Solution 1

#1 : UNDERGRADUATE MECHANICS

PROBLEM: A rope of uniform linear density μ and total length L is suspended from one end. It hangs vertically under its own weight. It is lightly tapped at the lower end. How long does it take for the perturbation to reach the top of the rope?

#1 : UNDERGRADUATE MECHANICS

SOLUTION: The tension at height z is $T = \mu g z$, so the local wave velocity is

$$v(z) = \sqrt{\frac{T(z)}{\mu}} = \sqrt{gz}$$

The propagation time is

$$t = \int_0^L \frac{\mathrm{d}z}{v(z)} = \int_0^L \frac{\mathrm{d}z}{\sqrt{gz}} = 2\sqrt{\frac{2L}{g}}$$

ALTERNATE SOLUTION: Consider the time reversed process, which has the same propagation time. Now the only force involved is gravity, and the time of propagation coincides with the time for free fall through height L. This gives the same result as above.

★ note corrections ★

Solution 2

#2 : UNDERGRADUATE MECHANICS

PROBLEM: A bead of mass m is constrained to move along a rigid, frictionless wire attached to a rotating disk at an angle $\theta \leq \frac{\pi}{2}$ to the plane of the disk, and at a radius R from the origin. The wire rotates with the disk with angular velocity ω. At the origin of the disk is a mass M which exerts a gravitational force on m. Ignore the gravitational force of the Earth.

(a) If x is the position of the bead along the wire, write down the Lagrangian function for the bead.

(b) Derive the equation of motion for the bead using Lagrange's equation.

#2 : UNDERGRADUATE MECHANICS

SOLUTION: a)

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m[\omega(R + x\cos\theta)]^2$$

$$U = -\frac{GMm}{[(R + x\cos\theta)^2 + (xsin\theta)^2]^{\frac{1}{2}}} = -\frac{GMm}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{1}{2}}}$$

Therefore

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m[\omega(R + x\cos\theta)]^2 + \frac{GMm}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{1}{2}}}.$$

b)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}.$$

$$\frac{\partial L}{\partial x} = m\omega^2(R + x\cos\theta)\cos\theta - \frac{GM(R\cos\theta + x)m}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{3}{2}}}.$$

Therefore

$$m\ddot{x} = m\omega^2(R + x\cos\theta)\cos\theta - \frac{GM(R\cos\theta + x)m}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{3}{2}}}.$$

Solution 3

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: A conducting sphere, of radius R, is in an external electric field $\mathbf{E}_{ext} = E_0\hat{\mathbf{z}}$.

3 (a) Find the potential $\phi(r,\theta)$ everywhere.

3 (b) Find the electric field $\mathbf{E}$ (in spherical coordinates) just outside the conductor, at $r = R$.

4 (c) Suppose that a cut is made at the equator of the conductor (at $z = 0$). How much force is required to hold the two hemispheres together?

#3 : UNDERGRADUATE ELECTROMAGNETISM

SOLUTION: (a) The potential inside the conductor ($r \leq R$) is $\phi = 0$, and the potential outside ($r \geq R$) is $\phi(r,\theta) = -E_0(r - \frac{R^3}{r^2})\cos\theta$.

(b) The electric field just outside, at $r = R$, is $\vec{E} = -\hat{r}\frac{\partial\phi}{\partial r}|_{r=R} = 3E_0\cos\theta$. $\hat{r}$.

(c) The energy density is $\frac{1}{2}\epsilon_0\vec{E}^2$, which is non-zero outside the conductor and zero inside. This gives a pressure

$$\frac{d\vec{F}}{dA} = \frac{1}{2}\epsilon_0\vec{E}^2\hat{r} = \frac{9}{2}\epsilon_0 E_0^2\cos^2\theta\hat{r}.$$

Integrating this over the hemisphere, we have

$$\vec{F} = \hat{z}\frac{9}{2}\epsilon_0 E_0^2 R^2 \int_{hemisphere} \cos^3\theta d\Omega.$$

This gives for the force required to hold the hemispheres together:

$$F = \frac{9\pi}{4}\epsilon_0 E_0^2 R^2.$$

Solution 4

#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: There is one conducting sheet in the plane $z = 0$ and another in the plane $z = L$. The lower sheet (i.e. $z = 0$) carries charge per unit area σ and the upper one carries $-\sigma$, where $\sigma > 0$. Also, the lower sheet carries current per unit length $\hat{y}K$ and the upper one carries $-\hat{y}K$. The sheets extend to $\pm\infty$ in x and y.

5 (a) Determine the scalar potential and the vector potential in the region between the two sheets.

5 (b) Suppose that the particle is ejected from the lower sheet with velocity $\vec{v} = \hat{z}v_0$. What is the minimum value of v_0 such that the electron will reach the upper sheet? *Hint: One way to solve this problem is to start with the Lagrangian.*

#4 : UNDERGRADUATE ELECTROMAGNETISM

SOLUTION: The fields are $\vec{E} = \hat{z}\sigma/\epsilon_0$ and $\vec{B} = \hat{x}\mu_0 K$ inside the sheets (and zero outside). Integrate to find the potentials, using the fact that they can only depend on z (symmetry): this gives $\phi = -z\sigma/\epsilon_0$ and $\vec{A} = -\hat{y}z\mu_0 K$.

The Lagrangian is $L = \frac{1}{2}m\vec{v}^2 + q\vec{v}\cdot\vec{A} - q\phi$. Translation invariance in the x and y directions imply that $p_x = mv_x$ and $p_y = mv_y - qz\mu_0 K$ are conserved. With the given initial velocity, this gives $v_x = 0$ and $mv_y = qz\mu_0 K$. Also $E = \frac{1}{2}m\vec{v}^2 + q\phi$ is conserved, so $\frac{1}{2}m(v_y^2 + v_z^2) - qz\sigma/\epsilon_0 = \frac{1}{2}mv_0^2$. The minimum v_0 is such that $v_z(z = L) = 0$, so $\frac{1}{2m}(qL\mu_0 K)^2 - qL\sigma/\epsilon_0 = \frac{1}{2}mv_0^2$.

Solution 5

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: A particle with mass m and energy E moves in the one-dimensional potential $V(x)$ given by $V(x) = 0$ for $x < 0$ and $V(x) = -V_0$ for $x \geq 0$ where $V_0 \geq 0$.

(a) Solve the time-independent Schödinger equation for the wavefunction $\psi(x)$ at all values of x with the boundary condition that the incident flux is from $x = -\infty$.

(b) Compute the transmission and reflection probabilities from your results in (a).

