PHYSICS DEPARTMENT

Spring 2004

Departmental Written Exam
&
Solutions

DEPARTMENT OF PHYSICS
DEPARTMENTAL EXAMINATION –SPRING 2004

PART I

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only.

Please indicate with a "check" which of the (7) questions you wish to be graded below:

Section 1:	**Problem 1**_____	**Problem 2** _____
Section 2:	**Problem 3**_____	**Problem 4** _____
Section 3:	**Problem 5**_____	**Problem 6**_____
Section 4:	**Problem 7** _____	**Problem 8** _____
Section 5:	**Problem 9**_____	**Problem 10** _____

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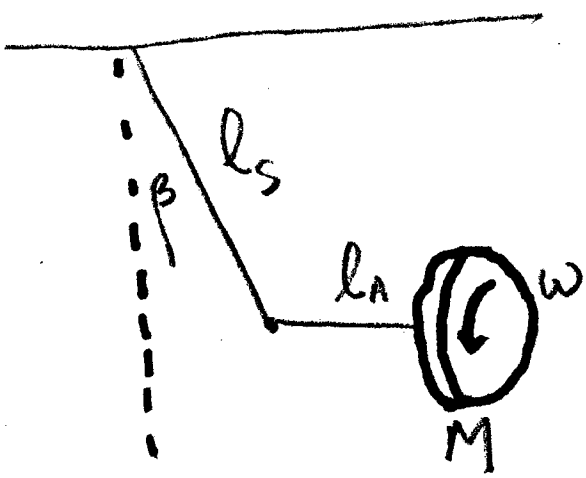
SECTION 1:

PROBLEM 1.

1. A solid disc of mass M and radius R is rotating at an angular frequency ω. As shown in the figure, the disc is attached to an axle of length ℓ_A which is being held up by a string of length ℓ_s. If the disc executes uniform precession in the horizontal plane and the axle is horizontal, find an equation for the angle β that the string makes with the vertical axis



Solution 1

Moment of Inertia

$I = \frac{1}{2} M R^2$ for the disc. We can write down three equations, from force considerations (2) and from torque (1). In detail

$$T \cos\beta = Mg \qquad \text{(Force in vertical direction)}$$

$$T \sin\beta = M(L \sin\beta + l)\Omega^2 \qquad \text{(horizontal)}$$

if Ω is the precession rate

and finally, torque equals rate of change of angular momentum. Here

$$\frac{dL}{dt} = \Omega L = I\omega\Omega$$

This must equal the torque (with respect to ~~point~~ the center of the circle along which the disk is precessing) O)

$$I\omega\Omega = Mg(l + L\sin\beta) - T L \cos\beta \sin\beta$$
$$= Mgl$$

Solving these equations gives $\Omega = \frac{Mgl}{I\omega}$

and

$$M(L\sin\beta + l)\Omega^2 = Mg \tan\beta$$

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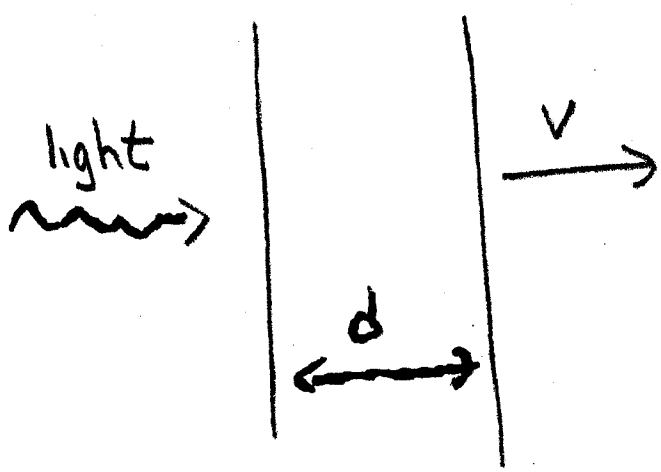
SECTION 1:

PROBLEM 2.

2. A piece of glass of width d with index of refraction n is sent into relativistic speed v, as measured in the laboratory frame of reference, in the direction normal to its surface. How long does it take does light (at normal incidence) propagate through the glass as determined by a stationary observer and how fast is the light observed to move?



SOLUTION (2)

In the frame of the glass, light goes at speed

$$V_0 = \frac{c}{n}$$

Transforming back to the lab frame requires the velocity transformation law

$$u = \frac{V + V_0}{1 + VV_0/c^2}$$

Substituting,

$$u = \frac{\frac{c}{n} + V}{1 + \frac{V}{nc}} = \frac{c}{n}\left(\frac{1 + \frac{nV}{c}}{1 + \frac{V}{nc}}\right)$$

The width of the glass suffers a Lorentz contraction by a factor of $\sqrt{1 - V^2/c^2}$. Finally, the time it takes to go across is given by the width over the relative speed between the light and the glass

$$T = \frac{d\sqrt{1 - V^2/c^2}}{u - V} = \frac{\frac{nd}{c}\left(1 + \frac{V}{nc}\right)}{\sqrt{1 - V^2/c^2}}$$

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SECTION 2:

PROBLEM 3.

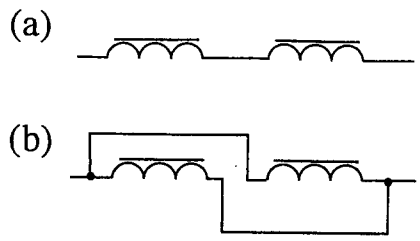

FIG. 1:

3. Two solenoids with ferrite cores are positioned in a close proximity to each other and are connected in parallel - see Fig. 1(a). Their individual inductances are equal to 1 and $2\,mH$, while the total measured inductance is $3.6\,mH$. Without changing the positions of the solenoids, they are switched to a parallel connection, see Fig. 1(b). What total inductance can be measured after the switch?

