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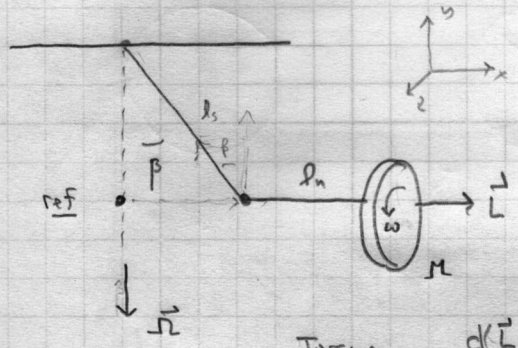
Problem 1 sp 04



Force y: $T\cos\beta = mg$

x: $T\sin\beta = m(l_s\sin\beta + l_n)\Omega^2$

↑ precession rate

Torque $\dfrac{d(\vec{L}_{cm} + \vec{L}')}{dt}_{\text{fixed}} = \dfrac{d\vec{L}'}{dt}_{\text{fixed}} = \left(\dfrac{d\vec{L}}{dt}\right)^{0}_{\text{body}} + \vec{\Omega}\times\vec{L}$

$= \Omega L\,\hat{z}$

Torque : $\vec{T} = mg(l_s\sin\beta + l_n)(-\hat{z}) + (T\cos\beta)\, l_s\sin\beta\,(+\hat{z})$

$\vec{T} = -mg\,l_n\,\hat{z}$

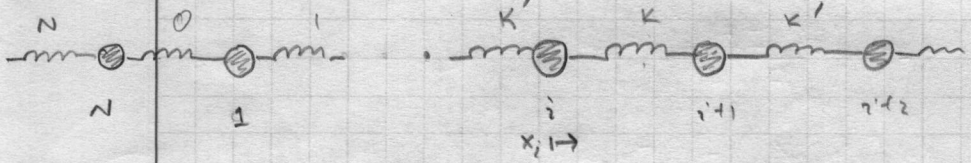
$\Omega L = -mg\,l_n$

$\Omega = \dfrac{-mgl_n}{L} = -\dfrac{mgl_n}{I\omega} = -\dfrac{mgl_n}{\frac{mR^2}{2}\omega} = -\dfrac{2gl_n}{R^2\omega}$

$$\boxed{\tan\beta = \frac{(l_s\sin\beta + l_n)\Omega^2}{g} = \frac{(l_s\sin\beta + l_n)}{g}\,\frac{4g^2 l_n^2}{R^4\omega^2}}$$

fall 03 2003

infinite conditions → assume periodic B.C.



1 more spring than masses

$$L = \sum_{i=1}^{N} \tfrac{1}{2} m \dot{x}_i^2 - \sum_{i=0}^{N} \frac{K_i}{2}(x_{i+1} - x_i)^2$$

$$- \sum_{\substack{i=0 \\ \text{odd}}}^{N} \frac{K}{2}(x_{i+1} - x_i)^2 - \sum_{\substack{i=0 \\ \text{even}}}^{N} \frac{K'}{2}(x_{i+1} - x_i)^2$$

$$- \frac{K}{2}(x_{j+1} - x_j)^2 - \frac{K'}{2}(x_j - x_{j-1})$$

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$$L = \sum_{i=1}^{N} \frac{m}{2} \dot{x}_i^2 - \sum_{i=0}^{N} \frac{\kappa}{2} \left(x_{2i+2} - x_{2i+1} \right)^2 - \sum_{i=0}^{N} \frac{\kappa'}{2} \left(x_{2i+1} - x_{2i} \right)^2$$

$x_{n+N} = x_n \quad (\; Na =$ period

$$x_n = \sum \frac{1}{\sqrt{N}} \sum_j e^{iqna} \hat{x}_q$$ discrete fourier series

$$\hat{x}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan} x_n$$

$$x_n = \frac{1}{N} \sum_{n'} \sum_j e^{iq_j n a} e^{-iq_j n' a} x_{n'}$$

$$= \frac{1}{N} \sum_{n'} \sum_j \exp\left(iqa(n-n') \right) x_{n'}$$

$$\sum_{j=1}^{N} \exp\left(2a(n-n') \right) = \frac{1 - e^{2a(n-n')(N+1)}}{1 - e^{2a(n-n')}}$$

