Departmental Written Exam

Fall 2003

(Exam Only)

ID Number _____ PART I Score _____

PHYSICS DEPARTMENTAL EXAM – FALL 2003

DEPARTMENT OF PHYSICS
DEPARTMENTAL EXAMINATION –FALL 2003

PART I

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. **Please indicate with a "check" which of the (7) questions you wish to be graded below:**

Section 1:	**Problem 1**_____	**Problem 2** _____
Section 2:	**Problem 3**_____	**Problem 4** _____
Section 3:	**Problem 5**_____	**Problem 6**_____
Section 4:	**Problem 7** _____	**Problem 8** _____
Section 5:	**Problem 9**_____	**Problem 10** _____

PART I - Undergraduate

Fundamental constants you may need

$c = 3.0 \times 10^{8}$ m/sec

$\hbar = 1.1 \times 10^{-34}$ J sec

$k = 1.4 \times 10^{-23}$ J/K

$\sigma = 5.7 \times 10^{-8}$ W/m^2 K^4

$G = 6.7 \times 10^{-11}$ m^3 /kg sec^2

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SECTION 1:

Problem 1.

A homogeneous ball of mass M and radius R is struck impulsively at its center, causing it to go from rest to a horizontal speed of v_0. Assuming a constant coefficient of friction μ, find the distance traveled by the ball before it begins to roll without slipping.

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SECTION 1:

Problem 2.

A cannon at (north) latitude θ is pointed due east and fires a projectile with initial speed v_0 and elevation angle α. Find the change in range of the projectile due to the earth's rotation, to linear order in the rotation rate Ω.

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SECTION 2:

Problem 3.

Consider a metallic wire of length L and radius a, which runs parallel to a conducting plane a distance of D away; assume that $L >> D >> a$. Find an approximate expression for the capacitance of this system.

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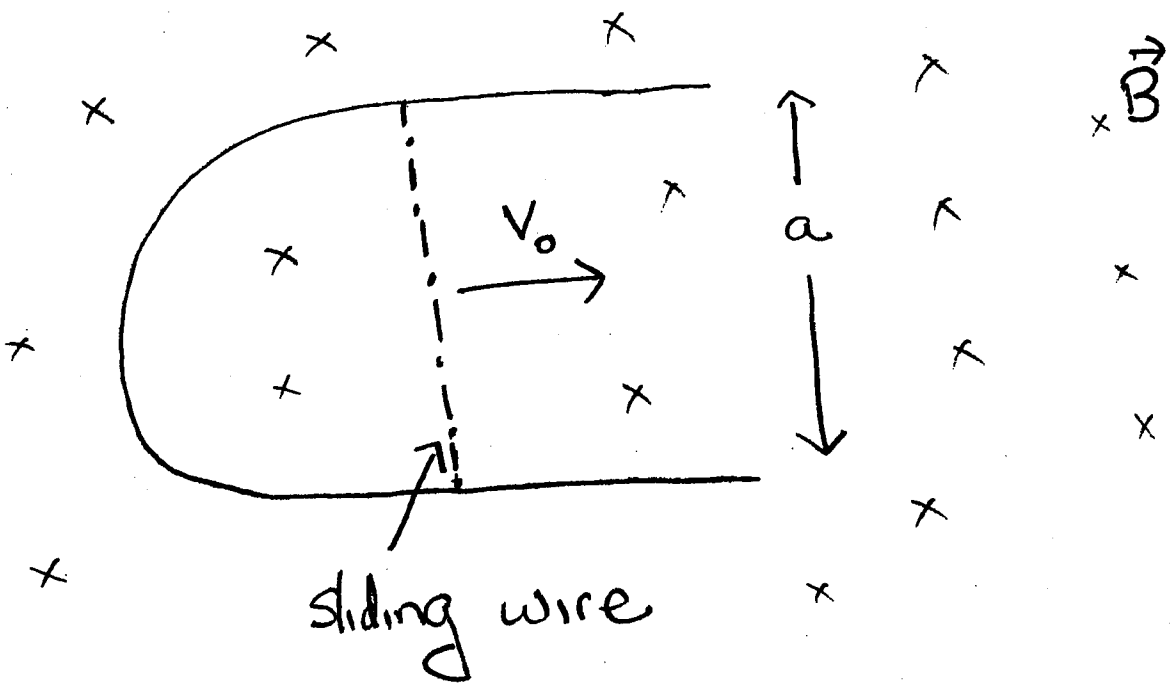
SECTION 2

PROBLEM 4.

A metallic "horseshoe-shaped" loop of width a is attached to a sliding wire, as shown in the figure. A uniform magnetic field is perpendicular to the plane of the system. If the wire has mass M and resistance R, how far will it travel if it is given an initial speed v_0?



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SECTION 3:

PROBLEM 5:

A particle with charge q and mass m is bound by a three-dimensional harmonic oscillator potential

$$V(x,y,z) = \frac{1}{2}k(x^2+y^2+z^2)$$

a. What is the degeneracy of the energy eigenstate with $E = \frac{7}{2}\hbar\sqrt{k/m}$?

b. Now, a (non-constant) electric field in the $\hat{x}$ direction is applied to this system. How much of the above degeneracy is lifted?

c. Now assume that the electric field is given by $\vec{E} = \hat{x}E_0 x e^{-Ax^2}$. Find the change in the energy of the ground state to lowest non-vanishing order in E_0.

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SECTION 3:

PROBLEM 6:

A pair of spin 1/2 particles is prepared in the entangled state

$$|\psi> = \frac{1}{\sqrt{2}}(|\uparrow\uparrow> + |\downarrow\downarrow>)$$

where the arrows refer to the $\hat{z}$ component of the particles' spins.

a. The spin component of the first particle is measured along an axis $\hat{u}_a$ making angles θ_a, ϕ_a with the $\hat{z}$ axis. What is the probability of obtaining $+\hbar/2$?

b. After the first measurement along $\hat{u}_a$ yields spin up, the second particle's spin component is measured along $\hat{u}_b$ with angles θ_b, ϕ_b. What is the probability of obtaining $+\hbar/2$?

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SECTION 4:

Problem 7.

To heat a cup of water of volume $250cm^3$, a heater at $T = 120°C$ is immersed therein. Assume that there is no heat loss to the cup and that the heating element remains at $120°C$. What is the change in the system's entropy as the water temperature increases from $20°C$ to $T = 50°C$? (note: the heat capacity of the water can be taken to be temperature-independent and equal $4190J/K \cdot kg$).

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SECTION 4:

Problem 8.