SOLUTION 3

In addition to the given self-inductances, there is a mutual inductance. For I_1, I_2 through the inductors, the induced EMFs are

$$\epsilon_1 = L_1 \frac{dI_1}{dt} + L_M \frac{dI_2}{dt}$$

$$\epsilon_2 = L_M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt}$$

for the series configuration, $\epsilon = \epsilon_1 + \epsilon_2$, $\dot{I}_1 = \dot{I}_2 = \dot{I}$

and hence the total inductance $= L_1 + L_2 + 2L_M$

$$\Rightarrow L_M = .3 \text{ mH}$$

In parallel, $\epsilon_1 = \epsilon_2 \equiv \epsilon$, $\dot{I} = \dot{I}_1 + \dot{I}_2$

The former gives $\dot{I}_1 (L_1 - L_M) = \dot{I}_2 (L_2 - L_M)$

yielding

$$\dot{I} = \dot{I}_1 \left(1 + \frac{L_1 - L_M}{L_2 - L_M}\right)$$

$$\epsilon = \dot{I}_1 L_1 + L_M \dot{I}_1 \left(1 + \frac{L_1 - L_M}{L_2 - L_M}\right)$$

$$= \dot{I} \left[\frac{L_1 + L_M \left(\frac{L_1 - L_M}{L_2 - L_M}\right)}{1 + \frac{L_1 - L_M}{L_2 - L_M}} \right]$$

SOLUTION 3.1.

simplifying, we get

$$L_{total} = \frac{L_1 L_2 - L_M^2}{L_1 + L_2 - 2L_M} \simeq .8\,mH$$

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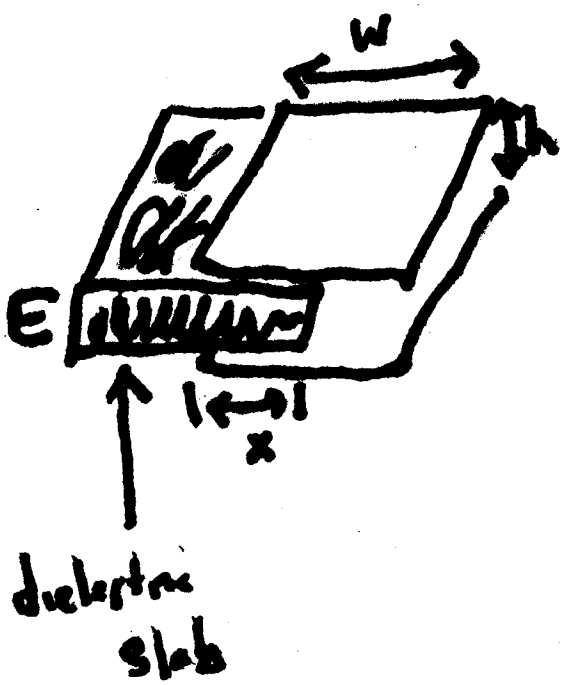
SECTION 2

PROBLEM 4.

4. A slab of material with dielectric constant ϵ is partially inserted in a square parallel place capacitor, as shown in the figure. Assuming a small gap thickness, find the force on the dielectric is the capacitor is isolated. What changes if instead the capacitor is held at constant voltage?



SOLUTION. 4

The energy can be written as $\frac{1}{2}\frac{Q^2}{C}$ where

$$C = \frac{\epsilon_0 W}{h}\left[(W-X) + \epsilon X\right] \qquad \text{(note: MKS)}$$

The force at constant Q is therefore

$$F = -\frac{dE}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

$$= \frac{1}{2}\frac{Q^2}{C^2}\frac{\epsilon_0 W}{h}(\epsilon - 1)$$

The force is positive and hence the slab is pulled inward.

If instead voltage is held constant, we have to include the work done by the battery so as to keep the voltage constant. This work is just

$$W_{battery} = V_0 \Delta Q = +V_0^2 \Delta C$$

so

$$\frac{d\,\text{Mechanical work}}{dx} = \Delta E - (\text{electrical work})/dx$$

$$= \frac{1}{2}V_0^2\frac{dC}{dx} - V_0^2\frac{dC}{dx} = -\frac{1}{2}\frac{dC}{dx}V_0^2$$

hence $F = \frac{1}{2}V_0^2\frac{dC}{dX} = \frac{1}{2}V_0^2\frac{\epsilon_0 W}{h}(\epsilon - 1)$, again inward.

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SECTION 3:

PROBLEM 5:

5. Consider a one-dimensional electron of energy E and mass m incident from the left on a repulsive δ function scatterer

$$V(x) = V_0\delta(x - x_0)$$

Find the transmission coefficient, defined as the probability that the electron makes it through the scatterer.

SOLUTION 5.

To the left of x_0, the wave function can be written as

$$\psi_L(x) = e^{ik(x-x_0)} + r e^{-ik(x-x_0)}$$

where $\frac{\hbar^2 k^2}{2m} = E$

to the right, $\psi_R = t e^{ik(x-x_0)}$

Matching at $x = x_0$ gives

$$1 + r = t$$

$$-ik(1-r) + ikt = \frac{2mV_0}{\hbar^2} t$$

Solving, gives

$$t = \frac{1}{1 + \frac{imV_0}{\hbar^2 k}}$$

Transmission coefficient $= |t|^2 = \dfrac{1}{1 + \left(\frac{mV_0}{\hbar^2 k}\right)^2}$

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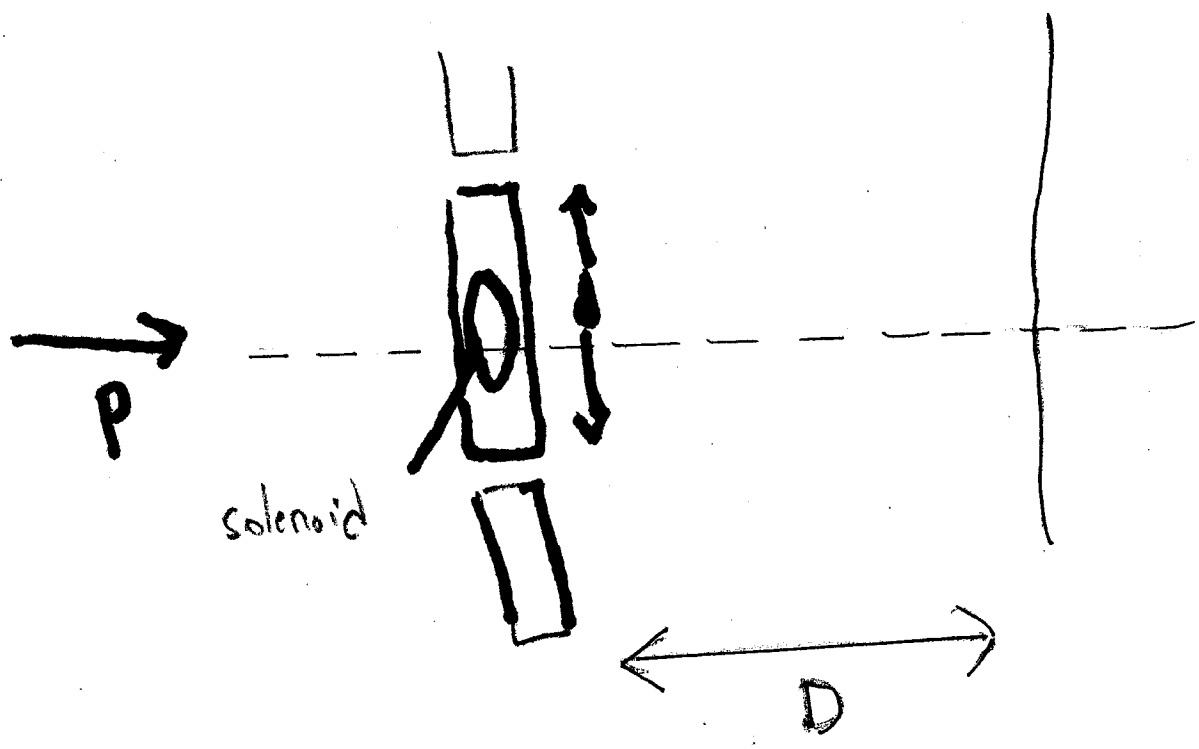
SECTION 3:

