Qual Review, Meeting 3: Electricity and Magnetism

1. A dielectric sphere of radius a and dielectric constant ϵ_1 is placed in a dielectric liquid of infinite extent and dielectric constant ϵ_2. Within the liquid (before the sphere was inserted) was a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$. Find the resultant electric field both inside and outside the sphere after it has been placed in the liquid.

2. A classical hydrogen atom has the electron circling the proton at a radius equal to the first Bohr radius. Derive an expression for the amount of time it takes the electron to smash into the proton in the center due to radiative losses. You may assume that the motion of the electron is nonrelativistic.

3. Find the magnetic field both inside and outside a uniformly magnetized sphere. Take the magnetization of the sphere to be $\mathbf{M} = M\hat{\mathbf{z}}$. Notice that since the sphere is a hard magnet, you may not assume $\mathbf{M} = \mu\mathbf{H}$.

4. A plane electromagnetic wave of angular frequency ω is incident normally from the vacuum on a slab of some material. The surface lies in the xy plane. The material is anisotropic with dielectric tensor $\epsilon_{xx} = n_x^2$, $\epsilon_{yy} = n_y^2$, and $\epsilon_{zz} = n_z^2$, and all off-diagonal elements equal to zero. You may not assume $n_x = n_y$. If the incident plane wave is linearly polarized with its electric field at 45 degrees to the x and y axes, what will be the state of polarization of the reflected wave for an infinitely thick slab?

E&M Sols

1] B.C.'s: 1) $\underline{\Phi} \to -Ez$ as $r \to \infty$
2) $\underline{\Phi}$ finite @ $r=0$.
3) Normal $\underline{D}$ continuous:
4) Tangential $\underline{E}$ continuous.

Separate into 2 regions: (1) inside (2) outside

Azimuthal symmetry, so no ϕ dependence

$$\Phi_1 = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta) \quad \& \quad \Phi_2 = \sum_\ell \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Now, just use b.c.'s! $B_\ell = 0 \ \forall \ell$ (b.c #2)

$C_1 = -E$, all others 0 (b.c #1)

Normal $\underline{D}$ continuous: $\varepsilon_1 \left.\frac{\partial \Phi_1}{\partial r}\right|_{r=a} = \varepsilon_2 \left.\frac{\partial \Phi_2}{\partial r}\right|_{r=a}$

Tangential $\underline{E}$ continuous: $\left.\frac{\partial \Phi_1}{\partial \theta}\right|_{r=a} = \left.\frac{\partial \Phi_2}{\partial \theta}\right|_{r=a}$

The latter is just saying $A_\ell a^\ell = C_\ell a^\ell + \frac{D_\ell}{a^{\ell+1}} \ \forall \ell$.

The former yields $\varepsilon_1 \ell A_\ell r^{\ell-1} = \varepsilon_2 \left(C_\ell \ell r^{\ell-1} - (\ell+1)\frac{D_\ell}{r^{\ell+2}}\right)\Big|_{r=a}$

Can set arbitrary constant to $0 \Rightarrow D_0 = 0$.

$$A_1 a = C_1 a + \frac{D_1}{a^2} \quad ; \quad \varepsilon_1 A_1 = \varepsilon_2 \left(C_1 - \frac{2D_1}{a^3} \right)$$

$$\Rightarrow A_1 = C_1 + \frac{D_1}{a^3} = \frac{\varepsilon_2}{\varepsilon_1} \left(C_1 - \frac{2D_1}{a^3} \right)$$

$$\Rightarrow \frac{D_1}{a^3} \left(1 + \frac{2\varepsilon_2}{\varepsilon_1} \right) = C_1 \left(\frac{\varepsilon_2}{\varepsilon_1} - 1 \right)$$

$$D_1 = \frac{C_1 a^3 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + 2\varepsilon_2} = \frac{E a^3 \varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} ,$$

$$A_1 = E \left(1 + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \right) = E \left(\frac{-3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \right)$$

$$= - \frac{3E\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} .$$

others: $A_\ell a^\ell = \frac{D_\ell}{a^{\ell+1}} \Rightarrow A_\ell = \frac{D_\ell}{a^{2\ell+1}}$

$$\varepsilon_1 \ell A_\ell a^{\ell-1} = -\varepsilon_2 (\ell+1) \frac{D_\ell}{a^{\ell+2}} \Rightarrow \varepsilon_1 \ell \frac{D_\ell}{a^{2\ell+1}} a^{\ell-1} = -\varepsilon_2 (\ell+1) \frac{D_\ell}{a^{\ell+2}}$$

$$\Rightarrow D_\ell \left(\varepsilon_1 \ell + \varepsilon_2 (\ell+1) \right) = 0 \Rightarrow D_\ell = 0 . \Rightarrow A_\ell = 0 .$$

$$\therefore \Phi_1 = \frac{-3E\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} r \cos\theta , \quad \Phi_2 = -E \left(r + \frac{a^3 (\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 + 2\varepsilon_2) r^2} \right) \cos\theta .$$

$$\Rightarrow \underline{E} = \frac{3E\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2} \hat{z} \text{ for } r < a \text{ and } \underline{E} = E\hat{z} + \frac{a^3 (\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_2} \left[\frac{3\underline{r}(\underline{E} \cdot \underline{r})}{r^5} - \frac{\underline{E}}{r^3} \right]$$

2] Energy: $E = \frac{1}{2}mv^2 + V = \frac{1}{2}mv^2 - \frac{e^2}{r}$

But for circular motion, $v^2 = \frac{r}{m}\frac{e^2}{r^2} = \frac{e^2}{rm}$

$$\Rightarrow \boxed{E = \frac{1}{2}m\frac{e^2}{rm} - \frac{e^2}{r} = -\frac{e^2}{2r}}$$

Formula for power loss due to radiation: $P = \frac{2}{3}\frac{e^2}{c^3}|a|^2 = \frac{-dE}{dt}$