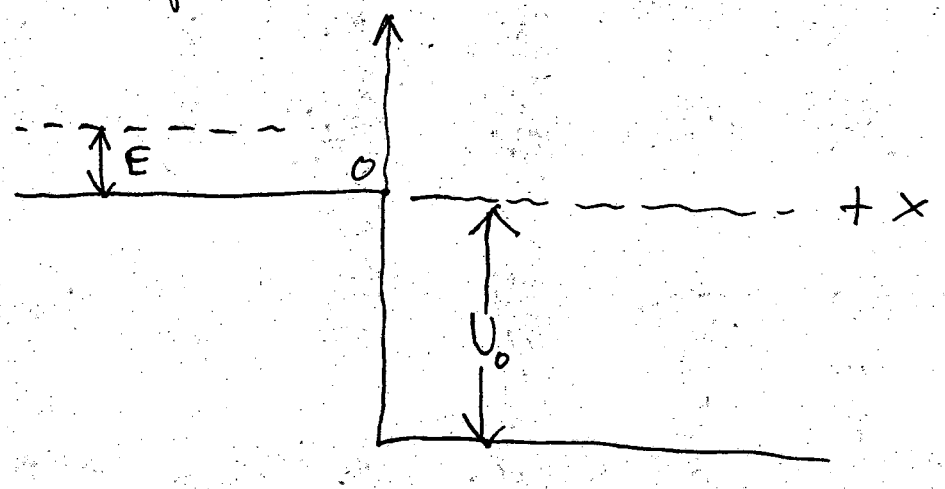
(c) What are the transmission and reflection probabilities in the limits $V_0 \to 0$ and $V_0 \to \infty$?

#5 : UNDERGRADUATE QUANTUM MECHANICS

SOLUTION: SEE NEXT PAGE,

SOLUTION 5.
FA06

Undergrad Quantum: Potential Well



a) $x<0$ $\quad \psi(x) = \underbrace{A e^{ik_0 x}}_{\text{right traveling}} + \underbrace{B e^{-ik_0 x}}_{\text{left traveling}}$

$x>0$ $\quad \psi(x) = \underbrace{C e^{ikx}}_{\text{right traveling}} + \underbrace{0}_{\text{no left traveling wave possible (no reflection at } x=+\infty)}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

(E is conserved across $x=0$)

$x<0$ $\quad V=0$ $\quad -\frac{\hbar^2}{2m}(ik_0)^2 = E$ $\quad k_0 = \sqrt{\frac{2mE}{\hbar^2}}$

$x>0$ $\quad V = -U_0$ $(U_0>0)$ $\quad -\frac{\hbar^2}{2m}(ik)^2 - U_0 = E$ $\quad k = \sqrt{\frac{2m(E+U_0)}{\hbar^2}}$

$k > k_0$

Set ψ and $\frac{d\psi}{dx}$ equal across $x=0$:

$$A + B = C$$

$$ik_0(A-B) = ikC$$

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Solution 5.2

Solve for $\frac{B}{A}$ and $\frac{C}{A}$ (A is "known" from incident intensity)

$$\frac{B}{A} = \frac{k_0 - k}{k_0 + k} \qquad \frac{C}{A} = \frac{2k_0}{k_0 + k}$$

$$x < 0 \qquad \psi(x) = A\left[e^{ik_0x} + \frac{k_0 - k}{k_0 + k} e^{-ik_0x}\right]$$

$$x > 0 \qquad \psi(x) = A \frac{2k_0}{k_0 + k} e^{ikx}$$

Two ways to define Transmission and Reflection coefficients

I: Amplitude $R = \frac{B}{A}$ $\quad T = \frac{C}{A}$

R and T may be complex in general

II: Probability $R = \left|\frac{B}{A}\right|^2$ $\quad T = \left|\frac{C}{A}\right|^2$

Either one is OK. Take I:

$$R = \frac{k_0 - k}{k_0 + k} \qquad T = \frac{2k_0}{k_0 + k}$$

b) There is reflection from an <u>attractive</u> potential (not so classically)

$U_0 \to 0 \quad k = k_0 \quad R = 0 \quad T = 1 \quad$ OK

$U_0 \to \infty \quad (V = -\infty) \quad k \gg k_0 \quad R = -1 \quad T = 0$

c) $\hbar$ not involved: classical limit is wavelength much smaller than typical size of object. This is true here because potential well is ∞ long (from $x = 0$ to $x = +\infty$)

#6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: A one dimensional Harmonic Oscillator has momentum p, mass m, and angular frequency ω. It is subject to a perturbation with a potential energy $U = \lambda x^4$ where λ is suitably small so that perturbation theory is applicable.

(a) Derive the expresssions for a and $a^\dagger$ in terms of x and p using the fact that they satisfy $[a, a^\dagger] = 1$, $H = \hbar\omega(a^\dagger a + 1/2)$.

(b) Calculate the energy shift ΔE_n of the state $|n\rangle$ due to the perturbation to first order in λ, using creation and annihilation operators.

#6 : UNDERGRADUATE QUANTUM MECHANICS

SOLUTION: SEE NEXT PAGE

SOLUTION Problem 6
FA06 Undergrad Quantum: Harmonic Oscillator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{p}{\sqrt{2m\omega\hbar}}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, x - i \frac{p}{\sqrt{2m\omega\hbar}}$$

$$a + a^\dagger = 2\sqrt{\frac{m\omega}{2\hbar}}\, x = \alpha x \quad \text{with } \alpha = 2\sqrt{\frac{m\omega}{2\hbar}}$$

$$x^4 = \left(\frac{a + a^\dagger}{\alpha}\right)^4 \qquad a, a^\dagger] = 1$$

$$\alpha^4 x^4 = (a + a^\dagger)^4 = 6aaa^\dagger a^\dagger - 12aa^\dagger + 3$$

(used $[a, a^\dagger] = 1$ numerous times)

$$\langle n | \alpha^4 x^4 | n \rangle = \left[6(n+1)(n+2) - 12(n+1) + 3\right]$$

using $a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$

$a |n\rangle = \sqrt{n}\, |n-1\rangle$

$$\Delta E = \langle n | \lambda x^4 | n \rangle = \frac{\lambda}{\alpha^4}\left[6(n+1)(n+2) - 12(n+1) + 3\right]$$

$$= \frac{\lambda}{4}\left(\frac{\hbar}{m\omega}\right)^2 \left[6n^2 + 6n + 3\right]$$

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Solution 6.1

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: An ideal heat engine is powered by two reservoirs of equal heat capacity C, which is temperature independent. As the engine works, the reservoirs gradually equilibrate. Find the overall efficiency of the engine from the starting point where the reservoirs are at temperatures T_1 and T_2 ($T_2 < T_1$) to the moment of complete equilibration. Efficiency is defined as the work done divided by the heat supplied by the hotter reservoir

#7 : UNDERGRADUATE STATISTICAL MECHANICS

SOLUTION: Let T_f be the final equilibrium temperature, Q_1 the heat removed from the hotter reservoir, and Q_2 the heat transferred to the cooler reservoir. Then the efficiency is given by the work done divided by the heat supplied by the hotter reservoir,

$$e = \frac{Q_1 - Q_2}{Q_1} = \frac{C(T_1 - T_f) - C(T_f - T_2)}{C(T_1 - T_f)} = 1 - \frac{T_f - T_2}{T_1 - T_f}$$ 4 pt.

