must be constant in x, so

$$dE_1 = dE_2 \implies \sigma_1 = \varepsilon\sigma_2$$

$$\implies \sigma_2 = \frac{Q}{Lx\varepsilon + L(L-x)}\ ,\ \sigma_1 = \frac{\varepsilon Q}{Lx\varepsilon + L(L-x)}.$$

The electrostatic energy of the system is

$$\begin{aligned} W_E &= Ldx\frac{E_1 D_1}{8\pi} + Ld(L-x)\frac{E_2 D_2}{8\pi} \\ &= Ldx\frac{2\pi\sigma_1^2}{\varepsilon} + Ld(L-x)2\pi\sigma_2^2 \\ W_E &= \frac{2\pi d Q^2}{Lx\varepsilon + L(L-x)}. \end{aligned}$$

So, the force on the slab is

$$\begin{aligned} F_x &= -\left.\frac{\partial W_E}{\partial x}\right)_Q \\ F_x &= \frac{2\pi d L Q^2(\varepsilon - 1)}{[Lx(\varepsilon-1) + L^2]^2}. \end{aligned}$$

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Consider a Lorentzian *scalar* electrodynamics with Lagrangian Density.

$L = 1/2\nabla_i\phi\nabla^i\phi - \mu^2\phi^2/2.$

The field-particle interaction action is, for a single charge, given by:

$S_{fp} = \frac{-g}{c}\int ds\phi.$

Here ϕ is a coupling co-efficient and ds is the infinitesimal interval.

(a) Calculate the force that one stationary point charge exerts on another. Assume the two charges are at $x = (\pm a, 0, 0)$. You may leave your answer in terms of the potential ϕ.

(b) Now calculate the potential of interaction, explicitly.

SOLUTION:

$S = -mc \int ds - \frac{g}{c} \int ds\ \phi$

$ds = cdt' = cdt\sqrt{1 - v^2/c^2}$

$\Rightarrow S = \int dt(-mc^2\sqrt{1 - v^2/c^2})[1 + \frac{g\phi}{mc^2}]$

$L = -mc^2\sqrt{1 - v^2/c^2}[1 + \frac{g\phi}{mc^2}]$

For dynamics:

$\frac{d}{dt}(\frac{\partial L}{\partial \underline{v}}) = \frac{\partial L}{\partial \underline{x}}$

$\Rightarrow \frac{d}{dt}(\frac{mc^2\underline{v}/c^2}{\sqrt{1-v^2/c^2}}[1 + \frac{g\phi}{mc^2}]) = \frac{-mc^2\sqrt{1-v^2/c^2}*}{(g\nabla\phi/mc^2)}$

$\therefore \frac{d}{dt}(\frac{m\underline{v}}{\sqrt{1-v^2/c^2}}[1 + \frac{g\phi}{mc^2}]) = -mc^2\sqrt{1 - v^2/c^2}\ (g\underline{\nabla}\phi/mc^2)$

$\Rightarrow [1 + \frac{g\phi}{mc^2}]\frac{d}{dt}(\frac{m\underline{v}}{\sqrt{1-v^2/c^2}}) + \frac{m\underline{v}}{\sqrt{1-v^2/c^2}}\frac{g}{mc^2}(\frac{d\phi}{dt})$

$= -mc^2\sqrt{1 - v^2/c^2}(g\underline{\nabla}\phi/mc^2)$

For stationary charges, fields: $\underline{p} = m\underline{v}/\sqrt{1 - v^2/c^2}$

$[1 + g\phi/mc^2]\frac{d}{dt}\underline{p} = -g\underline{\nabla}\phi,$

Thus, have $\underline{F} = \frac{d}{dt}\underline{p} = -g\nabla\phi/[1 + g\phi/mc^2]$

so

$F_{12} = -g_2\nabla\phi_1/[1 + g\phi_1/mc^2]$

Now, for potential:

$S = \frac{1}{c}\int d\Omega[\frac{1}{2}\nabla_i\phi\nabla^i\phi - \frac{\mu^2\phi^2}{2} - g\rho\phi]$,

where $\rho \equiv$ charge density

$\delta S = \frac{1}{c}\int d\Omega[2\frac{\partial\mathcal{L}}{\partial\nabla_i\phi}\delta(\nabla_i\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi]$

$= \frac{1}{c}\int d\Omega[-2\nabla_i(\frac{\partial\mathcal{L}}{\partial\nabla_i\phi}) + \frac{\partial\mathcal{L}}{\partial\phi}]\delta\phi$, where end point contributions vanish.

