DEPARTMENTAL EXAM & SOLUTIONS

SPRING 2002

ID Number _____ PART I Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

DEPARTMENT OF PHYSICS
DEPARTMENTAL EXAMINATION – SPRING 2002

PART I

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. **Please indicate with a "check" which of the (7) questions you wish to be graded below:**

Section 1:	**Problem 1**_____	**Problem 2** _____
Section 2:	**Problem 3**_____	**Problem 4** _____
Section 3:	**Problem 5**_____	**Problem 6**_____
Section 4:	**Problem 7** _____	**Problem 8** _____
Section 5:	**Problem 9** _____	**Problem 10** _____

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SECTION 1:

Problem 1. **[Classical Mechanics]** Brad Majors and Janet Weiss go bowling. Ever observant, Brad notes that his bowling ball initially skids down the alley, and only slowly begins to roll.

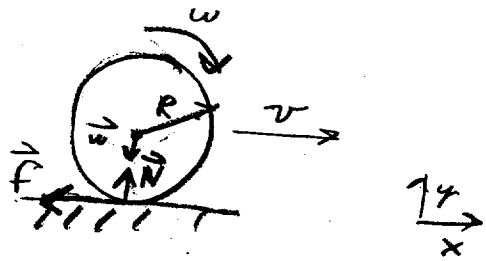
a) Briefly, and in words, why does a ball that initially skids down the alley eventually start to roll down the alley?
b) The bowling ball has radius R. What mathematical constraint holds when the ball begins to roll without slipping?
c) The ball is released with an initial translational speed of v_0 and an initial rotational speed of zero. The coefficient of kinetic friction between the ball and the alley is μ and the mass of the ball is M. Derive an expression for the time, denoted T, between the release of the ball and the onset of rolling without slipping [minor help: $I_{\text{sphere}} = (2/5)MR^2$].

Problem 1

a) Torque caused by Friction leads to rolling

b) $V = \omega R$

c) Eqns of C.O.M.



① $m\ddot{y} = W - N = 0$

$\quad '' \quad = Mg - N$

$\therefore N = Mg$

② $m\ddot{x} = -f$

$\quad '' \quad = -\mu N$

$\quad '' \quad = -\mu Mg$

$\therefore V = \dot{x} = \int_0^t dt' \ddot{x} = v_0 - \mu g t$

Eqn about C.O.M

$I\ddot{\theta} = Rf$

or $\frac{2}{5} MR^2 \ddot{\theta} = R\mu Mg$

$\ddot{\theta} = 5/2 \frac{\mu g}{R}$

$\omega = \dot{\theta} = \int_0^t dt' \ddot{\theta} = 5/2 \frac{\mu g}{R} t$

Rolling w/o slipping when $V = \omega R$, or

$v_0 - \mu g T = 5/2 \mu g T$

or $T = 2/7 \, v_0/\mu g$

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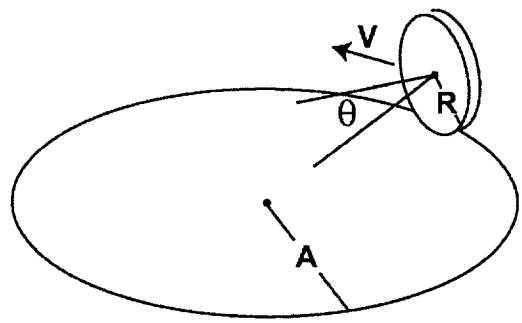
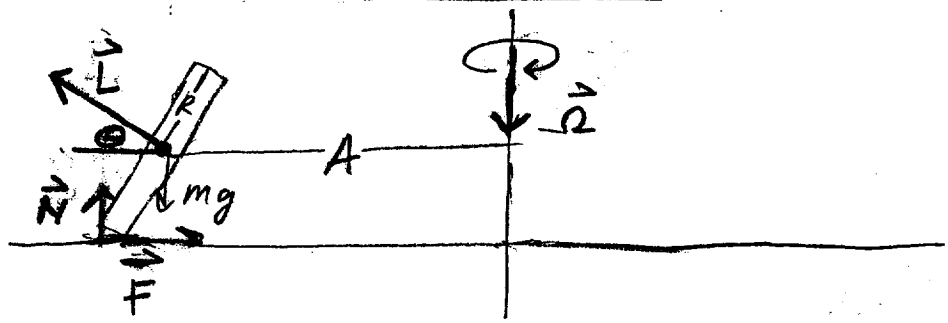
SECTION 1:

Problem 2. [Classical Mechanics] Dr. Frank N. Furter's latest creation, Rocky Horror, takes a keen interest in the motion of rigid bodies. He carefully rolls a coin on a table such that it moves in a circular path. Rocky notices that the coin leans inward, so that the axis of the coin is tilted by an angle, denoted θ, relative to the horizontal.



Find an expression for the angle θ. The speed of the center of mass of the coin is V, the radius of the coin is R, the mass of the coin is M, and radius of the circular path is A [hint: recall that there are potentially two contributions to a change in angular momentum] [minor help: $I_{\text{disk}} = (1/2)MR^2$].

Problem 2



Use C.O. of Mom. $\Rightarrow \vec{\Omega} \times \vec{L} + \cancel{\frac{\partial \vec{L}}{\partial t}}^{\,0} = \vec{\tau}$

$$\vec{\Omega} \times \vec{L} = \Omega L \left[-\sin(\pi/2 + \theta)\right] \hat{\theta}$$

$$= -\Omega L \cos\theta \, \hat{\theta}$$

$$= -\Omega I_{Disk} \omega \cos\theta \, \hat{\theta}$$

$$= -\frac{V}{A} \frac{1}{2} M R^2 \frac{V}{R} \cos\theta \, \hat{\theta}$$

$$= -\frac{1}{2} M V^2 \frac{R}{A} \cos\theta \, \hat{\theta}$$

$$\vec{\tau} = -N R \sin\theta \, \hat{\theta} + F R \cos\theta \, \hat{\theta}$$

$$= \left[-Mg R \sin\theta + \frac{M V^2}{A} R \cos\theta\right] \hat{\theta}$$

$$= \left[-Mg R \sin\theta + M V^2 \frac{R}{A} \cos\theta\right] \hat{\theta}$$

$$\therefore \; -\frac{1}{2} M V^2 \frac{R}{A} \cos\theta = -Mg R \sin\theta + M V^2 \frac{R}{A} \cos\theta$$

$$-\frac{3}{2} \cancel{M} V^2 \frac{\cancel{R}}{A} \cos\theta = -\cancel{M} g \cancel{R} \sin\theta$$

$$\theta = \tan^{-1}\left[3/2 \, \frac{V^2}{Ag}\right]$$

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SECTION 2:

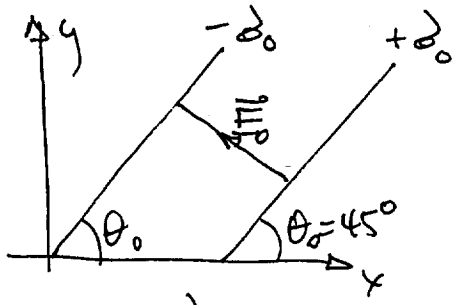
Problem 3. **[Electromagnetism]** A parallel-plate capacitor, at rest in S_0 and tilted at a 45° angle to the x-axis, carries charge densities $\pm\sigma_0$ on the two plates. System S is moving along the x-axis at speed v relative to S_0.

(a) Find $\mathbf{E}_0$, the field in S_0.
(b) Find $\mathbf{E}$, the field in S.
(c) What angle do the plates make with the x axis?
(d) Is the field perpendicular to the plates in S?

