Qual Review, Meeting 8: Potpourri 4

1. The bosonic low-energy excitations of a **two-dimensional** system of dimensions $L \times L$ are described by the monster wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} + C\nabla^4 u = 0.$$

(The scalar $u(\mathbf{r}, t)$ might, for example, represent height fluctuations transverse to a plane).

1a. Find the dispersion relation for this system.

1b. Find the density of states $g(E)$.

1c. Compute, to within a numerical constant, the low-temperature specific heat $C(T)$.

2. Consider a good conductor of finite conductivity σ and permeability μ that takes up the region of space $z < 0$. A low-frequency electromagnetic wave propagates in the $\hat{x}$ direction with the electric field normal to the surface of the conductor. Calculate the skin depth, which is the distance inside the conductor it takes for the electric and magnetic fields to fall to $1/e$ of their magnitudes outside.

3a. Consider a spin-1/2 particle with magnetic moment $\vec{\mu} = \gamma\vec{\sigma}$, where $\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$ with the σ_i the usual Pauli spin matrices. The particle is immersed in a sexy magnetic field $\mathbf{B}$ that points in the direction (θ, ϕ). Taking the spin quantization axis to be the z-axis (i.e. $\theta = 0$), find the energy eigenvalues and eigenstates.

3b. For a particle in each of these two eigenstates, find the probability for its spin to be measured along the $+x$ direction.

3c. Four spin-1/2 particles interact pairwise according to the Hamiltonian

$$H = J \sum_{i<j} \mathbf{S}_i \cdot \mathbf{S}_j$$

where J is a constant and i and j label the particles. The sum is over all possible pairs. Find all the energy eigenvalues.

Potpourri 4 Solns

1] a] Just go Fourier!

$$\rho \frac{\partial^2 u}{\partial t^2} + C\nabla^4 u = 0 \Rightarrow -\rho\omega^2 + Ck^4 = 0$$
$$\Rightarrow \omega = \sqrt{\frac{c}{\rho}}\, k^2.$$

] Total # of states in volume V of space & up to magnitude P in momentum space:

$$\Omega = \int \frac{d^2x}{h^2} \int_0^P d^2p = \frac{L^2}{(2\pi\hbar)^2}\pi P^2 \Rightarrow \text{density of states is } \frac{L^2}{(2\pi\hbar)^2} 2\pi P\, dP$$

From part a, $\varepsilon = \sqrt{\frac{c}{\rho}}\frac{p^2}{\hbar} \Rightarrow p = \left(\frac{\rho}{c}\right)^{1/4}\hbar^{1/2}\varepsilon^{1/2}$, so $dp = \frac{1}{2}\left(\frac{\rho}{c}\right)^{1/4}\hbar^{1/2}\varepsilon^{-1/2}d\varepsilon$

$$\Rightarrow g(\varepsilon) = \frac{L^2}{(2\pi\hbar)^2} 2\pi \left(\frac{\rho}{c}\right)^{1/4}\hbar^{1/2}\varepsilon^{1/2}\frac{1}{2}\left(\frac{\rho}{c}\right)^{1/4}\hbar^{1/2}\varepsilon^{-1/2}$$

$$\Rightarrow g(\varepsilon) = \left(\frac{\rho}{c}\right)^{1/2}\frac{L^2}{2\pi\hbar}\frac{1}{2} = \boxed{\frac{L^2}{4\pi\hbar}\left(\frac{\rho}{c}\right)^{1/2}}$$

] Mean energy:

$$E = \int \langle n_\varepsilon \rangle \varepsilon g(\varepsilon) d\varepsilon,$$

Using $\langle n_\varepsilon \rangle = \dfrac{1}{e^{\beta\varepsilon}-1}$ (with $\mu = 0$, as is normal for $T \to 0$ bosons).

$$\Rightarrow E = \int_0^\infty \frac{\varepsilon}{e^{\beta\varepsilon}-1} \frac{L^2}{4\pi\hbar}\left(\frac{\rho}{c}\right)^{1/2} d\varepsilon$$

