PHYSICS DEPARTMENTAL EXAM

FALL 2000

EXAM AND SOLUTIONS

ID Number _____ PART I Score _____

PHYSICS DEPARTMENTAL EXAM – Fall 2000

DEPARTMENT OF PHYSICS
DEPARTMENTAL EXAMINATION – FALL 2000

PART I

Section 1:	**Problem 1**_____	**Problem 2** _____
Section 2:	**Problem 3**_____	**Problem 4** _____
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Dept. Exam Fall 2000

Part One of the Dept.Exam contains 10 problems from 5 areas of undergraduate physics: mechanics, E&M, quantum mechanics, thermodynamics, and experimental physics.

You should attempt 7 of these problems. You must select at least one problem from each of the five categories.

Good luck!

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1. Undergraduate Mechanics

You are attending the graduate student BBQ, which is held on the beach. Someone hands you an open can, which is still full. Before you put it down on the uneven sand, you reason: "Now the center of gravity is at the center of the can. If I drink a little, the center of gravity will be lowered, and the stability of the can will be improved. If I empty the can, the center of gravity will be at the center of the can again. So there must be an optimum in between". You know that an empty can has a mass $m_c = 100$ g, and that it holds $m_b = 300$ g of beer. How much beer should you leave in the can for optimum stability?

Solution:

Let $H :=$ height of can, $h :=$ level of beer,
$\eta :=$ height of center of gravity.

$$\eta = \frac{1}{2}\,\frac{\frac{h}{H} m_b \cdot h + m_c \cdot H}{\frac{h}{H} m_b + m_c} = \frac{1}{2}\,\frac{m_b h^2 + m_c H^2}{m_b h + m_c H}$$

$$\frac{d\eta}{dh} = \frac{1}{2}\,\frac{2 m_b h (m_b h + m_c H) - m_b (m_b h^2 + m_c H^2)}{(m_b h + m_c H)^2}$$

For minimum: $\frac{d\eta}{dh} = 0 \Rightarrow$

$$2h(m_b h + m_c H) - m_b h^2 - m_c H^2 = 0 \Rightarrow$$

$$m_b h^2 + 2 m_c h H - m_c H^2 = 0 \Rightarrow$$

$$h = -\frac{m_c H}{m_b} + \sqrt{\left(\frac{m_c}{m_b}\right)^2 H^2 + \frac{m_c}{m_b} H^2} \Rightarrow$$

$$\boxed{h = \frac{m_c}{m_b} H \left(-1 + \sqrt{1 + \frac{m_b}{m_c}}\right)}$$

For $m_b = 300\,g$, $m_c = 100\,g$:

$\boxed{h = \frac{1}{3} H}$, one should leave 100g of beer.

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2. Undergraduate Mechanics

Two stars, with masses m_1 and m_2, rotate around each other, with separation $\vec{r} = r_0(\cos\omega t, \sin\omega t, 0)$, with r_0 and ω constant.

a) Verify that this is indeed a solution of the equations of motion and find ω in terms of the other quantities.

b) Suppose that $m_1 = m_2 = M_s$, the mass of our sun, and $r_0 = \frac{1}{2}R_{se}$, half of the distance between the earth and the sun. Find the approximate rotation period T, in units of years.

$$\cancel{m_1}\ddot{\vec{r}}_1 = +\frac{G\cancel{m_1}m_2\vec{r}}{r^3} \qquad \vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

$$\cancel{m_2}\ddot{\vec{r}}_2 = -\frac{G\cancel{m_1}\cancel{m_2}\vec{r}}{r^3}$$

$$\therefore \ddot{\vec{r}} = -\frac{G(m_1+m_2)\vec{r}}{r^3}$$

Plug in $\vec{r} = r_0(\cos\omega t, \sin\omega t, 0)$

$\ddot{\vec{r}} = -\omega_0^2\vec{r}$ solves EOM if

1) $\omega_0^2 = \sqrt{\dfrac{G(m_1+m_2)}{r_0^3}}$

2) period $T \sim \sqrt{\dfrac{r_0^3}{(m_1+m_2)}}$ so $\dfrac{T}{T_e} = \sqrt{\dfrac{(\frac{1}{2})^3}{2}}$

$T \approx \frac{1}{\ }$ year $\qquad T_e = 1$ year

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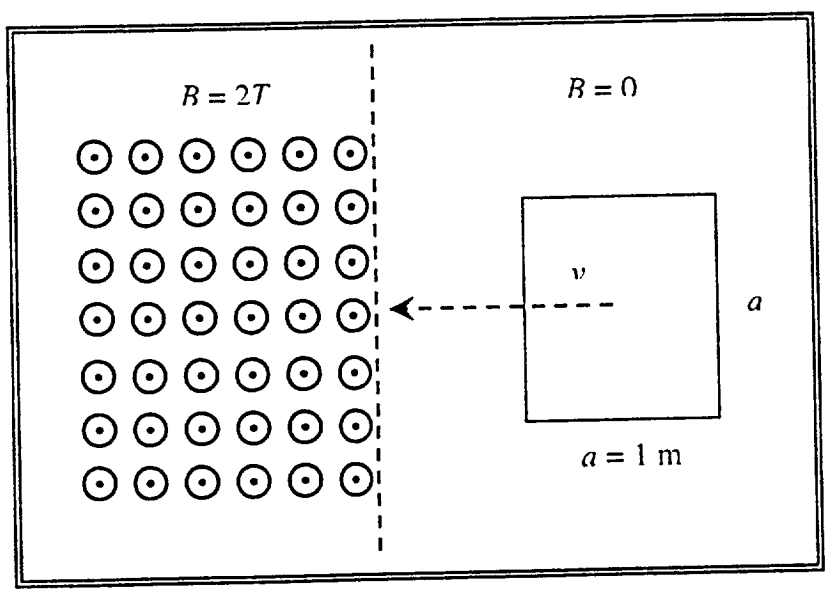
3. Undergraduate E & M

At room temperature (20° C), a square frame welded together from four 1 m long iron bars is pushed with velocity $v = 20$ m/s into a homogeneous sharply localized magnetic field B = 2T.

(a) What is the temperature of the iron frame immediately after the experiment?

(b) The frame is pulled out of the field region with the same velocity v. What is the temperature now?

(density of iron $\rho = 7870\,\mathrm{kg\,m^{-3}}$, specific heat $C = 460\,\mathrm{J\,kg^{-1}\,K^{-1}}$, resistivity $\rho_E = 0.098\times10^{-6}\,\Omega\,\mathrm{m}$).