Here, $a^2 = \left(\frac{v^2}{r}\right)^2 = \left(\frac{e^2}{r^2 m}\right)^2 = \frac{e^4}{r^4 m^2}$

$$\Rightarrow \frac{dE}{dt} = -\frac{2e^2}{3c^3}\frac{e^4}{r^4 m^2} = \frac{-2e^6}{3c^3 r^4 m^2}$$

But $\frac{dE}{dt} = \frac{dE}{dr}\frac{dr}{dt} = \frac{e^2}{2r^2}\frac{dr}{dt}$ $\Rightarrow$ $\frac{e^2}{2r^2}\frac{dr}{dt} = \frac{-2e^6}{3c^3 r^4 m^2}$

$$\Rightarrow \frac{dr}{dt} = \frac{-4e^4}{3c^3 r^2 m^2}$$

$$\Rightarrow \frac{-3c^3 m^2}{4e^4}\int_{a_0}^{0} r^2\,dr = \Delta t \Rightarrow \boxed{\Delta t = \frac{a_0^3 c^3 m^2}{4e^4}}$$

Note: Plugging in $a_0 = .5\times10^{-8}$ cm, $c = 3\times10^{10}$ cm/s, $m = 9.11\times10^{-28}$ g

$e = 4.8\times10^{-10}$ esu, $4 = 4$ yields

$\boxed{\Delta t = 1.3\times10^{-11}\text{ sec}}$, which is not long.

3]

$\nabla \times \underset{\sim}{H} = 0$, since there are no free currents

$\Rightarrow \underset{\sim}{H} = -\nabla \Phi$.

Outside the sphere, $\underset{\sim}{B} = \underset{\sim}{H}$, so $\nabla^2 \Phi = 0$.

Inside, $\underset{\sim}{B} = \underset{\sim}{H} + 4\pi \underset{\sim}{M} \Rightarrow \nabla \cdot \underset{\sim}{B} = \nabla \cdot \underset{\sim}{H} \Rightarrow \nabla^2 \Phi = 0$.

Inside: $\Phi_1 = \sum_\ell \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$

Outside: $\Phi_2 = \sum_\ell \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$

B.C.'s: $\Phi \to 0$ as $r \to \infty \Rightarrow C_\ell = 0 \ \forall \ell$.

Φ finite at $r = 0 \Rightarrow B_\ell = 0 \ \forall \ell$.

Tangential $\underset{\sim}{H}$ continuous @ $r = a$

Normal $\underset{\sim}{B}$ cont. @ $r = a$.

Tangential H continuous: $\left. \frac{\partial \Phi_1}{\partial \theta} \right|_{r=a} = \left. \frac{\partial \Phi_2}{\partial \theta} \right|_{r=a}$.

$\Rightarrow A_\ell a^\ell = \frac{D_\ell}{a^{\ell+1}}$

Normal B: $-\frac{\partial \Phi_1}{\partial r} + 4\pi M \hat{r} = -\frac{\partial \Phi_2}{\partial r} \Rightarrow -A_\ell \ell r^{\ell-1} P_\ell(\cos\theta) + 4\pi M \cos\theta =$

$+ (\ell+1) \frac{D_\ell}{r^{\ell+2}} P_\ell(\cos\theta)$

$\Rightarrow$

$A_1 + 4\pi M = \frac{2D_1}{a^3}$, $A_\ell \ell r^{\ell-1} = -(\ell+1)\frac{D_\ell}{r^{\ell+2}}$ $\ell \neq 1$

$\Rightarrow A_1 = \frac{D_1}{a^3}$, $A_1 - 4\pi M = \frac{-2D_1}{a^3}$

$-4\pi M = \frac{-3D_1}{a^3} \Rightarrow D_1 = \frac{4\pi}{3} M a^3$.

$\Rightarrow A_1 = \frac{4\pi M}{3}$, other D's & A's vanish.

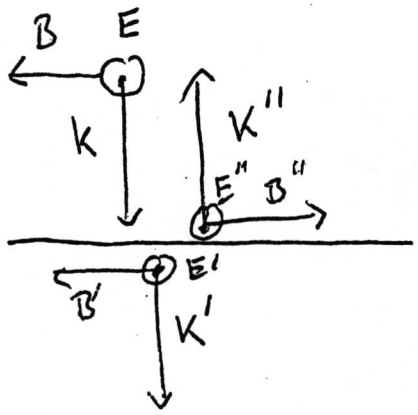
- $\Phi_1 = \frac{4\pi M}{3} r\cos\theta$, $\Phi_2 = \frac{4\pi M}{3}\frac{a^3}{r^2}\cos\theta$.

- $\underset{\sim}{B}_{inside} = -\frac{4\pi M}{3}\hat{z} + 4\pi M\hat{z} = \frac{8\pi}{3} M\hat{z}$

$\underset{\sim}{B}_{outside} = \underset{\sim}{H}_{outside} = -\nabla\left(\frac{4\pi M a^3}{3}\left(\frac{\underset{\sim}{r}\cdot\cos\theta}{r^3}\right)\right) = \frac{3(\underset{\sim}{m}\cdot\underset{\sim}{r})\underset{\sim}{r} - m r^2}{r^5}$

with $\underset{\sim}{m} = \frac{4\pi M a^3}{3}\hat{z}$.

So it's a freaky dipole field!



Boundary conditions:

H_t, E_t, B_n, D_n continuous

where it's at — nobody cares.

The incident electric field is $\underset{\sim}{E} = \frac{E}{\sqrt{2}}(\hat{x}+\hat{y}) = E_x\hat{x} + E_y\hat{y}$

The reflected one is $\underset{\sim}{E}'' = E_x''\hat{x} + E_y''\hat{y}$

and the transmitted one is $\underset{\sim}{E}' = E_x'\hat{x} + E_y'\hat{y}$

Since E_t is continuous, $E_x + E_x'' = E_x'$ & $E_y + E_y'' = E_y'$

Similarly, $H_x + H_x'' = H_x'$ & $H_y + H_y'' = H_y'$.