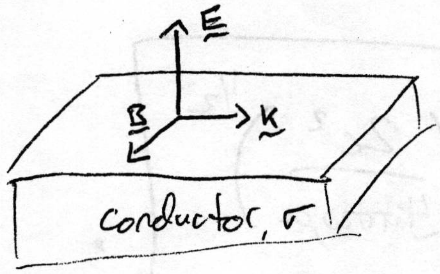
$$\Rightarrow E = \frac{L^2}{4\pi\hbar}\left(\frac{\rho}{c}\right)^{1/2}(k_B T)^2 \int_0^\infty \frac{x\,dx}{e^x-1}$$

$$\Rightarrow E = \frac{L^2}{4\pi\hbar}\left(\frac{\rho}{c}\right)^{1/2} k_B^2 T^2 \int_0^\infty \frac{x\,dx}{e^x-1}$$

So $\boxed{C(T) = \dfrac{\partial E}{\partial T} = \dfrac{2k_B^2 T L}{4\pi\hbar}\left(\dfrac{\rho}{c}\right)^{1/2} \displaystyle\int_0^\infty \frac{x\,dx}{e^x-1}}$

$$\left(\text{FYI} \int_0^\infty \frac{x\,dx}{e^x-1} = \frac{\pi^2}{6}. \right)$$

2a]



Maxwell's eqns: $\nabla\times\underset{\sim}{H} = \frac{4\pi}{c}\underset{\sim}{j} + \cancel{\frac{1}{c}\frac{\partial \underset{\sim}{D}}{\partial t}}$ low freq.

$$\nabla\times\underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\Rightarrow \underset{\sim}{E} = \frac{c}{4\pi\sigma}\nabla\times\underset{\sim}{H}$$

$$\underset{\sim}{H} = \frac{c}{i\omega\mu}\nabla\times\underset{\sim}{E}.$$

Only changes will be in $\hat{z}$ direction, so write $\nabla = \hat{z}\frac{\partial}{\partial z}$.

$$\Rightarrow \nabla\times\underset{\sim}{E} = \frac{c}{4\pi\sigma}\nabla\times(\nabla\times\underset{\sim}{H}) = \frac{-c}{4\pi\sigma}\nabla^2\underset{\sim}{H}$$

$$\Rightarrow \frac{i\omega\mu}{c}\underset{\sim}{H} = -\frac{c}{4\pi\sigma}\nabla^2\underset{\sim}{H} \Rightarrow \frac{\partial^2\underset{\sim}{H}}{\partial z^2} + \frac{4\pi\sigma\omega\mu i}{c^2}\underset{\sim}{H} = 0.$$

$$\Rightarrow \underset{\sim}{H}_{in} = Ae^{ikz}, \text{ where } k^2 = \frac{4\pi\sigma\omega\mu}{c^2}i \Rightarrow k = \left(\frac{4\pi\sigma\omega\mu}{c^2}\right)^{1/2}\begin{Bmatrix} e^{i\pi/4} \\ e^{5\pi i/4}\end{Bmatrix}$$

We need the solution that dies as $z\to-\infty$. $e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$e^{5\pi i/4} = -\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

outside surface

$$\Rightarrow \underset{\sim}{H}_{inside} = \underset{\sim}{H}_0 e^{i\left(\frac{4\pi\sigma\omega\mu}{c^2}\right)^{1/2}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)z} = \underset{\sim}{H}_0 e^{\left(\frac{4\pi\sigma\omega\mu}{2c^2}\right)^{1/2}z} e^{-i\left(\frac{4\pi\sigma\omega\mu}{2c^2}\right)^{1/2}z}.$$

Can also write this as

$$\underline{H}_{in} = \underline{H}_0\, e^{z/\delta} e^{-iz/\delta}, \text{ with } \boxed{\delta = \left(\frac{2c^2}{4\pi\sigma\omega\mu}\right)^{1/2}}.$$

3a] $H = -\underset{\sim}{\mu}\cdot\underset{\sim}{B} = -\gamma\underset{\sim}{\sigma}\cdot\underset{\sim}{B}$

$$= -\gamma B\left[\sin\theta\cos\phi\begin{pmatrix}0&1\\1&0\end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$$

$$\Rightarrow H = -\gamma B\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

Clearly, since this is just a rotation of $\underset{\sim}{B} = B\hat{z}$,

Eigenvals will be $\pm\gamma B$.