To find T_f, we can use the conservation of entropy.

$$\Delta S_i = \int_{T_i}^{T_f} \frac{C\,\mathrm{d}T}{T} = C \ln\left(\frac{T_f}{T_i}\right)$$

$$\Delta S_1 + \Delta S_2 = 0 \Rightarrow \ln\left(\frac{T_f}{T_1}\right) + \ln\left(\frac{T_f}{T_2}\right) = 0 \Rightarrow T_f = \sqrt{T_1 T_2}$$ 4 pt.

$$e = 1 - \frac{\sqrt{T_1 T_2} - T_2}{T_1 - \sqrt{T_1 T_2}} = 1 - \sqrt{\frac{T_2}{T_1}}$$ 2 pt.

1) No account for work done ~~by~~ at the lab side (-4pt.)

2) —" — heat needed to heat to T_f (-2pt.)

3)

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Solution

8: UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: A mass $M = 1\,\mathrm{g}$ of water at 20°C is forced through an insulated porous plug under a pressure of 10^4 atm into a lab where the pressure is 1 atm. Find the amount of water converted into steam. Assume that the density of water is constant, its specific heat is $C = 1\,\mathrm{cal}/(\mathrm{g} \times^\circ \mathrm{C})$, and the latent heat of evaporation is $L = 540\,\mathrm{cal/g}$. Here $1\,\mathrm{cal} = 4.18\,\mathrm{J}$.

8: UNDERGRADUATE STATISTICAL MECHANICS

SOLUTION: What is described is the usual Joule-Thompson process (also called "throttling" in chemistry and engineering). This process conserves enthalpy $H = U + PV$ because the entropy is unchanged and because the two pressures on each side of the plug are constant. If, in the final state, a fraction $x < 1$ of the liquid is converted into steam, we must have

$$h_i^l = (1-x)h_f^l + xh_f^s,$$

where i, f, l, and s stand for "initial," "final," "liquid," and "steam," respectively, and $h = H/M$ is the enthaply per unit mass. Solving for x, we get

$$x = \frac{h_i^l - h_f^l}{h_f^s - h_f^l} = \frac{h_i^l - h_f^l}{L}.$$

The numerator here is given by

$$h_i^l - h_f^l = \left(\frac{U_i}{M} + \frac{P_i}{\rho}\right) - \left(\frac{U_f}{M} + \frac{P_f}{\rho}\right) = C(T_i - T_f) + \frac{P_i - P_f}{\rho},$$

where $\rho = 10^3\,\mathrm{kg/m^3}$ is the density of the liquid. The final temperature is the boiling point $T_f = 100°\mathrm{C}$. Therefore, using $1\,\mathrm{atm} \approx 10^5\,\mathrm{Pa}$, we get

$$x \approx \frac{1}{4.18 \times 10^3 \times 540}\left[4.18 \times 10^3 \times (20 - 100) + 10^5(10^4 - 1)/10^3\right] \approx 0.29,$$

and so the answer is 0.29 g.

#9 : UNDERGRADUATE MATH METHODS

PROBLEM: Consider the function $F_k(\eta)$ defined by the integral

$$F_k(\eta) = \int_0^\infty \mathrm{d}x \, \frac{x^k}{e^{x-\eta}+1}, \qquad k = 0, 1, 2, 3, \ldots.$$

Find an analytic (polynomial in η) expression for

$$F_2(\eta) - F_2(-\eta)$$

Hint: By integrating by parts, show that

$$\frac{\mathrm{d}F_k(\eta)}{\mathrm{d}\eta} = kF_{k-1}(\eta),$$

and start with $F_0(\eta)$. You are given

$$F_1(0) = \frac{\pi^2}{12}$$

#9 : UNDERGRADUATE MATH METHODS

SOLUTION: SEE NEXT PAGE

solution: UG math

1.

Prob 9.

first prove differential recursion relation. Integrate by parts

$$\begin{cases} F_k(\eta) = \displaystyle\int_0^\infty \frac{x^k\,dx}{e^{x-\eta}+1} \\ \dfrac{dF_k(\eta)}{d\eta} = \displaystyle\int_0^\infty \frac{-x^k\left(-e^{x-\eta}\right)dx}{\left(e^{x-\eta}+1\right)^2} \\ = \dfrac{-x^k}{e^{x-\eta}+1}\Bigg|_0^\infty + k\displaystyle\int_0^\infty \frac{x^{k-1}\,dx}{e^{x-\eta}+1} \\ = k\,F_{k-1}(\eta) \end{cases}$$

$$\Rightarrow \boxed{F_k(\eta) - F_k(0) = k\int_0^\eta F_{k-1}(\eta')\,d\eta'}$$

now, integrate $F_0(\eta)$ analytically,

$$F_0(\eta) = \int_0^\infty \frac{dx\, e^{\eta-x}}{e^{\eta-x}+1} \qquad \text{let } u = e^{\eta-x}+1$$

then $du = -e^{x-\eta}$ $\Rightarrow$

$$F_0(\eta) = \int_{e^\eta+1}^{1} \frac{-du}{u} = \int_1^{e^\eta+1} \frac{du}{u} = \ln(1+e^\eta)$$

$$\Rightarrow F_0(\eta) = \ln(1+e^\eta) = \ln\left[e^\eta\left(e^{-\eta}+1\right)\right]$$

$$= \eta + \ln(1+e^{-\eta})$$

Solution 9.2

SOLUTION UG math prob 1 page 2.

note that ...

$$F_0(\eta) - F_0(-\eta) = \eta$$

$$\Rightarrow 1 \cdot \int_0^{\eta} F_0(\eta')\,d\eta' - 1 \cdot \int_0^{-\eta} F_0(-\eta')\,d\eta' = \tfrac{1}{2}\eta^2 \qquad \eta'' = -\eta'$$

$$\Rightarrow \underbrace{1 \cdot \int_0^{\eta} F_0(\eta')\,d\eta'}_{F_1(\eta) - F_1(0)} + \underbrace{\int_0^{-\eta} F_0(\eta'')\,d\eta''}_{F_1(-\eta) - F_1(0)} = \tfrac{1}{2}\eta^2$$

$$\Rightarrow F_1(\eta) + F_1(-\eta) = \tfrac{1}{2}\eta^2 + 2F_1(0)$$

$$\Rightarrow \underbrace{2 \cdot \int_0^{\eta} F_1(\eta')\,d\eta'}_{F_2(\eta) - F_2(0)} - \underbrace{2\int_0^{-\eta} F_1(\eta'')\,d\eta''}_{F_2(-\eta) - F_2(0)} = \tfrac{1}{3}\eta^3 + 4F_1(0)\eta$$

$$\Rightarrow \boxed{\begin{aligned} F_2(\eta) - F_2(-\eta) &= \tfrac{1}{3}\eta^3 + 4F_1(0)\eta \\ &= \tfrac{1}{3}\eta^3 + \tfrac{\pi^2}{3}\eta \end{aligned}}$$

where $F_1(0) = \pi^2/12$

#10 : UNDERGRADUATE OTHER

PROBLEM: Nucleons (neutrons and protons) can be regarded as composed of three *massless* relativistic quarks freely moving inside a spherical volume with radius r and uniform energy per unit volume B inside the sphere. (The energy density outside the sphere is zero). Given that the experimentally measured rest energy of a nucleon is $M \sim 1000$ MeV and the radius of a nucleon is ~ 1 fm (10^{-15} m), estimate the magnitude of B in MeV/m^3.