Solution:

The induced voltage is:

$$U = \dot{\Phi} = avB$$

The corresponding power is:

$$P = UI = \frac{U^2}{R} = \frac{a^2 v^2 B^2}{4 \rho_E a A^{-1}}$$ (A = cross section of bars)

The energy generated is:

$$E = P \cdot \Delta t = P \cdot \frac{a}{v} = \frac{a^2 v B^2 A}{4 \rho_E}$$

The temperature change is:

$$\boxed{\Delta T = \frac{E}{mC} = \frac{E}{4 \cdot \rho a A C} = \frac{avB^2}{16 \rho \rho_E C} = 14\ K}$$

The temperature after the experiment is $(20+14)\,°C = 34\,°C$

b) When the frame is pulled out of the field, the current is reversed, but the dissipated energy is the same, $\Delta T = 14\,K$, and the final temperature is $48\,°C$.

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4. Undergraduate E&M

An electromagnetic plane wave moves, in vacuo, with angular frequency ω and electric field polarized along the $\hat{z}$ axis, with maximum amplitude E_0. At $\vec{r} = \vec{0}$, $t = 0$, the electric and magnetic fields are measured to be

$$E_x = 0, \quad E_y = 0, \quad E_z = E_0,$$

$$B_x = 0, \quad B_y = ? \ (B_y > 0), \quad B_z = ?$$

Solve for $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ in terms of ω, E_0, and c (the speed of light).

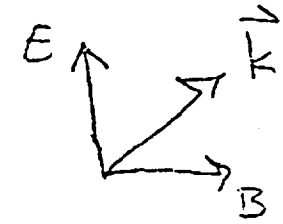
Write $\vec{E} = E_0 \hat{z} \cos\left(\vec{k}\cdot\vec{r} - \omega t\right)$

gives $\vec{E} = E_0 \hat{z}$ @ $\vec{r} = t = 0$

Maxwell's eqns solved by

$$\vec{B} = \vec{B}_0 \cos\left(\vec{k}\cdot\vec{r} - \omega t\right)$$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\hat{z} = 0$



$\nabla\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times E_0\hat{z} = \frac{\omega}{c}\vec{B}_0$

$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{k}\times\vec{B}_0 = -\frac{\omega}{c}E_0\hat{z}$

$\vec{E}\cdot\vec{B} = 0 \Rightarrow B_z = 0 \quad \therefore \vec{B}_0 = E_0\hat{y}$

& $\vec{k} = -\frac{\omega}{c}\hat{x}$

$$\therefore \vec{E} = E_0\hat{z}\cos\left(\frac{\omega}{c}x + \omega t\right)$$

$$\vec{B} = E_0\hat{y}\cos\left(\frac{\omega}{c}x + \omega t\right)$$

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5. Undergraduate Quantum Mechanics

A beam of silver atoms is emitted from an oven, which contains silver vapor at a temperature of 1200K. The beam is collimated by a small circular aperture of diameter a. The silver atoms create a spot of diameter D on a screen at a distance l from the oven.

(a) Use the uncertainty principle to show that D cannot be made arbitrarily small by reducing a.

(b) Calculate the minimum diameter D_{min} and the optimum aperture size a_{opt} for $l = 1$ m.

(Assume for simplicity that all atoms have identical momentum in the direction of the beam. $m_{Ag} = 1.8 \cdot 10^{-25}$ kg, $k = 1.38 \cdot 10^{-23}$ J/K, $\hbar = 1.05 \cdot 10^{-34}$ Js.)

Solution:

) $p_x :=$ momentum in beam direction

By equipartition, $\frac{p_x^2}{2m} = \frac{1}{2} kT \Rightarrow p_x = \sqrt{mkT}$

Uncertainty of momentum in y direction:

$$\Delta p_y \cdot a = \frac{\hbar}{2}$$

This leads to a beam spread with angle α, where

$$\tan\alpha = \frac{\frac{1}{2}\Delta p_y}{p_x} = \frac{\hbar}{4a\sqrt{mkT}}$$

The spot diameter is:

$$D = a + 2L\tan\alpha = a + \frac{\hbar L}{2a\sqrt{mkT}}$$, which cannot be made arbitrarily small by narrowing the aperture.

b)

$$\frac{dD}{da} = 1 - \frac{\hbar L}{2a^2\sqrt{mkT}} \Rightarrow \left(\frac{dD}{da} = 0\right)$$

$$\boxed{a_{opt} = \frac{\sqrt{\hbar L}}{\sqrt[4]{4mkT}} = 1\mu m, \quad D_{min} = 2a_{opt} = \frac{\sqrt{2\hbar L}}{\sqrt[4]{mkT}} = 2\mu m}$$

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6. Undergraduate Quantum Mechanics

Consider a one dimensional quantum mechanics scattering problem, involving a particle of mass m moving in the potential

$$V(x) = \begin{cases} V_0 & 0 \leq x \leq L \\ 0 & \text{otherwise.} \end{cases}$$

The incoming particle comes from $x = -\infty$, moving toward $x = +\infty$. Suppose that its energy is chosen to be precisely $E = V_0$. Find the transmission and reflection probabilities.

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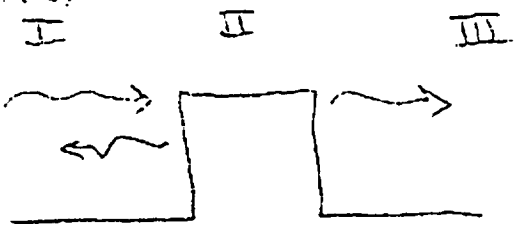

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

sol'n:

I: $\psi = A e^{ikx} + A' e^{-ikx}$ $\qquad k = \frac{\sqrt{2mE}}{\hbar}$

II: $\psi = B + cx$

III: $\psi = D e^{ikx}$

cont of ψ & ψ' $\Longrightarrow$

$$A + A' = B \qquad B + CL = De^{ikL}$$

$$ik(A - A') = C \qquad C = ikDe^{ikL}$$

$$\hookrightarrow \quad R = \left|\frac{A'}{A}\right|^2 = \frac{(kL)^2}{4 + (kL)^2}$$

$$T = \left|\frac{D}{A}\right|^2 = \frac{4}{4 + (kL)^2}$$

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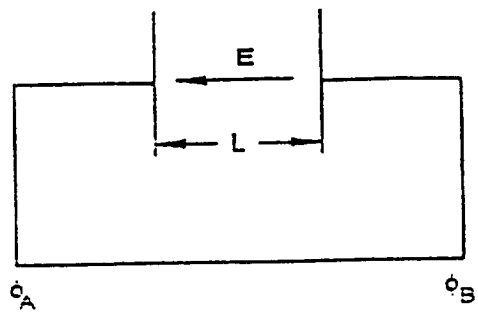
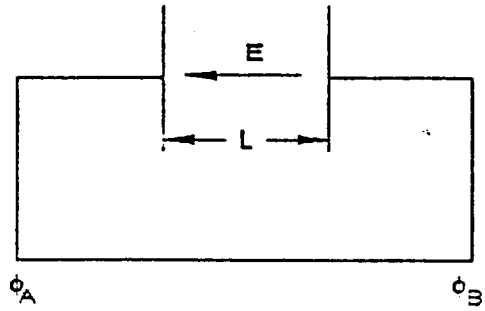
7. Undergraduate Thermodynamics