Hint: If you don't remember how electric field components transform, the electromagnetic field tensor is $F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix}$

PART I : PROBLEM #3

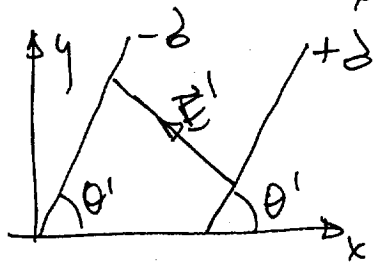
a)



In S_0, $|\vec{E}_0| = \frac{\sigma_0}{\epsilon_0}$, where

$$E_{0x} = -\frac{\sigma}{\sqrt{2}\,\epsilon_0}, \quad E_{0y} = +\frac{\sigma}{\sqrt{2}\,\epsilon_0}$$

b)



In S, $E_x' = E_{0x} = -\frac{\sigma_0}{\sqrt{2}\,\epsilon_0}$ ($\parallel$ component preserved)

but $E_y' = \gamma E_{0y} = +\frac{\gamma\sigma_0}{\sqrt{2}\,\epsilon_0}$ ($\perp$ component due to relative motion)

and $E' = \sqrt{\frac{1+\gamma^2}{2}}\,\frac{\sigma_0}{\epsilon_0}$.

c) $dy' = dy_0$ $\perp$ to relative motion

$dx' = dx_0/\gamma$ length contracted along motion.

$\therefore \tan\theta' = \frac{dy'}{dx'} = \gamma\frac{dy_0}{dx_0}$, but $\frac{dy_0}{dx_0} = \tan 45° = 1$

$\therefore \theta' = \tan^{-1}\gamma$ is angle of plates wrt x-axis

d) $$\frac{d\vec{r}'\cdot\vec{E}'}{r'E'} = \frac{\left(\frac{1}{\gamma}, 1\right)(-1, \gamma)}{\sqrt{1+\frac{1}{\gamma^2}}\sqrt{1+\gamma^2}} = \frac{\gamma^2-1}{\gamma^2+1} \neq 0$$

$\therefore$ $\vec{E}'$ is <u>not</u> perpendicular to plates in S.

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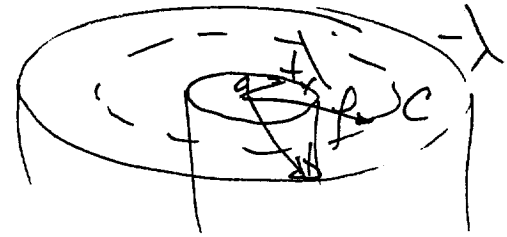
SECTION 2:

Problem 4. **[Electromagnetism]** A coaxial cylindrical capacitor has a dielectric material placed between its plates for which the dielectric constant varies as $\kappa_e = \varepsilon / \varepsilon_0 = \kappa_0 \rho^n$, where κ_0 and n are positive constants. Find the capacitance of a length L of this system by finding the energy in the fields between the plates.

PART I: PROBLEM #4



Place free charge/length $+\lambda, -\lambda$ on inner/outer conductors at $\rho = a, b$.

$$\oint \bar{D}\cdot d\bar{s} = +\lambda \;, a<\rho<b \longrightarrow \bar{D} = \frac{\lambda}{2\pi\rho}\hat{\rho} \quad a<\rho<b .$$

$$\therefore \bar{E} = \frac{\bar{D}}{\epsilon} = \frac{\lambda}{2\pi\rho}\cdot\frac{1}{\epsilon_0 k_0 \rho^n}\hat{\rho} = \frac{\lambda}{2\pi\epsilon_0 k_0 \rho^{n+1}}\hat{\rho} .$$

Energy $$U_e = \frac{1}{2}\int \bar{D}\cdot\bar{E}\,d\tau = \frac{1}{2}\int_0^L dz \int_0^{2\pi} d\varphi \int_a^b \left(\frac{\lambda}{2\pi}\right)^2 \frac{1}{\epsilon_0 k_0}\frac{\rho\,d\rho}{\rho^{n+2}}$$

$$= \frac{1}{2}\frac{L\lambda^2}{2\pi\epsilon_0 k_0}\left[-\frac{1}{n\rho^n}\right]_a^b$$

$$= \frac{1}{2}\frac{(L\lambda)^2}{2\pi L\epsilon_0 k_0 n}\left[\frac{1}{a^n}-\frac{1}{b^n}\right] .$$

But $U_e = \dfrac{Q^2}{2C}$ where $Q = \lambda L$ is charge on length L.

$$\therefore C = \frac{2\pi L\epsilon_0 k_0 n}{\left[\frac{1}{a^n}-\frac{1}{b^n}\right]}$$ is capacitance of length L.

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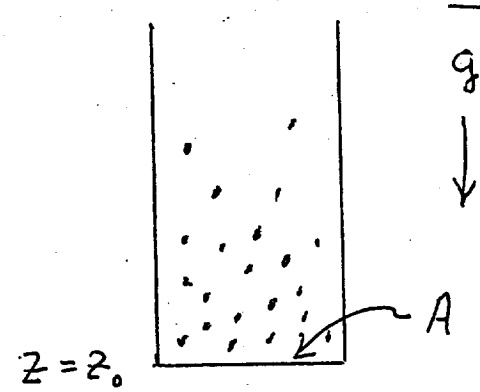
SECTION 3:

Problem 5. **[Statistical Mechanics/Thermodynamics]** Consider an ideal gas of N identical particles confined to a tall cylinder (*i.e.*, $mgL >> kBT$) of cross-sectional area A, oriented parallel to a gravitational field of strength g. The bottom of the container is at $z = z_0$, and the system is at temperature T.

a) Write down the partition function for this system;
b) Use this partition function to calculate the mean energy and the heat capacity, C_v, of the gas;
c) Use the partition function to compute the force on the bottom of the container

PART I: PROBLEM #5

5. Consider an ideal gas of N identical particles confined in a tall cylinder (i.e, mgL >> k_BT) of cross-sectional area A, oriented parallel to a gravitational field of strength g. The bottom of the container is at $z = z_0$, and the system is at temperature T.



(a) Write down the partition function for this system.

$$Z = \frac{1}{N!}\left(\frac{2\pi m k T}{h^2}\right)^{\frac{3N}{2}} A^N \left[\int_{z_0}^{\infty} e^{-\beta m g z}\, dz\right]^N$$

$$= \frac{1}{N!}\left(\frac{2\pi m k T}{h^2}\right)^{\frac{3N}{2}} \left(\frac{A e^{-\beta m g z_0}}{\beta m g}\right)^N$$

(b) Use this partition function to calculate the mean energy and the heat capacity, C_v, of the gas.

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \frac{3N}{2\beta} + \frac{N}{\beta} + m g z_0$$

$$= \frac{5}{2} N k_B T + N m g z_0$$

$$C = \frac{5}{2} N k_B$$

5. (cont'd)

(c) Use the partition function to compute the force on the bottom of the container.

$$\underline{X_{bott}} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial z_0} = \underline{-Nmg}$$

-- just the weight of the gas

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SECTION 3:

Problem 6. **[Statistical Mechanics/Thermodynamics]** Suppose that in a certain range of temperature, the force required to stretch a rod a length x beyond its natural length is of the form:

$$F = -Ax / T^2$$

where T is the absolute temperature and A is a constant. [In answering the following, assume that the temperature is in the range where this expression is valid.]

a) Derive the fundamental thermodynamic relation between the internal energy E, the entropy S, and the force F;
b) Using the answer to part a), develop an expression for the Maxwell relation for dS/dx (constant T) and evaluate it in terms of A, x, and T;
c) Develop an expression for dE/dx (constant T) and evaluate it in terms of A, x, and T;
d) Use the expression for dE/dx (constant T) derived above to calculate the change in internal energy ΔE when the rod is stretched an amount x_0 beyond its natural length at constant temperature T.

PART I: PROBLEM 6

2. Suppose that in a certain range of temperature, the force required to stretch a rod a length x beyond its natural length is of the form:

$$F = -Ax/T^2,$$

where T is the absolute temperature and A is a constant. [In answering the following, assume that the temperature is in the range where this expression is valid.]