To relate $\underset{\sim}{E}$ & $\underset{\sim}{H}$, use $\nabla \times \underset{\sim}{H} = \frac{1}{c}\frac{\partial \underset{\sim}{D}}{\partial t}$, $\underset{\sim}{D} = \underset{\approx}{\varepsilon}\cdot\underset{\sim}{E}$

$\Rightarrow \underset{\sim}{k}\times\underset{\sim}{H} = -\frac{\omega}{c}\underset{\approx}{\varepsilon}\cdot\underset{\sim}{E}$.

Incident: $\underset{\sim}{k}\times\underset{\sim}{H} = -\frac{\omega}{c}\underset{\sim}{E}$. Reflected: $\underset{\sim}{k}''\times\underset{\sim}{H}'' = -\frac{\omega''}{c}\underset{\sim}{E}''$

Transmitted: $\underset{\sim}{k}'\times\underset{\sim}{H}' = -\frac{\omega'}{c}(\varepsilon_{xx}E_x'\hat{x} + \varepsilon_{yy}E_y'\hat{y})$.

Using $\underset{\sim}{k}\times\underset{\sim}{H} = k(\hat{n}\times\underset{\sim}{H})$, where $\hat{n}$ is the normal, and noting that $\hat{n}\times\underset{\sim}{H} = H_t$, we have

$$-\frac{\omega}{ck}\underset{\sim}{E} + \frac{\omega''}{ck''}\underset{\sim}{E}'' = \frac{-\omega'}{ck'}(\varepsilon_{xx}E_x'\hat{x} + \varepsilon_{yy}E_y'\hat{y})$$

↑ b/c $k''\times H = -k''(\hat{n}\times\underset{\sim}{H})$

$$\Rightarrow \frac{\omega}{ck} E_x + \frac{\omega''}{ck''} E_x'' = \frac{\omega'}{ck'} \varepsilon_{xx} E_x'$$

$$\frac{\omega}{ck} E_y - \frac{\omega''}{ck''} E_y'' = \frac{\omega'}{ck'} \varepsilon_{yy} E_y'$$

Now, $\omega = \omega' = \omega''$, so since $n_2 = \sqrt{\mu\varepsilon} = \frac{ck}{\omega}$, $k = k''$.

$$\Rightarrow \begin{cases} E_x - E_x'' = \frac{k}{k'} \varepsilon_{xx} E_x' \\ E_x + E_x'' = E_x' \end{cases} \quad \& \quad \begin{cases} E_y - E_y'' = \frac{k}{k'} \varepsilon_{yy} E_y' \\ E_y + E_y'' = E_y' \end{cases}$$

Using $\varepsilon_{xx} = n_x^2$, $\varepsilon_{yy} = n_y^2$, and $\frac{k}{k'} = \frac{1}{n_z} \Rightarrow k' = n_z k$.

$$\Rightarrow 2E_x = \left(\frac{n_x^2}{n_z} + 1\right) E_x', \quad 2E_y = \left(\frac{n_y^2}{n_z} + 1\right) E_y'$$

$$\Rightarrow E_x' = \frac{2n_z}{n_x^2 + n_z} E_x, \quad E_y' = \frac{2E_y n_z}{n_y^2 + n_z}$$

and $E_x'' = E_x' - E_x = \left(\frac{2n_z}{n_x^2 + n_z} - 1\right) E_x = \left(\frac{n_z - n_x^2}{n_z + n_x^2}\right) E_x$

& $E_y'' = \left(\frac{n_z - n_y^2}{n_z + n_y^2}\right) E_y$

$\therefore$ the reflected wave has $\underline{E}' = \left(\frac{n_z - n_x^2}{n_z + n_x^2}\right) E_x \hat{x} + \left(\frac{n_z - n_y^2}{n_z + n_y^2}\right) E_y \hat{y}$

Shanes Review

$$L = -mc^2\sqrt{1-\left(\frac{v^2}{c^2}\right)} - q\Phi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

$$= -\frac{mc^2}{\gamma} - q\Phi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

find relativistic H

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = -mc^2 \cdot \frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \cdot \left(-\frac{2}{c^2}\right)\vec{v} + \frac{q}{c}\vec{A}$$

$$= m\vec{v}\gamma + \frac{q}{c}\vec{A}$$

$$H = \vec{p}\cdot\vec{v} - L$$

$$= m\gamma v^2 + \frac{q}{c}\vec{v}\cdot\vec{A} + \frac{mc^2}{\gamma} + q\Phi - \frac{q}{c}\vec{v}\cdot\vec{A}$$

$$\frac{1}{1-\beta^2} - 1$$

$$= mc^2\left(\gamma\frac{v^2}{c^2} + \frac{1}{\gamma}\right) + q\Phi$$

$$m\gamma\vec{v} = \left(\vec{p} - \frac{q}{c}\vec{A}\right) = \vec{p}_k \quad \rightarrow \quad \frac{\vec{p}_k}{mc} = \frac{\gamma\vec{v}}{c} \rightarrow \frac{p_k^2}{m^2c^2} = \gamma^2\left(\frac{v}{c}\right)^2$$

$$= \frac{\beta^2}{1-\beta^2} = \gamma^2 - 1$$

$$\vec{v}\cdot\vec{v} = \frac{1}{m^2\gamma^2}\left\{\left(p^2 - \frac{q}{c}\vec{A}\right)^2\right\}$$

$$\frac{v^2}{c^2} = \frac{1}{E^2}\left\{\left(p - \frac{q}{c}\vec{A}\right)^2\right\} \qquad 1 - \frac{v^2}{c^2} = \frac{E^2 - \left(p - \frac{q}{c}\vec{A}\right)^2}{E^2}$$

$$\gamma^2\frac{v^2}{c^2} + 1 = \frac{E^2}{E^2 - (p-\frac{q}{c}A)^2}\cdot\frac{(p-\frac{q}{c}A)^2}{E^2} + \frac{E^2 - (p-\frac{q}{c}A)^2}{E^2 - (p-\frac{q}{c}A)^2}$$