The capacitor and leads shown above are constructed from two different metals, A and B. Metal A, shown in red, has work function ϕ_A, and metal B, shown in blue, has work function ϕ_B. (The work function of a metal is the work required to remove an electron from the metal to a state of rest infinitely far from the metal: it is related to the chemical potential of electrons in the metal.) Clearly, there is no voltage difference between A and B. Find the electric field, E, between the capacitor plates.

7. Undergraduate Thermal Physics Problem

Debra Boman



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Solution

The work required to carry an electron from inside the B plate around the circuit is

(a) ϕ_B to remove the electron from the plate to vacuum just outside the plate.

(b) ϵEL to move the electron to the vacuum just outside the A plate.

(c) $-\phi_A$ to move the electron into the A plate.

No work is required to move the electron from inside the A plate to inside the B plate via the all metallic route. Therefore,

$$\phi_B + \epsilon EL - \phi_A = 0,$$

and the answer is

$$E = \frac{\phi_A - \phi_B}{\epsilon L}.$$

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8. Undergraduate Thermodynamics

The following describes a method to measure the specific heat ratio $\gamma = c_p/c_v$ of a gas. The gas, assumed ideal, is confined within a vertical cylindrical container and supports a freely moving piston of mass m. The piston and cylinder both have the same cross-sectional area A. Atmospheric pressure is p_0, and when the piston is in equilibrium under the influence of gravity and the gas pressure, the volume of the gas is V_0. The piston is now displaced slightly from its equilibrium position and is found to oscillate about this position with frequency ν. The oscillations are slow enough that the gas always remains in internal equilibrium, but fast enough so that the gas cannot exchange heat with the outside. Express γ in terms of m, A, p_0, V_0, and ν.

Solution

The oscillations are adiabatic, $pV^\gamma = p_0 V_0^\gamma$.

At equilibrium, $p_0 A = mg$.

Force at any position $x = x_0 + dx$:

$$F = pA - mg = pA - p_0 A = Ap_0\left(\frac{p}{p_0} - 1\right) = Ap_0\left(\left(\frac{V}{V_0}\right)^{-\gamma} - 1\right)$$

$$= -Ap_0\left(1 - \left(\frac{x}{x_0}\right)^{-\gamma}\right) = -Ap_0\left(1 - \left(\frac{x_0 + dx}{x_0}\right)^{-\gamma}\right) = -Ap_0\left(1 - \left(1 + \frac{dx}{x_0}\right)^{-\gamma}\right).$$

$$\approx -Ap_0\left(1 - \left(1 - \gamma\frac{dx}{x_0}\right)\right) = -\gamma A\frac{p_0}{x_0}dx$$

Frequency of harmonic oscillator with $F = -Dx$:

$$\omega = \sqrt{\frac{D}{m}} = \sqrt{\frac{\gamma A p_0}{x_0 m}} = \sqrt{\frac{\gamma A^2 p_0}{V_0 m}} \Longrightarrow$$

$$\boxed{\gamma = \frac{\omega^2 V_0 m}{p_0 A^2} = 4\pi^2 \nu^2 \frac{V_0 m}{p_0 A^2}}$$

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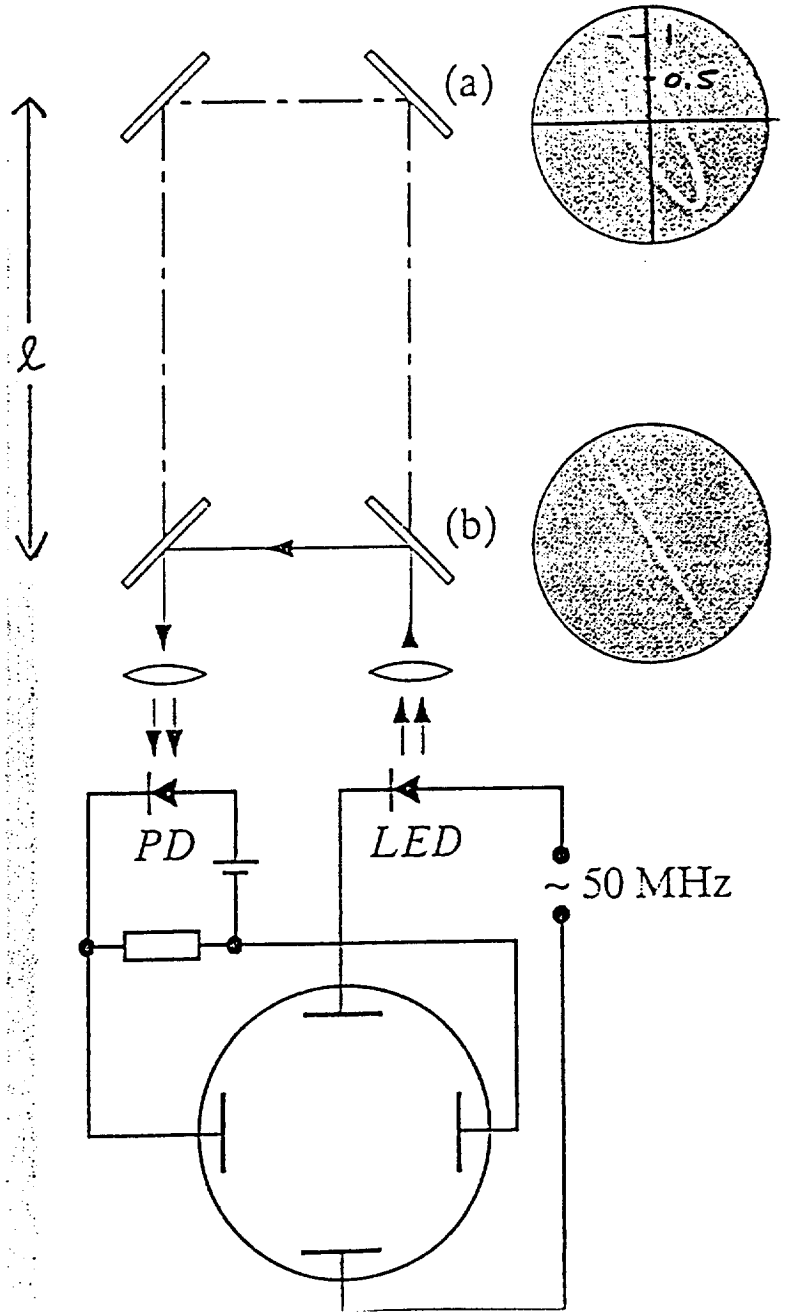
9. Undergraduate Experiment