#10 : UNDERGRADUATE OTHER

SOLUTION: The quarks are confined to with radius r, so

$$E \sim 3\left(\frac{\hbar c}{r}\right) + Br^3$$

$\partial E/\partial r = 0$ implies that

$$\left(\frac{\hbar c}{r^4}\right) \sim B$$

$\hbar c = 1.97 \times 10^{-13}$ MeV-m so

$$B = \frac{1.97 \times 10^{-13}}{(10^{-15})^4} = 1.97 \times 10^{47}\,\text{MeV/m}^3.$$

Comments on problem #10.

The problem, as stated, provided a model for the energy of the nucleon as consisting of a kinetic energy part ($3\frac{\hbar c}{r}$) and a confinement energy, ($\frac{4}{3}\pi r^3$) so that the total energy, E, of the nucleon is $E \approx 3\frac{\hbar c}{r} + \frac{4}{3}\pi r^3 B$

If the energy density is uniform throughout the interior of the nucleon then

$$\frac{\partial E}{\partial r} = 0$$

and

$$B = \frac{3\hbar c}{4\pi r^4}$$

$$\hbar c = 1.97 \times 10^{-13}$$

so that

$$B = 4.7 \times 10^{46}$$

Grading:

1) If answer set $\frac{4}{3}\pi r^3 B = 1000 Mev$ 2/10

2) If answer recognized kinetic contribution $\frac{\hbar c}{r}$ 4/10

3) If answer shows $B \propto \frac{\hbar c}{r^4}$ and answer is correct order of magnitude 10/10

   If numerical answer is of wrong order 9/10

4) If set $1000 Mev = 3\frac{\hbar c}{r} + \frac{4}{3}\pi B r^3$ and answer is of correct order 10/10

   If numerical answer is of wrong order 9/10

Solution 11

#11 : GRADUATE MECHANICS

PROBLEM: If the solar system were immersed in a uniformly dense spherical cloud of weakly-interacting massive particles (WIMPs) then objects in the solar system would experience gravitational forces from both the Sun and the cloud of WIMPs such that

$$F_r = -\frac{k}{r^2} - br$$

Assume that the extra force due to the WIMPs is very small (that is $b \ll k/r^3$). Work to first order in b.

(a) Find the frequency of radial oscillations for a nearly circular orbit. 5

(b) Find the average angular velocity. 3

(c) Find the rate of precession of the perihelion to lowest order in b using the results of (a) and (b). 2

#11 : GRADUATE MECHANICS

SOLUTION: The force corresponds to a potential energy

Graded out of 10

'he kinetic energy, which separates into a term due to the bead's mo-
on along the wire and a term due to the rotation of the bead with the
ire, is

$$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2. \tag{10.80}$$

'he Lagrangian is $L = T - V$. Using Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{10.81}$$

e find that

$$a\ddot{\theta} + g\sin\theta - a\omega^2\cos\theta\sin\theta = 0. \tag{10.82}$$

t an equilibrium point $\ddot{\theta} = 0$, so $g = a\omega^2\cos\theta$, or $\omega^2 = g/a\cos\theta$. This
uation has a solution for ω only if $\omega^2 \geq g/a$, so the critical angular
locity is

$$\omega_c = \sqrt{\frac{g}{a}}, \tag{10.83}$$

nd the equilibrium angle is

$$\theta_0 = \cos^{-1}\left(\frac{g}{a\omega^2}\right). \tag{10.84}$$

b) If the mass makes small oscillations around the equilibrium point
), then we can describe the motion in terms of a small parameter
$= \theta - \theta_0$. The equation of motion (10.82) becomes

$$a\ddot{\phi} + g\sin(\theta_0 + \phi) - a\omega^2\cos(\theta_0 + \phi)\sin(\theta_0 + \phi) = 0. \tag{10.85}$$

sing standard trigonometric identities, the small angle approximations
n $\phi \approx \phi$ and $\cos\phi \approx 1$, and our solution for θ_0 (10.84), it is easy to
now that

$$\ddot{\phi} + \omega^2\left(1 - \frac{g^2}{a^2\omega^4}\right)\phi = 0. \tag{10.86}$$

'his has the general solution

$$\phi = A\cos\Omega t + B\sin\Omega t, \tag{10.87}$$

where

$$\Omega = \omega\sqrt{1 - \frac{g^2}{a^2\omega^4}}, \tag{10.88}$$

and A and B are arbitrary constants. The period of oscillation is $2\pi/\Omega$.

SOLUTION: Prob. 11

Solution 1.10. In plane-polar coordinates, the Lagrangian for a particle moving in a central potential $V(r)$ is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{10.89}$$

where m is the mass of the particle. The potential is given in the question as

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2. \tag{10.90}$$

The θ-component of Lagrange's equation is

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} \equiv l. \tag{10.91}$$

The hamiltonian of our system is then

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{eff}(r), \tag{10.92}$$

with $p_r = m\dot{r}$ and

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V(r). \tag{10.93}$$

The term $l^2/2mr^2$ is referred to as an "angular momentum barrier." Solving the equations of motion for this hamiltonian is equivalent to solving Lagrange's equations for the Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 - V_{eff}(r). \tag{10.94}$$

This is a completely general result for the motion of a particle in a central potential and could easily have been our starting point in this problem (e.g., Goldstein, Chapter 3).

It may seem unnecessarily long-winded to go through this procedure, but note that the sign of the angular momentum barrier in (10.94) is *opposite* to what we would have gotten if we had naively replaced $\dot{\theta}$ with l/mr^2 in the Lagrangian (10.89). This is due to the fact that the Lagrangian is a function of the time derivative of the position, and not of the canonical momentum.

The equation of motion from (10.94) is

$$m\ddot{r} = -\frac{d}{dr}V_{eff}(r). \tag{10.95}$$

If the particle is in a circular orbit at $r = r_0$ we require that the force on it at that radius should vanish,

$$\left.\frac{dV_{eff}}{dr}\right|_{r=r_0} = 0. \tag{10.96}$$

Using our expression for V_{eff} (10.93), we derive an expression relating the angular momentum l to the radius of the orbit r_0:

$$\frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - br_0 = 0. \tag{10.97}$$

We are interested in perturbations about this circular orbit. Provided the perturbation remains small, we can expand $V_{eff}(r)$ about r_0,

$$V_{eff}(r) = V_{eff}(r_0) + (r - r_0)V'_{eff}(r_0) + \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0) + \cdots. \tag{10.98}$$

If we use this expansion in the Lagrangian (10.94) together with the condition (10.96), we find