A zipper has N links each of which can be in two possible states - closed with energy 0 and open with energy ϵ. The zipper can only open from the left, i.e. a link j can be open only if all links to its left with indices i, $1 \leq i < j$ are open.

a. Find the partition function of this system.

b. Find the average number of open links

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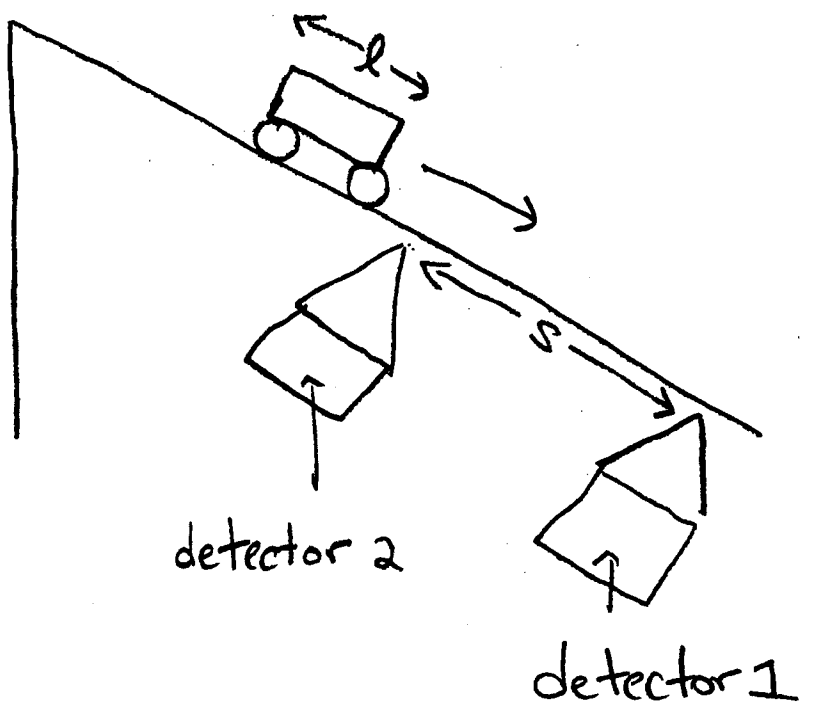
SECTION 5:

Problem 9.

Consider a laboratory experiment which aims to measure the acceleration of a cart down an inclined plane (see figure). The cart has length ℓ and is detected by two photocells separated by distance s along the slope, Assuming that one can neglect the acceleration during the periods of time that it takes the cart to pass by each of the detectors (t_1 and t_2), one can easily see that an estimate of the acceleration is

$$a = \frac{(\ell/t_1)^2 - (\ell/t_2)^2}{2s}$$

Assuming that the lengths are determined to be $\ell = 5.00 \pm .05$ cm, $s = 100.00 \pm .2$ cm, the times are $t_1 = 0.054$ sec and $t_2 = 0.031$ sec and finally that times can be measured with an absolute accuracy of 0.001 seconds, find the percentage uncertainty in the acceleration measurement. Would pushing harder make the error larger or smaller?



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SECTION 5:

Problem 10:

Assume that the total amount of energy radiated by the sun is 4×10^{26} J/sec.

a. The surface temperature of the sun is $5800^{o}K$. Find the radius of the sun.

b Assume that the earth is 1.5×10^{11} m from the sun and is in thermal equilibrium with it, i.e. re-radiates as much thermal energy as it receives. If the radius of the earth is 7×10^{8} m, find the surface temperature of the earth.

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PART II

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Please indicate with a "check" which of the (7) questions you wish to be graded below:

Section 1:	**Problem 11**_____	**Problem 12** _____
Section 2:	**Problem 13**_____	**Problem 14** _____
Section 3:	**Problem 15**_____	**Problem 16**_____
Section 4:	**Problem 17** _____	**Problem 18** _____
Section 5:	**Problem 19**_____	**Problem 20** _____

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SECTION 1:

Problem 11.

A point mass slides without friction inside a surface of revolution described by $z(r) = \alpha \sin r/R$. The mass is subject to a uniform gravitational field $-g\hat{z}$.

a. Construct the Lagrangian in terms of the coordinates r and ϕ.

b. Find the horizontal circular orbits.

c. Which of these orbits is linearly stable under perturbations applied transverse to the direction of motion? What is the small oscillations frequency around the stable orbits?

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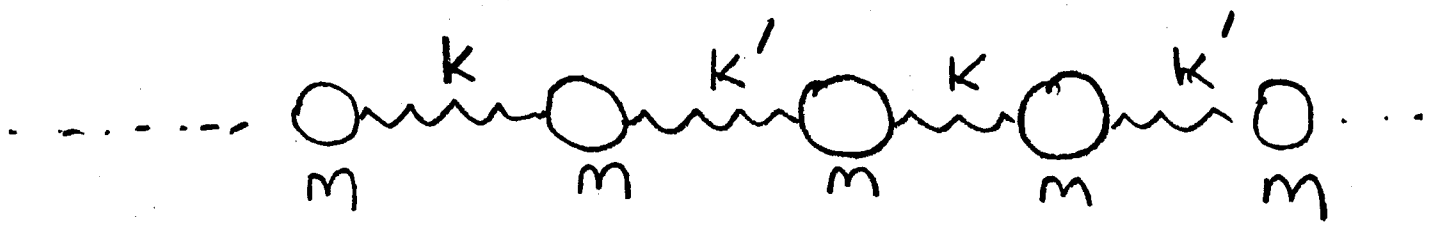
SECTION 1:

Problem 12.

Consider waves propagating on an infinite spring-mass system (see figure below) where the masses m are equal but the spring constants alternate in strength. The equilibrium distance between the masses is a. Find the dispersion relation and explain physically what happens in the limit $k >> k'$..



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SECTION 2:

Problem 13.

A plasma that is characterized by the frequency dependent dielectric constant $\epsilon(\omega) = 1 - \frac{\omega_p^3}{\omega^3}$ occupies the half-space $z > 0$. The half-space $z < 0$ is vacuum. An electromagnetic wave

$$\vec{E}(\vec{r},t) = Re\ E_0\hat{x}e^{ikz-i\omega t}$$
$$\vec{B}(\vec{r},t) = Re\ B_0\hat{y}e^{ikz-i\omega t}$$

with $\omega < \omega_p$ is incident on the plasma; here $k = \omega/c$ and $E_0 = B_0$. Determine the reflection coefficient and give a physical interpretation of your answer.

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SECTION 2

PROBLEM 14.

A superconducting slab occupies the region $z < 0$. A long straight wire is parallel to the $\hat{y}$ axis and lies at at $x = 0$ and $z = h$; assume that the wire carries current I in the $+\hat{y}$ direction.

a. Determine the magnetic field in the region above the superconductor, taking into account the fact that the superconductor excludes any magnetic field from its interior.

b. Determine the surface current on the superconductor.

c. Determine the force per unit length on the wire, stating explicitly whether the wire is attracted to or repelled from the surface.