Eigenvectors: $-\cancel{\gamma B}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \pm\begin{pmatrix}a\\b\end{pmatrix}\cancel{\gamma B}$

$$\Rightarrow \begin{matrix} a\cos\theta + b\sin\theta e^{-i\phi} = \mp a \\ a\sin\theta e^{i\phi} - b\cos\theta = \mp b\end{matrix}$$ (let's just do $+\gamma B$ now)

$$b = \frac{a\sin\theta e^{i\phi}}{\cos\theta - 1} = \frac{a\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi}}{1-2\sin^2(\frac{\theta}{2})-1} = -a\cot(\tfrac{\theta}{2})e^{i\phi}$$

So can write this eigenstate as $|\psi_-\rangle = \begin{pmatrix}-\sin(\frac{\theta}{2})e^{-i\phi/2}\\ \cos(\frac{\theta}{2})e^{i\phi/2}\end{pmatrix}$ $(E = \mp\gamma B)$

& the other as $|\psi_+\rangle = \begin{pmatrix}\cos(\frac{\theta}{2})e^{i\phi/2}\\ \sin(\frac{\theta}{2})e^{-i\phi/2}\end{pmatrix}$ with the $\phi/2$ phase just there to make stuff look nice. Overall phase arbitrary.

b] $\chi_{+,x} = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$$\Rightarrow P_1 = |\langle \chi_{+,x} | \psi_+ \rangle|^2 = \frac{1}{2}(1 + \sin\theta\cos\phi)$$

$$P_2 = |\langle \chi_{+,x} | \psi_- \rangle|^2 = \frac{1}{2}(1 - \sin\theta\cos\phi)$$

c] $H = J \sum_{i<j} \underset{\sim}{S_i} \cdot \underset{\sim}{S_j} = \frac{J}{2}\left[(\underline{S_1} + \underline{S_2} + \underline{S_3} + \underline{S_4})^2 - (S_1^2 + S_2^2 + S_3^2 + S_4^2) \right]$

$$= \frac{J}{2}\left[S_{TOT}^2 - 4(s+1)s\hbar^2 \right] = \frac{J}{2}\left[S_{TOT}^2 - 3\hbar^2 \right]$$

So what's S_{TOT}? $S_{TOT}^2 \Rightarrow \hbar^2 (S)(S+1)$

~~We~~ So need to find total S's for $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$:

$\frac{1}{2} \,\&\, \frac{1}{2} \Rightarrow 0, 1$

$\frac{1}{2} \,\&\, \frac{1}{2} \,\&\, \frac{1}{2} \Rightarrow \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$\frac{1}{2} \,\&\, \frac{1}{2} \,\&\, \frac{1}{2} \,\&\, \frac{1}{2} \Rightarrow 0, 1;\ 0, 1;\ 2, 1,$ ~~states~~ 3 states

1 states, 5 states

$\therefore$ Energy eigenvalues: $\frac{J}{2}\hbar^2 [s(s+1) - 3]$:

$$\frac{3J\hbar^2}{2} \text{ (5 states)}, \quad \frac{-J\hbar^2}{2} \text{ (9 states)}, \quad \frac{-3J\hbar^2}{2} \text{ (2 states)}.$$

Qual Review, Meeting 9: Potpourri 5

1. A perfectly conducting sphere of radius R moves with constant velocity $\mathbf{v} = v\hat{x}$ ($v << c$) through a uniform magnetic field $\mathbf{B} = B\hat{y}$. Find the surface charge density induced on the sphere.

2. (Warning: You will need a calculator to completely answer this question.) Consider a system of N noninteracting free fermions of mass m and spin $1/2$ in a box of volume V at zero temperature. The system is thermally insulated from everything else in the entire universe. Suddenly, without warning or provocation, the fermions all turn into spin 0 bosons with the same mass. Don't ask. Will the system Bose condense? You may want to know that

$$\int_0^\infty \frac{x^{1/2}dx}{e^x - 1} = 2.32$$

$$\int_0^\infty \frac{x^{3/2}dx}{e^x - 1} = 1.78$$

(Hint: Once the fermions turn into bosons, the temperature of the system changes, since now some particles are in excited states.)