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(r - r_0)^2 V''_{eff}(r_0), \tag{10.99}$$

where we have dropped a constant term. This is just the Lagrangian for a simple harmonic oscillator, describing a particle undergoing radial oscillations with frequency

$$\omega^2 = \frac{1}{m}V''_{eff}(r_0). \tag{10.100}$$

Differentiating $V_{eff}(r)$ twice gives us

$$\frac{3l^2}{mr_0^4} - \frac{2k}{r_0^3} + b = m\omega^2. \tag{10.101}$$

We can eliminate l between equations (10.101) and (10.97) to give the frequency of radial oscillations:

$$\omega = \left(\frac{k}{mr_0^3} + \frac{4b}{m}\right)^{1/2}. \tag{10.102}$$

To find the rate of precession of the perihelion, we need to know the period of the orbit. From the definition of angular momentum l equation (10.91), we have an equation for the orbital angular velocity ω_1,

$$\omega_1 \equiv \frac{d\theta}{dt} = \frac{l}{mr^2}. \tag{10.103}$$

Let us write $r(t) = r_0 + \epsilon(t)$, where $\epsilon(t)$ is sinusoidal with frequency ω and average value zero. We substitute $r(t)$ into equation (10.103) and expand in $\epsilon(t)$:

$$\frac{d\theta}{dt} = \frac{l}{mr_0^2}\left(1 - \frac{2\epsilon}{r_0} + \mathcal{O}(\epsilon^2)\right). \tag{10.104}$$

To zeroth order in the small quantities br_0^3/k and ϵ/r_0, the period of the orbit T_1 is the same as the period of oscillations $T_2 = 2\pi/\omega$. Therefore we can average ϵ over T_1 rather than T_2 and still get zero, to within terms of second order, which we are neglecting. The average angular velocity is therefore

$$\bar{\omega}_1 = \frac{2\pi}{T_1} \approx \frac{l}{mr_0^2} = \sqrt{\frac{k}{mr_0^3} + \frac{b}{m}}, \tag{10.105}$$

where we have made use of (10.97).

Now consider one complete period of the radial oscillation. This takes place in time $T_2 = 2\pi/\omega$. In this time the particle travels along its orbit through an angle of

$$\theta = 2\pi\frac{\bar{\omega}_1}{\omega} = 2\pi\frac{\sqrt{k/mr_0^3 + b/m}}{\sqrt{k/mr_0^3 + 4b/m}}$$

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$$\approx 2\pi\left(1-\frac{3br_0^3}{2k}\right). \tag{10.106}$$

In other words, the particle does not quite orbit through 2π before the radial oscillation is completed. Each time around the perihelion precesses backwards through an angle

$$\delta\theta = 3\pi\frac{br_0^3}{k}, \tag{10.107}$$

and it gets around in time T_2, so the precession rate is

$$\alpha = \frac{\delta\theta}{T_2} = \frac{3\pi br_0^3}{k}\frac{\sqrt{k/mr_0^3+4b/m}}{2\pi}$$

$$\approx \frac{3b}{2}\sqrt{\frac{r_0^3}{mk}}. \tag{10.108}$$

b) When r is large enough that $F_r \approx -br$, we see that the force is like that of a linear spring. In this case the planar motion of the orbit can be resolved into simple harmonic motion in each of its three cartesian components. Thus the orbits will in general be ellipses; however, in each case the sun will be at the *center* of the ellipse rather than at one of the foci (as is the case for Newtonian gravity).

#12 : GRADUATE MECHANICS

PROBLEM: A spherical marble of mass m and radius a rolls without slipping near the bottom of a perfectly rough spherical bowl of radius b, which is fixed in position. Calculate the angular frequency for small oscillations.

#12 : GRADUATE MECHANICS

SOLUTION: The no slip condition is $b\dot{\theta} = a\dot{\phi}$. The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 + mg(b-a)\cos\theta \\ &= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{5}mb^2\dot{\theta}^2 + mg(b-a)\cos\theta \end{aligned}$$

Expanding for small θ gives

$$L = \frac{1}{2}\left[m(b-a)^2 + \frac{2}{5}mb^2\right]\dot{\theta}^2 + mg(b-a) - \frac{1}{2}mg(b-a)\theta^2$$

so that

$$\omega^2 = \frac{mg(b-a)}{m(b-a)^2 + \frac{2}{5}mb^2} = \frac{5g(b-a)}{5(b-a)^2 + 2b^2}$$

Solution 13

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: Consider a thin spherical shell of radius R centered on the spherical coordinate system r, θ, ϕ. On the surface of the sphere is a current density $\mathbf{K}$ which produces a uniform magnetic field inside the shell directed along the polar axis, that is $\mathbf{B} = B_0\hat{\mathbf{z}}$ inside the shell. Find $\mathbf{K}$.

#13 : GRADUATE ELECTROMAGNETISM

SOLUTION: SEE NEXT ~~TWO~~ 3 PAGES

new solution

SOLUTION 13

The only currents are on the sphere, so inside and outside: $\bar{\nabla} \times \bar{B} = 0$. Hence we can express $\bar{B}$ as the gradient of a scalar potential φ: $\bar{B} = -\nabla \varphi$.

Since $\bar{\nabla} \cdot \bar{B} = 0 \Rightarrow \nabla^2 \varphi = 0$

So the problem reduces to the solution of Laplace's equation for φ (one of the boundary conditions involve the surface currents, so it will give us what we are looking for).

We know the solution of $\nabla^2 \varphi = 0$ in spherical symmetry can be expanded in spherical harmonics. Because $\bar{B}_{in} = B_0 \hat{z}$, the solution must have azimuthal symmetry, so we are left with:

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

We then impose the following 4 BCs:

1) $\bar{B} = B_0 \hat{z}$ as $r \to 0$
2) $\bar{B} \to 0$ as $r \to \infty$
3) $B_{\perp,out} - B_{\perp,in} = 0$
4) $B_{\parallel,out} - B_{\parallel,in} = \frac{4\pi}{c} K$

<u>B.C. 1</u>: $\varphi_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$

$$-\nabla \varphi_{in} = B_0 \hat{z}$$

$$-\frac{\partial}{\partial r}\left[\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial \theta}\left[\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)\right]\hat{\theta} =$$

$$= B_0 \cos\theta\, \hat{r} - B_0 \sin\theta\, \hat{\theta}$$

Clearly this is satisfied only by $A_\ell = 0$ for $\ell \neq 1$,
and $A_\ell = -B_0$ for $\ell = 1$

$$\Rightarrow \quad \varphi_{in}(r,\theta) = -B_0 r \cos\theta \qquad (r < R)$$

<u>B. C. 2:</u>

$$\varphi_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{C_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad (r > R)$$

<u>B. C. 3:</u>

$$\bar{B}_{out}\hat{r}\Big|_{R^+} - \bar{B}_{in}\cdot\hat{r}\Big|_{R^-} = 0$$

$$\Rightarrow \quad -\nabla\varphi_{out}\cdot\hat{r}\Big|_{R^+} + \nabla\varphi_{in}\cdot\hat{r}\Big|_{R^-} = 0$$

$\nabla\varphi = \frac{\partial}{\partial r}\varphi\,\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\varphi\,\hat{\theta}$, but since we are projecting on $\hat{r}$, we only keep the first component: $\frac{\partial}{\partial r}\varphi\,\hat{r}$

$$\nabla\varphi_{out}\cdot\hat{r} = \sum_{\ell=0}^{\infty}(-\ell-1)\frac{C_\ell}{r^{\ell+2}}P_\ell(\cos\theta)$$

$$\nabla\varphi_{in}\cdot\hat{r} = -B_0\cos\theta$$

$$\Rightarrow \quad \sum_{\ell=0}^{\infty}(\ell+1)\frac{C_\ell}{R^{\ell+2}}P_\ell(\cos\theta) = B_0\cos\theta$$