In the experimental setup (see figure) the light emitted by an LED is reflected back to a photo diode (PD). The light of the LED is modulated with a 50 MHz sine wave. This signal, and a signal that is proportional to the current through the PD, are connected to the horizontal and vertical plates of an oscilloscope. The phase is adjusted such that a straight line (b) is seen. Now the mirrors are moved a distance $l = 25$ cm, and the ellipse (a) is observed. From these data, determine the speed of light c.



solution

The x-signal is $x = x_0 \sin(\omega t)$.
The y-signal is $y = y_0 \sin(\omega t + \varphi)$.

The Lissajous ellipse gives $y_0 = 1$ and $y(t=0) = y_0 \sin\varphi = 0.5$.
Thus $\varphi = 0.52$ rad $\Rightarrow$ $\Delta t = \frac{\varphi}{2\pi\nu} = 1.67 \cdot 10^{-9}$ s $\Rightarrow$
$c = \frac{2\ell}{\Delta t} = 3 \cdot 10^8$ m/s.

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10. Undergraduate Experiment

^{40}K decays through β emission with a decay constant $\lambda_1 = 4.75 \cdot 10^{-10}$ yr^{-1} in ^{40}Ca, and through K capture with $\lambda_2 = 0.585 \cdot 10^{-10}$ yr^{-1} in ^{40}Ar. A mineral sample contains 4.21% K. Of the total K content only 0.0119% is the isotope ^{40}K. In addition, 0.000088% ^{40}Ar are found in the sample. How old is the mineral? (Assume that the sample did not contain any Ar when it was formed.)

Solution:

The total ^{40}K content is $4.21\% \cdot 0.0119\% = 0.000501\%$.

The amount of ^{40}K that has decayed into ^{40}Ca is

$$\frac{\lambda_1}{\lambda_2} \cdot 0.000088\% = 0.000715\%$$

The initial amount of ^{40}K is the sum of current ^{40}Ca, ^{40}K, ^{40}Ar:

$0.000715\% + 0.000501\% + 0.000088\% = 0.001304\%$

The age t of the sample can now be obtained from the law of radioactive decay:

$$e^{-(\lambda_1+\lambda_2)t} = \frac{0.000501\%}{0.001304\%} = 0.3842 \Rightarrow$$

$$\boxed{t = \frac{-\log 0.3842}{(4.75+0.585)\cdot 10^{-10}\,yr^{-1}} = 1.79 \cdot 10^{9}\,yr}$$

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PART II

Section 1:	**Problem 11**_____	**Problem 12** _____
Section 2:	**Problem 13**_____	**Problem 14** _____
Section 3:	**Problem 15**_____	**Problem 16**_____
Section 4:	**Problem 17** _____	**Problem 18** _____
Section 5:	**Problem 19**_____	**Problem 20** _____

Part Two of the Dept.Exam contains 10 problems from 5 areas of graduate physics: classical mechanics, E&M, quantum mechanics, statistical physics, and mathematical methods.

You should attempt 7 of these problems. You must select at least one problem from each of the five categories.

Good luck!

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11. Graduate Classical Mechanics

A small block, of mass m, rests on top of a frictionless sphere, of radius R. The sphere is fixed to the ground and does not move. The gravitational acceleration is g (constant and downward, as usual). Let r and ϕ be the coordinates of the block, with r its radial distance from the center of the sphere (clearly $r \geq R$) and ϕ its angle, with $\phi = 0$ at the top of the sphere.

a) Write the Lagrangian and equations of motion for the block, in terms of $r(t)$ and $\phi(t)$, for general $r \geq R$.

b) Suppose the block starts at rest near the top, at $r(t = 0) = R$ and $\phi(t = 0) = \epsilon \approx 0$. Find the critical value ϕ_* of ϕ where the block loses contact with the sphere.

c) Find the time T that it takes for the block to lose contact with the sphere, i.e. solve $\phi(t = T) = \phi_*$, as a function of g, R, ϵ, and ϕ_* (you don't need to numerically evaluate it). As a check, you should find $T \to \infty$ for $\epsilon \to 0$, since the block could be in a meta-stable equilibrium if it rests precisely at the top of the sphere.

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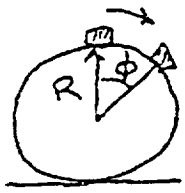

For $r \geq R$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr\cos\phi$$

EOM : $\ddot{r} = r\dot{\phi}^2 - g\cos\phi$

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = gr\sin\phi$$

Initially $r = R$ stays constant, since RHS of $\ddot{r}$ eqn is negative, but can't have $r < R$. Loses contact when RHS of $\ddot{r}$ eqn passes through zero, on way to becoming positive. Before then, we set $r = R$ and the conserved energy is

$$E = \frac{1}{2}mR^2\dot{\phi}^2 + mgR\cos\phi = mgR\cos\epsilon$$
$$\approx mgR$$

$$\hookrightarrow \dot{\phi}^2 = \frac{2g}{R}\left(1-\cos\phi\right) = \frac{4g}{R}\sin^2\frac{\phi}{2}$$

Loses contact when $R\dot{\phi}^2 = g\cos\phi$

$$\hookrightarrow \quad 2\cancel{g}(1-\cos\phi) = \cancel{g}\cos\phi$$

$$\hookrightarrow \quad \cos\phi_* = \frac{2}{3}$$

Find time via $\dot{\phi} = 2\sqrt{\frac{g}{R}}\sin\phi/2$

$$\int_0^T dt = \sqrt{\frac{R}{g}}\int_\varepsilon^{\phi_*}\frac{d(\phi/2)}{\sin(\phi/2)} = \sqrt{\frac{R}{g}}\log\left|\tan\left(\frac{\phi}{4}\right)\right|\Bigg|_\varepsilon^{\phi_*}$$

$$= \sqrt{\frac{R}{g}}\log\left(\frac{4\tan(\phi_*/4)}{\varepsilon}\right) = \sqrt{\frac{R}{g}}\log\left(\frac{4}{\varepsilon}\sqrt{\frac{1-\sqrt{5/6}}{1+\sqrt{5/6}}}\right)$$

$$T \approx \sqrt{\frac{R}{g}}\log(\varepsilon^{-1}) \quad \text{for } \varepsilon \to 0$$