$$= \frac{E^2}{E^2 - (p-\frac{q}{c}A)^2} = \gamma^2$$

$$\left(\gamma^2\frac{v^2}{c^2} + 1\right)\cdot\frac{1}{\gamma} = \gamma\,\frac{E^2}{E^2}$$

also

$$\gamma\frac{v^2}{c^2} + 1 = \frac{\gamma^2 - 1}{\gamma} + \frac{1}{\gamma} = \gamma$$

so

$$\frac{p_k^2 + m^2c^2}{m^2c^2} = \gamma^2 \rightarrow \gamma = \frac{1}{mc}\sqrt{p_k^2 + m^2c^2}$$

$$H = \gamma mc^2 + q\Phi$$

$$= \sqrt{(cP_k)^2 + m^2c^4} + q\Phi$$

$$= \sqrt{\left(\vec{p} + \frac{q}{c}\vec{A}\right)^2 c^2 + m^2c^4} + q\Phi$$

Non-relativistic

$$L = \frac{1}{2}mv^2 - q\Phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

from $\sqrt{1-\left(\frac{v}{c}\right)^2} \cong 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots$

$$-mc^2\sqrt{1-\left(\frac{v}{c}\right)^2} \cong -mc^2 + \frac{mv^2}{2} + \cdots$$

irrelevant constant!

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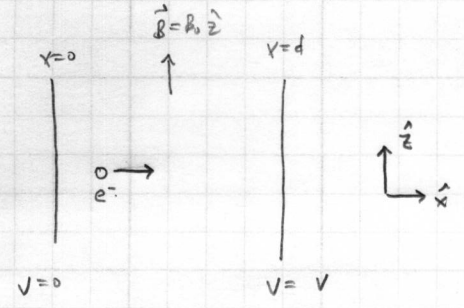

$E = -\nabla V$

$F = -eE = -(-e)(-\nabla V) = e\nabla V$

find min V s.t. e^- hits plate

$\vec{B} = B_0\hat{z} \rightarrow \vec{A} = \frac{\vec{B}\times\vec{r}}{2}$ if constant

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & B_0 \\ x & y & z \end{vmatrix} = -yB_0\hat{x} - \hat{y}(-xB_0)$$

$$= (-yB_0, xB_0, 0)$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -yB_0 & xB_0 & 0 \end{vmatrix}$$

$$= \hat{z}\,\frac{B_0 - (-B_0)}{2} = B_0\hat{z}$$

$$L = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) - (-e)V\left(\frac{x}{d}\right) + \frac{(-e)}{c}(-yv_x + xv_y)B_0$$

conserved quantities

$$\frac{\partial L}{\partial z} = 0 \qquad m v_z = \text{const} = 0$$

$$\frac{\partial L}{\partial t} = 0 \qquad \frac{m|\dot{\vec{r}}|^2}{2} - e V\left(\frac{x}{d}\right) = E = \text{const}$$

$t=0,\ v=0 \quad x=0 \rightarrow \quad E=0$

at $x=d$; $\quad \frac{m}{2}(v_x^2 + v_y^2) - eV = 0$

eq of motion: $\quad \frac{\partial L}{\partial x} = \left(\frac{eV}{d} - \frac{e}{c} v_y B_0\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\dot{v}_x + \frac{e}{c} y \dot{v}_x - \frac{e}{c} v_x v_y$$

$$\left(m + \frac{e}{c} y\right)\ddot{x} - \frac{e}{c}(v_x + B_0) v_x + \frac{eV}{d} = 0$$

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try $\quad \vec{A} = -x B_0 \hat{y}$

$$L = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) + eV\left(\frac{x}{d}\right) - \frac{e}{c}(x v_y B_0)$$

$$\frac{\partial L}{\partial z} = 0 \qquad m v_z = \text{const} = 0$$

$$\frac{\partial L}{\partial t} = 0 \qquad \frac{m v^2}{2} - eV\left(\frac{x}{d}\right) = E = 0$$

$$\frac{\partial L}{\partial y} = 0 \qquad m v_y - \frac{e}{c} x B_0 = \text{const} = 0 \qquad v_y = \frac{e}{c}\frac{x B_0}{m}$$

at $x=d$; $\quad \frac{m}{2}(v_x^2 + v_y^2 + \cancel{v_z^2}^{\,0}) - eV = 0$

$$V = \frac{m}{2e}\left\{v_x^2 + \left(\frac{e}{c}\frac{d B_0}{m}\right)^2\right\}$$

$V_{min} \Rightarrow v_x = 0$

$$V_{min} = \frac{m}{2e}\left(\frac{e}{c}\right)^2 \frac{(dB_0)^2}{m^2} = \frac{1}{2}\frac{e}{mc^2}(dB_0)^2$$

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$$L = -\frac{mc^2}{\gamma} + eV\left(\frac{x}{d}\right) = \frac{e}{c}\left(\dot{x} B_0 \hat{y}\right)$$

$$\frac{\partial L}{\partial t} = 0 \qquad -mc^2\gamma - eV\left(\frac{x}{d}\right) = mc^2$$

$$\frac{\partial L}{\partial z} = 0 \qquad \frac{\partial L}{\partial \dot{z}} = -mc^2 \cdot \tfrac{1}{2}\left(1-\beta^2\right)^{-1/2} \cdot \left(-\frac{2v_z}{c^2}\right)$$

$$= m\gamma v_z = \text{const} = 0 \rightarrow v_z = 0$$

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial \dot{y}} = m\gamma v_y - \frac{e}{c} x B_0 = \text{const} = 0$$

at $x = d$, min V corresponds to $v_x = 0$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2}{0} + \frac{v_y^2}{c^2} + \frac{v_z^2}{0}}} = \frac{1}{\sqrt{1 - \frac{v_y^2}{c^2}}}$$