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SECTION 3:

PROBLEM15:

Consider the anti-linear time reversal operator

$$\hat{T} = Ke^{-i\pi S_y/\hbar}$$

where K is complex conjugation $K[\psi] = \psi^*$ and S_y is the $\hat{y}$ component of the spin operator (for arbitrary spin s, not necessarily 1/2).

a. Which of these systems is time-reveral invariant, i.e. has $[H, \hat{T}] = 0$? Spinless charged particle in an electric field, spin 1/2 particle in a central field with spin-orbit coupling, spinless charged particle in a constant magnetic field?

b. What is $\hat{T}^2$ equal to?

c. Show that $\hat{T}[\psi_n]$ is orthogonal to ψ_n for a spin-1/2 particle in an energy eigenstate ψ_n. What does this imply about the energy spectrum?

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SECTION 3:

PROBLEM 16:

Use the Born approximation to find an approximate expression for the differential and total cross sections of a spinless particle of mass m scattering off of a screened Coulomb potential

$$V(r) = -\frac{Q}{r}e^{-r/a}$$

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SECTION 4:

Problem 17.

Consider a system of $N \gg 1$ impenetrable beads of mass m and diameter a on a semi-infinite frictionless wire. Let the coordinates of the particles be $x_1 > 0$, $x_2 > x_1 + a$, ..., $x_N > x_{N-1} + a$. The system is in thermal equilibrium at a temperature T, and the rightmost bead is subjected to a constant force F towards the origin.

(a) Write down the Hamilton of the system in terms of the momenta and positions of the particles. For convenience, take the zero of the potential energy U to be that at the minimal position of the rightmost bead, $x_N = (N-1) \cdot a$.

(b) Find the partition function Z of this system in terms of N, $\beta = 1/kT$, and F. Note that since the beads cannot pass through one another, they should be treated as distinguishable particles.
[Hint: It may be useful to use the 'displacement' coordinates $u_i = x_{i+1} - x_i$ to perform some of the integrals.]
[Gaussian integral: $\int_{-\infty}^{+\infty} dy\, e^{-y^2/2\sigma^2} = (2\pi\sigma^2)^{1/2}$.]

(c) Calculate the average energy and specific heat of the system. Find the average position of the rightmost bead $\langle x_N \rangle$ as a function of temperature. For what temperatures can $\langle x_N \rangle$ be much larger than the minimal position $(N-1) \cdot a$?

(d) For a one-dimensional system, the "pressure" can be simply taken to be the force F applied to the rightmost particle. Find the equation of state $p(n, T)$, where $n = N/\langle x_N \rangle$ is the "density" of the system.

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SECTION 4:

Problem 18.

Consider an ultra-relativistic electron gas where the electron kinetic energies are typically much larger than the electron rest mass energy mc^2 and where interactions are negligible.

a. Find the Fermi energy in terms of the density N/V where N is the number of electrons in volume V.

b. Find the energy of the ground state.

c. Some white dwarf stars can be thought of as a sphere of radius R composed of ionized Hydrogen gas in which the electrons can be treated as being ultra-relativistic. Using the virial theorem to relate the gravitational potential energy of the protons to the electronic kinetic energy, find the value of N.

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SECTION 5:

Problem 19.

Evaluate the integral

$$I(\alpha) = 2\int_0^\infty dx\, \frac{x^{2\alpha-1}}{1+x^2}$$

for $0 \leq Re\,\alpha \leq 1$

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SECTION 5.:

Problem 20.

Solve for the function $u(x)$:

$$P\int_{-\infty}^{+\infty} \frac{dx}{\pi}\frac{u(x)}{x-t} = \frac{1}{1+t^2}$$

where P denotes a principal value integral.

Fall 2003 Departmental Exam Solutions

N.J. Miller
Friday, August 12, 2005

Problem #1

The requirement for rolling without slipping is $\omega R = v_{x,cm}$. We write down the equations of motions for x_{cm}, y_{cm}, and θ_{cm} and solve. This problem is essentially the same as one done in a previous year. Applying all forces and torques, we get for y, x, and θ motions:

$$\begin{aligned} -Mg + N &= Ma_y = 0 \\ -\mu N &= Ma_x \\ R\mu N &= I\alpha = \frac{2}{5}MR^2\alpha \end{aligned}$$

Solving for v_x, we get

$$v_x = v_0 - \mu g t$$

Solving for ω, we get

$$\omega = \frac{5\mu g}{2R}t$$

Setting $\omega R = v_x$ at time T, we get

$$\begin{aligned} \frac{5\mu g}{2}T &= v_0 - \mu g T \\ \Rightarrow T &= \frac{2v_0}{7\mu g} \end{aligned}$$

To find the distance it rolls in this period, we must find the position of the center of mass at this time

$$x_{cm} = -\frac{1}{2}\mu g t^2 + v_0 t$$

at time T, this is

$$x(T) = \frac{12}{49}\frac{v_0^2}{\mu g}$$

Problem #3

For this problem we want to find the difference in potential between the plane and the wire. The potential for each is due to the potential of the charge on the wire and the potential of the image wire. Using Gauss's Law, for an infinite cylinder with charge plastered on it, we get

$$E_{\rho_1} 2\pi\rho_1 L = 4\pi\lambda L$$

Therefore

$$\vec{E} = \frac{2\lambda}{\rho_1}\hat{\rho}_1$$

Which gives ther potential as

$$\phi = 2\lambda \log \rho_1$$

For the image wire, we stick charge of the opposite sign on, for

$$\phi = -2\lambda \log \rho_2$$

Note that ρ_1 and ρ_2 refer to distance we are from the specific wire. The potential at the plane is just 0. The two terms just cancel each other. The potential at the surface of the real wire is

$$\phi_{wire} = 2\lambda \log a - 2\lambda \log D$$

where since $D \gg a$ we can assume the surface of the real wire is a distance D away (approximating a line charge for that part). Therefore the difference in potential between the plane and the wire is

$$\Delta\phi = \frac{2Q}{L} \log \frac{D}{a}$$

Since $Q = C\Delta\phi$, we can solve for C and we get

$$C = \frac{L}{2}\left(\log \frac{D}{a}\right)^{-1}$$

If we instead tried to calculate the capacitance of the two wires (real and image), we would have gotten

$$C = \frac{L}{4}\left(\log \frac{D}{a}\right)^{-1}$$

The way we can see that the capacitance of the plane-wire system is doubled is that it has half the total energy (upper plane and lower plane have the same energy in wire-wire system). Since the charge on the wire is constant when switching to the plane-wire system (and difference in potential is halved), we see that the capacitance must be doubled.