PROBLEM 6:

6. Consider a modified version of the double slit experiment where a plane wave electron beam of momentum p is incident on a wall with two slits separated by distance d and with an embedded solenoid with coil radius R, number of turns N- note that the electrons never actually reach the interior of the solenoid.

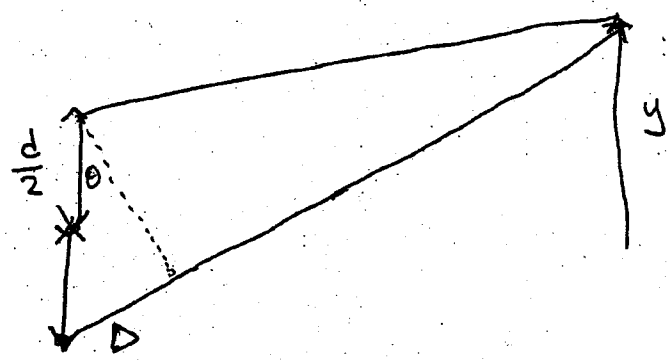
   i. If there is no current through the solenoid, find the first position (above the center-line) on an observation wall D away that exhibits a maximum intensity.

   ii. Now, the current is turned on and a field B_0 is generated. How big should this field be so that this point becomes an intensity minimum?



SOLUTION 6

Without the solenoid, we only care about the path length difference. We use the usual far field formula



where the path difference Δ is just $d\sin\theta$

$$\Delta = \frac{dy}{D}$$

The first maximum is then add $\quad K\Delta = 2\pi$

$$\text{or} \quad y = \frac{2\pi D k}{dp}$$

If we add a magnetic field, we pick up a Aharonov-Bohm phase factor of

$$\Theta \equiv \frac{e\Phi_M}{\hbar c}, \quad \Phi_M = \pi R^2 B_0$$

To get a minimum, we need

$$\Theta = \pi, \Rightarrow B_0 = \frac{\hbar c}{eR^2}$$

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SECTION 4:

PROBLEM 7.

7. Two equal containers, each of volume V, contain ideal gases at temperature T, pressure p. In container 1, the gas consists of N_1 molecules of gas α and M_1 of gas β in container 2, N_2 and M_2 respectively - note that $N_1 + M_1 = N_2 + M_2$. Derive an expression for the entropy of mixing, i.e. the entropy gain obtained by allowing the containers to freely mix. Evaluate the two limiting cases (1) $N_1 = N_2$ and (2) $N_1 = M_2$.

SOLUTION 7.

Solution

The relevant part of the ideal gas partition function for the unmixed and mixed cases are

unmixed: $Z = Z_1 Z_2$

$$Z_1 \sim \frac{V^{N_1}}{N_1!}\frac{V^{M_1}}{M_1!} \qquad Z_2 = \frac{V^{N_2}}{N_2!}\frac{V^{M_2}}{M_2!}$$

$$S \sim -\frac{\partial F}{\partial T} \simeq k N_1 \ln \frac{V}{(N_1 \lambda_\alpha^3)} + k M_1 \ln \frac{V}{(M_1 \lambda_\beta^3)}$$

$$+ k N_2 \ln \frac{V}{(N_2 \lambda_\alpha^3)} + k M_2 \ln \frac{V}{(M_2 \lambda_\beta^3)}$$

after mixing (volume is now $2V$), where λ_α and λ_β are the thermal wavelengths for the gases α and β,

$$S' = k(N_1 + N_2) \ln \frac{2V}{(N_1+N_2)} \qquad \lambda_\alpha = \text{const.} \frac{\hbar}{\sqrt{2 m_\alpha k_B T}}$$

$$+ k(M_1 + M_2) \ln \frac{2V}{(M_1+M_2)}$$

So

$$\Delta S = S' - S = k\left[N_1 \ln \frac{2N_1}{(N_1+N_2)} + N_2 \ln \frac{2N_2}{(N_1+N_2)} + M_1 \ln \frac{2M_1}{(M_1+M_2)} + M_2 \ln \frac{2M_2}{(M_1+M_2)} \right]$$

if $N_1 = N_2 = \frac{1}{2}(N_1+N_2)$ $\Delta S = 0$: if $M_1 = N_2 = 0$ $\Delta S = 2N_1 k \ln 2$

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SECTION 4:

PROBLEM 8.

8. A spectral line with wavelength λ from a hot gas is observed by a spectrometer. Find an expression (in term of the temperature T and the mass of the gas molecules m) for the Doppler broadening of the line, i.e. the root mean-square line-width caused by observing an ensemble of molecules with different velocities.

Solution: 8

The Doppler shift of a line due to motion along the line of sight is (for small speeds)

$$\frac{\delta\lambda}{\lambda} = \frac{v_x}{c}$$

The distribution of velocities is Maxwellian, so the variation

$$\langle(\delta\lambda)^2\rangle = \lambda^2 \frac{\int \frac{v_x^2}{c^2} e^{-\frac{1}{2}mv_x^2/KT}\, dv_x}{\int e^{-1/2\, mv_x^2/KT}\, dv_x}$$

This immediately gives

$$\langle(\delta\lambda)^2\rangle^{1/2} \simeq \lambda\sqrt{\frac{KT}{mc^2}}$$

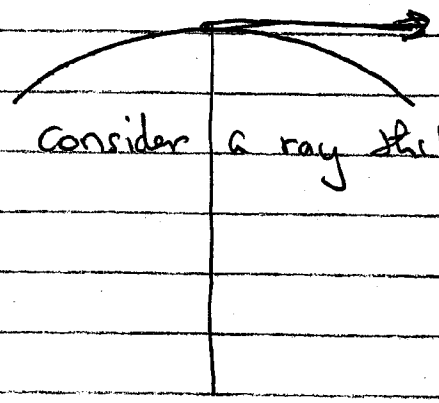
SECTION 5:

PROBLEM 9.