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12. Graduate Classical Mechanics

Consider the motion of a particle, of mass m, in an anharmonic oscillator potential. Its position is governed by

$$\frac{d^2x}{dt^2} + \omega_0^2 x + \alpha x^2 = 0, \quad \alpha > 0.$$

Do not assume α is small.

a) Sketch the potential energy, labelling all extrema.

b) Suppose that the particle has $x(t=0) = 0$ and $\dot{x}(t=0) = v$. What is the condition on v in order to have stable oscillations? What happens if v does not satisfy this condition? Does the sign of the initial v matter for the issue of stability?

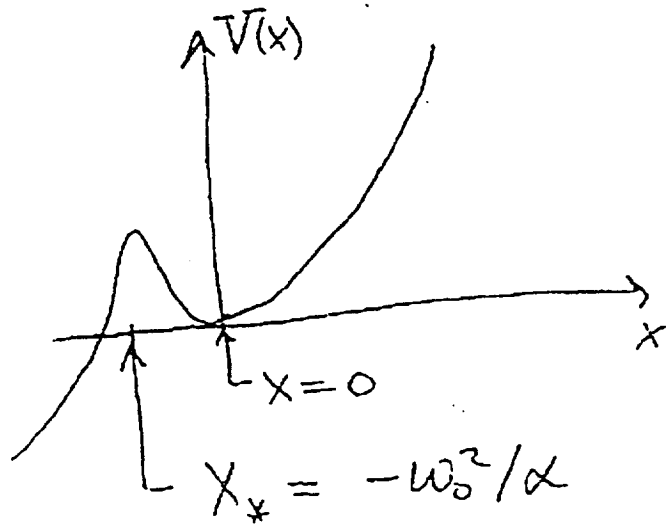
c) Suppose that the particle is exhibiting small stable oscillations. Its position $x(t)$ can be found via perturbation theory, $x = x^{(1)} + x^{(2)} + x^{(3)} + \ldots$, with $x^{(1)} = \epsilon \sin \omega t$, at the oscillation frequency $\omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + \ldots$. The perturbative expansion is order-by-order in the small parameter ϵ, with $\omega^{(n)}$ and the magnitude of $x^{(n)}$ both order ϵ^n. Find the time averaged displacement, $\langle x \rangle$, to order ϵ^2.

The conserved energy is

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 + \frac{1}{3} m \alpha x^3$$

with potential

a)



b) The potential energy at the top of the hill is

$$V(x_*) = \frac{1}{6} \frac{m \omega_0^6}{\alpha^2} \equiv E_{crit.} \qquad \text{If } E > E_{crit}$$

the particle will eventually oscillate over the hill to $x \to -\infty$. The sign of $v = \dot{x}(0)$ does not matter, if it's positive it would still oscillate over the hill on the rebound. So stability requires $|v| < \sqrt{\frac{2}{m} E_{crit}} = \frac{\omega_0^3}{\sqrt{3}\,\alpha}$

c) Average the EOM: $\underbrace{\cancel{\langle \ddot{x} \rangle}}_{0} + \omega_0^2 \langle x \rangle + \alpha \langle x^2 \rangle = 0$

So $\langle x \rangle = -\dfrac{\alpha}{\omega_0^2} \langle x^2 \rangle$

In the perturbative expansion, $\langle x^{(1)} \rangle = 0$.

and $\langle x^{(2)} \rangle = -\dfrac{\alpha}{\omega_0^2} \langle (x^{(1)})^2 \rangle = -\dfrac{\alpha \varepsilon^2}{2\omega_0^2}$

so, to order ε^2, $\langle x \rangle = -\dfrac{\alpha \varepsilon^2}{2\omega_0^2}$

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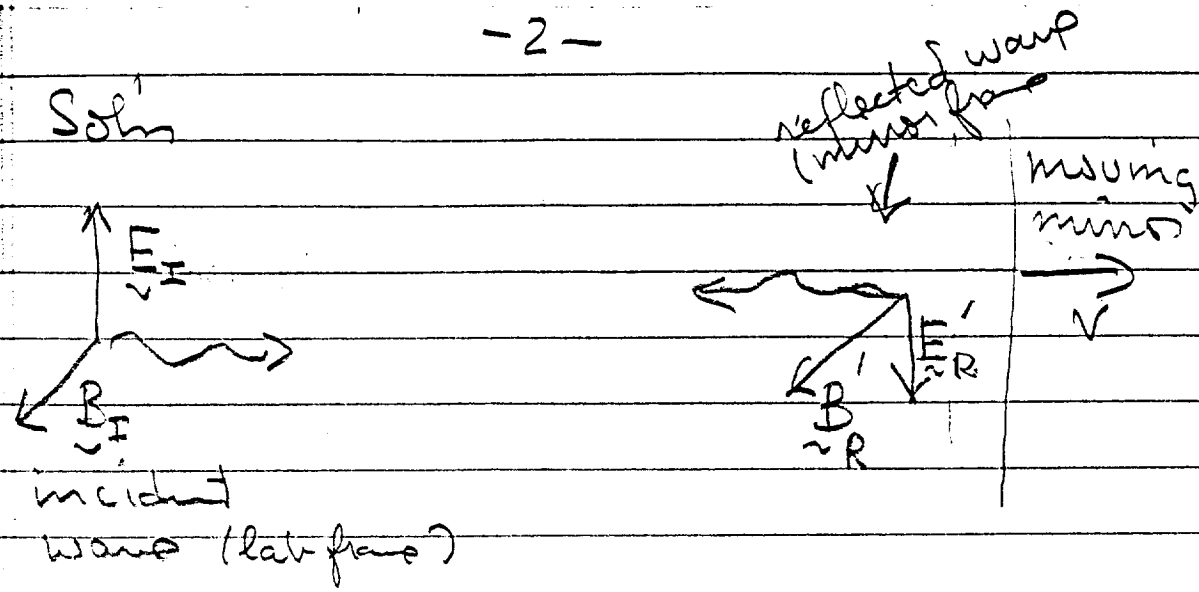
13. Graduate Electricity and Magnetism

When an electromagnetic wave is incident normally on a perfectly conducting plane (a mirror) at rest, the reflection coefficient is unity. What is the reflection coefficient (as measured in the laboratory frame) when the mirror moves with velocity v in the direction of propagation of the incident wave?