Problem #4

When the wire moves, there will be an EMF produced, which creates a current. The magnetic field provides a force on the current slowing it down. The EMF is $EMF = -\frac{1}{c}\frac{d\Phi}{dt}$, where $\Phi = BA$, where B is the magnitude of the magentic field inside the loop and A is the area of the loop. $A = K + x(t)a$, whee K is the area of the horseshoe part, $x(t)$ is the distance from the start for the wire. Then $\frac{d\Phi}{dt} = B\frac{dA}{dt} = Bav(t)$. We therefore get

$$EMF = -\frac{Bav(t)}{c} = IR$$

and the force on the wire is

$$F = \frac{IaB}{c} = -\frac{B^2a^2}{c^2}\frac{v(t)}{R} = Ma$$

We can solve this equation for $x(t)$ as a function of time, with initial conditions $x(0) = 0$ and $v(0) = v_0$, giving

$$x(t) = -\frac{Rc^2}{B^2a^2}v_0 \exp\left(-\frac{B^2a^2}{Rc^2}t\right) + \frac{Rc^2}{B^2a^2}v_0$$

The distance the loop travels is $x(\infty) = \frac{Rc^2}{B^2a^2}v_0$

Problem #5

Part a)

$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 + \frac{p_y^2}{2m} + \frac{1}{2}ky^2 + \frac{p_z^2}{2m} + \frac{1}{2}kz^2$$

The solution to this is just the product of 3 one-dimensional harmonic oscillators, and the energy is the just the sum of the energies of these 3 oscillators. We therefore get $E = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})$. In order to get $E = \frac{7}{2}\hbar\omega$, we must have $n_x + n_y + n_z = 2$. There are 6 possibilities that allow this.We therefore have a degeneracy of 6 for this energy

Part b)

The change in potential energy is $q\Phi$, where $-\vec{\nabla}\Phi = \vec{E}$. Since $\vec{E}$ only points in the $\hat{x}$ directions, $\Phi = \Phi(x)$. The change in energy of the state is

$$< n_x n_y n_z | q\Phi | n_x n_y n_z > = \int d^3\vec{r}|\Psi|^2 q\Phi(x)$$

Since this integral depends only on x, the states $n_x n_y n_z$ of 0 2 0 and 0 0 2 will still have the same energy. Also 1 1 0 and 1 0 1 also will have the same energy. We therefore split the single energy (6 different states) state into 4 energies with the degeneracies as described in the previous sentence.

Part c)

The ground state of the harmonic oscillator is $N\exp(-\frac{m\omega}{2\hbar}x^2)$. If you don't remember this, you can derive it by knowing that $a = \frac{1}{\sqrt{2}}(\lambda x + \frac{ip}{\hbar\lambda})$, where $\lambda = \sqrt{\frac{m\omega}{\hbar}}$, and $a|0>=0$. Solving for the normalization (just need to integrate a gaussian which we can do), we get the ground state of the three-dimensional oscillator as

$$\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\exp\left(-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)\right)$$

If $\vec{E} = \hat{x}E_0 x\exp(-Ax^2)$, then $\Phi = \frac{E_0}{2A}\exp(-Ax^2)$, so the change in energy to the ground state is

$$\int dxdydz\left(\frac{m\omega}{\pi\hbar}\right)^{3/2}\exp\left(-\frac{m\omega}{\hbar}(x^2+y^2+z^2)\right)\frac{E_0}{2A}\exp(-Ax^2)$$

The y and z integrals can be done and just get rid of their normalization factor. We then get the change in energy as

$$\frac{E_0}{2A}\int dx\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\left(\frac{m\omega}{\hbar}+A\right)x^2\right) = \frac{E_0}{2A}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{\pi}{\frac{m\omega}{\hbar}+A}\right)^{1/2}$$

Problem #6

Part a)

It doesn't matter what ϕ_a is for this part of the problem. In spherical coordinates, ϕ_a measures the rotation about the z-axis (making a circle for $0 < \phi < 2\pi$ has the center of the circle as the z-axis for all ϕ). Therefore all we need to do is calculate the probability of being spin up when rotating about the y-axis by an angle θ_a. This requires the roation matrices. We calculate what each spin-1/2 particle rotates to and then multiple the 2 particles together. For 1-particle the new state will be

$$\begin{aligned}
\chi' &= \exp\left(i\,\theta_a\frac{S_y}{\hbar}\right)\chi = \exp\left(i\,\frac{\theta_a}{2}\sigma_y\right)\chi \\
&= \sum_{n=0}^{\infty}\frac{\left(i\,\frac{\theta_a}{2}\sigma_y\right)^n}{n!}\chi = \left(\mathbf{1}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\theta_a}{2}\right)^{2n}}{2n!} + i\,\sigma_y\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\theta_a}{2}\right)^{2n+1}}{(2n+1)!}\right)\chi \\
&= \left(\mathbf{1}\cos\frac{\theta_a}{2} + i\,\sigma_y\sin\frac{\theta_a}{2}\right)\chi \\
&= \begin{pmatrix} \cos\frac{\theta_a}{2} & \sin\frac{\theta_a}{2} \\ -\sin\frac{\theta_a}{2} & \cos\frac{\theta_a}{2}\end{pmatrix}\chi
\end{aligned}$$

Therefore $|\uparrow>$ goes to $\cos\frac{\theta_a}{2}|\uparrow> - \sin\frac{\theta_a}{2}\,|\downarrow>$ and $|\downarrow>$ goes to $\sin\frac{\theta_a}{2}|\uparrow> + \cos\frac{\theta_a}{2}\,|\downarrow>$. Therefore our new entangled state is now

$$
\begin{aligned}
\chi' &= \frac{1}{\sqrt{2}}[\left(\cos\frac{\theta_a}{2}|\uparrow>_1 - \sin\frac{\theta_a}{2}|\downarrow>_1\right)\left(\cos\frac{\theta_a}{2}|\uparrow>_2 - \sin\frac{\theta_a}{2}|\downarrow>_2\right) \\
&+ \left(\sin\frac{\theta_a}{2}|\uparrow>_2 + \cos\frac{\theta_a}{2}|\downarrow>_2\right)\left(\sin\frac{\theta_a}{2}|\uparrow>_2 + \cos\frac{\theta_a}{2}|\downarrow>_2\right)] \\
&= \frac{1}{\sqrt{2}}\left(|\uparrow>_1|\uparrow>_2 + |\downarrow>_1|\downarrow>_1\right)
\end{aligned}
$$

If we find the amplitude the first particle is spin up we get

$$
\frac{1}{\sqrt{2}}|\uparrow>
$$

Therefore the probability of spin up for the first particle is $\frac{1}{2}$

Part b)