9. The index of refraction of a planet's atmosphere decreases with height h above the surface according to the law $n = n_0 - \alpha h$. The radius of the planet is R. Find at which height h_0 there exists a natural optical waveguide that supports the propagation of light at a constant-height orbit around the planet. (Hint: Consider a small portion of the orbit and use Snell's law of refraction).

SOLUTION 9

Consider an orbit at height h_0, which corresponds to a circular orbit at radius $R_0 = R + h_0$



Consider a ray that moves horizontally. after some distance δl, it will see a change in index of refraction

$$dn = -\alpha R_0 \left[\frac{(\delta l)^2}{2R_0^2} \right]$$

$$= -\alpha R_0 \frac{(\delta\theta)^2}{2}$$

The condition for the orbit is that this change in index of refraction gives rise to total reflection of the ray. This condition is

$$\frac{-\alpha h_0 + n_0 + dn}{n_0 - \alpha h_0} = \sin\left(\frac{\pi}{2} - \delta\theta\right)$$

or $$1 + \frac{dn}{(n_0 - \alpha h_0)} = 1 - \frac{(\delta\theta)^2}{2}$$

$$\frac{-\alpha R_0 (\delta\theta)^2}{2(n_0 - \alpha h_0)} = -\frac{(\delta\theta)^2}{2}$$

$$\Rightarrow \alpha R_0 = n_0 - \alpha h_0 \Rightarrow \boxed{h_0 = \frac{1}{\alpha}\left(\frac{n_0}{\alpha} - R\right)}$$

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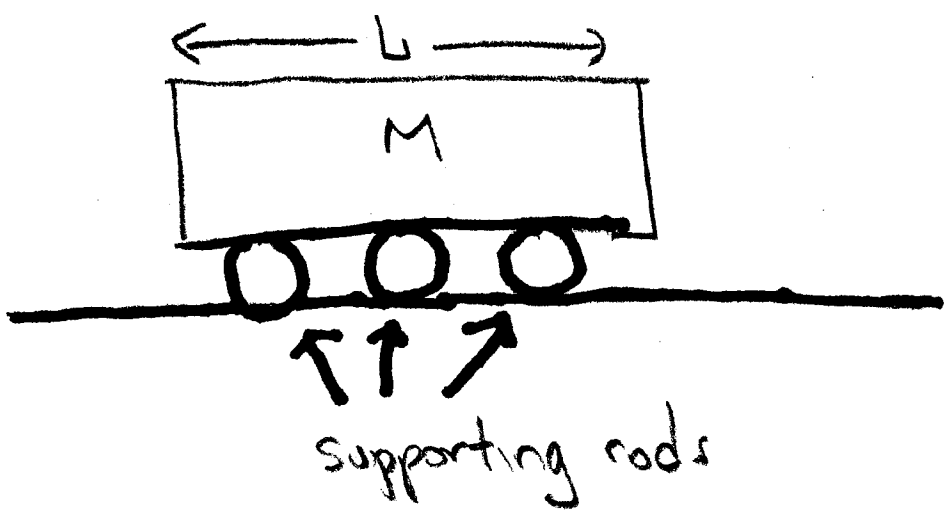
SECTION 5:

PROBLEM 10.

10. A rigid box of length L is supported by N horizontal bars located at positions x_i, $i = 1, N$. The bars and the box have the same depth, so the problem is two-dimensional.

    i. Above some critical N, the problem of finding the forces on each rod cannot be solved by assuming rigid rods. Find the critical N.

    ii. Assume the rods can be treated as Hookean springs with a purely vertical displacement. Find an expression for the tilt angle of the box.



Solution Prob. 10

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SOLUTION 10

i) If the rods are rigid, we have just two conditions; force balance and torque balance. The system is incomplete, then, for

$$N \geq 3$$

ii) now force due to i^{th} rod is $-K y_i$

Since the box is rigid, the lower edge is described by

$$y(x) = A + Bx$$

Force balance is thus

$$K \sum_i (A + Bx_i) - Mg = 0$$

torque balance is

$$-K \sum_i (A + Bx_i) x_i - \frac{MgL}{2} = 0$$

eliminating A, we derive the tilt angle $\approx B/L$ via

$$-KB\left[\frac{(\sum x_i)^2}{N} - \sum x_i^2\right] = Mg\left(\frac{L}{2} - \frac{\sum x_i}{N}\right)$$

$$\frac{B}{L} = \frac{Mg}{KL}\left[\frac{\frac{L}{2} - \frac{\sum x_i}{N}}{\frac{1}{N}(\sum x)^2 - \sum_i x_i^2}\right]$$

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PART II

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Section 1:	**Problem 11**_____	**Problem 12** _____
Section 2:	**Problem 13**_____	**Problem 14** _____
Section 3:	**Problem 15**_____	**Problem 16**_____
Section 4:	**Problem 17** _____	**Problem 18** _____
Section 5:	**Problem 19**_____	**Problem 20** _____

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SECTION 1.

PROBLEM 11.

11. The infinite series

$$S(z) = 1 + 2z + 3z^2 + 4z^3 + \ldots.$$

converges for $|z| < 1$. Find an analytic continuation of $S(z)$ to the largest possible domain in the complex z plane and describes its singularities.

Problem 11

The series

$$S(z) = 1 + 2z + 3z^2 + 4z^3 + \ldots$$

converges for $|z| < 1$. Find the analytic continuation of $S(z)$ to the largest possible domain in the complex z plane and describe its singularities.

Solution 11

We compute

$$S(z) = \frac{d}{dz}(1 + z + z^2 + z^3 + z^4 + \ldots) = \frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}.$$

This defines the analytic continuation to the whole z plane, and there is a double pole at $z = 1$ with residue zero.

Solution Prob. 11

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SECTION 1.

PROBLEM 12.