$$\frac{m}{\sqrt{1-\frac{v_y^2}{c^2}}} v_y = \frac{e}{c} d B_0$$

$$v_y^2 = \left(\frac{e d B_0}{mc}\right)^2 \left(1 - \frac{v_y^2}{c^2}\right) = \alpha^2\left(1 - \frac{v_y^2}{c^2}\right)$$

$$v_y^2\left(1 + \frac{\alpha^2}{c^2}\right) = \alpha^2$$

$$v_y^2 = \frac{\alpha^2}{1 + \left(\frac{\alpha}{c}\right)^2} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v_y^2}{c^2}}} = \sqrt{\frac{1}{1 + \left(\frac{\alpha}{c}\right)^2 - \left(\frac{\alpha}{c}\right)^2}}$$

$$= \sqrt{1 + \left(\frac{\alpha}{c}\right)^2}$$

$$E = mc^2\gamma - \frac{e}{c} V_{min} = mc^2$$

$$V_{min} = mc^2(\gamma - 1)$$

$$= mc^2\left(\sqrt{1 - \left(\frac{\alpha}{c}\right)^2} - 1\right)$$

$$= mc^2\left(\sqrt{1 - \left(\frac{e d B_0}{mc^2}\right)^2} - 1\right)$$

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suppose quantities vary like $e^{-i\omega t}$

$$\vec{J}(\vec{r},t) = \mathrm{Re}\left(J(\vec{r})\, e^{-i\omega t} \right)$$

$$\vec{A}(\vec{r},t) = \mathrm{Re}\left(\vec{A}(\vec{r})\, e^{-i\omega t} \right)$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3\vec{r}' \cdot \frac{\vec{J}(\vec{r}')\, e^{-i\vec{k}|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

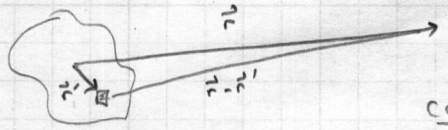
$$\varphi(\vec{r},t) = \int d^3\vec{r}' \quad \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, e^{i\vec{k}|\vec{r}-\vec{r}'|}$$

Multipole expansion for small source ($d \sim$ size of source)

$\lambda = \frac{c}{f}$

$c = \left(\frac{\lambda}{2\pi}\right)(2\pi f) = \frac{\omega}{k}$

$\lambda : \quad d << \frac{c}{\omega} = \frac{1}{k} = \frac{\lambda}{2\pi}$



case 1 : near zone

$$d << |\vec{r}| << \frac{c}{\omega}$$

- $\frac{1}{|\vec{r}'-\vec{r}|} \simeq \frac{1}{r} - (\vec{r}'\cdot\nabla)\frac{1}{r} + \cdots$
- $e^{i\vec{k}|\vec{r}-\vec{r}'|} \simeq 1 \qquad k|\vec{r}-\vec{r}'| \sim \frac{\omega}{c} r << 1$

recover electrostatics

case 2 : far zone (radiation zone) $\qquad d << \frac{c}{\omega} = \frac{1}{k} << |\vec{r}|$

$$|\vec{r}-\vec{r}'| \simeq r - \hat{r}\cdot\vec{r}' + \cdots$$

↑ expand in $\frac{r'}{r} \sim \frac{d}{r}$ (small)

$$= r\left|\hat{r} - \frac{r'}{r}\hat{r}'\right|$$

expand exponential

$$\vec{A}(\vec{r}) = \frac{e^{ikr}}{cr} \cdot \sum_{n=0}^{\infty} \int d^3\vec{r}' \quad J(\vec{r}') \quad \frac{(-ik\hat{r}\cdot\vec{r}')^n}{n!}$$

↑ only $\vec{r}$ dependence

Radiation

fall 03	14
fall 02	15
fall 02	14
Spring 01	16
fall 01	16
Sp 200	19
fall 97	10

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Radiation Review

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho$$

in Lorentz gauge: $\partial_\mu A^\mu = 0$

$$\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -4\pi\vec{J}$$

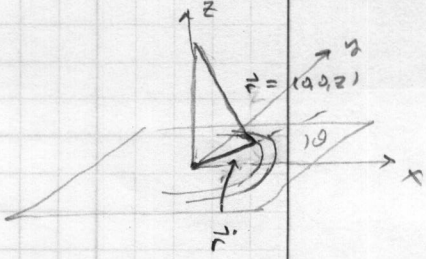
$$\frac{1}{c}\frac{\partial\varphi}{\partial t} - \nabla\cdot\vec{A} = 0$$

solutions:

$$\varphi(\vec{r},t) = \int d^3\vec{r}' \; \frac{\rho\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3\vec{r}' \; \frac{\vec{J}\left(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\vec{r}-\vec{r}'|}$$

ex) $t=0$, $\vec{K}(t) = \begin{cases} K_0(t)\,\hat{i} & t>0 \\ 0 & t<0 \end{cases}$ in x-y plane, find E & B.



$$\vec{A}(\vec{r},t) = \int \frac{ds'}{|\vec{r}-\vec{r}'|}\,\vec{K}\left(\vec{r}', t - r/c\right)$$

let $\vec{r} = (0,0,z)$ by symmetry

$$ds' = r'\,dr'\,d\theta$$

$$|\vec{r}-\vec{r}'| = |(-x', -y', z)| = \sqrt{r'^2 + z^2}$$

$$A(\vec{r},t) = \hat{i}\, 2\pi \int_0^\infty dr'\, r' \, \frac{K_0\left(t - \sqrt{r'^2+z^2}/c\right)}{\sqrt{r'^2+z^2}}$$

change variables, $r'^2 + z^2 = u^2$

$$r'\,dr' = u\,du$$

$$= 2\pi\,\hat{i} \int_{|z|}^\infty \cancel{u}\,du \, \frac{K_0(t-u/c)}{\cancel{u}}$$

but $t - u/c > 0$. for K_0 to be non zero $\rightarrow$ $t > u/c$

or $u < tc$

$$= 2\pi\,\hat{i} \int_{|z|}^{tc} du\, K_0(t-u/c)$$

$\xi = t - u/c$ $\qquad$ $\xi(|z|) = t - |z|/c$

$d\xi = -\frac{du}{c}$ $\qquad$ $\xi(tc) = 0$

$$\vec{A}(\vec{r},t) = \frac{2\pi}{c}\,\hat{i} \int_0^{t-|z|/c} d\xi\, K_0(\xi)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{2\pi}{c^2}\, K_0\left(t - |z|/c\right)\,\hat{i}$$