3. An electromagnetic plane wave of angular frequency ω propagates through an optically active dextrose solution. The solution is non-conducting and non-magnetizable but has a polarization given by $\mathbf{P} = \alpha\mathbf{E} + \beta\nabla \times \mathbf{E}$ when a plane wave propagates through the medium. You may assume that α and β are positive real constants, and that $\frac{\beta\omega}{c} << 1$ so that you can give all answers to first order in $\frac{\beta\omega}{c}$. Find the possible indices of refraction for the plane wave and the corresponding polarizations of the electric field.

4. Find both solutions to the differential equation

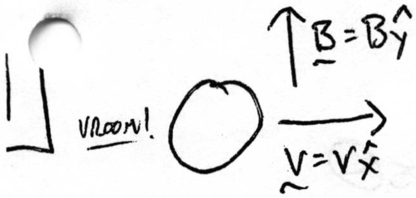
$$x^4 y'' - x^2 y' + \frac{1}{4} y = 0$$

in the limit as x approaches 0 to leading order (that is, until you obtain a logarithmic term in the exponent of your solution).

A Benediction for the Qual

Dear Baal,
I do not hold Brian responsible
For anything he said wouldn't be on the qual
That shows up.
Amen.

Potpourri 5 Solns



First, transform fields to sphere's frame $\mathcal{O}'$

$\underset{\sim}{E}_{\parallel}' = \underset{\sim}{E}_{\parallel}, \quad \underset{\sim}{B}_{\parallel}' = \underset{\sim}{B}_{\parallel} \Rightarrow E_x' = 0, \; B_x' = 0.$

$$\underset{\sim}{E}_{\perp}' = \gamma\left(\underset{\sim}{E}_{\perp} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}_{\perp}}{c}\right) = \gamma\left(\frac{vB}{c}\hat{x} \times \hat{y}\right) = \frac{\gamma v B}{c}\hat{z} \simeq \frac{vB}{c}\hat{z}.$$

$$\underset{\sim}{B}_{\perp}' = \gamma\left(\underset{\sim}{B}_{\perp} - \frac{\underset{\sim}{v} \times \underset{\sim}{E}_{\perp}}{c}\right) = \gamma B\hat{y} \simeq B\hat{y}$$

So it's in a static $\underset{\sim}{E}$ field in the $\hat{z}$ direction

$\Rightarrow$ Do b.c. problem. $\phi = \sum_{\ell}\left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

B.C.'s: 1) $\phi = -Ez$ as $r \to \infty$

2) $\phi = 0$ on surface (conducting sphere)

1: $a_1 = -E$, all others 0.

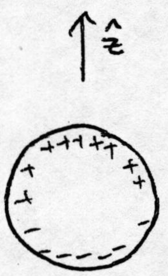
$$\Rightarrow \phi = \left(-Er + \frac{b_1}{r^2}\right)\cos\theta \Rightarrow -Ea + \frac{b_1}{a^2} = 0 \Rightarrow b_1 = Ea^3$$

$$\Rightarrow \phi = -E\left(r - \frac{a^3}{r^2}\right)\cos\theta.$$

Across surface, $(\underset{\sim}{E}_{out} - \underset{\sim}{E}_{in}) \cdot \hat{n} = 4\pi\sigma$

$$\Rightarrow -\frac{\partial \phi}{\partial r}\bigg|_{r=a} = 4\pi\sigma \Rightarrow \sigma = -\frac{1}{4\pi}\left(-E - \frac{2a^3 E}{a^3}\right)\cos\theta$$

$$= \frac{3E}{4\pi}\cos\theta = \frac{3vB}{4\pi c}\cos\theta$$



Need to transform back to lab frame. But since the motion is non-relativistic, there are no length contractions etc. → charge density the same.

$$\Rightarrow \boxed{\sigma = \frac{3vB}{4\pi c}\cos\theta}$$

2| First, find total energy of the fermions:

$$N = \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon = \int_0^{\varepsilon_F} 2 \cdot \frac{4\pi V}{(2\pi\hbar)^3}\frac{(2m)^{3/2}}{2}\varepsilon^{1/2}\,d\varepsilon = \frac{4\pi V}{(2\pi\hbar)^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\varepsilon_F^{3/2}$$

$$\Rightarrow \varepsilon_F = \left(\frac{3N\cancel{8}\pi^3}{\cancel{8}\pi V}\right)^{2/3}\left(\frac{\hbar^2}{2m}\right)$$