$\Rightarrow -\nabla_i(\nabla^i\phi) - \mu^2\phi - g\rho = 0$

$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi + \nabla^2\phi - \mu^2\phi = g\rho$

For stationary point charge: $\partial^2\phi/\partial t^2 = 0$

$\Rightarrow \nabla^2\phi - \mu^2\phi = g\rho$

$= g\delta(\underline{r} - a\widehat{x})$

so $\phi_1 = g\ \exp(-\mu|\underline{r} - a\widehat{x}|)/|(\underline{r} - a\widehat{x})|$

$F_{12} = -g\nabla\phi_1/[1 + g\phi_1/mc^2]\Big|_{r=-a\widehat{x}}$

#15 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` A classical gas of non-interacting atoms is in thermal equilibrium at temperature T in a container of volume V and surface area A. The potential energy of the atoms in the bulk is zero. Atoms adsorbed on the surface have a potential energy $V = -E_a$ and behave as an ideal two-dimensional gas.

Find an analytic expression for the surface density $\sigma(n, T) \equiv N_{\text{surface}}/A$ in terms of the bulk density $n \equiv N_{\text{bulk}}/V$ and the temperature. Be sure to "correct Boltzmann counting".

The following mathematical results may be useful.

$$\ln(N!) \approx N \ln N - N$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2)\, dx = 1$$

`SOLUTION:`

<u>First do the bulk gas.</u>

Use $N_{\text{bulk}} = N$ for brevity,
and let the energy of a particle be $\epsilon = (p_x^2 + p_y^2 + p_z^2)/(2m)$.

$$Z_{\text{bulk}}(N,T,V) = \frac{V^N}{N!\,h^{3N}} \left[\int_{-\infty}^{\infty} e^{-p_x^2/(2mkT)}\,dp_x\right]^{3N} = \frac{(2\pi mkT)^{\frac{3N}{2}} V^N}{N!\,h^{3N}}.$$

Use $\ln(N!) \approx N\ln N - N$, we get

$$F_{\text{bulk}} = -kT\ln Z_{\text{bulk}} \approx -NkT\ln\left[\frac{(2\pi mkT)^{\frac{3}{2}}}{h^3}\,\frac{V}{N}\right] - NkT.$$

$$\mu_{\text{bulk}} = \left.\frac{\partial F_{\text{bulk}}}{\partial N}\right|_{T,V} = -kT\ln\left[\frac{(2\pi mkT)^{\frac{3}{2}}}{h^3}\,\frac{V}{N}\right].$$

<u>Next do the surface.</u>

Use $N_{\text{surface}} = N'$ for brevity,
and let the energy of a particle be $\epsilon = -E_0 + (p_x^2 + p_y^2)/(2m)$.

$$Z_{\text{surface}}(N',T,A) = \frac{A^{N'}}{N'!\,h^{2N'}}\left\{e^{E_0/kT}\left[\int_{-\infty}^{\infty} e^{-p_x^2/(2mkT)}\,dp_x\right]^2\right\}^{N'} = \frac{1}{N'!}\left[\frac{e^{E_0/kT}\,(2\pi mkT)A}{h^2}\right]^{N'}.$$

$$F_{\text{surface}} = -kT\ln Z_{\text{surface}} \approx -N'kT\ln\left[\frac{2\pi mkT}{h^2}\,\frac{A}{N'}\,e^{E_0/kT}\right] - N'kT.$$

$$\mu_{\text{surface}} = \left.\frac{\partial F_{\text{surface}}}{\partial N'}\right|_{T,V} = -kT\ln\left[\frac{2\pi mkT}{h^2}\,\frac{A}{N'}e^{E_0/kT}\right].$$

Now put the bulk and surface pieces in equilibrium.

Set $\mu_{\text{bulk}} = \mu_{\text{surface}}$ at equilibrium.
Let $n \equiv N_{\text{bulk}}/V$ and $\sigma \equiv N_{\text{surface}}/A$ be the bulk and surface density, respectively.

$$\ln\left[\frac{(2\pi mkT)^{\frac{3}{2}}}{h^3}\,\frac{V}{N_{\text{bulk}}}\right] = \ln\left[\frac{2\pi mkT}{h^2}\,\frac{A}{N_{\text{surface}}}e^{E_0/kT}\right].$$

$$\frac{\sqrt{2\pi mkT}}{h}\,n = \frac{1}{\sigma}\,e^{E_0/kT}.$$

$$\sigma(n,T) = n\,e^{E_0/kT}\frac{h}{\sqrt{2\pi mkT}}.$$

#16 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider the statistical equilibrium of a white dwarf star, for which the motion of the electrons in the star must be treated as *relativistic*. Using the approximation

$$\varepsilon = (p^2c^2 + m^2c^4)^{1/2} - mc^2 \simeq pc - mc^2 + (m^2c^3/2p) \qquad (p \gg mc),$$

evaluate the pressure of the electron gas and obtain thereby the "mass-radius relationship" for such stars — thus establishing the existence of a limiting mass M_0 that signals the gravitational collapse of the star.