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & 0 & 0 \end{vmatrix} = +\partial_z A_x\,\hat{y}$$

$$= +\frac{2\pi}{c}\,\hat{j}\, K_0\left(t \mp \frac{z}{c}\right)\cdot\left(\mp\frac{1}{c}\right)$$

$$= \mp\frac{2\pi}{c^2}\,\hat{j}\, K_0\left(t \mp z/c\right) \qquad \begin{matrix} - & z>0 \\ + & z<0 \end{matrix}$$

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when $Kr >> 1$ (

$\nabla \times \vec{A}(\vec{r}) = \vec{B}$; $\nabla e^{iKr} = iK\hat{r}\, e^{iKr}$

$\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ $\quad \nabla \times \frac{\hat{r}}{r^2} \sim \frac{1}{r^2}$

neglect $1/r^2$ terms

$$\vec{B}(\vec{r}) \sim iK\, \hat{r} \times \vec{A}(\vec{r})$$

$$-i\left(\frac{\omega}{c}\right)\vec{E} = \nabla \times \vec{B}$$

$$\vec{E}(\vec{r}) = -iK\, \hat{r} \times \hat{r} \times \vec{A}$$

electric dipole: $n=0$ $\qquad \vec{A}(\vec{r}) = \frac{e^{iKr}}{cr}\int d^3\vec{r}'\, \vec{J}(\vec{r}')$

continuity: $-i\omega\rho(\vec{r}') + \nabla' \cdot \vec{J}(\vec{r}') = 0$

use $\int d^3\vec{r}'\, \vec{r}' \nabla' \cdot \vec{J}(\vec{r}') \overset{\text{parts}}{=} -\int d^3\vec{r}'\, \vec{J}(\vec{r}')$

$\downarrow$

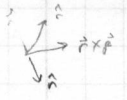
$\int d^3\vec{r}'\, \vec{r}'\, (-i\omega\rho(\vec{r}'))$

$\vec{p}$

$$\vec{A}(\vec{r}) = -i\left(\frac{\omega}{c}\right)\frac{e^{iKr}}{r}\int d^3\vec{r}'\, \vec{r}'\, \rho(\vec{r}')$$

$$= -iK\,\vec{p}\,\frac{e^{iKr}}{r}$$ (Dipole radiation) $\quad \left\{ \text{more general } \vec{A} \sim \frac{1}{cr}\frac{\partial \vec{p}}{\partial t} \right\}$

$$\vec{B} \approx K^2 (\hat{r} \times \vec{p})\,\frac{e^{iKr}}{r}$$



$$\vec{E} = -\hat{r} \times \vec{B}(\vec{r}) = -K^2\frac{e^{iKr}}{r}\,\hat{r} \times (\hat{r} \times \vec{p}) = -K^2\frac{e^{iKr}}{r}\,|\hat{r} \times \vec{p}|\,\hat{n}$$

$$\langle \vec{S} \rangle = \frac{1}{2}\,\mathrm{Re}\left\{\frac{c}{4\pi}\,\vec{E} \times \vec{B}^*\right\} = \mathrm{Re}\,\frac{c}{8\pi}\left\{-\frac{K^4}{r^2}\,\underbrace{\hat{n} \times (\hat{r} \times \vec{p})}_{-\hat{r}}\right\}|\hat{r} \times \vec{p}|$$

$$= \frac{c}{8\pi}\,\mathrm{Re}\left\{-\frac{K^4}{r^2}\right\}\left\{+\hat{r}\,|\hat{r} \times \vec{p}|^2 \rightarrow\right\}$$

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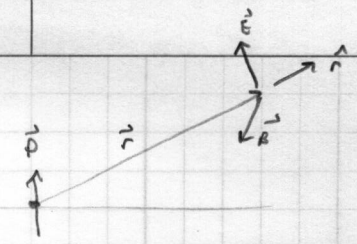
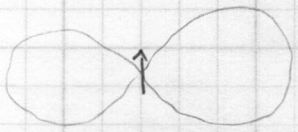
$$\langle \vec{S} \rangle = \hat{r} \left(\frac{c k^4}{8\pi} \right) \frac{|\hat{r} \times \vec{p}|}{r^2}$$

$$= \hat{r} \frac{c k^4}{8\pi} \frac{p \sin\theta}{r^2}$$



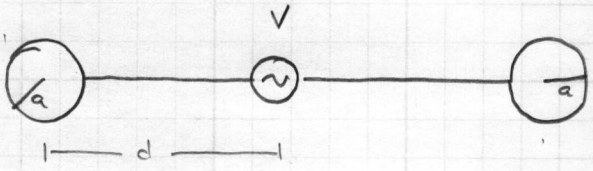
$$d\langle P \rangle = d\vec{A} \cdot \langle \vec{S} \rangle$$

$$\frac{d\langle P \rangle}{d\Omega} = r^2 \hat{r} \cdot \langle \vec{S} \rangle = \frac{c k^4}{8\pi} p^2 \sin^2\theta$$



$$\boxed{\langle P \rangle = \int d\Omega \frac{d\langle P \rangle}{d\Omega} = \frac{\omega^4 |\vec{p}|^2}{3 c^3}}$$