Hence $C_\ell = 0$ for $\forall\, \ell \neq 1$
and $C_1 = \frac{B_0 R^3}{2}$

$$\Rightarrow \quad \varphi_{out}(r,\theta) = \frac{B_0 R^3}{2r^2}\cos\theta \qquad (r > R)$$

(Notice that φ is NOT continuous on the surface of the sphere)

B.C. 4: Let's write it in vector form, so we get $\bar{K}$:

$$\hat{r} \times \bar{B}_{out}\Big|_{R^+} - \hat{r} \times \bar{B}_{in}\Big|_{R^-} = \frac{4\pi}{c}\bar{K}$$

$$\bar{B}_{out} = -\nabla \varphi_{out} = \frac{B_0 R^3}{r^3}\cos\theta\,\hat{r} + \frac{B_0 R^3}{2r^3}\sin\theta\,\hat{\theta}$$

$$\hat{r} \times \bar{B}_{out}\Big|_{R^+} = \frac{B_0}{2}\sin\theta\,\hat{\phi}$$

$$\hat{r} \times \bar{B}_{in}\Big|_{R^-} = -B_0 \sin\theta\,\hat{\phi}$$

$$\Rightarrow \quad \frac{B_0}{2}\sin\theta\,\hat{\phi} + B_0\sin\theta\,\hat{\phi} = \frac{4\pi}{c}\bar{K}$$

$$\Rightarrow \quad \boxed{\bar{K} = \frac{c}{4\pi}\,\frac{3}{2}\,B_0 \sin\theta\,\hat{\phi}}$$

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: An electromagnetic wave

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$$

$$\mathbf{B}(\mathbf{r},t) = \mathrm{Re}[\mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$$

is incident on a plane conducting surface. The wave vector $\mathbf{k}$ makes an angle θ relative to the normal $\hat{\mathbf{n}}$ to the surface. The wave is polarized perpendicular to the plane of incidence, that is $\mathbf{E}_0$ is perpendicular to $\mathbf{k}$ and $\hat{\mathbf{n}}$. Determine the radiation pressure (the time-averaged force per unit area) exerted on the surface.

#14 : GRADUATE ELECTROMAGNETISM

SOLUTION:

O'Neil

Graduate E+M

14.1

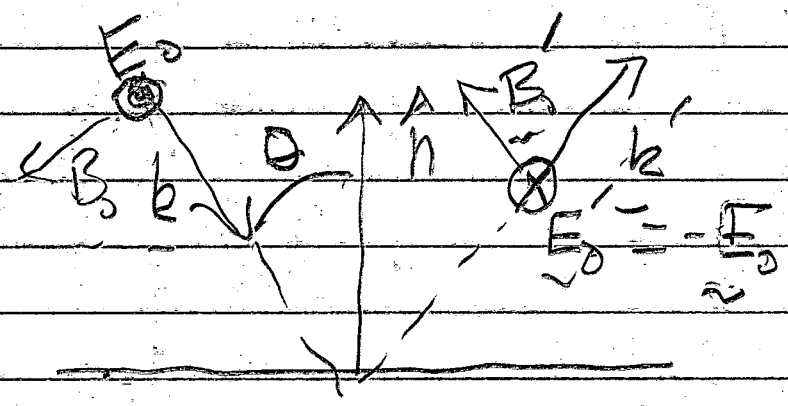
(1) An electromagnetic wave,

$$\underline{E}(\underline{r},t) = \mathrm{Re}\, \underline{E}_0\, e^{i\underline{k}\cdot\underline{r} - i\omega t}$$

$$\underline{B}(\underline{r},t) = \mathrm{Re}\, \underline{B}_0\, e^{i\underline{k}\cdot\underline{r} - i\omega t},$$

is incident on a plane conducting surface. The wave vector $\underline{k}$ makes an angle θ relative to the normal $\hat{n}$ to the surface. The wave is polarized perpendicular to the plane of incidence, that is, $\underline{E}_0$ is perpendicular to $\underline{k}$ and $\hat{n}$. Determine the time-average force/area exerted on the conducting surface (the radiation pressure).

18.8 solution



$|B_0| = |E_0|$

at the surface $\underset{\sim}{E} = \underset{\sim}{E_0} + \underset{\sim}{E_0'} = 0$

$$B_n = (B_0 + B_0')\cdot\hat{n} = 0$$

$$B_t = (B_0 + B_0')_t = 2B_0\cos\theta$$

$\left\langle \frac{\underset{\sim}{F}}{A} \right\rangle_t$ determined by stress tensor:

$$\left\langle \frac{\underset{\sim}{F}}{A} \right\rangle_t = \frac{1}{2}\mathrm{Re}\left\{ \frac{\hat{n}}{4\pi}\cdot\left[\underset{\sim}{E}\underset{\sim}{E}^* + \underset{\sim}{B}\underset{\sim}{B}^* - \frac{1}{2}\underset{\sim}{\mathbb{1}}\left(|E|^2 + |B|^2\right)\right]\right\}$$

$$\hat{n}\cdot\underset{\sim}{B} = 0$$

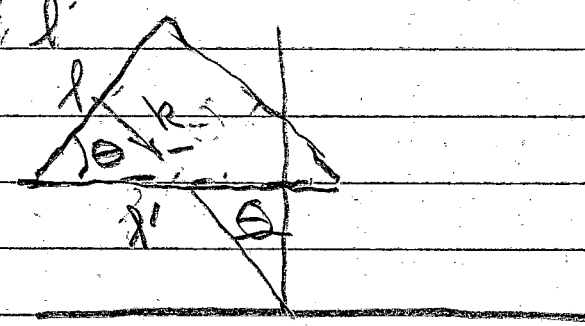
$$\therefore \left\langle \frac{\underset{\sim}{F}}{A} \right\rangle_t = \hat{n}\,\frac{1}{8\pi}\left(2B_0\cos\theta\right)^2 = \hat{n}\,\frac{E_0^2\cos^2\theta}{4\pi}$$

<u>alternate</u> soln (not using stress tensor)

time average momentum density in wave

$$\langle \underset{\sim}{g} \rangle_t = \frac{1}{2} \mathrm{Re} \frac{1}{4\pi c} \underset{\sim}{E}_0 \times \underset{\sim}{B}_0^* = \frac{1}{8\pi c} |\underset{\sim}{E}_0|^2 \hat{k}$$

$$\left\langle \frac{F_t}{A} \right\rangle_t = \hat{n} \, 2 \, \hat{n} \cdot \langle \underset{\sim}{g} \rangle_t \frac{l}{l'} c$$



$$l' = l' \cos\theta$$

$$\left\langle \frac{F_t}{A} \right\rangle_t = \hat{n} \frac{2|E_0|^2}{8\pi} \overbrace{\hat{n} \cdot \hat{k}}^{\cos\theta} \overbrace{\frac{l}{l'}}^{\cos\theta}$$

$$= \hat{n} \frac{|E_0|^2}{4\pi} \cos^2\theta$$

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: Two electrons each of mass m are placed in a one-dimensional box of width L placed in an external magnetic field B in the z direction. The interaction Hamiltonian of the electrons is

$$H_{\text{int}} = A\,\mathbf{S}_1 \cdot \mathbf{S}_2 - (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) \cdot \mathbf{B} \tag{1}$$

where the magnetic moment is $\boldsymbol{\mu}_{1,2} = -\mu_0\,\mathbf{S}_{1,2}$, where μ_0 is a constant.