12. Let R be the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$. Find the eigenvalues and eigenfunctions that satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \lambda u = 0$$

within R with the boundary condition $u = 0$ on its boundary.

Problem 12

Let $\mathcal{R}$ be the rectangle $0 \le x \le 2$, $0 \le y \le 1$. Find the eigenvalues and eigenfunctions that satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \lambda u = 0$$

in $\mathcal{R}$ with the boundary condition $u = 0$ on the boundary of $\mathcal{R}$.

Solution 12

Separate variables, so that

$$u(x, y) = f(x) \sin n\pi y$$

with integer $n \ge 1$ and

$$f'' + f' + (\lambda - n^2\pi^2) f = 0.$$

The possible solutions are linear combinations of $e^{-x/2} e^{\pm irx}$, where $r = \sqrt{\lambda - 1/4 - n^2\pi^2}$. The boundary conditions require

$$f(x) = e^{-x/2} \sin rx$$

and $\sin 2r = 0$, so $r = m\pi/2$ for some integer $m \ge 0$. This means that

$$\lambda - 1/4 - n^2\pi^2 = m^2\pi^2/4$$

or

$$\lambda = 1/4 + n^2\pi^2 + m^2\pi^2/4.$$

The corresponding eigenfunction is

$$u(x, y) = e^{-x/2} \sin(m\pi x/2) \sin(n\pi y).$$

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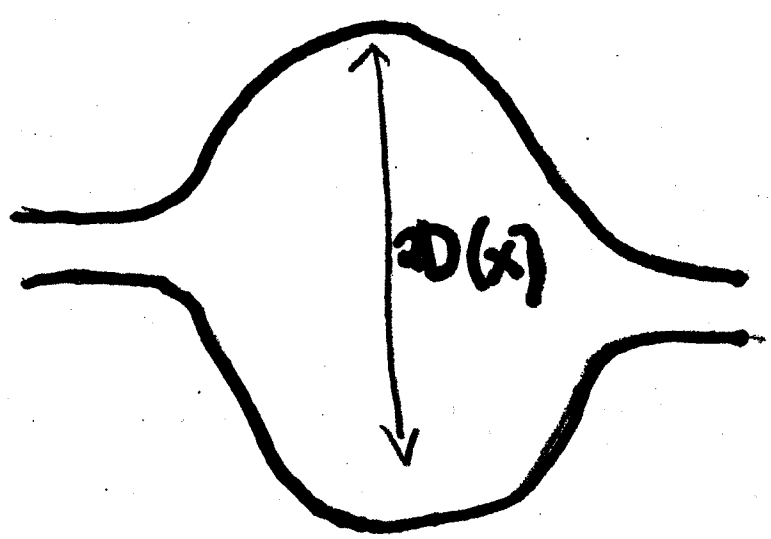
SECTION 2:

PROBLEM 13

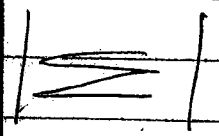
13. Consider the "mechanical mirror" shown in the figure below. What is the condition for the containment of particles by the mirror, in terms of the particle's energy and $\hat{x}$ component of its velocity? Assume throughout that $D(x)$ varies slowly. Do this problem by following the steps below

    i. Derive the adiabatic invariant for a particle bouncing between plates at $\pm D$

    ii. Use the result from i. to determine v_x^2 and the condition for trapping. Express your answer in terms of the particle's position when injected into the mirror and its initial perpendicular velocity.



Solution 13

i) 

$$2\pi \underset{\text{action}}{I} = \int_{-D}^{D} (m v_\perp)\,dy + \int_{D}^{-D} (-m v_\perp)\,dy$$

$$= 4mDv_\perp$$

so $I = \dfrac{2Dmv_\perp}{\pi}$ is the adiabatic invariant

(ii) now $D = D(x)$

$$E = \tfrac{1}{2}m\left(v_\perp^2 + v_\parallel^2\right)$$, conserved as no work done on particle

$$= \frac{1}{2}m\left(\frac{\pi^2 I^2}{4D^2 m^2} + v_\parallel^2\right)$$

so $$v_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x)^2}\frac{I^2}{m^2}$$

so if $v_\parallel^2(L/2) \leq 0 \Rightarrow$ particle confined.

so need $\left.\dfrac{\pi^2 I^2}{4D(x)^2 m^2}\right|_{x=\pm L/2} > \dfrac{2F}{m}$

for confinement of the particle

Now $I = \dfrac{2}{\pi} D(x_0)\, m v_{\perp 0}$

d separation at injection point

so $\left(\dfrac{D(x_0)}{D(L/2)}\right)^2 v_{\perp 0}^2 > 2F/m$

for confinement.

SECTION 2

PROBLEM 14.

14. Consider a free asymmetric top with $I_3 > I_2 > I_1$.

    i. Show that energy and total angular momentum are conserved by the equations of motion.

    ii. Assume the top is set spinning about the axis corresponding to I_2. Describe qualitatively what happens.

    iii. Describe the ultimate state of the phenomenon described in ii. Why does the linearized theory break down?

SOLUTION 14

Solution Prob. 14

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2.) $\frac{\partial \underline{L}}{\partial t} + \underline{\Omega} \times \underline{L} = \underline{f}_{ext}$

$\underline{f}_{ext} = 0$, so energy conserved

$\underline{L} \cdot \left(\frac{\partial \underline{L}}{\partial t} + \underline{\Omega} \times \underline{L} \right) = 0$

$\therefore \frac{\partial \underline{L}^2}{\partial t} = 0$ $\qquad \underline{L}^2 = \text{const.}$

so $\underline{L}^2 = L_1^2 + L_2^2 + L_3^2$

$$E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

2.) $\underline{\Omega} = \begin{pmatrix} \delta\Omega_1 \\ \Omega_2 + \delta\Omega_2 \\ \delta\Omega_3 \end{pmatrix}$ $\qquad \delta \Rightarrow$ perturbation

$I_1 \frac{d}{dt} \delta\Omega_1 + (I_3 - I_2)\Omega_2 \delta\Omega_3$

$I_2 \frac{d}{dt}(\Omega_2 + \delta\Omega_2) + \cancel{(I_1 - I_3)\delta\Omega_1 \delta\Omega_3}$ h.o.