$$x_{N+n} = \frac{1}{\sqrt{N}} \sum_{q=1}^{N} e^{iq_j (N+n) a} \frac{1 - e^{2a(n-n')}}{1 - e^{2N a}} \hat{x}_q$$

$$= \frac{1}{\sqrt{N}} \sum_{q=1}^{N} e^{iqan} \hat{x}_n \quad \rightarrow \quad e^{iqNa} = 1$$

$$qNa = 2\pi j$$

$$q = \frac{2\pi}{Na} j \quad \rightarrow \quad j = 1 \cdots N$$

$$x_n = \frac{1}{N} \sum_{j=1}^{N} \sum_{n'} \exp\left(iqa(n-n') \right) x_{n'}$$

$$= \frac{1}{N} \sum_{j=1}^{N} e^{iq_j an} \frac{1 - e^{iq_j aN}}{1 - e^{-iq_j a}} \left(\frac{1 + (e^{-iq_j a})^{N+1}}{1 - e^{-iq_j a}} \right) x_n$$

$$q \cdot N = \frac{2\pi}{a} j$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}_j} \right) = \frac{\partial L}{\partial x_j} = + \frac{\kappa}{2} 2 (x_{j+1} - x_j) - \frac{\kappa'}{2} 2 (x_j - x_{j-1})$$

$$m \ddot{x}_j = \kappa x_{j+1} + \kappa' x_{j-1} - (\kappa + \kappa') x_j$$

$$\ddot{\hat{x}}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \ddot{x}_n$$

$$= \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(K x_{n+1} + K' x_{n-1} - (K+K') x_n \right)$$

shift indices $\quad n+1 = n' \qquad n-1 = n'$
$\qquad\qquad\qquad\quad n = n'-1$

$$= \frac{1}{\sqrt{N}} \sum_{n=2}^{N+1} e^{-iqa(n-1)} x_n K + \sum_{n=0}^{N-1} e^{-iqa(n+1)} K' x_n - \sum_{n=1}^{N} e^{iqan} (K+K') x_n$$

$$= \frac{1}{\sqrt{N}} \sum_{n=} e^{-iqan} \left(K e^{+iqa} + K' e^{-iqa} - (K+K') \right) x_n$$

$$= \left\{ K e^{iqa} + K' e^{-iqa} - (K+K') \right\} \hat{x}_q$$

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Rigid Bodies

Euler's eq $\left(\frac{d\vec{L}}{dt}\right)_{fixed} = \vec{N}$ (torque)

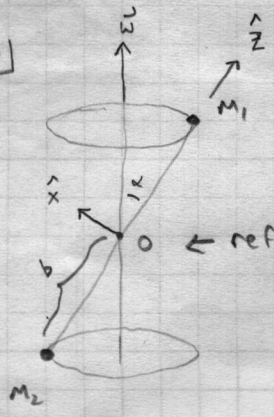
$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = \vec{N}, \quad \vec{L} = I_i\omega_i \quad \text{(body frame)}$$

$$\vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}$$

$$= \hat{x}\,(I_3 - I_2)\,\omega_2\omega_3 - \hat{y}\,(I_3 - I_1)\,\omega_1\omega_3 + \hat{z}\,(I_2 - I_1)\,\omega_1\omega_2$$

$$\left.\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_1\omega_3 &= N_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\,\omega_1\omega_2 &= N_3 \end{aligned}\right\} \text{ Euler's eq}$$

ex



body axis: $I = \begin{pmatrix} (m_1+m_2)b^2 & & \\ & (m_1+m_2)b^2 & \\ & & 0 \end{pmatrix}$

$\vec{\omega} = \omega\left(\hat{z}\cos\alpha + \hat{x}\sin\alpha\right)$, $\dot{\omega} = 0$

$\vec{L} = L_1\,\hat{x}$ (by choice)

$= (I_1\omega_1)\,\hat{x}$

$= (m_1+m_2)b^2\,\omega\sin\alpha$

E-q: $(I_2 - I_3)\,\omega_2\omega_3 = 0 = N_1$

$-(I_3 - I_1)\,\omega_1\omega_3 = (m_1+m_2)b^2\,\omega^2\sin\alpha\cos\alpha = N_2$ ← required for dynamic stability

$(I_1 - I_2)\,\omega_1\omega_2 = 0 = N_3$

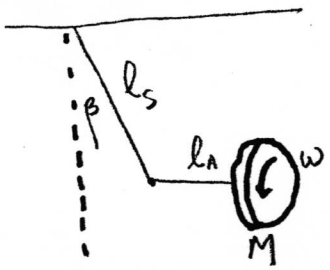
ID Number _____ PART I Score _____

PHYSICS DEPARTMENTAL EXAM – SPRING 2004

SECTION 1:

PROBLEM 1.