Sol'n



first transform to mirror frame

$$\underset{\sim}{E}'_I = \gamma\left[\underset{\sim}{E}_I + \frac{\underset{\sim}{V}\times\underset{\sim}{B}_I}{c}\right] = \gamma\left[1-\frac{V}{c}\right]\underset{\sim}{E}_I$$

In the mirror frame, the reflect wave is given by

$$\underset{\sim}{E}'_R = -\underset{\sim}{E}'_I \quad , \quad \underset{\sim}{B}'_R = \underset{\sim}{B}'_I$$

Transform back to lab frame

$$\underset{\sim}{E}_R = \gamma\left[\underset{\sim}{E}'_R - \frac{\underset{\sim}{V}\times\underset{\sim}{B}'_R}{c}\right] = -\gamma\left[1-\frac{V}{c}\right]E'_R$$

$$\left|\frac{E_R}{E_I}\right|^2 = \gamma^2\left[1-\frac{V}{c}\right]^2 = \frac{1-V/c}{1+V/c}$$

$\therefore$ Reflection Coefficient $= \left(\dfrac{1-V/c}{1+V/c}\right)^2$

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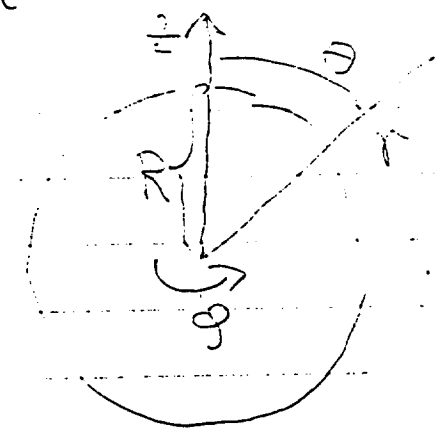
14. Graduate Electricity and Magnetism

A solid superconducting sphere of radius R is placed in a uniform external P magnetic field, B. Assuming that surface currents exclude the magnetic field from the interior of the sphere, determine the magnetic field in the region outside the sphere. Also, determine the surface current.

Outside the sphere $\underline{\nabla} \times \underline{B} = 0$

$$\therefore \quad \underline{B} = -\underset{\sim}{\nabla} \Phi \qquad 0 = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = -\nabla^2 \Phi$$

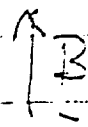


$$\Phi = A r \cos\theta + \frac{C}{r^2} \cos\theta$$

as $r \to \infty$

$$\Phi \to -B_0 z = -B_0 r \cos\theta$$

$$\therefore \quad A = -B_0$$



$$\nabla \times \underline{B} = 0 \Rightarrow \hat{r} \cdot \underline{B} = 0 \quad \text{at } r = R$$

$$\therefore \quad 0 = -A \cos\theta + \frac{C}{R^3} \cos\theta$$

$$\therefore \quad C = R^3 A = -\frac{B_0 R^3}{2}$$

$$\underset{\sim}{B} = -\underset{\sim}{\nabla} \Phi = +B_0 \hat{z} + \frac{B_0 R^3}{2} \left[-\frac{\hat{r}\, 2 \cos\theta}{r^3} - \frac{\hat{\theta} \sin\theta}{r^3} \right]$$

$$\frac{4\pi}{c} K_\varphi = B_\theta = -\frac{B_0}{2}\sin\theta + B_0 \underbrace{\hat{z}\cdot\hat{\theta}}_{-\sin\theta}$$

$$\frac{4\pi}{c} K_\varphi = -\frac{3}{2} B_0 \sin\theta$$

surface current in $\hat{\varphi}$ direction

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15. Graduate Quantum Mechanics

The two-particle states K^0 and $\overline{K}^0$ can be considered as a basis set for the two-state system with the Hamiltonian:

$$H = \hbar \begin{bmatrix} A - i\Gamma & A - i\Gamma \\ A - i\Gamma & A - i\Gamma \end{bmatrix},$$

where the decay term $\Gamma > 0$ makes the Hamiltonians non-hermitian.

(a) If the particle is initially K^0, find the probability for it being found as the same particle in a subsequent time t.

(b) Discuss the time evolution of the total probability in both states K^0 and $\overline{K}^0$.

Solution

(a) By changing the energy zero, we remove the A terms from the diagonal elements of the Hamiltonian

$$\mathcal{H} = \begin{bmatrix} -i\Gamma & A - i\Gamma \\ A - i\Gamma & -i\Gamma \end{bmatrix},$$

where the energy is measured in units of $\hbar = 1$.

If the state at time t is given by

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix},$$

with the initial state

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

the equation of motions are

$$\begin{aligned} i\frac{dc_1}{dt} &= -i\Gamma c_1 + (A - i\Gamma)c_2 \\ i\frac{dc_2}{dt} &= -i\Gamma c_2 + (A - i\Gamma)c_1. \end{aligned}$$

A transformation

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} e^{-\Gamma t},$$

leads to the simplified equations

$$\begin{aligned} \frac{da_1}{dt} &= (-iA - \Gamma)a_2 \\ \frac{da_2}{dt} &= (-iA - \Gamma)a_1. \end{aligned}$$

Solution of these equations with the initial conditions yields

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \frac{e^{-\Gamma t}}{2} \begin{bmatrix} e^{(-iA-\Gamma)t} + e^{(iA+\Gamma)t} \\ e^{(-iA-\Gamma)t} - e^{(iA+\Gamma)t} \end{bmatrix}.$$

The probability in state K^0 is

$$|c_1(t)|^2 = \frac{1}{2}e^{-2\Gamma t}\left[\cosh(2\Gamma t) + \cos(2At)\right].$$

(b) The probability for the $\overline{K}^0$ state is

$$|c_2(t)|^2 = \frac{1}{2}e^{-2\Gamma t}\left[\cosh(2\Gamma t) - \cos(2\Delta t)\right].$$

The total probability is

$$|c_1(t)|^2 + |c_2(t)|^2 = e^{-2\Gamma t}\cosh(2\Gamma t)$$

The model decay process reduces the total particle density to half of its initial value.

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16. Graduate Quantum Mechanics

A hydrogen atom is placed in a static electric field of 10^6 V/m pointing in the z direction.