(a) Find the possible energies of the system, and the quantum numbers (i.e. spatial and spin quantum numbers) and multiplicities of the allowed states.

(b) Write the energy as a multiple of $\hbar^2/(2mL^2)$ in terms of the dimensionless parameters $a = mL^2A$ and $b = \mu_0 BmL^2/\hbar$.

(c) Find the ground state quantum numbers as a function of a and b. Give your answer by marking the quantum numbers in the $a - b$ plane.

#15 : GRADUATE QUANTUM MECHANICS

SOLUTION: If the electrons are in spatial states $n_{1,2}$, then the energy is

$$\begin{aligned} E &= \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2\right) + A\left(\frac{1}{2}\mathbf{S}^2 - \frac{1}{2}\mathbf{S}_1^2 - \frac{1}{2}\mathbf{S}_2^2\right) + \mu_0 B\left(S_{1z} + S_{2z}\right) \\ &= \frac{\hbar^2}{2mL^2}\left[\pi^2\left(n_1^2 + n_2^2\right) + a\left(s(s+1) - \frac{3}{2}\right) + 2bs_z\right] \end{aligned}$$

where s is the total spin of the system, and s_z is the total z component of spin.

The states are given by n_1, n_2, s, s_z. By Fermi statistics, the net wavefunction must be antisymmetric, so for $n_1 = n_2$ the state must have $s = s_z = 0$. For $n_1 \neq n_2$, one can have $s = 1$ with $s_z = 1, 0, -1$ or $s = 0, s_z = 0$, and each state has multiplicity one. Note that (n_1, n_2) and (n_2, n_1) don't give different states, because the wavefunction has to be antisymmetrized in $1 \leftrightarrow 2$. So

15.1

the energies are

$$\frac{\hbar^2}{2mL^2}\left[\pi^2\left(n_1^2+n_2^2\right)+\frac{1}{2}a+2bs_z\right],\ s=1,\ s_z=\pm1,0,\ n_1\neq n_2$$
$$\frac{\hbar^2}{2mL^2}\left[\pi^2\left(n_1^2+n_2^2\right)-\frac{3}{2}a\right],\ s=0,\ n_1=n_2$$

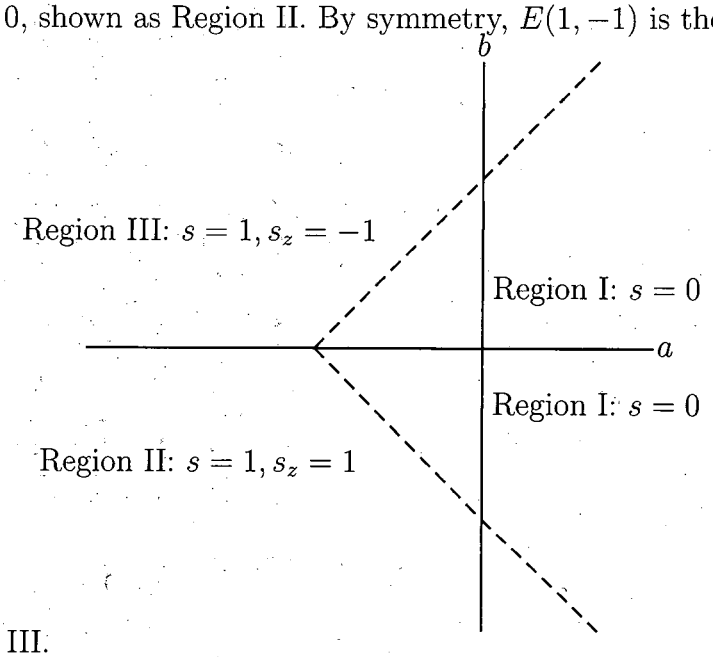
The lowest spin-0 state has $n_1=n_2=1$, with energy

$$E(0) = \frac{\hbar^2}{2mL^2}\left[2\pi^2-\frac{3}{2}a\right]$$

The lowest spin-1 state has $n_1=1, n_2=2$ with energy

$$E(1,s_z) = \frac{\hbar^2}{2mL^2}\left[5\pi^2+\frac{1}{2}a+2bs_z\right]$$

$E(0)$ is the ground state if $E(0)\leq E(1,s_z)$, so that $2a+3\pi^2\geq 0$, $2a+3\pi^2\pm 2b\geq 0$. This is shown as Region I in the figure. $E(1,1)$ is the ground state if $b\leq 0$, $2a+2b+3\pi^2\leq 0$, shown as Region II. By symmetry, $E(1,-1)$ is the



ground state in Region III.

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: In studying the hydrogen atom one takes the proton to be a point charge with mass M. Suppose instead that the proton charge is distributed uniformly within the volume of a sphere with radius $r_0 = 10^{-15}$ m.

(a) Using perturbation theory, calculate the shift in energy of the $1s$ level of hydrogen to first order in the perturbation.

(b) Give an order of magnitude estimate of the ratio of the $2p$ and $1s$ level shifts.

#16 : GRADUATE QUANTUM MECHANICS

SOLUTION:

The potential due to a uniformal spherical volume with net charge e and radius r_0 is

$$\begin{aligned} U &= -\frac{er^2}{2r_0^3} + \frac{3e}{2r_0}, \qquad (r < r_0) \\ &= \frac{e}{r}, \qquad (r > r_0) \end{aligned}$$

where the constant of integration in the first expression has been chosen to make U continuous at r_0.

The perturbation V in the Hydrogen atom potential is

$$\begin{aligned} \Delta V &= \frac{e^2r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}, \qquad (r < r_0) \\ &= 0, \qquad (r > r_0) \end{aligned}$$

The energy shift is

$$\Delta E = \langle \Delta V \rangle$$

to first order in perturbation theory. For the $1s$ state,

$$\Delta E = \int |\psi|^2 \Delta V = \int_0^{r_0} 4\pi r^2 \mathrm{d}r \; |\psi|^2 \Delta V$$

Since $r_0 \ll a_0$, the typical scale of variation of the wavefunction, $\psi \approx \psi(0)$, and

$$\Delta E \approx |\psi(0)|^2 \int_0^{r_0} 4\pi r^2 \mathrm{d}r \left[\frac{e^2 r^2}{2r_0^3} - \frac{3e^2}{2r_0} + \frac{e^2}{r}\right] = \frac{2}{5} e^2 \pi r_0^2 |\psi(0)|^2$$

For the $1s$ state,

$$\psi = \sqrt{\frac{1}{\pi a_0^3}} e^{-r/a_0}$$

so

$$\Delta E \approx \frac{2}{5}\frac{e^2 r_0^2}{a_0^3} = \frac{2}{5}\frac{e^2}{a_0}\frac{r_0^2}{a_0^2}$$

The ground state energy of H is $-e^2/(2a_0) = -13.6$ eV, and $a_0 = 0.529 \times 10^{-10}$ m, so

$$\Delta E = \frac{2}{5}(2 \times 13.6\,\mathrm{eV})\frac{r_0^2}{a_0^2} = 3.9 \times 10^{-9}\,\mathrm{eV}$$

The wavefunction of the $2p$ state vanishes at the orgin. This suppresses ΔE by an additional factor of $r^2/a_0^2 \sim 10^{-10}$

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: A system of N bosons in two dimensions has an energy-momentum relation

$$\epsilon_p = c p^f$$

and density $n = N/A$ (A=area). c and f are constants.