(a) Derive the fundamental thermodynamic relation between the internal energy, E, the entropy, S, and the force F.

$$dE = TdS - Fdx$$

(b) Using the answer to part (a), develop an expression for the Maxwell relation for dS/dx(const. T) and evaluate it in terms of A, x and T.

$$d(E - TS) = -SdT - Fdx$$

$$S = -\left.\frac{\partial(E-TS)}{\partial T}\right|_x \qquad F = -\left.\frac{\partial(E-TS)}{\partial x}\right|_T$$

see sectio
5.5 of Re
esp. 5.5.1
5.5.15

$$\left.\frac{\partial S}{\partial x}\right|_T = \left.\frac{\partial F}{\partial T}\right|_x = \frac{2Ax}{T^3}.$$

2. (cont'd)

(c) Develop an expression for dE/dx(const. T), and evaluate it in terms of A, x, and T.

$$dS = \left.\frac{\partial S}{\partial x}\right|_T dx + \left.\frac{\partial S}{\partial T}\right|_x dT$$

so

$$dE = T\left.\frac{\partial S}{\partial x}\right|_T dx + T\left.\frac{\partial S}{\partial T}\right|_x dT - F\,dx$$

$$\left.\frac{dE}{dx}\right|_T = T\left.\frac{\partial S}{\partial x}\right|_T - F = \underline{\underline{\frac{3Ax}{T^2}}}$$

(d) Use the expression for dE/dx(const. T) derived above to calculate the change in internal energy, ΔE, when the rod is stretched an amount x_0 beyond its natural length at constant temperature, T.

$$E = \int_0^{x_0} \frac{3A}{T^2} x\,dx = \underline{\underline{\frac{3Ax_0^2}{2T^2}}}$$

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SECTION 4:

Problem 7. **[Quantum Mechanics]** Calculate the energy shifts to the four hyperfine ground states of hydrogen in a constant magnetic field B. Assume that the field strength is such that the spin-field and the spin-spin interactions are about the same size so that neither can be assumed to be small. You may assume that the hyperfine interaction in hydrogen is given by

$$H_{\text{hyperfine}} = \frac{4}{3}(Z\alpha)^4 \left(\frac{m}{M_N}\right)(mc^2) g_p \frac{1}{n^3}\frac{\mathbf{S}\cdot\mathbf{I}}{\hbar^2} \equiv A\frac{\mathbf{S}\cdot\mathbf{I}}{\hbar^2}$$

where $\mathbf{I}$ is the proton spin operator.

PART I: PROBLEM #7

Solution to Hyperfine Splitting in an Intermediate B Field

We are to work the full problem with no assumptions about which perturbation is stronger. Assume the total angular momentum is given by

$$\vec{F} = \vec{S} + \vec{I}.$$

We work with the basis states of definite total angular momentum $|fm_f\rangle$. The matrix for $\langle fm_f|H_{hf} + H_B|f'm'_f\rangle$ is

$$\begin{array}{cc} 1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 0 \end{array} \begin{pmatrix} \frac{A\hbar^2}{4} + \mu_B B & 0 & 0 & 0 \\ 0 & \frac{A\hbar^2}{4} - \mu_B B & 0 & 0 \\ 0 & 0 & \frac{A\hbar^2}{4} & \mu_B B \\ 0 & 0 & \mu_B B & \frac{-3A\hbar^2}{4} \end{pmatrix}.$$

The top part is already diagonal so we only need to work in bottom right 2 by 2 matrix, solving the eigenvalue problem.

$$\begin{pmatrix} A & B \\ B & -3A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{where} \qquad \begin{array}{l} A \equiv \frac{A\hbar^2}{4} \\ B \equiv \mu_B B \end{array}$$

Setting the determinant equal to zero, we get

$$(A - E)(-3A - E) - B^2 = 0.$$

$$E^2 + 2AE - 3A^2 - B^2 = 0$$

$$E = \frac{-2A \pm \sqrt{4A^2 + 4(3A^2 + B^2)}}{2} = -A \pm \sqrt{A^2 + (3A^2 + B^2)}$$

$$= -A \pm \sqrt{4A^2 + B^2}$$

The eigenvalues for the $m_f = 0$ states, which mix differently as a function of the field strength, are

$$E = -\frac{A\hbar^2}{4} \pm \sqrt{\left(\frac{A\hbar^2}{2}\right)^2 + (\mu_B B)^2}.$$

The eigenvalues for the other two states which remain eigenstates independent of the field strength are

$$\frac{A\hbar^2}{4} + \mu_B B$$

and

$$\frac{A\hbar^2}{4} - \mu_B B.$$

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SECTION 4:

Problem 8. **[Quantum Mechanics]** Particles with angular momentum 1 are passed through a Stern-Gerlach apparatus, which separates them according to the z-component of their angular momentum. Only the $m = -1$ component is allowed to pass through the apparatus. A second apparatus separates the beam according to its angular momentum component along the u-axis. The u-axis and the z-axis are both perpendicular to the beam direction but have an angle θ between them. Find the relative intensities of the three beams separated in the second apparatus.

PART I: PROBLEM #8

Solution to Stern-Gerlach Problem After the first apparatus, the state of the system is $\chi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Lets define the beam direction to be along the y axis. There are many ways to solve this problem. One way is to find the eigenstates of S_u then dot them into χ.

$$S_u = \cos\theta S_z + \sin\theta S_x = \hbar \left[\cos\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \right] = \hbar \begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & 0 & \frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix}$$

We know the eigenvalues of S_u will be 0, $\pm\hbar$ so we have the eigenvalue equation.

$$\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & 0 & \frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This is straightforward but tedious.

We may use a rotation operator to solve the problem in one step. The rotation operator is

$$R_y(\theta) = e^{i\theta L_y/\hbar} = e^{i\theta L_y/\hbar} = \sum_{n=0}^{\infty} \frac{\left(\frac{i\theta L_y}{\hbar}\right)^n}{n!}$$

$$e^{i\theta L_y/\hbar} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin(\theta)\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \frac{1}{2}(\cos(\theta) - 1) \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ -\frac{1}{\sqrt{2}}\sin\theta & \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ \frac{1}{2}(1-\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{pmatrix}$$

Using the u-axis to define the basis states we then have

$$\chi' = R_y(\theta)\chi = \frac{1}{2} \begin{pmatrix} 1-\cos\theta \\ \sqrt{2}\sin\theta \\ 1+\cos\theta \end{pmatrix}$$

The sum of the squares is 1 as it should be. So, the relative intensities are the squares of the amplitudes.

$$Intensity \propto \frac{1}{4} \begin{pmatrix} (1-\cos\theta)^2 \\ 2\sin^2\theta \\ (1+\cos\theta)^2 \end{pmatrix}$$

Its easier to rotate about the z-axis since the rotation matrix is just.

$$R_z(\theta) = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}$$

If we redefine the beam direction to be the z direction, say the intitial apparatus separation is along x, then we just need to find the spin down state along the x axis.

$$\chi = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{2} \end{pmatrix}$$

$$\chi' = \begin{pmatrix} \frac{-1}{2}e^{i\theta} \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{2}e^{-i\theta} \end{pmatrix}$$

Now dot this back into three S_x eigenstates.

$$\psi_+ = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \qquad \psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad \psi_- = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{2} \end{pmatrix}$$

$$I_+ = \left|\frac{1}{2} - \frac{1}{2}\cos\theta\right|^2$$
$$I_0 = \left|\frac{i\sin\theta}{\sqrt{2}}\right|^2$$
$$I_- = \left|\frac{1}{2} + \frac{1}{2}\cos\theta\right|^2$$

Same answer.

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SECTION 5:

Problem 9. [Experiment]

a) How long does it take a photon created at the center of the sun to random walk its way to the surface? The mass of the sun is $M_\odot \approx 2\times10^{33}$ g and the solar radius is $R_\odot \approx 7\times10^{10}$ cm. (Hint: Estimate the mean free path of the photon using the Thomson cross section $\sigma_T \approx 10^{-24}$ cm^2.)

b) Say that all sources of nuclear energy in the sun were suddenly shut off. At its current luminosity ($L_\odot \approx 4\times10^{33}$ erg-s^{-1}), how long would it take the sun to radiate away all its internal energy? The central temperature of the sun is ~ 1 keV.

c) Why is there such a large disparity between the timescales in parts a) and b)?