(a) Find the change in energies of the $n = 2$ levels in electron volts and the new eigenfunctions in terms of the old ones in the zero field. [Hint: If $|n\ell m\rangle$ denotes the energy eigenstates of hydrogen in zero field, then

$$\langle 210|z|200\rangle = -3a_0,$$

where $a_0 \approx 0.5 \times 10^{-10}$ m is the Bohr radius.]

(b) If a circularly polarized light propagating along the z direction is used with the electric vector rotating in the clockwise direction if sighted along the positive z direction, deduce which of the new $n = 2$ states can be excited.

Solution – The ratio of the perturbation to the binding energy is approximately,

$$e\mathcal{E}_z a_0/E_0 = 10^6 \text{ eV/m } 0.5 \times 10^{-10} \text{ m}/13.6 \text{ eV} \approx 4 \times 10^{-6},$$

which is sufficiently small for the first-order degenerate perturbation theory to be valid. The perturbation Hamiltonian for the degenerate levels $|200\rangle, |211\rangle, |210\rangle, |21\bar{1}\rangle$ is

$$H_\mathcal{E} = - - 3e\mathcal{E}_z a_0 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues can be found by symmetry, giving the energy shifts:

$$\Delta E_{21\pm1} = 0,$$

with the states unchanged; and the (200) and (210) states split into

$$\Delta E_\pm = \pm 3e\mathcal{E}_z a_0 = \pm 1.5 \times 10^{-4} \text{ eV},$$

with the states,

$$\frac{1}{2}\left[|200\rangle + |210\rangle\right].$$

(b) The polarization vector is given by

$$\mathcal{E} = (\hat{x} + \hat{y})e^{-i\omega t}.$$

Since this transforms like the Y_{10} component, the matrix element $\langle 2\ell m|\mathcal{E}|100\rangle$ is non-vanishing only if $\ell = 1, m = 1$.

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17. Graduate Stat. Mech.

Consider a Fermi gas of N electrons, at temperature $T = 0$, in a 3d volume V. Ignore electric repulsion. The momentum levels are filled to the Fermi surface p_F, with $p > p_F$ empty.

a) Evaluate p_F for general N and V.

b) Find the energy as a function of the mass m_e and N and V.

c) Find the number of collisions of the electrons with the walls per unit time, per unit area of the wall.

d) Explain why your answers for the above parts behave as they do in the limit $\hbar \to 0$.

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Fermi gas: # of states density is

$$dN = \underbrace{2}_{\text{spin dof.}} \frac{V}{(2\pi\hbar)^3} p^2\, dp\, d\Omega$$

so $$N = \frac{2V(4\pi)}{(2\pi\hbar)^3}\int_0^{p_F} p^2 dp = \frac{V p_F^3}{3\pi^2\hbar^3}$$

a)

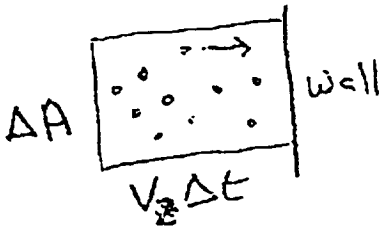
$$\therefore \quad p_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}\hbar$$

$\boxed{\begin{array}{l} p \le p_F \text{ filled} \\ p > p_F \text{ empty} \end{array}}$

b) $$E = \int_0^{p_F} \frac{p^2}{2m}\, dN = \frac{V}{2m\pi^2\hbar^3}\int_0^{p_F} p^4 dp$$

$$= \frac{V p_F^5}{10 m\pi^2\hbar^3} = \frac{3}{10}\frac{\hbar^2}{m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} N$$

c)



In time Δt, $\frac{dN}{V} V_z \Delta t \Delta A$ hit the wall for $V_z > 0$.

c cont'd:

Thus the # collisions / $\Delta t \, \Delta A$ =

$$\int_0^{p_F} dp \int_0^1 d(\cos\theta) \int_0^{2\pi} d\phi \; \frac{2}{(2\pi\hbar)^3} \; p^2 \underbrace{\left(\frac{p}{m}\cos\theta\right)}_{(V_z)}$$

$(V_z \geq 0)$

$$= \frac{2}{(2\pi\hbar)^3}(2\pi)\left(\frac{1}{2}\right)\frac{p_F^4}{4m} = \hbar \cdot \frac{3(3\pi^2)^{1/3}}{16m}\left(\frac{N}{V}\right)^{4/3}$$

d) All the above $\longrightarrow 0$ for $\hbar \longrightarrow 0$.

Without quantum statistics repulsion, the electrons all just crowd into $p = 0$ state & nothing happens.

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18. Graduate Stat. Mech.

A gas of classical identical particles, interacting by the two-body potential

$$U(\mathbf{r}) = U_0 \ln(a/r)\Theta(a-r)$$

where $\Theta(x)$ is the set function, is confined to a container.

a) Write a formal expression for the grand partition function Ξ, to second order in the fugacity $z \equiv e^{\mu/k_B T}$, as a function of the temperature T, volume V, and potential $U(r)$.

b) Determine the pressure $p = p(z, T)$ to second order in the fugacity. (Hint: recall that $\ln \Xi = \Omega/k_B T = -pV/k_B T$, with Ω extensive.)

c) Determine the density $n = n(z, T)$ to second order in the fugacity.

d) Derive the equation of state $p = p(n, T)$ to second order in the density n.

Solution

To second order in the density, one has

$$p/k_B T = n - B_2 n^2 + \cdots ,$$

where

$$B_2 = -\tfrac{1}{2} \int d^3r \left[e^{-U(r)/k_B T} - 1 \right] .$$

Interlude – This result is easily derived by taking the logarithm of the grand partition function,

$$\begin{aligned} \Xi &= \sum_{N=0}^{\infty} \frac{1}{N!} e^{N\mu/k_B T} Z_N(T,V) \\ &= 1 + zV\lambda_T^{-3} - \tfrac{1}{2} z^2 V \lambda_T^{-6} \int d^3r\, e^{-U(r)/k_B T} + \cdots , \end{aligned}$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_B T}$ is the thermal wavelength, which appears upon performing the momentum integrals, *viz.*

$$\int \frac{d^3p}{(2\pi\hbar)^3} \exp(-p^2/2mk_B T) = \lambda_T^{-3}$$

Recalling that $\ln \Xi = \Omega/k_B T = -pV/k_B T$ and carrying out the Taylor series expansion to second order in the fugacity, one immediately obtains the expansion for $p(z,T)$:

$$p/k_B T = z\lambda_T^{-3} - \tfrac{1}{2} z^2 \lambda_T^{-6} \int d^3r\, f(r) + \cdots ,$$

where $f(r) \equiv e^{-U(r)/k_B T} - 1$ is the Mayer function. Next, one invokes $n = -V^{-1}(\partial\Omega/\partial\mu)_{T,V} = \partial(p/k_B T)/\partial \ln z$ to obtain a series expansion for $n(z,T)$:

$$n = z\lambda_T^{-3} + z^2 \lambda_T^{-6} \int d^3r\, f(r) + \cdots .$$

Finally, one inverts $n(z,T)$ to find $z(n,T)$ and substitutes into the pressure equation, yielding

$$p/k_B T = n + B_2 n^2 + \cdots ,$$

with $B_2 = -\frac{1}{2} \int d^3r\, f(r)$.