In determining M_0, do not worry about *exact* numerical factors — just focus on the main characters of the play!

For $p \gg mc$, where m is the rest mass of the electron,

$$\varepsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2 \simeq pc - mc^2 + \frac{m^2c^3}{2p},$$

whence the speed of the electron turns out to be

$$u = \frac{d\varepsilon}{dp} = c - \frac{m^2c^3}{2p^2}.$$

The (ground-state) pressure of the electron gas is then given by

$$P = \frac{1}{3}\frac{N}{V}\langle pu\rangle = \frac{1}{3}\int_0^{p_F}\left(pc - \frac{m^2c^3}{2p}\right)\frac{2 \cdot 4\pi p^2 dp}{h^3}$$

$$= \frac{8\pi}{3h^3}\left[\frac{1}{4}p_F^4 c - \frac{m^2c^3}{2}\cdot\frac{1}{2}p_F^2\right]$$

$$= \frac{2\pi m^4 c^5}{3h^3}\left(x^4 - x^2\right),$$

where

$$x = \frac{p_F}{mc} = \left(\frac{3n}{8\pi}\right)^{1/3}\frac{h}{mc} = \left(\frac{9}{64\pi^2}\frac{M}{m_p}\right)^{1/3}\frac{h/mc}{R}.$$

The "equilibrium" configuration of the white dwarf star is now determined by equating the (outward) pressure of the electron gas with the (inward) pressure caused by gravitation, viz. $\frac{\alpha}{4\pi}\frac{GM^2}{R^4}$, where α is a number $O(1)$.

We thus get

$$A\frac{M^{4/3}}{R^4} - B\frac{M^{2/3}}{R^2} = C\frac{M^2}{R^4},$$

where $A \sim \frac{m^4 c^5}{h^3}\left(\frac{1}{m_p}\right)^{4/3}(h/mc)^4 \sim \frac{hc}{m_p^{4/3}}$,

$B \sim \frac{m^4 c^5}{h^3}\left(\frac{1}{m_p}\right)^{2/3}(h/mc)^2 \sim \frac{m^2 c^3}{h\, m_p^{2/3}}$, and

$C \sim G$.

The net result is that, under this approximation, the mass-radius relationship for a white dwarf star turns out to be

$$R = \sqrt{\frac{A}{B}}\, M^{1/3}\left[1-\left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2},$$

where $M_0 = (A/C)^{3/2} \sim \frac{(hc/G)^{3/2}}{m_p^2}$.

Clearly, the star collapses as its mass M approaches the limiting mass M_0.

#17 :GRADUATE QUANTUM MECHANICS

`PROBLEM:` Two identical spin $\frac{1}{2}$ fermions of mass M are in a cubical box of length L. They interact via the attractive potential

$$V_1(\vec{r}_1, \vec{r}_2) = -\epsilon L^3 \delta^3(\vec{r}_1 - \vec{r}_2),$$

where ϵ is small and positive. Treat the problem non-relativisitically, treating V_1 as a perturbation, and use $\sin\theta = Im(e^{i\theta})$ to do the integral.

(a) Compute the energy eigenvector and eigenvalue of the groundstate, working to $\mathcal{O}(\epsilon)$. Include the spin degrees of freedom.

(b) What is the total spin S of the groundstate, and the first excited state, and their total degeneracies (don't forget to include that of the spin)?

(c) What is the total spin and total degeneracy of the first excited state if the potential were instead repulsive ($\epsilon < 0$)?

SOLUTION:

(a) The basis states for a particle in a box are

$$\phi_{\vec{n}}(\vec{r}) = (\frac{2}{L})^{3/2} \sin(n_1 \pi x/L) \sin(n_2 \pi y/L) \sin(n_3 \pi z/L).$$

For two identical particles, the spatial part of the wavefunction is $\phi_{\vec{n}}(\vec{r}_1)\phi_{\vec{m}}(\vec{r}_2) \pm \phi_{\vec{m}}(\vec{r}_1)\phi_{\vec{n}}(\vec{r}_2)$, with energy $E = (\vec{n}^2 + \vec{m}^2)/2mL^2$. The spin part is the antisymmetric state $|S = 0\rangle$, or the symmetric state $|S = 1\rangle$. The full wavefunction must be antisymmetric, since they're fermions. Finally, the perturbation shifts the energy by

$$\Delta E = \int d^3\vec{r}_1 d^3\vec{r}_2 \psi(\vec{r}_1, \vec{r}_2)^* V(\vec{r}_1, \vec{r}_2)\psi(\vec{r}_1, \vec{r}_2) = -\epsilon L^3 \int d^3\vec{r} |\psi(\vec{r}, \vec{r})|^2,$$

which is non-zero for the spatially symmetric wavefunction, and zero for the spatially antisymmetric.