SECTION I: PROBLEM #9

random walk $\Rightarrow$

a.) $R_\odot \simeq \sqrt{N}\,\lambda_{mfp} \Rightarrow N = \dfrac{R_\odot^2}{\lambda_{mfp}^2}$

N = number of scatterings

λ_{mfp} = mean free path of photon

$\bar{\rho}$ = mean density $= \dfrac{M_\odot}{4/3\,\pi R_\odot^3} \simeq \dfrac{2\times10^{33}\,g}{4/3\,\pi\,(7\times10^{10})^3} \sim \dfrac{2\times10^{33}}{1.4\times10^{33}}\,g/cm^3$

$\sim 1\,g/cm^3$

$$\boxed{\lambda_{mfp} = \frac{1}{n_e\,\sigma_T} \simeq \frac{1}{10^{24}\,cm^{-3}\,10^{-24}\,cm^2} \sim 1\,cm}$$

$n \simeq \bar{\rho}\,N_A \simeq 10^{24}\,\dfrac{amu}{cm^3}$

$\Rightarrow n_e \sim n \sim 10^{24}\,cm^{-3}$

$\Rightarrow N = \left(\dfrac{7\times10^{10}\,cm}{1\,cm}\right)^2 \sim 5\times10^{21}$

time between scatterings $\Rightarrow t = \dfrac{\lambda_{mfp}}{c} \simeq \dfrac{1\,cm}{3\times10^{10}\,cm/s}$

$\simeq \frac{1}{3}\times10^{-10}\,s = 3\times10^{-11}\,s$

$t_{\substack{random\\walk}}$ = time total $= N\cdot t = 5\times10^{21}\,(3\times10^{-11}\,s)$

$\simeq 15\times10^{10}\,s = \dfrac{15\times10^{10}\,s}{\pi\times10^{7}\,s/yr} \simeq 5\times10^{3}\,yrs$

$t_{\substack{random\\walk}} \simeq 5000\,yrs$

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b.) internal energy $U = \frac{3}{2}\,N_{part}\,kT$

$N_{part} = 2\times10^{33}\,g \cdot 6\times10^{23}\,\dfrac{amu}{g} \sim 12\times10^{56}\,cm$

$\sim 10^{57}$

$U \sim 3\,(10^{57})(\ldots)$

$\therefore U \sim 2\times10^{60}$ eV

but 1.602×10^{-19} C per elementary charge

$\Rightarrow 1.6\times10^{-19}$ Joule $= 1$ eV

1.6×10^{-12} erg $= 1$ eV

$\Rightarrow U \simeq 3\times10^{48}$ erg

$$t = \frac{U}{L_\odot} = \frac{3\times10^{48}\ \text{ergs}}{4\times10^{33}\ \text{erg/s}} \sim 10^{15}\ \text{s} \sim \frac{10^{15}}{3\times10^{7}\ \text{s/yr}} = \tfrac{1}{3}\times10^{8}\ \text{yr}$$

$$\simeq 10^{7}\ \text{yrs} \gg t_{\text{random walk}}$$

c.) ~~The [illegible]~~

$\Rightarrow$ Very little of the internal energy of the sun resides in the radiation field

e.g., compare aT^4 with $\frac{3}{2} N_{part}\, kT / R_\odot^3$

use central temperature $\Rightarrow$ overestimate of T^4

$$T^4 = (1\ \text{keV})^4 = (10^{-3}\ \text{MeV})^4 \simeq \frac{10^{-12}\ \text{MeV}^4}{(200\ \text{MeV-f})^3}$$

$$\simeq \frac{10^{-12}\ \text{MeV}}{8\times10^{6}\ \text{f}^3} \simeq 10^{-19}\ \frac{\text{MeV}}{\text{f}^3} = 10^{-19}\times10^{39}\ \frac{\text{MeV}}{\text{cm}^3} = 10^{20}\ \frac{\text{MeV}}{\text{cm}^3}$$

vs. 2×10^{60} eV $= 2\times10^{54}$ MeV $= U$ $\qquad R_\odot^3 \simeq 3\times10^{33}\ \text{cm}^3$

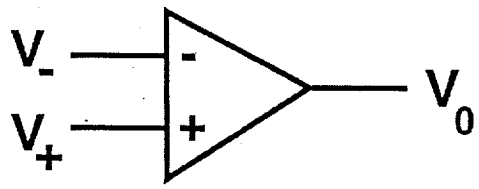
TR.

ID Number _____ PART I Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 5:

Problem 10. **[Experiment]** Electronic feedback circuits can be used to transform the impedance of a passive component. We consider two examples in the context of operational amplifiers (op amps). The op amp is a three terminal device, with two inputs, denoted V_+ and V_-, and one output, denoted V_0, whose characteristics can be idealized as:



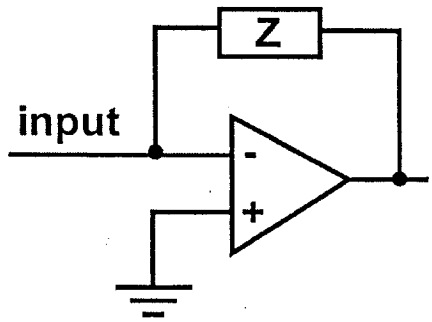
i) $V_0 = G(V_+ - V_-)$ where G is the gain (typically $G >> 1$);

ii) The inputs act as ideal voltmeters, *i.e.*, the input impedances can be taken as infinite or, formally, $Z_+ \approx Z_- \rightarrow \infty$; and

iii) The output acts as an ideal voltage source, *i.e.*, the output impedance can be taken as zero or, formally, $Z_0 \approx 0$.

Practical issues, such as constraints on the maximum and minimum value of V_0, are not of concern for these questions.

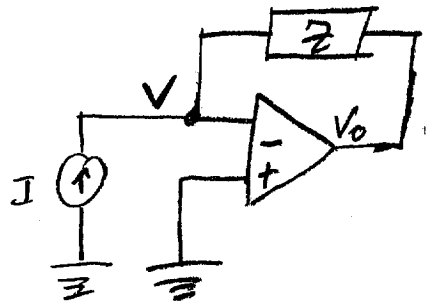
a) Consider the effect of inserting an impedance, denoted Z, in a negative feedback loop as shown below. What is the input impedance to the composite device as seen from the input line relative to ground [hint: apply an input current and calculate the voltage at the input]?



b) Consider the effect of a symmetric feedback circuit placed between the input and an impedance, denoted Z. What is the input impedance to the composite device as seen from the input line relative to ground? For simplicity, you may evaluate the input impedance in the limit of $G \rightarrow \infty$.

Problem 10

a)



Since $Z_- \to \infty$

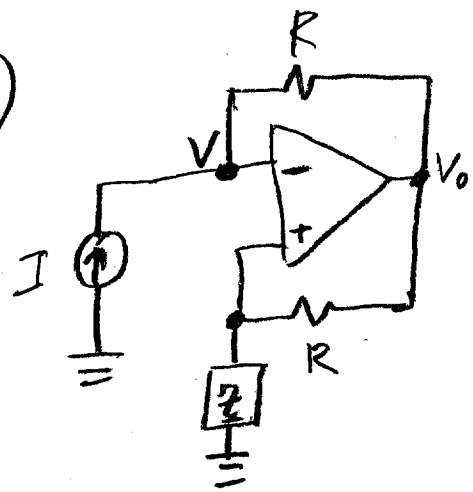
① $V_o = G(0-V) = -GV$

② and $\frac{V-V_o}{Z} = I$

$\therefore \frac{V+GV}{Z} = \frac{V(1+G)}{Z} = I$

$Z_{in} = \frac{V}{I} = \frac{Z}{1+G} \simeq \frac{Z}{G}$

b)



① $I = \frac{V-V_o}{R}$

② $0 = \frac{V_+}{Z} + \frac{V_+ - V_o}{R}$

③ $V_o = G(V_+ - V)$

$\therefore V_+\left(\frac{R+Z}{Z}\right) = V_o$ or $V_+ = \frac{Z}{R+Z} V_o$

as $G \to \infty$ $V_+ = \frac{Z}{R+Z} V_o = V$

$\therefore V_o = \frac{R+Z}{Z} V$

$I = \frac{V - \frac{R+Z}{Z} V}{R} = -\frac{1}{Z} V$

$Z_{in} = \frac{V}{I} = -Z$

ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 1:

Problem 11. **[Mathematics]** You want to solve the modified inhomogeneous Helmholtz equation

$$\nabla^2 u + 2i\mathbf{k}\cdot\mathbf{r}u + k^2 u = -\rho \qquad \text{[PDE]}$$

in the interior of the square $|x| < a$, $|y| < a$, with the periodic boundary condition on the square

$$\left.\begin{aligned} u(a,y) &= u(-a,y) \\ u(x,a) &= u(x,-a) \end{aligned}\right\} \qquad \text{[BC]}$$

A benefactor gives you a Green's function $G(\mathbf{r},\mathbf{r}')$ which, for any $\mathbf{r}'$, satisfies (PDE) and (BC) with $\rho(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$.