(a) The Mayer function for our problem is

$$f(r) = \left[\left(\frac{r}{a}\right)^{U_0/k_B T} - 1 \right] \Theta(a - r) .$$

This is continuous and strictly nonpositive for $r \in [0,\infty)$. Integrating, we find

$$B_2(T) = \tfrac{2}{3}\pi a^3 \frac{U_0}{U_0 + 3k_B T} .$$

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19. Graduate Phys. Methods

The Fourier transform F(s) is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x s} dx \quad ;$$

note that

$$f(x) = \int_{-\infty}^{\infty} F(s)\, e^{2\pi i x s} ds \quad .$$

The functions Π, Λ, and sinc are defined as follows:

$$\Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 & |x| \le \frac{1}{2} \end{cases}$$

$$\Lambda(x) = \begin{cases} 0 & |x| > 1 \\ 1 - |x| & |x| < 1 \end{cases}$$

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

a) Show that the Fourier transform of $\Pi(x)$ is sinc(s).

b) Show that $\Pi(x) * \Pi(x) = \Lambda(x)$

(* denotes the convolution operator $f * g = \int_{-\infty}^{\infty} f(u) g(x-u) du$.)

c) Using the results from a) and b), and Parseval's theorem (a.k.a. the power theorem), show that $\int_{-\infty}^{\infty} \mathrm{sinc}^4(x) dx = \frac{2}{3}$

Solution

a)
$$F(s) = \int_{-\infty}^{\infty} \Pi(x)\, e^{-2\pi i x s}\, dx$$
$$= \int_{-1/2}^{1/2} e^{-2\pi i x s}\, dx$$
$$= \int_{-1/2}^{1/2} \cos 2\pi x s\, dx$$
$$= \frac{\sin \pi s}{\pi s}$$
$$= \operatorname{sinc}(s)$$

b)
$$\Pi(x) * \Pi(x) = \int_{-\infty}^{\infty} \Pi(u)\, \Pi(x-u)\, du$$
$$= \begin{cases} 0 & x < -1 \\ \int_{-1/2}^{x+1/2} du = 1+x & -1 \le x < 0 \\ \int_{x-1/2}^{1/2} du = 1-x & 0 \le x < 1 \\ 0 & x > 1 \end{cases}$$
$$= \Lambda(x)$$

c) ~~$\int_{-\infty}^{\infty} \operatorname{sinc}^4(x)\, dx =$~~ From a), b), ~~we have that~~ and the convolution theorem we know the Fourier transform of $\operatorname{sinc}^2(x) = \operatorname{sinc}(x) \cdot \operatorname{sinc}(x)$ is $\Pi(s) * \Pi(s) = \Lambda(s)$. Therefore:

$$\int_{-\infty}^{\infty} \operatorname{sinc}^4(x)\,dx = \int_{-\infty}^{\infty} \left(\operatorname{sinc}^2(x)\right)^2 dx$$

$$\overset{\text{Parseval}}{=} \int_{-\infty}^{\infty} |\Lambda(x)|^2\,dx$$

$$= 2\int_0^1 (1-x)^2\,dx$$

$$= 2\int_0^1 (1-2x+x^2)\,dx$$

$$= 2\left[x - x^2 + \tfrac{1}{3}x^3\right]_0^1$$

$$= \frac{2}{3}$$

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20. Graduate Math Methods

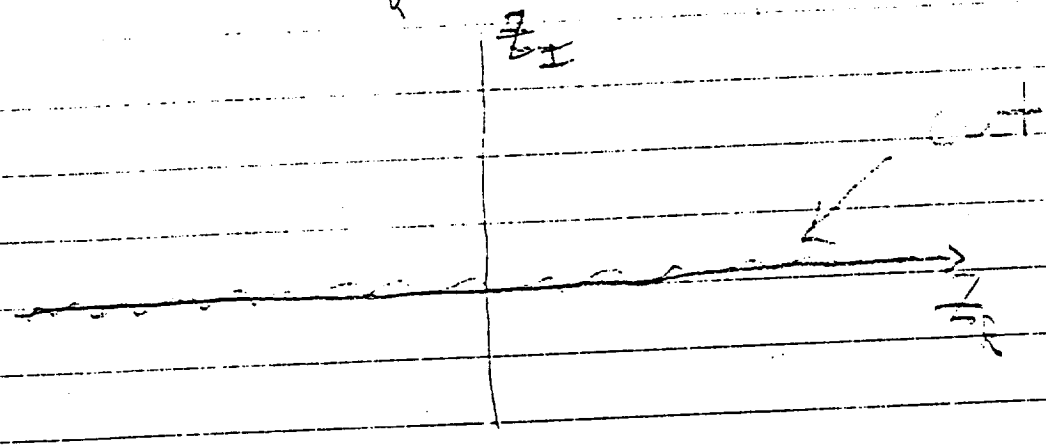
The function

$$f(z) = \int_{-\infty}^{\infty} dx \, \frac{e^{-x^2}}{x - z}$$

has no poles in the finite z - plane, but does have a branch cut. Locate the branch cut and determine the change in the value of f(z) across the cut.

Sol'n

The branch cut is along the real z-axis (i.e. for $z = z_R$)



$$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \frac{dx\, e^{-x^2}}{x - (z_R + i\epsilon)} = P \int_{-\infty}^{+\infty} \frac{dx\, e^{-x^2}}{x - z_R} + \pi i\, e^{-z_R^2}$$

$$\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \frac{dx\, e^{-x^2}}{x - (z_R - i\epsilon)} = P \int_{-\infty}^{+\infty} \frac{dx\, e^{-x^2}}{x - z_R} - \pi i\, e^{-z_R^2}$$

$$\lim_{\epsilon \to 0} \left[f(z_R + i\epsilon) - f(z_R - i\epsilon) \right] = 2\pi i\, e^{-z_R^2}$$