$$= \left(\frac{3\pi^2}{V}\right)^{2/3} N^{2/3}\left(\frac{\hbar^2}{2m}\right)$$

$$\Rightarrow E_{TOT} = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon)\,d\varepsilon = \int_0^{\varepsilon_F} \frac{4\pi V}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{3/2}\,d\varepsilon = \frac{\cancel{2}}{5}\frac{V}{\cancel{2}\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon_F^{5/2}$$

$$= \frac{V}{5\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left(\frac{3\pi^2}{V}\right)^{5/3}N^{5/3}\left(\frac{\hbar^2}{2m}\right)^{5/2}$$

$$= \frac{V}{5\pi^2}N^{5/3}\left(\frac{3\pi^2}{V}\right)^{5/3}\left(\frac{\hbar^2}{2m}\right)$$

$$= \boxed{\frac{3^{5/3}}{5}\left(\frac{\pi^2}{V}\right)^{2/3}N^{5/3}\left(\frac{\hbar^2}{2m}\right)}$$

Now, we know that the total energy of the bosons if they condense ($\mu = 0$) is

$$E = \int \varepsilon g(\varepsilon)\frac{1}{e^{\beta\varepsilon}-1} = \int \frac{\overset{\text{no longer } 2s+1 \text{ degeneracy}}{2}\,\cancel{1}\cdot 4\pi V}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\varepsilon^{3/2}}{e^{\beta\varepsilon}-1}\,d\varepsilon$$

$$E = \frac{2\pi V}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int\frac{\varepsilon^{3/2}d\varepsilon}{e^{\beta\varepsilon}-1} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{5/2}\int\frac{x^{3/2}dx}{e^x-1}$$

For condensation, $N_{TOTAL} > N_{excited} = \int\frac{g(\varepsilon)d\varepsilon}{e^{\beta\varepsilon}-1}$

$$N_{excited} = \int\frac{2\pi V}{(2\pi)^3}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\varepsilon^{1/2}d\varepsilon}{e^{\beta\varepsilon}-1} = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{3/2}\int\frac{x^{1/2}dx}{e^x-1}.$$

$$\Rightarrow N > \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{3/2}\int\frac{x^{1/2}dx}{e^{x}-1}$$

$$\Rightarrow k_BT < \frac{(4\pi^2)^{2/3}}{V^{2/3}}\left(\frac{1}{\int\frac{x^{1/2}dx}{e^x-1}}\right)^{2/3}\left(\frac{\hbar^2}{2m}\right)N^{2/3} = k_BT_c$$

From cons. of energy, $\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_BT)^{5/2}\int\frac{x^{3/2}dx}{e^x-1} = \frac{3^{5/3}}{5}\left(\frac{\pi^2}{V}\right)^{2/3}N^{5/3}\left(\frac{\hbar^2}{2m}\right)$

$$\Rightarrow k_BT = \frac{1}{\left(\int\frac{x^{3/2}dx}{e^x-1}\right)^{2/5}}\frac{3^{2/3}}{5^{2/5}}\frac{(4\pi^2)^{2/5}(\pi^2)^{4/15}}{V^{2/3}}N^{2/3}\left(\frac{\hbar^2}{2m}\right)$$

Is $k_BT < k_BT_c$? $\Rightarrow$ Is $\underbrace{\frac{1}{\left(\int\frac{x^{3/2}dx}{e^x-1}\right)^{2/5}}\frac{3^{2/3}(4\pi^2)^{2/5}\pi^{8/15}}{5^{2/5}}}_{6.95} < \underbrace{\frac{(4\pi^2)^{2/3}}{\left(\int\frac{x^{1/2}dx}{e^x-1}\right)^{2/3}}}_{6.61}$?

No $\rightarrow$ The system will not condense.

5] Maxwell: $\nabla \times \underset{\sim}{H} = \frac{4\pi}{c}\underset{\sim}{j} + \frac{1}{c}\frac{\partial D}{\partial t}$ $\qquad$ $\nabla \cdot \underset{\sim}{B} = 0$

$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$ $\qquad$ $\nabla \cdot \underset{\sim}{D} = 4\pi\rho$

No sources, so $\underset{\sim}{j} = \rho = 0$. And since it's non-magnetizable,

$$\underset{\sim}{B} = \underset{\sim}{H}$$

$\Rightarrow \nabla \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial D}{\partial t}$ $\qquad$ Assume $\underset{\sim}{E} = \underset{\sim}{E_0} e^{i(kz-\omega t)}$.