The groundstate is wavefunction is $\phi_{111}(\vec{r}_1)\phi_{111}(\vec{r}_2) \otimes |S = 0\rangle$. The perturbation shifts the energy by

$$\Delta E = -\epsilon L^3 (\frac{2}{L})^6 (\frac{L}{\pi})^3 \left(\int_0^\pi dx \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^4 \right)^3 = -\frac{27}{8}\epsilon$$

So the groundstate energy to $\mathcal{O}(\epsilon)$ is

$$E_0 = \frac{3\pi^2\hbar^2}{ML^2} - \frac{27}{8}\epsilon.$$

(b) The groundstate has $S = 0$, and is not degenerate.

The first excited state is $\vec{n} = (2, 1, 1)$ and permutations. Since the interaction is attractive, the symmetric spatial state has lower energy, so the spin part must be antisymmetric, i.e. $S = 0$. The degeneracy is 3.

(c) If the interaction were repulsive, the spatially antisymmetric state would have lower energy. So the spin is symmetric. The degeneracy is thus the product of the spatial and spin degeneracies: $3 \cdot 3 = 9$.

#18 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a particle moving in the one-dimensional infinitely deep potential well. Let $V(x) = 0$ for $0 < x < a$, and $V(x) = \infty$ everywhere else. The eigenstates of the Hamiltonian $H_0 = \frac{p^2}{2m} + V(x)$ are $\phi_n(x) = \sqrt{\frac{2}{a}} sin(\frac{n\pi x}{a})$ with eigenvalues $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. Assume that at t=0 you put an obstacle at the bottom of the well which is represented by the potential barrier $W(x) = W_0$ for $\frac{a}{4} < x < \frac{3a}{4}$ and $W(x) = 0$ everywhere else. The time-dependent total Hamiltonian is $H = H_0 + h(t)$ where $h = 0$ for $t < 0$ and $h = W(x)$ for $t \geq 0$.

Answer the following question:

If at t=0 the particle is in the state $\phi_3(x)$, what is the probability of finding it in the state $\phi_1(x)$ at time t, if $W_0 \ll E_1$?

Useful relation:

$2sin(z_1)sin(z_2) = cos(z_1 - z_2) - cos(z_1 + z_2)$

SOLUTION:

The problem governed by H_0 is already solved since $h = 0$ for $t < 0$. The probability $P(E_k, t)$ of finding the particle in the eigenstate $|k>$ of H_0 at time t, i.e. the probability of measuring the eigenvalue E_k is given by

$$P(E_k, t) = \frac{1}{\hbar^2}\left|\int_0^t exp(\frac{i}{\hbar}(E_k - E_m)t')\langle k|h(t')|m\rangle dt'\right|^2 ,$$

with $m = 3$ in our case. This is the result of first order time-dependent perturbation theory where $P(E_k, t)$ is the transition probability from $|m\rangle$ to $|k\rangle$.

If $h(t)$ is independent of time for $t \geq 0$, i.e. if a constant small term is added to the Hamiltonian at $t = 0$, then

$$\begin{aligned}
P(E_k, t) &= \frac{1}{\hbar^2}\Big|\langle k|W|m\rangle\Big|^2 \cdot \Big|\frac{exp(\frac{i}{\hbar}(E_k - E_m)t) - 1}{\frac{i}{\hbar}(E_k - E_m)}\Big|^2 \\
&= \frac{1}{\hbar^2}\Big|\langle k|W|m\rangle\Big|^2 \cdot \Big|\frac{e^{-\frac{i}{\hbar}E_m t} - e^{-\frac{i}{\hbar}E_k t}}{\frac{i}{\hbar}(E_k - E_m)}\Big|^2 \\
&= \frac{|\langle k|W|m\rangle|^2}{(E_k - E_m)^2} \cdot \Big|2sin(\frac{1}{2}\frac{E_k - E_m}{\hbar}t)\Big|^2 \\
&= \frac{|\langle k|W|m\rangle|^2}{(\hbar\omega_{km})^2} \cdot 4sin^2(\frac{1}{2}\omega_{km}t) \ .
\end{aligned}$$

Here $\omega_{km} \equiv \frac{E_k - E_m}{\hbar}$ and we have used $|e^{i\theta} - e^{i\phi}| = 2sin(\frac{1}{2}(\theta - \phi))$.