As with the last case, it might be easier to create a similar problem with the same answer. In this case $\theta'_a = \theta_a$, $\phi'_a = 0$, $\theta'_b = \theta_b$, and $\phi'_b = \phi_b - \phi_a$. The new primed rotations have the same orientation with each other as with the original problem. The reason we are doing this is because we have already found the exact state the second particle is in after the measurement of spin up. The state is $|\uparrow>_1|\uparrow>_2$. In general if we make θ and ϕ, then S_u along that axis is

$$
\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\exp(-i\,\phi) \\ \sin\theta\exp(i\,\phi) & -\cos\theta \end{pmatrix}
$$

with

$$
\begin{aligned}
|\uparrow>_z &= \cos\frac{\theta}{2}\exp\left(i\,\frac{\phi}{2}\right)|\uparrow>_u - \sin\frac{\theta}{2}\exp\left(i\,\frac{\phi}{2}\right)|\downarrow>_u \\
|\downarrow>_z &= \sin\frac{\theta}{2}\exp\left(-i\,\frac{\phi}{2}\right)|\uparrow>_u + \cos\frac{\theta}{2}\exp\left(-i\,\frac{\phi}{2}\right)|\downarrow>_u
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|\uparrow\rangle_{u_a} &= \cos\frac{\theta_a}{2}|\uparrow\rangle_z + \sin\frac{\theta_a}{2}|\downarrow\rangle_z \\
&= \cos\frac{\theta_a}{2}\left(\cos\frac{\theta_b}{2}\exp\left(i\,\frac{\phi_b-\phi_a}{2}\right)|\uparrow>_{u_b} - \sin\frac{\theta_b}{2}\exp\left(i\,\frac{\phi_b-\phi_a}{2}\right)|\downarrow>_{u_b}\right) \\
&+ \sin\frac{\theta_a}{2}\left(\sin\frac{\theta_b}{2}\exp\left(-i\,\frac{\phi_b-\phi_a}{2}\right)|\uparrow>_{u_b} + \cos\frac{\theta_b}{2}\exp\left(-i\,\frac{\phi_b-\phi_a}{2}\right)|\downarrow>_{u_b}\right) \\
&= \left(\cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\exp\left(i\,\frac{\phi_b-\phi_a}{2}\right) + \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\exp\left(-i\,\frac{\phi_b-\phi_a}{2}\right)\right)|\uparrow>_{u_b} \\
&+ \left(\sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\exp\left(-i\,\frac{\phi_b-\phi_a}{2}\right) - \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\exp\left(i\,\frac{\phi_b-\phi_a}{2}\right)\right)|\downarrow>_{u_b}
\end{aligned}
$$

Plugging in, multipying the states of the two particles, simplifying, taking the amplitude of spin up for particle 2, and finding the probability it is in this state requires HUGE amounts of algebra (which makes it seem as if there is an easier way with less simplifcation or I am not seeing simple simplifications)

$$P = \frac{1}{4}\left(2 + \cos\left(\theta_a - \theta_b\right) + \cos\left(\theta_a + \theta_b\right) + 2\cos\left(\phi_b - \phi_a\right)\sin\theta_a \sin\theta_b\right)$$

When $\theta_a = \theta_b$ and $\phi_a = \phi_b$ we get 1 which is what we expect and if $\theta_a = \theta_b = \frac{\pi}{2}$ and $\phi_b - \phi_a = \pi$ we get 0 which is what we expect (assuming what I expect is correct)

Problem #7

The heat capacity of water is defined as $C = T\left(\frac{\partial S}{\partial T}\right)$. We can solve for ΔS and get $\Delta S = C \ln\left(\frac{T_f}{T_i}\right)$, where T is in kelvin. The density of water (at $20\,^\circ\mathrm{C}$) is $1\,\mathrm{g\,cm^{-3}}$, so the mass of the water is $250\,\mathrm{g} = 0.25\,\mathrm{kg}$. We therefore get the heat capacity as (they give specific heat capacity) $1047.5\frac{\mathrm{J}}{\mathrm{K}}$. The change in entropy is therefore

$$\Delta S = C \ln\left(\frac{T_f}{T_i}\right) = 1047.5\frac{\mathrm{J}}{\mathrm{K}} \ln\left(\frac{323}{293}\right) = 102.11\frac{\mathrm{J}}{\mathrm{K}}$$

Problem #8

Part a)

There are $N+1$ possible states. There are where $0, \ldots, N$ links are open. There is only one possibility for each number of open links, since every link to the left of a link must be open for it to be open. We therefore have the states specified as where the k left links are open. $0 < k < N$. The energy for each state is $k\epsilon$. Therefore the partition function is

$$Z = \sum_{n=0}^{N} \exp(-\beta n \epsilon) = \sum_{n=0}^{N} \exp(-\beta\epsilon)^n$$

A simple thing to know is that

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}$$

Therefore the partition function can be simplified to

$$Z = \frac{1 - \exp\left(-\beta\epsilon(N+1)\right)}{1 - \exp(-\beta\varepsilon)}$$

Part b)

The average number of open links is

$$
\begin{aligned}
\frac{1}{Z}\sum_{n=0}^{N} n\exp(-\beta n\epsilon) &= -\frac{\partial}{\partial(\beta\epsilon)}\ln Z \\
&= -\frac{\partial}{\partial(\beta\epsilon)}\ln\left(\frac{1-\exp\left(-\beta\epsilon(N+1)\right)}{1-\exp(-\beta\varepsilon)}\right) \\
&= -\frac{\partial}{\partial(\beta\epsilon)}\left[\ln\left(1-\exp\left(-\beta\epsilon(N+1)\right)\right)-\ln\left(1-\exp(-\beta\varepsilon)\right)\right] \\
&= \frac{\exp(-\beta\epsilon)}{1-\exp(-\beta\epsilon)} - \frac{(N+1)\exp(-\beta\epsilon(N+1))}{1-\exp(-\beta\epsilon(N+1))} \\
&= \frac{1}{\exp(\beta\epsilon)-1} + \frac{N+1}{1-\exp(\beta\epsilon(N+1))}
\end{aligned}
$$

The answer goes to 0 as $\epsilon \to \infty$ and $\frac{N}{2}$ as $\epsilon \to 0$ which is what we expect.

Problem #9

This problem is an exercise is error propagation. In general

$$
\begin{aligned}
\delta(x+y) = \delta\left(x-y\right) &= \left[(\delta x)^2 + (\delta y)^2\right]^{1/2} \\
\frac{\delta\left(xy\right)}{xy} = \frac{\delta\left(\frac{x}{y}\right)}{\frac{x}{y}} &= \left[\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2\right]^{1/2} \\
\frac{\delta\left(x^n\right)}{x^n} &= n\left(\frac{\delta x}{x}\right)
\end{aligned}
$$

We can use these relations for errors to find the uncertainty in the acceleration. Our calculated points and there errors are

$$
\begin{aligned}
l &= 5.00 \pm 0.05\,\mathrm{cm} \\
s &= 100 \pm 0.2\,\mathrm{cm} \\
t_1 &= 0.054 \pm 0.001\,\mathrm{s} \\
t_2 &= 0.031 \pm 0.001\,\mathrm{s}
\end{aligned}
$$