$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$.

$\Rightarrow ik\,\hat{z} \times \underset{\sim}{E_0} = \frac{i\omega}{c}\underset{\sim}{B}$ & $ik\,\hat{z} \times \underset{\sim}{B} = -\frac{i\omega}{c}\underset{\sim}{D}$

But $\underset{\sim}{D} = \underset{\sim}{E} + 4\pi \underset{\sim}{P} = \underset{\sim}{E} + 4\pi(\alpha \underset{\sim}{E} + \beta \nabla \times \underset{\sim}{E})$

$= \left[(1+4\pi\alpha)\underset{\sim}{E_0} + ik\beta\,\hat{z} \times \underset{\sim}{E_0}\right] e^{i(kz-\omega t)}$

$\Rightarrow \underset{\sim}{B} = \frac{ck}{\omega}\hat{z} \times \underset{\sim}{E_0}$ & $-\frac{ck}{\omega}\hat{z} \times \underset{\sim}{B} = (1+4\pi\alpha)\underset{\sim}{E_0} + ik\beta\,\hat{z} \times \underset{\sim}{E_0}\, 4\pi$

$\hat{z} \times (\hat{z} \times \underset{\sim}{E_0}) = -\underset{\sim}{E_0} \Rightarrow \frac{c^2k^2}{\omega^2}\underset{\sim}{E_0} = (1+4\pi\alpha)\underset{\sim}{E_0} + ik\beta\,\hat{z} \times \underset{\sim}{E_0}\, 4\pi$

$\nabla \cdot \underset{\sim}{D} = 0,\ \nabla \cdot \underset{\sim}{B} = 0 \Rightarrow B_z = D_z = 0$

$D_z = (1+4\pi\alpha)E_z \Rightarrow E_z = 0$

$$\Rightarrow \underset{\sim}{E_0} = E_x \hat{x} + E_y \hat{y} \;, \quad \hat{z} \times \underset{\sim}{E_0} = -\hat{x} E_y + \hat{y} E_x$$

$$\Rightarrow \frac{c^2 k^2}{\omega^2} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = (1 + 4\pi\alpha) \begin{pmatrix} E_x \\ E_y \end{pmatrix} + ik\beta \begin{pmatrix} -E_y \\ E_x \end{pmatrix} 4\pi$$

Noting $\frac{ck}{\omega} = n$ $\Rightarrow \begin{pmatrix} 1 + 4\pi\alpha - n^2 & -i4\pi k\beta \\ ik\beta 4\pi & 1 + 4\pi\alpha - n^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$

$$\Rightarrow (1 + 4\pi\alpha - n^2)^2 - (4\pi\beta k)^2 = 0$$

$$\Rightarrow 1 + 4\pi\alpha - n^2 = \pm 4\pi\beta K$$

$$\Rightarrow n^2 = 1 + 4\pi\alpha \mp 4\pi\beta K$$

$$\Rightarrow \begin{pmatrix} \pm 4\pi\beta K & -4\pi\beta k i \\ 4\pi\beta k i & \pm 4\pi\beta K \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0 \Rightarrow 4\pi\beta k \begin{pmatrix} \pm 1 & -i \\ i & \pm 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

$$\Rightarrow \pm E_x = i E_y \Rightarrow E_y = \mp i E_x$$

So polarization is $\hat{x} \mp i\hat{y}$ for $n^2 = 1 + 4\pi\alpha \mp 4\pi\beta K$ (circular)

Not totally done: $n^2 = \frac{c^2 k^2}{\omega^2} = 1 + 4\pi\alpha \mp 4\pi\beta K$