(a) Find the solution $u(\mathbf{r})$ in terms of $\rho(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$.

(b) Green's functions are often, but not always, symmetric functions of their two arguments. Is it necessarily true here that $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$? Justify your answer.

PART II: PROBLEM #11

New 11/26/01

Problems on Methods of Mathematical Physics

submitted by Donald R. Fredkin

Problem: You want to solve the modified inhomogeneous Helmholtz equation

$$\nabla^2 u + 2i\mathbf{k}\cdot\mathbf{r}\,u + k^2 u = -\rho \tag{PDE}$$

in the interior of the square $|x| < a$, $|y| < a$, with the periodic boundary condition on the square

$$\left.\begin{aligned} u(a,y) &= u(-a,y) \\ u(x,a) &= u(x,-a). \end{aligned}\right\} \tag{BC}$$

A benefactor gives you a Green's function $G(\mathbf{r},\mathbf{r}')$ which, for any $\mathbf{r}'$, satisfies (PDE) and (BC) with $\rho(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}')$.

(a) Find the solution $u(\mathbf{r})$ in terms of $\rho(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$.

(b) Green's functions are often, but not always, symmetric functions of their two arguments. Is it necessarily true here that $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$? Justify your answer.

Solution:

(a)

$$u(\mathbf{r}) = \int_S d^2r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'),$$

where S is the given square $||\mathbf{r}||_\infty < a$.

(b) For two solutions of (PDE) and (BC) with different source terms $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$, we have

$$\begin{aligned} &\nabla\cdot[\bar{u}_1(\mathbf{r})\nabla u_2(\mathbf{r}) - \nabla\bar{u}_1(\mathbf{r})u_2(\mathbf{r}) + i\mathbf{k}\bar{u}_1(\mathbf{r})u_2(\mathbf{r})] = \\ &\qquad \bar{u}_1(\mathbf{r})[\nabla^2 u_2(\mathbf{r}) + i\mathbf{k}\cdot\nabla u_2(\mathbf{r})] - \overline{\nabla^2 u_1(\mathbf{r}) + i\mathbf{k}\cdot\nabla u_1(\mathbf{r})}u_2(\mathbf{r}) = \bar{u}_1(\mathbf{r})\rho_2(\mathbf{r}) - \bar{\rho}_1(\mathbf{r})u_2(\mathbf{r}). \end{aligned}$$

Integrate over S, use the divergence theorem, and note that the boundary terms cancel because the integrands at opposite points are the same but the normals are opposite (opposite points mean $(\pm a, y)$ or $(x, \pm a)$). The result is the reciprocity relation

$$\int_S d^2r\,[\bar{u}_1(\mathbf{r})\rho_2(\mathbf{r}) - \bar{\rho}_1(\mathbf{r})u_2(\mathbf{r})] = 0.$$

Now choose $\rho_1(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_1)$ and $\rho_2(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_2)$. We then have

$$\bar{G}(\mathbf{r}_2,\mathbf{r}_1) = G(\mathbf{r}_1,\mathbf{r}_2),$$

so the Green's function is Hermitean. It will not, in general, be symmetric unless it happens to be real, which is unlikely because (PDE) has a complex coefficient.

ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 1:

Problem 12. [Mathematics] Calculate

$$\int_0^\infty \frac{x \sin x}{a^2 + x^2}\, dx$$

PART II : PROBLEM #12

Problem: Calculate

$$\int_0^\infty \frac{x \sin x}{a^2 + x^2} dx.$$

Solution: The integral in question can be written as $\Im I/2$, where

$$I = \int_{-\infty}^{\infty} \frac{x e^{ix}}{a^2 + x^2} dx.$$

By Jordan's lemma, the integration path can be closed with a giant semicircle in the upper half-plane. Then, using the residue theorem,

$$I = 2\pi i \operatorname*{Res}_{x=ia} \frac{x e^{ix}}{a^2 + x^2} = \pi i e^{-a}$$

and the desired integral is

$$\frac{\pi}{2} e^{-a}.$$

ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 2:

Problem 13. **[Classical Mechanics]** Find the conditions under which

$$Q = \frac{\alpha p}{x}, \quad P = \beta x^2$$

where α, β are constants, is a canonical transformation for a system with one degree of freedom. Find a generating function and apply the transformation to solve the harmonic oscillator.

PART II : PROBLEM #13

Find the conditions under which

$$Q = \frac{\alpha p}{x} \qquad P = \beta x^2$$

where α, β are constants is a canonical transformation for a system with 1 degree of freedom. Find a generating function and Apply the transformation to solve the harmonic oscillator

---

$$[Q, P] = \frac{\partial Q}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial x} = \left(-\frac{\alpha p}{x^2}\right)(0) - \left(\frac{\alpha}{x}\right)(2\beta x)$$

$$= -2\alpha\beta = 1 \quad \text{for a canonical transformation.}$$

$$\therefore \quad Q = \frac{\alpha p}{x}, \quad P = -\frac{x^2}{2\alpha}$$

For a type I generating function $F_1(q, Q, t)$, $p = \frac{\partial F}{\partial q}$ $\quad P = -\frac{\partial F}{\partial Q}$

here: $\quad p = \frac{xQ}{\alpha} = \frac{\partial F}{\partial x} \Rightarrow F = \frac{1}{2\alpha} x^2 Q$

Then $\quad P = -\frac{\partial F}{\partial Q} = -\frac{1}{2\alpha} x^2$ ✓

Harmonic oscillator: $\quad H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$K = H + \frac{\partial F}{\partial t} = H = \left(\frac{xQ}{\alpha}\right)^2 \frac{1}{2m} + \frac{1}{2} m\omega^2 (-2\alpha P)$$

$$= \frac{1}{2m\alpha^2} Q^2(-2\alpha P) - m\alpha\omega^2 P = -m\alpha\omega^2 P - \frac{Q^2 P}{m\alpha}$$

Let $\alpha = -\frac{1}{m}$: $\quad K = Q^2 P + \omega^2 P = (Q^2 + \omega^2) P$.

(1) $\quad \dot{P} = -\frac{\partial K}{\partial Q} = -2QP$

(2) $\quad \dot{Q} = \frac{\partial K}{\partial P} = Q^2 + \omega^2$

$$\therefore \quad \omega = \sqrt{\frac{g}{a} - \Omega^2} = \sqrt{\Omega_c^2 - \Omega^2} \qquad \Omega \leq \Omega_c$$

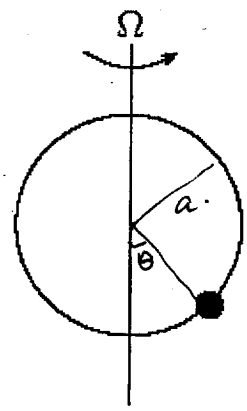
$$= \sqrt{\Omega^2 - \frac{g^2}{a^2\Omega^2}} = \sqrt{\Omega^2 - \frac{\Omega_c^4}{\Omega^2}} = \frac{1}{\Omega}\sqrt{\Omega^4 - \Omega_c^4} \qquad \Omega \geq \Omega_c$$

ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

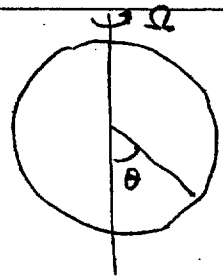
SECTION 2:

Problem 14. **[Classical Mechanics]** A particle of mass m is free to move along a vertical wire hoop of radius a. The hoop rotates at a constant angular frequency Ω about a vertical axis passing through its center. There is a downward gravitation field g. Write down the Lagrangian for the system. Find the frequency of small oscillations about the equilibrium as a function of Ω.