The equation we must propagate the errors through is

$$
a = \frac{\left(\frac{l}{t_2}\right)^2 - \left(\frac{l}{t_1}\right)^2}{2s}
$$

I switched the location of t_2 and t_1 in the equation since $v_f^2 = v_i^2 + 2a\Delta x$ and I want v_f to be at photocell 2 with Δx positive down the slope, leading to a positive acceleration. Using the measured values (without errors) we get an acceleration of

$$a = 87.2\,\mathrm{cm\,s^{-2}}$$

The error in the denominator is

$$\Delta\left(2s\right) = 0.4\,\mathrm{cm}$$

The first term in the numerator is (after combining two error propagations (dividing and raising to a power))

$$\begin{aligned}
\Delta\left(\left(\frac{l}{t_2}\right)^2\right) &= 2\left(\frac{l}{t_2}\right)^2\left[\left(\frac{\delta l}{l}\right)^2 + \left(\frac{\delta t_2}{t_2}\right)^2\right]^{1/2} \\
&= 2\left(\frac{5}{0.031}\right)^2\left[\left(\frac{0.05}{5}\right)^2 + \left(\frac{0.001}{0.031}\right)^2\right]^{1/2} \\
&= 1757\,\mathrm{cm^2\,s^{-2}}
\end{aligned}$$

So we get $\left(\frac{l}{t_2}\right)^2 = 26014 \pm 1757\,\mathrm{cm^2\,s^{-2}}$. Doing the same thing for the second term in the numerator, we get $\left(\frac{l}{t_2}\right)^2 = 8573 \pm 579\,\mathrm{cm^2\,s^{-2}}$. Putting these two together we get $\left(\frac{l}{t_2}\right)^2 - \left(\frac{l}{t_2}\right)^2 = 17441 \pm 1849\,\mathrm{cm^2\,s^{-2}}$. Dividing through by $2s$ gives

$$a = 87.2 \pm 9.3\,\mathrm{cm\,s^{-2}}$$

or a fractional uncertainty of 0.1 (assuming I still remember how to use my super HP 49G). I don't know what it means by pushing harder.

Problem #10

For a more complete discussion of essentially this problem go to http://en.wikipedia.org/wiki/Black-body_radiation

Part a)

We approximate the sun as a black-body. Then this is an application of the Stefan-Boltzmann Law.

$$P_{rad} = \sigma T^4 \times 4\pi R_s^2$$

The power radiated per unit area is proportional to T^4, where the proportionality is $\sigma = 5.7 \times 10^{-8}\,\mathrm{W\,/\,m^2\,K^4}$. If you didn't know the Stefan-Boltzmann

law, look at the fundamental constants that are given on the page before the first problem. It is the only constant that relates power to temperature and area (use dimensional analysis to get relation). Plugging in the values we get

$$4 \times 10^{26}\,\mathrm{W} = 5.7 \times 10^{-8}\,\mathrm{W}\,/\,\mathrm{m}^2\,\mathrm{K}^4 \times (5780\,\mathrm{K})^4 \times 4\pi R_s^2$$

or

$$R_s \approx 7.09 \times 10^8\,\mathrm{m}$$

which is close to the measured value of

$$R_s \approx 6.96 \times 10^8\,\mathrm{m}$$

Part b)

We must find the fraction of total energy absorbed by the earth as a function of time. At a distance of d, the power density is $\frac{P}{4\pi d^2}$. The earth effectively covers its cross sectional area πR_E^2. Therefore the total power hitting the earth is $P\frac{\pi R_E^2}{4\pi d^2}$. Using the Stefan-Boltzmann Law we have

$$P\frac{\pi R_e^2}{4\pi d^2} = \sigma T_E^4 \times 4\pi R_e^2$$

Plugging in numbers, we get

$$T_E = 281\,\mathrm{K}$$

Near the measured value of

$$T_E = 287\,\mathrm{K}$$

Problem #11

Part a)

For cylindrical coordinates, $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2)$. For $z(r)$ defined as in the problem, we get $\dot{z} = \frac{\alpha\cos r}{R}\dot{r}$, so $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \frac{\alpha^2\cos^2 r}{R^2}\dot{r}^2) = \frac{1}{2}m\dot{r}^2\left(1 + \frac{\alpha^2\cos^2 r}{R^2}\right) + \frac{1}{2}mr^2\dot{\phi}^2$. Since we just have a gravitational force, the potential energy is $V = mgz = \frac{mg\alpha}{R}\sin r$. Hence the Lagrangian is

$$L = \frac{1}{2}m\dot{r}^2\left(1 + \frac{\alpha^2\cos^2 r}{R^2}\right) + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{mg\alpha}{R}\sin r$$

Part b)

The horizontal circular orbits are when $\dot{r} = 0$. First we must find the equations of motion. For ϕ, we get

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0$$

which is just conservation of angular momentum. For r, we get

$$\frac{d}{dt}\left(m\left(1+\frac{\alpha^2}{R^2}\cos^2 r\right)\dot{r}\right) = mr\dot{\phi}^2 - \frac{1}{2}m\dot{r}^2\left(\frac{2\alpha^2}{R^2}\cos r\sin r\right) - \frac{mg\alpha}{R}\cos r$$

Setting $\dot{r} = 0$, we get

$$mr\dot{\phi}^2 - \frac{mg\alpha}{R}\cos r = 0$$

or

$$\dot{\phi} = \sqrt{\frac{g\alpha}{rR}\cos r}$$

We have horizontal circular orbits when $\dot{\phi}$ is real. This is when $0 < r < \frac{\pi}{2}$.

Part c)

We originally satisfy $\dot{\phi}_0 = \sqrt{\frac{g\alpha}{r_0 R}\cos r_0}$, and give a bump transverse to motion (i.e. essentially in r direction). We take $r \to r_0 + \delta r(t)$, $\phi \to \phi_0(t) + \delta\phi(t)$, and look at the resulting equation of motion to first order in δr. Expanding out the derivative in the EOM for r, we get

$$m\ddot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2 r\right) - m\dot{r}^2\left(\frac{2\alpha^2}{R^2}\cos r\sin r\right) = mr\dot{\phi}^2 - \frac{1}{2}m\dot{r}^2\left(\frac{2\alpha^2}{R^2}\cos r\sin r\right) - \frac{mg\alpha}{R}\cos r$$

The second equation says

$$mr^2\dot{\phi} = K$$

where K is a constant. Plugging in to the second equation, we get

$$m(r_0+\delta r)^2(\dot{\phi}_0 + \delta\dot{\phi}) \approx mr_0^2\dot{\phi}_0 + mr_0^2\delta\dot{\phi} + 2mr_0\delta r_0\dot{\phi}_0 = K$$

to first order in δr and $\delta\dot{\phi}$. Since $mr_0^2\dot{\phi}_0 = K$ by definition of K, we get

$$\delta\dot{\phi} = -2\frac{\dot{\phi}_0}{r_0}\delta r$$

Expanding the EOM to r to first order, we get

$$
\begin{aligned}
m\delta\ddot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2 r_0\right) &= mr_0\dot{\phi}_0^2 + m\delta r\dot{\phi}_0^2 + 2mr_0\dot{\phi}_0\delta\dot{\phi} - \frac{mg\alpha}{R}\left(\cos r_0 - \sin r_0\,\delta r\right) \\
&= m\frac{g\alpha}{r_0 R}\cos r_0\,\delta r - 4m\frac{g\alpha}{r_0 R}\cos r_0\,\delta r + \frac{mg\alpha}{R}\sin r_0\,\delta r \\
&= \left(-\frac{3mg\alpha}{r_0 R}\cos r_0 + \frac{mg\alpha}{R}\sin r_0\right)\delta r
\end{aligned}
$$