Easiest to write $k = \frac{n\omega}{c}$ & solve for n:

$$n^2 = 1 + 4\pi\alpha \mp \frac{4\pi\beta\omega}{c} n \Rightarrow n^2 \pm \frac{4\pi\beta\omega}{c} n - (1+4\pi\alpha) = 0$$

$$\Rightarrow n = \frac{\mp \frac{4\pi\beta\omega}{c} \pm \sqrt{\frac{16\pi^2\beta^2\omega^2}{c^2} + 4(1+4\pi\alpha)}}{2}$$

lose negative soln $\Rightarrow$ Want n positive. (i.e. k, ω positive)

$$\Rightarrow n = \mp \frac{\cancel{4}2\pi\beta\omega}{c} + \sqrt{\frac{4\pi^2\beta^2\omega^2}{c^2} + (1+4\pi\alpha)}$$

$$\simeq \mp \frac{2\pi\beta\omega}{c} + \sqrt{1+4\pi\alpha}\left(1 + \frac{4\pi^2\beta^2\omega^2}{c^2(1+4\pi\alpha)}\right)^{1/2}$$

$$\simeq \boxed{\sqrt{1+4\pi\alpha} \mp \frac{2\pi\beta\omega}{c}}$$ corresponding to $\hat{x} \mp i\hat{y}$.

7] $x^4y'' - x^2y' + \frac{y}{4} = 0$

Dominant Balance: $y = e^{S(x)}$, $y' = S'e^S$, $y'' = \left((S')^2 + S''\right)e$

$\Rightarrow x^4\left[(S')^2 + S''\right] - x^2S' + \frac{1}{4} = 0$

Balance: $x^4(S')^2 - x^2S' + \frac{1}{4} = 0 \Rightarrow \left(S' - \frac{1}{2x^2}\right)^2 = 0$

$\Rightarrow S' = \frac{1}{2x^2} \Rightarrow S = -\frac{1}{2x}$

(Note: $x^4S'' \sim x \ll x^4(S')^2, x^2S', \frac{1}{4}$)

Now try $y = e^{\overbrace{-\frac{1}{2x}}^{S} + \overbrace{C}^{C \ll S.}} \Rightarrow$

$y' = (S'+C')e^{S+C}$, $y'' = \left[(S'+C')^2 + S'' + C''\right]e^{S+C}$

$\Rightarrow x^4\left[\cancel{S'^2} + 2S'C' + C'^2 + S'' + C''\right] - x^2(\cancel{S'} + C') + \cancel{\frac{1}{4}} = 0$

$\Rightarrow x^4\left[\frac{1}{x^2}C' + C'^2 - \frac{1}{x^3} + C''\right] - x^2C' = 0$

$\Rightarrow x^4C'^2 - x + x^4C'' = 0$

$\Rightarrow x^4C'^2 = x \Rightarrow C' = \frac{\pm 1}{x^{3/2}} \Rightarrow C = \mp\frac{2}{x^{1/2}}$

$\Rightarrow y = e^{-\frac{1}{2x} \pm \frac{2}{\sqrt{x}}}$

Now try $y = e^{-\frac{1}{2x} \pm \frac{2}{\sqrt{x}} + D}$, $D \ll x^{-1/2}$

$$\Rightarrow x^4\left[(S'+C'+D')^2 + S''+C''+D''\right] - x^2\left[S'+C'+D'\right] + \frac{1}{4} = 0$$

$$\Rightarrow x^4\left[\cancel{S'^2}+\cancel{C'^2}+D'^2+\cancel{2S'C'}+2C'D'+2S'D'+\cancel{S''}+C''+D''\right] - \cancel{x^2}\left[\cancel{S'}+\cancel{C'}+D'\right] + \cancel{\frac{1}{4}} = 0$$

$$\Rightarrow x^4\left[D'^2 \pm \frac{2}{x^{3/2}}D' + \cancel{\frac{1}{x^2}D'} \mp \frac{3}{2}\frac{1}{x^{5/2}} + D''\right] - \cancel{x^2 D'} = 0$$

$$\Rightarrow x^4 D'^2 \pm 2x^{5/2}D' \mp \frac{3}{2}x^{3/2} + x^4 D'' = 0$$

$$\Rightarrow \pm 2x^{5/2}D' \mp \frac{3}{2}x^{3/2} = 0 \Rightarrow D' = \frac{3}{4x} \Rightarrow D = \frac{3}{4}\ln x \ \checkmark$$

$$\Rightarrow y = e^{-\frac{1}{2x} \pm \frac{2}{\sqrt{x}} + \frac{3}{4}\ln x} = x^{3/4} e^{-\frac{1}{2x} \pm \frac{2}{\sqrt{x}}}$$