$= 0$

$I_3 \frac{d}{dt}(\delta\Omega_3) + (I_2 - I_1)\Omega_2 \delta\Omega_1 = 0$

So

$$\delta\ddot{\Omega}_2 + \frac{(I_3 - I_1)(I_1 - I_2)}{I_1 I_3}\Omega_2^2\,\delta\Omega_1 = 0$$

for $I_3 > I_2 > I_1$,

$$\alpha^2 = \frac{(I_3 - I_2)(I_1 - I_2)}{I_1 I_3} > 0$$

instability at $\gamma = \alpha$.

iii) L is constrained to move on intersection of sphere of radius L and ellipsoid of axes $\sqrt{2T/I_{1,2,3}}$, aka Poinsot construction. Thus exponential growth must cease, and L move along intersection curves.

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SECTION 3:

PROBLEM15:

15. An electric dipole $\vec{p} = \hat{y}p_0$ is fixed at the origin of the (x, y, z, t) reference frame. Determine the magnetic field as seen in a frame (x', y', z', t') that moves with velocity $\hat{x}v_0$ with respect to the original frame. Assume the frames are coincident at time $t = t' = 0$. Express your answer in terms of the coordinates of the primed frame.

SOLUTION 15

Solution Prob. 15

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(3) soln

$$\Phi = \frac{\underset{\sim}{P} \cdot \underset{\sim}{r}}{r^3} = \frac{P_y y}{r^3} \quad , \quad \underset{\sim}{A} = 0$$

$$\begin{pmatrix} \bar{\Phi} \\ \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & & 0 \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} \Phi \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{A}_x = -\gamma\beta\Phi = -\gamma \frac{v}{c} \frac{P_y y}{r^3}$$

$$\bar{y} = \bar{y} \quad , \quad r^2 = \bar{y}^2 + \bar{z}^2 + \gamma^2(\bar{x} + v\bar{t})^2$$

$$z = \bar{z}$$

$$x = (\bar{x} + v\bar{t})\gamma$$

$$\underset{\sim}{\bar{B}} = \bar{\nabla} \times \bar{A} = \left(\hat{y}\frac{\partial}{\partial \bar{y}} + \hat{z}\frac{\partial}{\partial \bar{z}} \right) \times \hat{x} \left(-\gamma \frac{v}{c} \frac{P_y \bar{y}}{r^3} \right)$$

SOLUTION 15.1

$$\bar{B} = +\hat{z}\frac{\gamma V}{c} P_y \left(\frac{1}{r^3} - \frac{3\bar{y}^2}{r^5} \right)$$

$$+\hat{y}\frac{\gamma V}{c} P_y \frac{3\bar{y}\bar{z}}{r^5}$$

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SECTION 3:

PROBLEM 16:

16. A cylindrical solenoid of radius a and length L is tightly wound with N turns. The solenoid is driven by a voltage source $V(t) = \mathrm{Re} V_0 e^{-i\omega t}$. Determine the time average power radiated byb the solenoid assuming that $a << L << c/\omega$.

6. A cylindrical solenoid of radius a and length L is tightly wound with N-turns. The solenoid is driven by the voltage source $V(t) = \mathrm{Re}\, V_0 e^{-i\omega t}$. Determine the time average power radiated by the solenoid assuming that $a \ll L \ll c/\omega$.

magnetic dipole radiation since $L \ll \frac{c}{\omega}$

$$\langle P \rangle = \frac{1}{3}\frac{\omega^4}{c^3}|\underset{\sim}{m}_\omega|^2$$

can use magnetostatic solution to find $\underset{\sim}{m}_\omega$ since $L \ll c/\omega$

$$\mathrm{Re}\, V_0 e^{-i\omega t} = \frac{1}{c} N \frac{d\Phi}{dt} = \frac{N}{c} \underbrace{A}_{\pi a^2} \frac{dB}{dt} = \frac{N}{c} A \frac{4\pi N}{cL}\frac{dI}{dt}$$

$a \ll L$

$$\mathrm{Re}\left(V_0 e^{-i\omega t}\right) = \frac{N^2 A 4\pi}{c^2 L}\mathrm{Re}\left(-i\omega I_\omega e^{-i\omega t}\right)$$

$$m_\omega = \frac{NAI_\omega}{c} = \frac{NA}{c}\frac{c^2 L V_0}{4\pi N^2 A(-i\omega)} = \frac{V_0 c L}{4\pi N(-i\omega)}$$

$$\langle P \rangle = \frac{1}{3}\frac{\omega^4}{c^3}\frac{|V_0|^2 c^2 L^2}{16\pi^2 N^2 \omega^2}$$

$$\langle P \rangle = \frac{1}{48\pi^2}\frac{\omega^2 L^2 |V_0|^2}{N^2 c}$$

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SECTION 4:

PROBLEM 17

17. An atom has spin S and orbital angular momentum L. Strong spin-orbit coupling ensures that the low-energy states of the system are eigenstate of $\vec{J}^2$ (with eigenvalue $J(J+1)\hbar^2$, where $\vec{J} = \vec{S} + \vec{L}$ is the total angular momentum. If a weak magnetic field is applied to this system, find the splitting of the a state with fixed J. Also, find the expectation value of the magnetization at temperature T.

SOLUTION 17

Solution Prob. 17

Dept. Exam Solutions SP04

The coupling to a magnetic field is always given by

$$H = -\vec{M}\cdot\vec{B} = -M_z B_z$$

$$\vec{M} = -\mu_B(\vec{L}+2\vec{S}) = \mu_B(\vec{J}+\vec{S})$$

In a state with fixed $\vec{J}^2$, the expectation value of the J_z operator is obviously given by m_j which runs from $-j$ to j by integers. Similarly, the expectation value of S_z is given by

$$\langle S_z\rangle = \frac{\langle \vec{S}\cdot\vec{J}\rangle}{\langle \vec{J}^2\rangle}\langle J_z\rangle$$

$$= \left\langle \frac{1}{2}\left(\vec{J}^2+\vec{S}^2-(\vec{J}-\vec{S})^2\right)\right\rangle \frac{m_j}{j(j+1)}$$

$$= \frac{j(j+1)+s(s+1)-L(L+1)}{2j(j+1)}\, m_j$$

Hence for small field the energy levels are

$$E[m_j] = +m_j\mu_B\left[1+\frac{j(j+1)+s(s+1)-L(L+1)}{2j(j+1)}\right]$$

$$\equiv +m_j g_j \mu_B$$

and

$\langle\vec{M}\rangle =$

SOLUTION 11.1

$$\frac{\sum_{m_j} m_j \mu_B g_j \, e^{-m_j \mu_B g_j / kT}}{\sum e^{-m_j \mu_B g_j / kT}}$$

$$= -g_j \mu_B \frac{d}{dx}\left(\ln \sum_{m_j} e^{-m_j x}\right)\Bigg|_{x = \mu_B g_j / kT}$$

$$= -g_j \mu_B \frac{d}{dx} \ln \left[e^{jx} \sum_{n=0}^{2j} e^{-nx} \right]$$

$$= -g_j \mu_B \frac{d}{dx} \ln \left[e^{jx} \frac{1 - e^{(-2j-1)x}}{1 - e^{-x}} \right]$$

$$= +g_j \mu_B \frac{d}{dx} \ln \frac{\sinh (j + 1/2)x}{\sinh \frac{x}{2}}$$

$$= g_j \mu_B \left[(j+1/2) \coth (j+1/2)x - \frac{1}{2} \coth \frac{x}{2} \right]$$

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SECTION 4:

PROBLEM 18.