The coefficient in front of δr must be negative in order to have stable oscillations. This happens when

$$
-\frac{3}{r_0}\cos r_0 + \sin r_0 < 0
$$

or

$$
\tan r_0 < \frac{3}{r_0}
$$

Problem #13

This is just reflection at a dielectric boundary. In the plasma, $\vec{B} = \frac{c\vec{k}}{\omega}\times\vec{E}$. Using this we get for our 2 boundary conditions ($E_{||}$ and $H_{||}$ must be continuous across the boundary)

$$
\begin{aligned}
E_0\,\exp(-i\omega t) + RE_0\exp(-i\omega t) &= TE_0\exp(-i\omega t) \\
E_0\exp(-i\omega t) - RE_0\exp(-i\omega t) &= \frac{ck_2}{\omega}TE_0\exp(-i\omega t)
\end{aligned}
$$

These simplify to

$$
\begin{aligned}
1 + R &= T \\
1 - R &= nT
\end{aligned}
$$

where $n = \sqrt{\epsilon(\omega)}$ is the index of refraction of the plasma. We can solve this to get

$$
R = \frac{1-n}{1+n}
$$

The reflection probability is $|R|^2$. Since $\omega < \omega_p$ that means that $\epsilon < 0$ and n is imaginary. Therefore $R = 1$. All of the energy is reflected at the boundary. Since our index of refraction is imaginary, the wave in exponentially damped in the plasma and no energy can be transmitted.

Problem #14

Part a)

This problem requires the use of images. The superconductor resists any and all currents, so the magnetic field inside the superconductor must be zero. Since B_n must be continuous, it means that $B_n = 0$ just above the superconductor. We therefore need to find an image below the superconductor which gives a total magnetic field satisfying the boundary condition. We can get this by putting a image current I in the $-\hat{y}$ direction at $x = 0$ and $z = -h$. Therefore the magnetic field in the region above the superconductor is

$$\vec{B} = \frac{2I}{cr_1}\hat{\theta}_1 + \frac{2I}{cr_2}\hat{\theta}_2$$

where r_1 and r_2 are the distances from the current and image current respectively and $\hat{\theta}_1$ and $\hat{\theta}_2$ both point in the $-\hat{x}$ direction when we are at $z = 0$ and $x = 0$

Part b)

The free current can be gotten from $\frac{4\pi}{c}\vec{K}_f = \hat{n} \times \vec{B}|_{z=0}$. Before we can solve for the free current we must solve for $\vec{B}_x\,(z = 0)$. Both the fields due to the current and the image current point in the $-\hat{x}$ direction and have the same magnitude. Therefore $B_x = -\frac{4I}{cr_x}\cos\theta$, where $r_x = \sqrt{x^2 + h^2}$ (the distance from the currents) and $\theta = \frac{\pi}{2} - \tan^{-1}\frac{h}{x}$ (the angle each field makes with the x axis), so we get

$$\begin{aligned} B_x|_{z=0} &= -\frac{4I}{cr_x}\cos\left(\frac{\pi}{2} - \tan^{-1}\frac{h}{x}\right) \\ &= -\frac{4Ih}{cr_x^2} \end{aligned}$$

The last line comes from expanding out the cosine and simplifying. We therefore get

$$\begin{aligned} \vec{K}_f &= \frac{c}{4\pi}\hat{z} \times \vec{B}_{z=0} \\ &= \frac{c}{4\pi}(\hat{z} \times \hat{x})\left(-\frac{4Ih}{cr_x^2}\right) \\ &= -\frac{Ih}{\pi r_x^2}\hat{y} \end{aligned}$$

Part c)

The force per unit length on the wire is just $\vec{f} = \frac{\vec{I} \times \vec{B}}{c}$, where $\vec{B}$ is the field due to the image current. The field due to the image current at the real current is

$$\vec{B} = -\frac{I}{ch}\hat{x}$$

So the force per unit length is

$$\begin{aligned}\vec{f} &= \frac{\vec{I} \times \vec{B}}{c} = -\frac{I^2}{c^2 h}\hat{y} \times \hat{x} \\ &= \frac{I^2}{c^2 h}\hat{z}\end{aligned}$$

Problem #15

Part a)

Particles in electromagnetic fields are time reversal invariant. $\vec{E}$ is invariant under time-reversal, while both $\vec{v}$ and $\vec{B}$ change sign. Therefore the motion in these fields is time-reversal invariant. Angular momentum and spin both change sign under time reversal so spin-orbit coupling does not change sign under time-reversal. Kinetic energy is time reversal invariant since it involves p^2, which is invariant. Therefore all three cases are time-reversal invariant. (?)

Part b)

By definition, $\hat{T}^2 = K \exp\left(-\frac{i\pi S_y}{\hbar}\right) K \exp\left(-\frac{i\pi S_y}{\hbar}\right)$. Applying this to a state ψ, we get

$$\begin{aligned}\hat{T}^2\psi &= K \exp\left(-\frac{i\pi S_y}{\hbar}\right) K \exp\left(-\frac{i\pi S_y}{\hbar}\right)\psi \\ &= K \exp\left(-\frac{i\pi S_y}{\hbar}\right) \exp\left(\frac{i\pi S_y^\star}{\hbar}\right)\psi^\star \\ &= \exp\left(\frac{i\pi S_y^\star}{\hbar}\right) \exp\left(-\frac{i\pi S_y}{\hbar}\right)\psi\end{aligned}$$

Since $S_y = \frac{1}{2i}(S_+ - S_-)$, where S_+ and S_- are the raising and lowering operators (and are real since their normalization factors are real), we have that $S_y^\star = -S_y$. Therefore we get

$$\hat{T}^2\psi = \exp\left(-i\,2\pi\,\frac{S_y}{\hbar}\right)\psi$$

Hopefully this is right

Part c)

For a spin-1/2 particle,

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In order to calculate $\hat{T}^2$, then all we need is to calculate $\exp\left(-i\,2\pi\,\frac{S_y}{\hbar}\right)$. It is

$$\begin{aligned}
\exp\left(-i\,2\pi\,\frac{S_y}{\hbar}\right) &= \exp\left(\pi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
&= \sum_{n=0}^{\infty}\frac{\pi^n}{n!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^n \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{2n!} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\pi + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\pi \\
&= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore for a spin-1/2 particle $\hat{T}^2 = -1$. Therefore, $\hat{T}[\psi_n]$ is orthogonal to ψ_n. You can prove this by knowing that

$$(\hat{T}\psi, \hat{T}\phi) = (\phi, \psi)$$

so

$$(\hat{T}\psi, \psi) = (\hat{T}\psi, \hat{T}^2\psi) = (\hat{T}\psi, -\psi) = -(\hat{T}\psi, \psi)$$

so the inner product is zero and hence they are orthogonal. This implies that the energy spectrum is at least doubly degenerate.