(a) Show that at low temperatures the system will Bose condense, for some values of f. Find these values of f. (In what follows, assumes that f indeed takes these values.) Show that the Bose condensation temperature $T_c \sim n^\alpha$. Find α (in terms of f).

(b) Show that the entropy below T_c goes as $S \sim T^\beta$, and the "pressure" below T_c (i.e. its equivalent in two dimensions) goes as $P \sim T^\gamma$. Find β and γ (again in terms of f).

#17 : GRADUATE STATISTICAL MECHANICS

SOLUTION: Use

$$N = \sum_p \frac{1}{z^{-1}e^{\beta\epsilon_p} - 1} \to \frac{2\pi A}{h^2}\int_0^\infty p dp \frac{1}{z^{-1}e^{\beta c p^f} - 1} + \text{(groundstate term)}.$$

Let $x \equiv \beta c p^f$, so $p = (xkT/c)^{1/f}$

$$N \sim AT^{2/f}\int_0^\infty \frac{x^{2/f-1}dx}{z^{-1}e^x - 1} + \text{(groundstate term)}.$$

There is Bose condensation if the integral remains finite for $z = e^{\beta\mu} \to 1$. The relevant limit of the integral to check is $x \to 0$, where the integral $\sim \lim_{x\to 0} x^{2/f-1-1+1}$. So there is Bose condensation if $2/f - 1 > 0$, i.e. if $f < 2$. The temperature is $T_c \sim (N/A)^{f/2}$.

(b) Use $S = -(\frac{\partial F}{\partial T})_A$ and $P = -(\frac{\partial F}{\partial A})_T$, at $z = 1$ for T below T_c, with

$$F = -kT\ln\mathcal{Z} = kT\sum_p \ln(1 - e^{-\beta\epsilon_p}).$$

The sum is evaluated as in part (a). This gives $F \sim AT^{2/f+1}$, so the entropy $S \sim T^{2/f}$ and pressure $P \sim T^{2/f+1}$.

Solution 18

#18 : GRADUATE STATISTICAL MECHANICS

PROBLEM: Consider a gas that obeys a modified van der Waals equation of state

$$(P + \frac{a}{V^3})(V - b) = RT,$$

where V is the volume of one mole of gas, and a and b are constants. Determine the critical parameters P_c, V_c and T_c of this gas and evaluate the quantity RT_c/P_cV_c.

#18 : GRADUATE STATISTICAL MECHANICS

SOLUTION: Given that

$$P = \frac{RT}{V-b} - \frac{a}{V^3},$$

the critical point is determined by the conditions

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{-RT}{(V-b)^2} + \frac{3a}{V^4} = 0, \rightarrow RT = \frac{3a(V-b)^2}{V^4},$$

and

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_T = \frac{2RT}{(V-b)^3} - \frac{12a}{V^5} = 0, \rightarrow RT = \frac{6a(V-b)^3}{V^5}.$$

Equating the two, we get $2(V-b) = V$, or $V_c = 2b$. Substituting this result into either of the two expressions for RT, we get

$$RT_c = \frac{3a}{16b^2} \rightarrow T_c = \frac{3a}{16b^2R}.$$

Substituting V_c and T_c into the equation of state, we get

$$P_c = \frac{3a}{16b^3} - \frac{a}{8b^3} = \frac{a}{16b^3}.$$

It follows that the ratio $RT_c/P_cV_c = 3/2$.

#19 : GRADUATE MATHEMATICAL METHODS

PROBLEM: Evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin\theta}$$

where $0 < b < a$.

#19 : GRADUATE MATHEMATICAL METHODS

SOLUTION: SEE NEXT PAGE

II.3 J. Kuti Math Problem 3/14/95 19.1

3 PARTS

PART 2 Fall 95

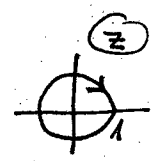
Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$, where $0 < b < a$

$\boxed{\text{solution}}$: $z = e^{i\theta}$ $\qquad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) =$

$d\theta = \frac{dz}{iz}$ $\qquad = \frac{1}{2i}\left(z - z^{-1}\right)$

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = 2\int_{C_+} \frac{1}{(2ia + bz - bz^{-1})}\frac{dz}{z}$$

$$= 2\int_{C_+} \frac{dz}{bz^2 + 2iaz - b}$$



where C_+ is the unit circle clockwise

$$z = -i\left(\frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)$$ singularities of integral

If $0 < b < a$, then $-i\left(\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)$ is inside unit circle; other is outside

by residue theorem:

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b}\int_{C_+} \frac{dz}{\left[z + i\left(\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)\right]\left[z + i\left(\frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right)\right]}$$

$$= 2\pi i\left[\frac{2}{b}\frac{1}{2i\sqrt{\left(\frac{a}{b}\right)^2 - 1}}\right] = \boxed{\frac{2\pi}{\sqrt{a^2 - b^2}}}$$

Solution 20

#20 : GRADUATE MATHEMATICAL METHODS

PROBLEM: Find the Green's function for the differential operator

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 1$$

in the interval $[0, 1]$ with the boundary condition that $f(0) = f(1) = 0$.

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SOLUTION: The Green's function equation is

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - f = \delta(x - x_0)$$

so the solution that satisfies the boundary condition is

$$\begin{aligned} f &= A\left(e^x - e^{-x}\right), \qquad 0 \le x \le x_0 \\ f &= B\left(\frac{e^x}{e} - ee^{-x}\right), \qquad x_0 \le x \le 1. \end{aligned}$$

The function must be continuous at x_0, and have a jump of 1 in its slope,

$$\begin{aligned} A\left(e^{x_0} - e^{-x_0}\right) &= B\left(\frac{e^{x_0}}{e} - ee^{-x_0}\right) \\ 1 + A\left(e^{x_0} + e^{-x_0}\right) &= B\left(\frac{e^{x_0}}{e} + ee^{-x_0}\right) \end{aligned}$$

so that

$$\begin{aligned} A &= \frac{e^{x_0} - e^2 e^{-x_0}}{2(e^2 - 1)} \\ B &= e\frac{e^{x_0} - e^{-x_0}}{2(e^2 - 1)} \end{aligned}$$