PART II : PROBLEM #14

1.



$$L = \frac{1}{2} m \left(a^2 \dot{\theta}^2 + a^2 \sin^2\theta\, \Omega^2 \right) + mga\cos\theta$$

$$= \underbrace{\frac{1}{2} ma^2 \dot{\theta}^2}_{\substack{\text{Kinetic} \\ T}} + \underbrace{\left(\frac{1}{2} ma^2 \sin^2\theta\, \Omega^2 + mga\cos\theta \right)}_{\substack{\text{Potential} \\ -V}}$$

$\therefore$ if θ = coordinate

ma^2 = mass,

$V = -\frac{1}{2} ma^2 \sin^2\theta\, \Omega^2 - mga\cos\theta$ = potential energy

Equilibrium: $\frac{\partial V}{\partial \theta} = 0 \Rightarrow -ma^2 \sin\theta \cos\theta\, \Omega^2 + mga\sin\theta = 0$

$\therefore \quad \sin\theta = 0 \quad$ or $\quad \cos\theta = \frac{g}{a\Omega^2}$

$\therefore \quad \theta = 0, \pi \quad$ or $\quad \theta_0$

$$\frac{\partial^2 V}{\partial \theta^2} = -ma^2\Omega^2 \left(\cos^2\theta - \sin^2\theta \right) + amg\cos\theta$$

$$= amg - ma^2\Omega^2 \qquad \theta = 0$$

$$= -amg - ma^2\Omega^2 \qquad \theta = \pi$$

$$= -ma^2\Omega^2 \left(2\left(\frac{g}{a\Omega^2}\right)^2 - 1 \right) + mga\frac{g}{a\Omega^2}$$

$$= ma^2\Omega^2 - \frac{g^2 m}{\Omega^2}, \quad \theta = \theta_0$$

$\therefore$ stable equilibrium is $\theta = 0$ for $\Omega \lesssim \Omega_c$ $\quad \Omega_c = \sqrt{g/a}$

$\theta = \theta_0$ for $\Omega \gtrsim \Omega_c$

ω = Oscillation frequency $= \sqrt{\frac{k}{m}}$ here $k = \frac{\partial^2 V}{\partial \theta^2}$ $\quad m \to ma^2$

$$(2) \Rightarrow \quad \frac{dQ}{Q^2+\omega^2} = dt \quad \Rightarrow \quad \frac{1}{\omega}\tan^{-1}\frac{Q}{\omega} = t + c$$

$$\therefore \quad Q = \omega \tan(\omega t + c')$$

$$\dot{P} = -2\omega \tan(\omega t + c')\, P$$

$$\therefore \quad \ln P = -2\omega \int \tan(\omega t + c')$$

$$= +2 \ln \cos(\omega t + c') + c''$$

$$\therefore \quad P = \frac{c'''}{\cancel{\cos(\omega t + c')}} \cos^2(\omega t + c')$$

$$\therefore \quad x = \sqrt{-2\alpha P} = \sqrt{\frac{2P}{m}} = \sqrt{\frac{2c''}{m}} \cos(\omega t + c')$$

$$p = \frac{xQ}{\alpha} = -mxQ = -m\omega\sqrt{\frac{2c''}{m}} \sin(\omega t + c')$$

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PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 3:

Problem 15. **[Electrodynamics]** Two conducting plates are separated by a distance d, where d is small compared to the dimensions of the plates. One of the plates has a small hemispherical boss of radius a, where $a << d$. This plate is grounded and the other plate is maintained at voltage V. Find the electric potential in the vicinity of the boss.

PART II: PROBLEM #1

3. Two conducting plates are separated by distance d, where d is small compared to the dimensions of the plates (see figure). One of the plates has a small hemispherical boss of radius a, where $a \ll d$. This plate is grounded and the other plate is maintained at voltage V. Find the electric potential in the vicinity of the boss.

Use spherical coordinates with origin at center of hemispherical boss.

$$\varphi(r,\theta) = A\frac{1}{r^2}\cos\theta + Br\cos\theta$$

$$Br\cos\theta = Bz \simeq V \quad \text{for } z = d$$

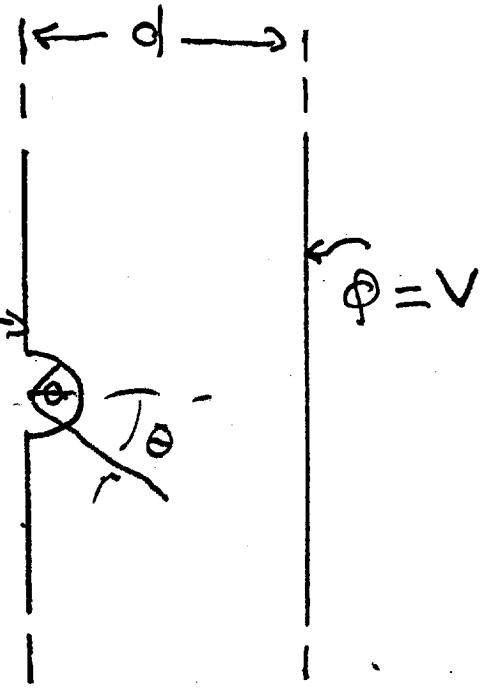
$$\therefore \; B \simeq V/d$$

$\varphi = 0$ on boss and for $\theta = \pi$

$$\therefore \; \frac{A}{a^2} = -Ba = -\frac{V}{d}a$$

$$A = -Va^3/d$$

$$\varphi(r,\theta) = \frac{V}{d}r\cos\theta - \frac{Va^3}{r^2 d}\cos\theta$$



ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 3:

Problem 16. **[Electrodynamics]** A long cylindrical copper rod is of radius R. For $t \leq 0$, the rod is immersed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, where (z, ρ, φ) is a cylindrical coordinate system with the z-axis coincident with the axis of the rod. At $t = 0$, the external field is switched off (or the rod is suddenly jerked out of the field). You may assume that $T = \sigma R^2 / c^2$ is large compared to the time for the external field to be switched off, where σ is the conductivity of the copper rod.

(a) Obtain an equation that governs the evolution of $B_z(\rho, \varphi, t)$ in a cross section of the rod that is far from either end. What boundary condition does $B_z(\rho, \varphi, t)$ satisfy at $\rho = R$?

(b) Find $B_z(\rho, \varphi, t)$ for $t > 0$ and $\rho < R$.

PART II: PROBLEM #16

7. A long cylindrical copper rod is of radius R. For $t \le 0$, the rod is immersed in a uniform external magnetic field $\mathbf{B} = \hat{z}B_0$, where (z, r, θ) is a cylindrical coordinate system with the z-axis coincident with the axis of the rod. At $t = 0$, the external field is switched off (or the rod is suddenly jerked out of the field). You may assume that $T = \sigma R^2/c^2$ is large compared to the time for the external field to be switched off, where σ is the conductivity of the copper rod.

a. Obtain an equation that governs the evolution of $B_z(r, \theta, t)$ in a cross section of the rod that is far from either end. What boundary condition does $B_z(r, \theta, t)$ satisfy at $r = R$?

$$\nabla \times \mathbf{B} \simeq \frac{4\pi}{c}\mathbf{J} = \frac{4\pi\sigma}{c}\mathbf{E}, \quad \underbrace{\nabla \times \nabla \times \mathbf{B}}_{-\nabla^2 \mathbf{B}} = \frac{4\pi\sigma}{c}\underbrace{\nabla \times \mathbf{E}}_{-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}}$$

$$\nabla_\perp^2 B_z - \frac{4\pi\sigma}{c^2}\frac{\partial B_z}{\partial t} = 0$$

$$B_z(r = R, t) = 0 \quad \text{for } t > 0$$

b. Find $B_z(r, \theta, t)$ for $t > 0$ and $r < R$.