18. a) Prove the quantum virial theorem which states that for a particle moving in one-dimension acted upon by potential $V(x) = ax^n$, the expectation value (in any stationary state) of the kinetic energy is $\frac{n}{2}$ times the expectation value of the potential energy (hint: consider $\frac{d}{dt} < xp >$)

b) Use this theorem to prove that the expectation value of x for a particle moving in the potential $V(x) = Fx$ for $x > 0$, $V(x) = \infty$ for $x < 0$ has the same value as the classical value for the average position for one orbit

$$\bar{x} = \frac{1}{T}\int_0^T x(t)dt$$

SOLUTION 18

Solution Prob. 18

Dept. Exam Solutions SP04

a) Consider

$$\frac{d}{dt}\langle xp\rangle = 0 \quad \text{(since state is stationary)}$$

$$= \langle [H, xp]\rangle$$

$$= \left\langle \left[\frac{p^2}{2m} + V(x), xp\right]\right\rangle$$

$$\Rightarrow \left\langle -\frac{p^2}{m} + \frac{\partial V(x)}{\partial x}\right\rangle = 0$$

for $V = x^n$, this obviously means

$$2\langle KE\rangle = n\langle PE\rangle$$

b) Here $n=1$ so if $V = Fx$

$$\langle PE\rangle = 2\langle KE\rangle$$

$$= \frac{2E}{3} \quad \text{(since } E = \langle PE\rangle + \langle KE\rangle\text{)}$$

$$\Rightarrow \langle x\rangle = \frac{2}{3}\frac{E}{F}$$

classically $x(t) = v_0 t - \frac{Ft^2}{2m}$, period $= 2v_0 m/F$, $E = \frac{1}{2}mv_0^2$

$$\langle \bar{x}\rangle = \frac{1}{T}\int_0^T dt\left(v_0 t - \frac{Ft^2}{2m}\right) = \frac{F}{2v_0 m}\left(\frac{1}{6}\frac{4v_0^3 m^2}{F^2}\right) = \frac{2}{3}\left(\frac{\frac{1}{2}mv_0^2}{F}\right)$$

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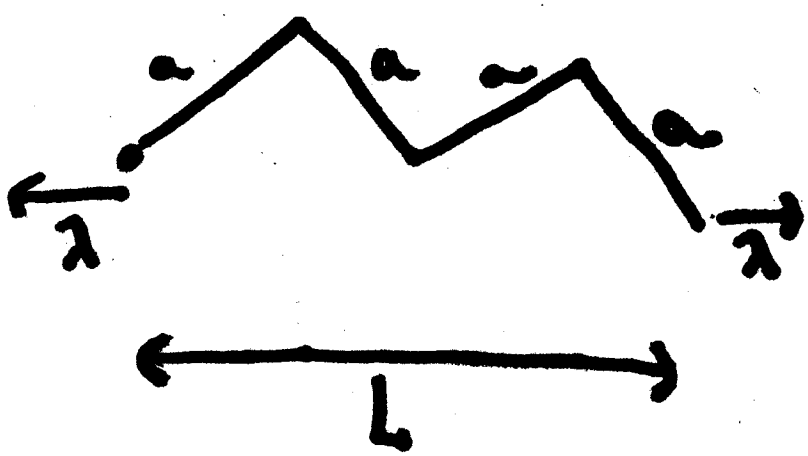
SECTION 5:

PROBLEM 19.

19. As shown in the figure, a chain molecule is taken to consist of N rigid rods each of length a. The units are joined so as to permit free rotation about the joints. Assuming the molecule is in contact with a reservoir at temperature T, derive the relationship connecting an applied tension λ between the ends of the molecule and the equilibrium distance L between those ends



SOLUTION 19

The partition function with tension λ can be written as

$$Z = \int \prod_i d\phi_i \sin\theta_i \, d\theta_i \; e^{-\lambda L/KT}$$

where $L = \sum_i a \cos\theta_i$

The expection value of L is clearly $-KT \dfrac{\partial \ln Z}{\partial \lambda}$

Since Z factors, it is easily found to be

$$Z = (2\pi)^N \left(\frac{2\sinh \frac{\lambda a}{KT}}{\frac{\lambda a}{KT}} \right)^N$$

Thus

$$\langle L \rangle = aN \left[\coth \frac{\lambda a}{KT} - \frac{KT}{\lambda a} \right]$$

20. Consider the following system defined on a square lattice of N sites. On each lattice site i, there is a discrete spin-like variable $s(i)$ which may take on one of q discrete values, e.g., $s(i) \in \{1, 2, ..., q\}$. The energetics of the system is given by the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \left[q\delta_{s(i),s(j)} - 1 \right]$$

such that if the nearest neighbor spins (indicated by $\langle i, j \rangle$ belong to the same state, there is an energy gain of $-(q-1)J$, while if the nearest neighbor states are different, there is an energy cost of $+J$. This model is known as the q-state Potts model; it is a generalization of the Ising model for $q > 2$. Here you are asked to provide a mean-field solution to the Potts model for $q = 3$.

(a) Let the fraction of sites belonging to state s be x_s. Find the mixing entropy $S(x_1, x_2, x_3)$ by computing first the number of configurations Ω that the N sites have N_1 sites in state 1, N_2 sites in state 2, and N_3 sites in state 3.

[Hint: You may assume that N_1, N_2, N_3 all much larger than 1 and use the Stirling approximation, $\ln(x!) \approx x\ln(x) - x$ valid for large x.]