Hertzian dipole



$a \ll d \ll c/\omega$

if sphere has charge Q, what is potential at surface

$$\varphi = \frac{Q}{r} \rightarrow \frac{Q}{a}$$

so $$V = \Delta\varphi_{sphere} = \frac{Q}{a} - \left(-\frac{Q}{a}\right) = \frac{2Q}{a}$$

so $$Q(t) = \frac{a}{2} V(t)$$

$$\vec{p}(t) = 2 d Q(t)$$

$$= \underbrace{2 d \left(\frac{a}{2} V_0 e^{-i\omega t} \right)}_{p(\vec{r})}$$

$$\langle P \rangle = \frac{\omega^4 (a d V_0)^2}{3 c^3}$$

From circuit analysis $\rightarrow$ $I(t) = \dot{Q} = \frac{a}{2} V_0 (-i\omega) e^{-i\omega t} = I_0 e^{-i\omega t}$

but

$$\langle P \rangle = \frac{\omega^4 a^2 d^2 V_0^2}{3 c^3} \cdot \frac{I_0^2}{\left(\frac{a^2 V_0^2}{4}\right)} \frac{2}{2} = \left(\frac{8 \omega^4 d^2}{3 c^3} \right) \frac{I_0^2}{2}$$

$\uparrow$ Resistance due to Radiation

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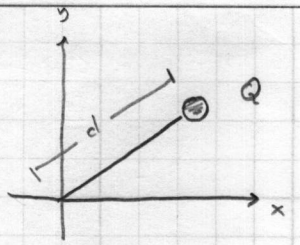

note :

$$\rho(\vec{r},t) \neq \mathrm{Re}\left(\rho(\vec{r})\, e^{-i\omega t}\right)$$

instead we formalism

$$P_x = Q d \cos\omega t$$

$$P_y = Q d \sin\omega t$$

$$P_z = 0$$

$$P(t) = \mathrm{Re}\left\{ Q d\, (\hat{x} + i\hat{y})\, e^{-i\omega t} \right\} = \mathrm{Re}\left(\vec{p}\, e^{-i\omega t}\right)$$ correct form

$$\langle P \rangle = \frac{\omega^4}{3c^3} |\vec{p}|^2 = \frac{\omega^4}{3c^2} (1^2 + 1^2)\, Q^2 d^2$$

$$= \frac{2}{3} \frac{\omega^4 d^2}{c^3} Q^2$$

$$= \frac{2}{3} \frac{Q^2}{c^3} a^2$$ Larmor formula

note only valid non-relativistically since we assumed $d \ll \frac{c}{\omega}$

$|\omega d| = |\vec{v}| \ll c$

Magnetic dipole + electric quadrapole $n = 1$

$$\vec{A}(\vec{r}') = -\frac{ik}{cr} e^{ikr} \int d^3\vec{r}'\, (\hat{r}\cdot\vec{r}')\, \vec{J}(\vec{r}')$$

$$= -\frac{ik}{cr} e^{ikr} \int d^3\vec{r}' \left\{ -\frac{1}{2}\, \hat{r} \times (\vec{r}' \times \vec{J}) \right\} + \left\{ \frac{1}{2} (\hat{r}\cdot\vec{r}')\vec{J} + (\hat{r}'\cdot\vec{J})\vec{r}' \right\}$$

Magnetic dipole electric quadrapole

$$\vec{m} = \frac{1}{2c} \int \vec{r}' \times \vec{J}(\vec{r}')\, d^3\vec{r}'$$

$$\vec{A}_M = \frac{ik}{r} e^{ikr}\, \hat{r} \times \vec{m}$$

as before

$$\frac{d\langle P \rangle}{d\Omega} = \frac{c}{8\pi} k^4 |\hat{r} \times \vec{m}|^2 = \frac{c k^4}{8\pi} \cdot |\vec{m}|^2 \sin^2\theta$$

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$\nabla \times \dot{d}(r) \Rightarrow \quad \partial_x \dot{d}(t - r/c) = -\frac{1}{c}\ddot{d}(t-r/c) \quad \left(-\frac{1}{c}, \partial_x r\right)$

$= -\frac{1}{c}\ddot{p}(t-r/c)\,\frac{x}{r}$

$\nabla \times \dot{d}(r) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \dot{p}_x & \dot{p}_y & \dot{p}_z \end{vmatrix} = \left(\ddot{p}_z \frac{y}{r} - \ddot{p}_y \frac{z}{r}\right)\hat{x} + \left(\ddot{p}_z \frac{x}{r} - \ddot{p}_x \frac{z}{r}\right)(-\hat{y}) + \left(\ddot{p}_y \frac{x}{r} - \ddot{p}_x \frac{y}{r}\right)\hat{z}$

$= +\frac{1}{c}\ddot{p}(t-r/c) \times \hat{r}$

$\vec{B} = \frac{q}{rc^2}\,\ddot{\vec{p}}\,(t-r/c) \times \hat{r} + \mathcal{O}(1/r^2)$

$\varphi(\vec{r},t) = \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|} \qquad \rho(\vec{r},t) = q\,\delta(\vec{r} - \vec{d}(t)) - q\,\delta(\vec{r} + \vec{d}(t))$

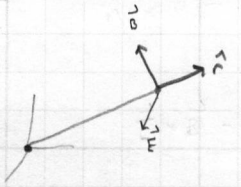
$= \frac{q}{|\vec{r}-\vec{d}|} - \frac{q}{|\vec{d}+\vec{r}|} = \frac{1}{b}\left(1 + \frac{\vec{d}}{b}\cdot\hat{r}\right) - \frac{1}{b}\left(1 - \frac{\vec{d}}{b}\cdot\hat{r}\right)$

$= \frac{q}{r^2}\,\hat{r}\cdot\vec{d}$

$\nabla\varphi = q\,\frac{\dot{d}(t-r/c)\left(-\frac{1}{c}\right)}{r^2}\,\hat{r} + \mathcal{O}(1/r^2)$

$\approx \mathcal{O}(1/r^2)$

$|\vec{E}| = |\vec{B}|$ in radiation zone, orthogonal



$\vec{E} \times \vec{B} = |\vec{B}|^2\,\hat{r}$

$\vec{p} = \int d^3\vec{r}' \, \vec{r}' \rho(\vec{r}')$ $\qquad \vec{m} = \frac{1}{2c} \int d^3\vec{r}' \, \vec{r}' \times \vec{J}(\vec{r}')$