Let $$B_z(r, t) = \sum_n A_n J_0\left(\frac{r}{R}\chi_{0n}\right) e^{-\nu_n t}$$

$$-\frac{\chi_{0n}^2}{R^2} + \frac{4\pi\sigma}{c^2}\nu_n = 0, \quad \nu_n = \frac{c^2\chi_{0n}^2}{4\pi\sigma R^2}$$

$$B_z(r, t = 0) = B_0 \quad \text{for } r < R$$

$$\therefore B_0 \underbrace{\int_0^R r\,dr\, J_0\left(\frac{r}{R}\chi_{0n}\right)}_{\left(\frac{R}{\chi_{0n}}\right)^2 J_1(\chi_{0n})\chi_{0n}} = A_n \underbrace{\int_0^R r\,dr\, J_0^2\left(\frac{r}{R}\chi_{0n}\right)}_{\frac{R^2}{2}J_1^2(\chi_{0n})}$$

$$A_n = \frac{2B_0}{J_1(\chi_{0n})\chi_{0n}}$$

ID Number _____ PART II Score _____

PHYSICS DEPARTMENTAL EXAM -- SPRING 2002

SECTION 4:

Problem 17. **[Quantum Mechanics]** Consider a neutrino created as an electron flavor neutrino, ν_e, in the sun at time $t=0$. We can describe the subsequent evolution of this neutrino as it propagates to earth through the vacuum of space either in terms of a flavor description or an energy eigenstate description. Suppose these two bases are related to one another as follows:

$$|\nu_e\rangle = \frac{1}{\sqrt{2}}\left(|\nu_1\rangle + |\nu_2\rangle\right)$$
$$|\nu_\mu\rangle = \frac{1}{\sqrt{2}}\left(-|\nu_1\rangle + |\nu_2\rangle\right)$$

Here $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are states of definite neutrino flavor and can be taken to be orthonormal, *e.g.*, $\langle\nu_e|\nu_\mu\rangle = 0$ and $\langle\nu_e|\nu_e\rangle = 1$, and $|\nu_1\rangle$ and $|\nu_2\rangle$ represent energy eigenstates corresponding to neutrino mass eigenvalues m_1 and m_1, respectively. This basis can be taken to be orthonormal. Assume the neutrino to be ultra-relativistic so that the relationship between energy and momentum for the first energy eigenstate is $E_1 = \sqrt{p^2c^2 + m_1^2c^4} \approx pc + m_1^2c^4/2E_\nu$, while the relationship for the second is $E_2 = \sqrt{p^2c^2 + m_2^2c^4} \approx pc + m_2^2c^4/2E_\nu$. Here we take the momentum p to be the same for each state and we define the approximate energy of the neutrino as $E_\nu \equiv pc$. The speed of light is c.

a) If the neutrino is created at time $t=0$ as a $|\nu_e\rangle$ in the sun, what is the probability that it is still a $|\nu_e\rangle$ after propagating a distance ct? Ignore the effects of the matter in the sun. Express your result in terms of the difference of the squares of the vacuum neutrino mass eigenvlaues, $\delta m^2 \equiv m_2^2 - m_1^2$ and the approximate overall neutrino energy E_ν.

b) If the earth is a distance 1.5×10^{13} cm from the sun, and the neutrino energy is $E_\nu = 10\,\text{MeV}$, what must δm^2 be in order that the probability for detecting the neutrino as a ν_e is minimal at the earth?

Quantum problem 1

PART II: PROBLEM #7

(a)

$$|\nu_e\rangle = \frac{1}{\sqrt{2}}\left(|\nu_1\rangle + |\nu_2\rangle\right)$$

$$|\nu_\mu\rangle = \frac{1}{\sqrt{2}}\left(-|\nu_1\rangle + |\nu_2\rangle\right)$$

$$|\psi(t=0)\rangle = |\nu_e\rangle = \frac{1}{\sqrt{2}}\left(|\nu_1\rangle + |\nu_2\rangle\right)$$

$$|\psi(t)\rangle \approx \frac{1}{\sqrt{2}}\left(|\nu_1\rangle e^{-i\, m_1^2/2E_\nu\, t} + |\nu_2\rangle e^{-i\, m_2^2/2E_\nu\, t}\right)$$

ignore overall common phase $E = (p^2+m^2) \simeq p + \frac{m^2}{2}$

$$\langle\nu_e|\psi(t)\rangle = \frac{1}{2}\left(\langle\nu_1| + \langle\nu_2|\right)\left(|\nu_1\rangle e^{-i\, m_1^2/2E_\nu\, t} + |\nu_2\rangle e^{-i\, m_2^2/2E_\nu\, t}\right)$$

$$= \frac{1}{2}\left(e^{-i\, m_1^2/2E_\nu\, t} + e^{-i\, m_2^2/2E_\nu\, t}\right)$$

$$\text{Prob}\,(\nu_e \to \nu_e) = |\langle\nu_e|\psi(t)\rangle|^2 \qquad \alpha_{1,2} = \frac{m_{1,2}^2}{2E_\nu}\, t$$

$$= \frac{1}{4}\left(e^{-i\,\alpha_1} + e^{-i\,\alpha_2}\right)\left(e^{i\,\alpha_1} + e^{i\,\alpha_2}\right)$$

$$= \frac{1}{4}\left(1 + \underbrace{e^{i(\alpha_2-\alpha_1)}}_{\cos(\alpha_2-\alpha_1)+i\sin(\alpha_2-\alpha_1)} + \underbrace{e^{i(\alpha_1-\alpha_2)}}_{\cos(\alpha_2-\alpha_1)-i\sin(\alpha_2-\alpha_1)} + 1\right)$$

$$= \frac{1}{2}\left(1 + \cos(\alpha_2-\alpha_1)\right)$$

$$= \frac{1}{2}\left(1 + \cos\left(\frac{m_2^2-m_1^2}{2E_\nu}\, t\right)\right) = \frac{1}{2}\left(1 + \cos\overbrace{\left(\frac{\delta m^2}{2E_\nu}\, t\right)}^{2\theta}\right)$$

$\delta m^2 \equiv m_2^2 - m_1^2$

$$= \frac{1}{2}\left(1 + \cos^2 2\theta\right) = \frac{1}{2}\left(1 + (c^2 - s^2)\right)$$

$$= \frac{1}{2}\left(2 - 2\sin^2\theta\right)$$

$$= 1 - \sin^2\left(\frac{\delta m^2}{4E_\nu}\, t\right)$$

Prob 1 [illegible] Prob$_{\nu_e\to\nu_e}$ = 1 again @ $t = \frac{4\pi E_\nu}{\delta m^2}$

(b.) $\frac{1}{2}$ cycle $\Rightarrow$ $\text{Prob}_{\nu_e \to \nu_e} = 0$

$$\frac{\delta m^2}{4E_\nu}(ct) = \pi/2 \quad \text{or}$$

$$\delta m^2 = \frac{4\pi E_\nu}{2\,(ct)} = \frac{4\pi\,(10\ \text{MeV})\,(\hbar c)}{2\,(1.5\times10^{13}\,\text{cm})\,(10^{13}\,\frac{f}{cm})}$$

$\simeq 197$ MeV-f

$$\simeq \frac{4\pi\,(10\ \text{MeV})\,(200\ \text{MeV-f})}{3\times10^{26}\ f}$$

$$\simeq 8.27\times10^{-23}\ \text{MeV}^2 \simeq 8.3\times10^{-11}\ \text{eV}^2$$

$$\simeq 10^{-10}\ \text{eV}^2$$

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SECTION 4:

Problem 18. **[Quantum Mechanics]** Consider two identical linear oscillators each with spring constant k. Suppose these oscillators additionally are coupled by an interaction term ax_1x_2, where x_1 and x_2 are the oscillator (displacement) variables for the first and second oscillator, respectively, and a is a constant. Find the energy levels for this system.