Problem #16

The first Born Approximation says $f(k', k) \approx -\frac{m}{2\pi\hbar^2}\tilde{V}(q)$, where $\tilde{V}(q)$ is the Fourier transform of the potential and $\vec{q} = \vec{k}' - \vec{k}$.

$$\tilde{V}(q) = \int d^3\vec{r}\, e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) = \frac{4\pi}{q}\int_0^\infty r\sin(qr)V(r)$$

$$= -\frac{4\pi}{q}\int_0^\infty r\sin(qr)\frac{Q}{r}e^{-r/a} = -\frac{4\pi Q}{q}\int_0^\infty \sin(qr)\, e^{-r/a}$$

$$= -\frac{4\pi Q}{q}\frac{1}{2i}\int_0^\infty \left(e^{iqr} - e^{-iqr}\right)\, e^{-r/a} = 4\pi Q\frac{1}{q^2 + \frac{1}{a^2}}$$

Therefore

$$f(k', k) \approx -\frac{2mQ}{\hbar^2}\frac{1}{q^2 + \frac{1}{a^2}}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k', k)|^2 = \frac{4m^2Q^2}{\hbar^4}\frac{1}{\left(q^2 + \frac{1}{a^2}\right)^2}$$

$$= \frac{4m^2Q^2}{\hbar^4}\frac{1}{\left(2k^2(1-\cos\theta) + \frac{1}{a^2}\right)^2}$$

Since $q^2 = (\vec{k}' - \vec{k})^2 = k'^2 + k^2 + 2\vec{k}'\cdot\vec{k} = 2k^2 + 2k^2\cos\theta$, since $k' = k$. The magnitude of the wave vectors are the same. The total cross section is:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega}$$

$$= \frac{8\pi m^2Q^2}{\hbar^4}\int_0^\pi d\theta \sin\theta \frac{1}{\left(2k^2(1-\cos\theta) + \frac{1}{a^2}\right)^2}$$

If we change variables to $x = \cos\theta$ we get

$$\sigma = \frac{8\pi m^2Q^2}{\hbar^4}\int_0^1 dx \frac{1}{\left(2k^2(1-x) + \frac{1}{a^2}\right)^2}$$

If we now change variables to $y = 2k^2(1-x)$, we get

$$\sigma = \frac{8\pi m^2Q^2}{\hbar^4}\frac{1}{2k^2}\int_0^{2k^2} dy \frac{1}{\left(y + \frac{1}{a^2}\right)^2}$$

$$= \frac{8\pi m^2Q^2}{\hbar^4}\frac{1}{2k^2}\left(\frac{1}{2k^2 + \frac{1}{a^2}} - \frac{1}{\frac{1}{a^2}}\right)$$

Problem #19

This integral actually proves Euler's Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

It comes from the fact that the integral is a representation of $B(y, 1-y)$, where $B(x, y)$ is the Beta function, which is usually calculated as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

To actually do the integral is first substitue $t = x^2$. The the integral is now

$$I(\alpha) = \int_0^\infty \frac{t^{\alpha-1}}{1+t} dt$$

Lets look at a different integral.

$$\int_C \frac{z^{\alpha-1}}{1-z} dz$$

We take the branch cut along the negative real axis direction. Using the Professor Pac-Man contour with the mouth pointing in the negative real axis direction (because of the branch cut), moving along the outer circle in the counterclockwise direction and radius of circles R and ϵ, we get

$$\int_C \frac{z^{\alpha-1}}{1-z} dz = 2\pi i \times (1)^{\alpha-1} = 2\pi i$$

Therefore, we have

$$2\pi i = \int_{-\pi}^{\pi} \frac{(R\ \exp(i\theta))^{\alpha-1}}{1 - R\ \exp(i\theta)} d\theta + \int_R^\epsilon \frac{(t\ \exp(i\pi))^{\alpha-1}}{1+t} dt + \int_\pi^{-\pi} \frac{(\epsilon \exp(i\theta))^{\alpha-1}}{1 - \epsilon \exp(i\theta)} d\theta + \int_\epsilon^R \frac{(t\ \exp(-i\pi))^{\alpha-1}}{1+t} dt$$

The first and third integrals go to zero when we take $R \to \infty$ and $\epsilon \to 0$. Simplifying the second and fourth integrals, we get

$$2\pi i = I(\alpha)\left(\exp(i\pi\alpha) - \exp(-i\pi\alpha)\right) = I(\alpha)\ 2i \sin \pi\alpha$$

Therefore

$$I(a) = \frac{\pi}{\sin \pi\alpha}$$

Problem #20

The thing to know for this problem is that if $P(x)$ and $Q(x)$ and polynomials of order n and m with $m \geq n+2$ then

$$PV \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum \text{Res}\left(\frac{P}{Q}, z_j\right) + \pi i \sum \text{Res}\left(\frac{P}{Q}, t_j\right)$$

where z_j are the poles in the upper half plane and t_j are the poles on the real axis. Since our function we are integrating is of the form $\frac{1}{\pi}\frac{u(x)}{x-t}$, it means that $u(x)$ is of the form $\frac{A(x)}{B(x)}$ where the two functions are polynomials and B's order is at least one greater than A's order. We can rule out B of order 1 and A of order zero, because that integral is zero and we want the poles of $u(x)$ to be imaginary. If they were real then for a certain t, the pole of $u(x)$ is t and we don't have a simple pole, which changes the integral. We need something that works for all t. The next to try would be $B(x) = x^2 + a^2$ and $A(x) = 1$. Constant multipliers can be ignored (added in if needed). We use $+a^2$ in B because we need imaginary poles. This integral is

$$\begin{aligned} PV \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{x-t}\frac{1}{x^2+a^2} dx &= \frac{1}{\pi}\left(\frac{\pi i}{t^2+a^2} + \frac{2\pi i}{(ia-t)(2ia)}\right) \\ &= -\frac{t}{a(t^2+a^2)} \end{aligned}$$

We have a t in the numerator which we don't want. The next thing to try would be $A(x) = x + b$, while keeping $B(x) = x^2 + a^2$. This integral is

$$\begin{aligned} PV \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{x-t}\frac{x+b}{x^2+a^2} dx &= \frac{1}{\pi}\left(\frac{\pi i(t+b)}{t^2+a^2} + \frac{2\pi i(ia+b)}{(ia-t)(2ia)}\right) \\ &= \frac{a^2-bt}{a(t^2+a^2)} \end{aligned}$$

We can get a solution of $\frac{1}{1+t^2}$ if $b = 0$ and $a = 1$. Therefore $u(x) = \frac{x}{x^2+1}$