(b) Write down an expression for the average energy $E(x_1, x_2, x_3)$ assuming that each site may belong to state s with probability x_s independently of the states of the other sites.

(c) Anticipating spontaneous symmetry breaking at low temperatures, one can describe the fraction x_s's in terms of a single order parameter m (with $0 \leq m \leq 1$) such that when $m = 0$, the x_s's are equal

(corresponding to the lack of ordering), and when $m = 1$, one of the x's, e.g., x_1 becomes 1, while the other two becomes 0. Find the linear functions $x_s(m) = a_s m + b_s$ satisfying the above requirements.

(d) Using $x_s(m)$ found in part (c) for the entropy S and average energy E obtained earlier to show that the *mean-field* free energy $F(m) \equiv E - TS$ is of the form

$$\frac{F(m)}{Nk_BT} \approx c_0 + \frac{T - T_0}{T} m^2 - \frac{1}{3} m^3 + \frac{1}{2} m^4$$

for small m, with c_0 being an unimportant constant, and T_0 being a characteristic temperature scale set by J/k_B. In what sense is what we have done the "mean-field approximation"? [`Hint:` $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$] .

Soln: 3-State Potts model

(a) fraction in state s: $x_s = \frac{N_s}{N}$; $\sum_s x_s = 1$

Config: $\Omega = \frac{N!}{N_1!\,N_2!\,N_3!} = \frac{N!}{(Nx_1)!\,(Nx_2)!\,(Nx_3)!}$

$$\ln\Omega \cong N\ln N - \cancel{N} - [(Nx_1)\ln(Nx_1) - \cancel{Nx_1} + (Nx_2)\ln(Nx_2) - \cancel{Nx_2} + (Nx_3)\ln(Nx_3) - \cancel{Nx_3}]$$

$$= -[Nx_1\ln x_1 + Nx_2\ln x_2 + Nx_3\ln x_3]$$

entropy $S = k_B \ln\Omega$

$$\boxed{\frac{S}{Nk_B} = -\sum_s x_s \ln x_s}$$

Solution Prob. 20

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(b). on a square lattice, there are 4 nearest neighbor for each lattice point

$E = \frac{4}{2} N \cdot \varepsilon$, where ε is avg energy per bond.

$$\varepsilon = -J\,[x_1^2 + x_2^2 + x_3^2] + J\,[2x_1x_2 + 2x_1x_3 + 2x_2x_3].$$

(c). Find $x_s(m)$:

$x_1 = a_1 m + b_1$

$x_2 = x_3 = a_2 m + b_2$.

Since $x_1 = x_2 = x_3 = \frac{1}{3}$ when $m = 0$.

we have $b_1 = b_2 = \frac{1}{3}$.

$m = 1$: $x_2 = x_3 = 0$. $a_2 = -b_2 = -\frac{1}{3}$.

$x_1 = 1$ $a_1 = \frac{2}{3}$.

$\Rightarrow$ $\boxed{x_1(m) = \frac{1+2m}{3}}$ $\boxed{x_2(m) = x_3(m) = \frac{1-m}{3}}$.

(d). $\mathcal{E}(m) = -J\left[(2x_1^2 + 4x_2^2) - (4x_1x_2 + 2x_2^2)\right]$

$= -2J(x_1 - x_2)^2 = -2Jm^2$.

$\rightarrow$ $\boxed{E(m) = -4NJm^2}$

$$-\frac{S(m)}{Nk_B} = \frac{1+2m}{3}\ln\left(\frac{1+2m}{3}\right) + 2\cdot\frac{1-m}{3}\cdot\ln\left(\frac{1-m}{3}\right)$$

$$= -\ln 3 + \frac{1}{3}\left[(1+2m)\cdot\left(2m - \frac{(2m)^2}{2} + \frac{(2m)^3}{3} - \frac{(2m)^4}{4}\cdots\right)\right.$$

$$\left. + (1-m)\cdot\left(-m - \frac{m^2}{2} - \frac{m^3}{3} - \frac{m^4}{4}\cdots\right)\right]$$

$$\boxed{-\frac{S(m)}{Nk_B} = -\ln 3 + m^2 - \frac{1}{3}m^3 + \frac{1}{2}m^4}$$

free energy $F(m) = E(m) - TS(m)$.

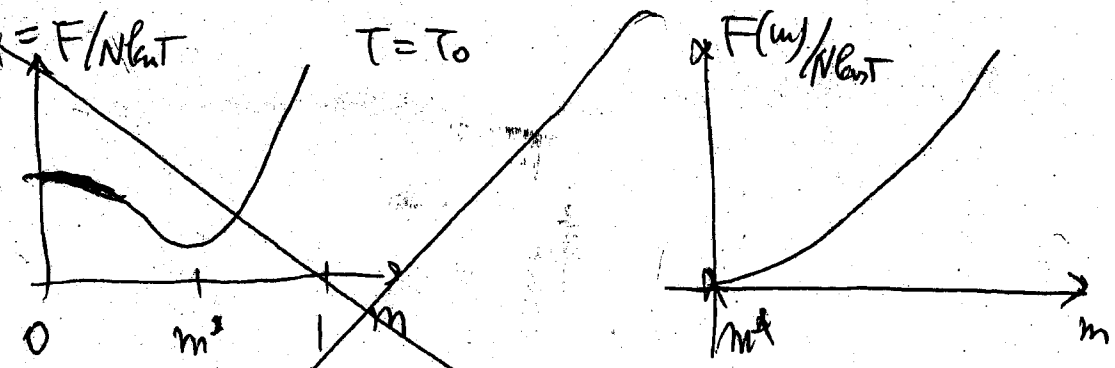
$$\frac{F(m)}{Nk_BT} = -\frac{4J}{k_BT}m^2 + \left[-\ln 3 + m^2 - \frac{1}{3}m^3 + \frac{1}{2}m^4\right]$$

$$= -\ln 3 + \frac{T-T_0}{T}m^2 - \frac{1}{3}m^3 + \frac{1}{2}m^4$$

where $\boxed{T_0 = 4J/k_B}$.

Mean-field approximation: We assumed that the the same probabilities x_s are used for each site independently of the states of the other site. This is the essence of the mean-field approximation.

(e) $\psi = F/Nk_BT$ $T = T_0$



$$\psi''(m) = \frac{T-T_0}{T} - 2m + 6m^2$$

$T = T_0$. $\boxed{m^* = 1/3}$.

$T \to \infty$. no sol'n for $\psi''(m) = 0$. $\boxed{m^* = 0}$