PART II: PROBLEM #1

Problem [4] (4) Consider two identical linear oscillators each with spring constant k. Additionally, These oscillators are coupled by an interaction term $a x_1 x_2$ (and a is a constant) where x_1 and x_2 are the oscillator variables. Find the energy levels for this system.

---

sol'n: total Hamiltonian is

$$H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}k\left(x_1^2 + x_2^2\right) + a\, x_1 x_2$$

define new variables $x_1 = \dfrac{\xi+\eta}{\sqrt{2}}$ $\quad x_2 = \dfrac{\xi-\eta}{\sqrt{2}}$

or $\xi = \dfrac{x_1+x_2}{\sqrt{2}}$ & $\eta = \dfrac{x_1-x_2}{\sqrt{2}}$

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x_1}\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial \xi}{\partial x_2}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right) = \left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}\right)\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}\right) = \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) + \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) - \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}$$

so $$\boxed{H = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) + \frac{1}{2}(k+a)\xi^2 + \frac{1}{2}(k-a)\eta^2}$$

so exact energy levels are

$$\boxed{E = n_1 \hbar \omega_1 + n_2 \hbar \omega_2 + \frac{\hbar}{2}(\omega_1 + \omega_2)}$$

with n_1 & n_2 positive integers and

$$\boxed{\omega_1^2 = \frac{k-a}{m}} \qquad \boxed{\omega_2^2 = \frac{k+a}{m}}$$

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SECTION 5:

Problem 19. **[Statistical Mechanics]** A surface consists of a number N of adsorption sites. Each site can accommodate at most one Q particle. Each surface-bound Q particle has energy $-W < 0$. (Since each adsorption site is distinct, the surface-bound Q particles may be regarded as distinguishable.)

(a) Working in the grand canonical ensemble, find an expression for the average occupancy $f(\mu, T)$ of each adsorption site.

Now consider a degenerate quantum gas of Q particles. The Q particles are spin-0 bosons obeying the dispersion relation $\varepsilon(\mathbf{p}) = A|\mathbf{p}|^3$.

(b) Find the equation of state for the gas, which relates μ, T, and the number density n.

(c) Under conditions of gas-surface equilibrium, find an expression for $f(n, T)$.

DAN AROVAS
~~(NOT USED)~~
PART II: PROBLEM 1

Graduate Statistical Mechanics Problem

A surface consists of a number N of adsorption sites. Each such site can accomodate at most one Q particle. Each surface-bound Q particle has energy $-W < 0$. (Since each adsorption site is distinct, the surface-bound Q particles may be regarded as distinguishable.)

(a) Working in the grand canonical ensemble, find an expression for the average occupancy $f(\mu, T)$ of each adsorption site.

Now consider a degenerate quantum gas of Q particles. The Q particles are spin-0 bosons obeying the dispersion $\varepsilon(\boldsymbol{p}) = A|\boldsymbol{p}|^3$.

(b) Find the equation of state for the gas which relates μ, T, and the number density n.

(c) Under conditions of gas-surface equilibrium, find an expression for $f(n, T)$.

<u>Solution</u>

(a) Treating each of the N adsorption sites as independent, and obeying the occupancy constraint of at most one particle per adsorption site, we have

$$f(T) = \frac{e^{\mu/k_{\rm B}T} e^{W/k_{\rm B}T}}{1 + e^{\mu/k_{\rm B}T} e^{W/k_{\rm B}T}} = \frac{1}{1 + e^{-(\mu+W)/k_{\rm B}T}} \ .$$

(b) The density $n(\mu, T)$ is given by

$$\begin{aligned} n(\mu, T) &= \int \frac{d^3p}{h^3} \frac{1}{z^{-1} e^{\varepsilon(p)/k_{\rm B}T} - 1} \\ &= \frac{4\pi k_{\rm B}T}{3Ah^3} \int_0^\infty \frac{du}{z^{-1}e^u - 1} \\ &= -\frac{4\pi k_{\rm B}T}{3Ah^3} \ln(1 - z) \ , \end{aligned}$$

where $z \equiv e^{\mu/k_{\rm B}T}$ is the fugacity, and where we've used

$$\frac{1}{z^{-1}e^u - 1} = z\, e^{-u} + z^2\, e^{-2u} + z^3\, e^{-3u} + \ldots = \sum_{k=1}^{\infty} z^k\, e^{-ku}$$

and

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1 - x) \ .$$

Thus, we obtain

$$n(\mu, T) = -\frac{4\pi k_{\rm B}T}{3Ah^3} \ln\left(1 - e^{\mu/k_{\rm B}T}\right) \ ,$$

or, equivalently,

$$\mu(n, T) = k_{\rm B}T \ln\left(1 - e^{-3Ah^3 n/4\pi k_{\rm B}T}\right) \ .$$

(c) Gas-surface equilibrium means $\mu_{\rm gas} = \mu_{\rm surface} \equiv \mu$. Thus, combining our previous results, we obtain

$$f(n, T) = \frac{1 - \exp(-3Ah^3 n/4\pi k_{\rm B}T)}{1 - \exp(-3Ah^3 n/4\pi k_{\rm B}T) + \exp(-W/k_{\rm B}T)} \ .$$

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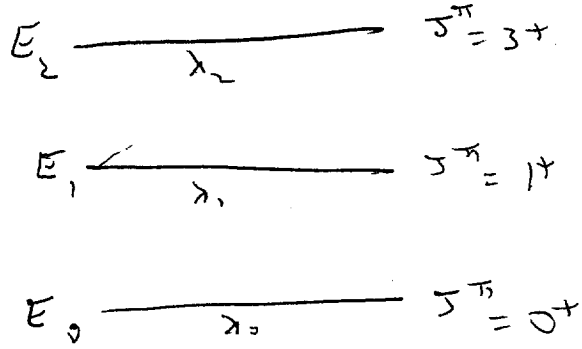
SECTION 5:

Problem 20. **[Statistical Mechanics]** Consider a fictitious three-level nucleus in thermal equilibrium with a heat bath at temperature T. In particular, all of the three levels of this nucleus have population ratios in accord with thermal equilibrium. The ground state energy of this nucleus is E_0 and the ground state spin and parity are $J^\pi = 0^+$. (The spin and parity quantum numbers label the overall angular momentum and parity of the wave function for a given level.) The first excited state has energy E_1 and spin/parity $J^\pi = 1^+$, while the third state has energy E_2 and spin/parity $J^\pi = 3^+$. The laboratory (zero temperature) decay rate of the ground state is measured to be λ_0; while the laboratory individual decay rates of the first and second excited states are measured to be λ_1 and λ_2, respectively. What is the decay rate of this nucleus in the thermal bath?

SECTION II:
PROBLEM 20



T

Partition function

$$Z = \sum_i (2J_i+1)\, e^{-E_i/kT}$$

$$= 1 + 3 \cdot e^{-E_1/kT} + 7\, e^{-E_2/kT}$$

or if you don't set E_0 to 0

$$\Rightarrow Z = 1 + 3\, e^{-(E_1-E_0)/kT} + 7\, e^{-(E_2-E_0)/kT}$$

Population factors $\Rightarrow P_i = \dfrac{(2J_i+1)\, e^{-E_i/kT}}{Z}$

total decay rate $\lambda = \sum_i P_i \lambda_i$

$$= \frac{\lambda_0 + 3\lambda_1 e^{-E_1/kT} + 7\lambda_2 e^{-E_2/kT}}{Z}$$

$$= \frac{\lambda_0 + 3\lambda_1 e^{-E_1/kT} + 7\lambda_2 e^{-E_2/kT}}{1 + 3\, e^{-E_1/kT} + 7\, e^{-E_2/kT}}$$