$\downarrow$

$\delta \vec{r}' \, m_2 -$

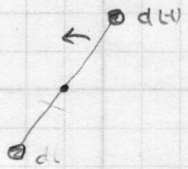
$\left|\frac{\vec{m}}{\vec{p}}\right| \sim \frac{\omega p c}{\omega c} \ll \frac{1}{r}$

Sp. 02 #16 $\qquad -q + q$ separated by $d(t)$, rotate about center

$|\vec{r}| \gg d$

Method 1:

$$\vec{A}(\vec{r},t) = \frac{1}{c} \int d^3\vec{r}' \, \frac{\vec{J}(\vec{r}', t - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|}$$



$$\vec{J}(\vec{r},t) = q\left(\frac{\vec{d}}{2}\right)\delta\left(\vec{r} - \frac{\vec{d}(t)}{2}\right) + (-q)\left(-\frac{\vec{d}}{2}\right)\delta\left(\vec{r} + \frac{\vec{d}(t)}{2}\right)$$

$\vec{J} = \hat{y} \, J\Delta = \vec{J}(\vec{r}) f$

$|\vec{r}+\vec{r}'| = r + \hat{r}\cdot\vec{r}' = r - \hat{r}\cdot\vec{r}'$

$$\frac{1}{|\vec{r}-\vec{r}'|} \approx \frac{1}{r - \hat{r}\cdot\vec{r}'} = \frac{1}{r}\frac{1}{(1-\frac{\hat{r}\cdot\vec{r}'}{r})} = \frac{1}{r}\left(\frac{\hat{r}\cdot\vec{r}'}{r} + 1\right)$$

$$\frac{1}{|\frac{\vec{r}}{2} - \vec{r}'|} \approx \frac{1}{r}\left(\frac{\vec{r}}{2}\cdot\frac{\vec{r}}{r} + 1\right)$$

$$\frac{1}{|\frac{\vec{r}}{2} + \vec{r}'|} \approx \frac{1}{r}\left(-\frac{\vec{r}}{2}\cdot\frac{\vec{r}}{r} + 1\right)$$

so $\quad \vec{A}(\vec{r},t) = \frac{q\dot{\vec{d}}}{2c}\left(\frac{1}{|\frac{\vec{r}}{2} - \vec{r}'|} + \frac{1}{|\frac{\vec{r}}{2} + \vec{r}'|}\right) + \frac{q\ddot{\vec{d}}}{2}\left(\frac{1}{|\frac{\vec{r}}{2} + \vec{r}'|}\right)$

$$\approx \frac{q\dot{\vec{p}}}{cr} + O\left(\frac{1}{r^2}\right)$$

$\vec{B} = \nabla \times \vec{A}$

$$= \frac{q}{c}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \dot{p}_x & \dot{p}_y & \dot{p}_z \end{vmatrix} = \frac{q}{c}\left(\frac{\partial \dot{p}_z}{\partial y} - \frac{\partial \dot{p}_y}{\partial z}\right)\hat{x} + \frac{q}{c}\nabla \times \dot{\vec{d}}(r) =$$

$$\hat{x}\left(\frac{z\dot{p}_y - y\dot{p}_z}{r^3}\right) - \hat{x}\left(\frac{\dot{p}_z}{r}\right) + \frac{q}{c}\nabla \times \dot{\vec{d}}(r) =$$

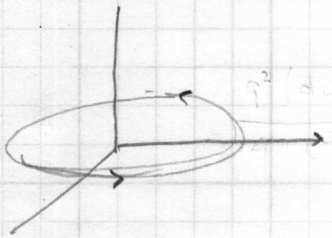
$$\hat{x}\left(\frac{\vec{r}}{r^2} \times \dot{\vec{p}}\right)$$

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \frac{q^2}{4\pi c^3} \left| \ddot{\vec{d}} \times \hat{r} \right|^2 \frac{\hat{r}}{r^2}$$

$$dP = (\vec{S} \cdot \hat{r})\, r^2 d\Omega$$

$$= \frac{q^2}{4\pi c^3} \left| \ddot{\vec{d}} \times \hat{r} \right|^2 d\Omega$$

Loop using Formalism



$$\vec{m}(t) = -\frac{I_0 A}{c} e^{-i\omega t} \hat{z}$$

$$\frac{d\langle P \rangle}{d\Omega} = \frac{c k^4}{8\pi} \left(\frac{I_0 A}{c} \right)^2 \sin^2\theta$$

$$\langle P \rangle = \frac{4}{3} \frac{\omega^4}{c^5} I_0^2 A^2$$

Scattering - small dielectric sphere in uniform $\vec{E}$-field

$$\vec{P} = \left(\frac{\varepsilon - 1}{\varepsilon + 1} \right) a^3 \vec{E}$$

if $\vec{E} = E_0 e^{-i\omega t}$, then

$$\vec{P}_0 = \left(\frac{\varepsilon - 1}{\varepsilon + 1} \right) a^3 \vec{E}_0$$

$$\frac{d\langle P \rangle}{d\Omega} = \frac{c k^4}{8\pi} \left(\frac{\varepsilon - 1}{\varepsilon + 1} \right)^2 a^6 \left| \vec{E} \times \hat{r} \right|^2$$

Cross section

$$\frac{d\langle P \rangle}{d\Omega} = \frac{d\sigma}{d\Omega} \times \langle I \rangle \quad \leftarrow \text{intensity of wave } \left(\frac{P}{A} \right)$$

$$= \frac{d\sigma}{d\Omega} \cdot \frac{c}{8\pi} |E|^2$$

$$= k^4 a^6 \left(\frac{\varepsilon - 1}{\varepsilon + 1} \right)^2 \sin^2\theta$$

Rayleigh Scattering