When applied to our problem we therefore have

$$P(E_1, t) = \frac{|\langle \phi_1|W|\phi_3\rangle|^2}{(\hbar\omega_{13})^2} \cdot 4sin^2(\frac{1}{2}\omega_{13}t) \ ,$$

$$E_1 - E_3 = -\frac{8\pi^2\hbar^2}{2ma^2} \ , \quad \omega_{13} = -\frac{8\pi^2\hbar}{2ma^2} \ .$$

We use the following integrals:

$$
\begin{aligned}
\langle\phi_1|W|\phi_3\rangle &= \int_{\frac{a}{4}}^{\frac{3a}{4}} dx\phi_1^*(x)W_0\phi_3(x) \\
&= \frac{2W_0}{a}\int_{\frac{a}{4}}^{\frac{3a}{4}} dx sin\frac{\pi x}{a} sin\frac{3\pi x}{a} \\
&= \frac{2W_0}{\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} dx' sin(x')sin(3x') \\
&= \frac{2W_0}{\pi}\left(\frac{sin(2x')}{4} - \frac{sin(4x')}{8}\right)_{x'=\frac{\pi}{4}}^{x'=\frac{3\pi}{4}} \\
&= -\frac{W_0}{\pi} \ .
\end{aligned}
$$

Therefore we find

$$
\begin{aligned}
P(E_1,t) &= \frac{4m^2a^4W_0^2}{64\pi^6\hbar^4}\cdot 4sin^2(\frac{2\pi^2\hbar}{ma^2}t) \\
&= \frac{m^2a^4W_0^2}{4\pi^6\hbar^4}\cdot sin^2(\frac{2\pi^2\hbar}{ma^2}t) \ ,
\end{aligned}
$$

where $P(E_1,t)$ oscillates in time.

#19 : GRADUATE MATH METHODS

`PROBLEM:` Evaluate by contour integration

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^2}$$

showing the contour and all poles and branch cuts.

`SOLUTION:`

The poles are at $z = \pm i$, and the squareroot branch cut can be taken along the positive real axis. Integrate along the entire real axis, and close the contour at infinity in the upper half plane, getting the residue at $z = i$. This gives

$$\oint \frac{z^{1/2}dz}{1+z^2} = \frac{2\pi i\sqrt{i}}{2i} = \frac{\pi(1+i)}{\sqrt{2}} = (1+i)\int_0^\infty \frac{x^{1/2}dx}{x^2+1},$$

so the desired integral is

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^2} = \frac{\pi}{\sqrt{2}}.$$

#20 : GRADUATE GENERAL PHYSICS

PROBLEM:

An elastic ball of radius a collides with a rigid wall and rebounds. Estimate the collision time τ assuming that the initial velocity of the ball v is much smaller than the speed of sound c in the material of which it is made. Neglect any dissipation, friction, or rotation.

SOLUTION:

In the course of the collision the front of the ball becomes flattened, which causes a restoring elastic force. The collision time is related to the average value F of this force and the initial momentum p of the ball:

$$F\tau = 2p = \frac{8\pi}{3}\rho v a^3,$$

where ρ is the density of the ball. The force is the product of the contact area πr^2, the Young modulus Y, and the elastic strain ε:

$$F \sim \pi r^2 Y \varepsilon\,.$$

The contact radius r can be crudely estimated as the radius of the circle obtained by flattening the ball to the depth $u \sim v\tau$:

$$r \sim (2au)^{1/2} \sim (2av\tau)^{1/2}.$$

(A rigorous treatment of the contact between a sphere and a rigid wall, which gives the missing numerical coefficient, can be found in the elasticity theory books.) The elastic deformation is concentrated in a region of linear size r; therefore, $\varepsilon \sim u/r$. Putting everything together and solving for τ, we obtain

$$\tau \sim a\left(\frac{\rho^2}{Y^2 v}\right)^{1/5} = \frac{a}{c}\left(\frac{c}{v}\right)^{1/5}.$$

Our analysis was quasistatic. In particular, we neglected the fact that the collision results in creation of sound waves bouncing inside the ball. However, our τ is much longer than the roundtrip time $4a/c$ of the sound waves. Hence, the effect of such dynamic vibrations should effectively average out, especially if a small dissipation is present, as is usually the case.