1. A solid disc of mass M and radius R is rotating at an angular frequency ω. As shown in the figure, the disc is attached to an axle of length ℓ_A which is being held up by a string of length ℓ_s. If the disc executes uniform precession in the horizontal plane and the axle is horizontal, find an equation for the angle β that the string makes with the vertical axis



Solution Prob. 1

Dept. Exam Solutions SP04

Solution 1

Moment of Inertia

$I = \frac{1}{2} M R^2$ for the disc. We can write down three equations, from force considerations (2) and from torque (1). In detail

$T\cos\beta = Mg$ (Force in vertical direction)

$T\sin\beta = M(L_A\sin\beta + \ell)\Omega^2$ (horizontal)

if Ω is the precession rate

and finally, torque equals rate of change of angular momentum. Here

$\frac{dL}{dt} = \Omega L = I\omega\Omega$

the center of the circle along which the disc is precessing

This must equal the torque (with respect to ~~part~~ O)

$I\omega\Omega = Mg(\ell + L_A\sin\beta) - T L_A \cos\beta \sin\beta$

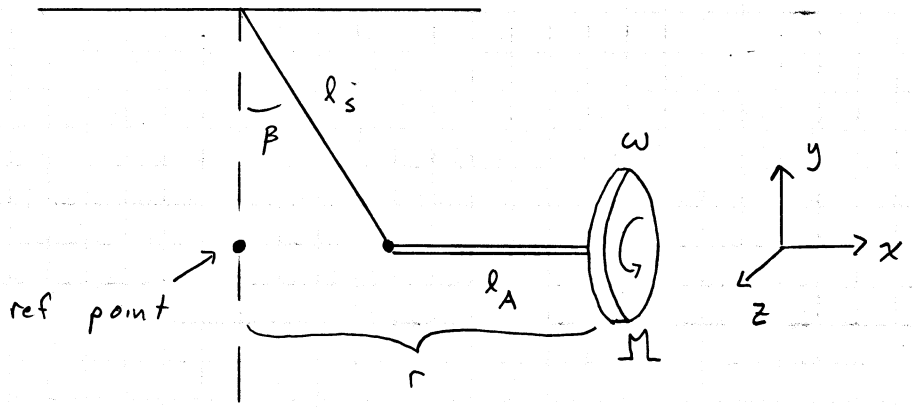
$= Mg\ell$

Solving these equations gives $\Omega = \frac{Mg\ell}{I\omega}$

and

$M(L_A\sin\beta + \ell)\Omega^2 = Mg\tan\beta$

A solid disc of mass M and radius R is rotating at an angular frequency ω. As shown in the figure, the disc is attached to an axle of length ℓ_A which is being held up by a string ℓ_s. If the disc executes uniform precession in the horizontal plane and the axle is horizontal, find an equation for the angle β that the string makes with the vertical axis.



First, balance the forces: two forces

$$\vec{T} + Mg(-\hat{y}) = \frac{Mv^2}{r}(-\hat{x}) = M\Omega^2(\ell_s \sin\beta + \ell_A)(-\hat{x})$$

$$\Rightarrow \quad T\cos\beta = Mg \qquad (y\text{-comp})$$

$$T\sin\beta = M\Omega^2 \qquad (x\text{-comp})$$

so,

$$\tan\beta = \frac{\Omega^2}{g}(\ell_s \sin\beta + \ell_A)$$

However, we are not done. We can now balance the torques to elliminate Ω.

Assumption: we can neglect the mass of the axle compared to the mass of the disc, i.e. $M_A \ll M$.

Balance torques:

$$\vec{N} = \sum_i \vec{r}_i \times \vec{F}_i$$

$$= \vec{r}_1 \times \vec{T} + \vec{r}_2 \times \vec{F}_g$$

$$\vec{r}_1 = l_s \sin\beta \, \hat{x}$$

$$\vec{r}_2 = (l_s \sin\beta + l_A)\, \hat{x}$$

$$\vec{T} = -T_x(-\hat{x}) + T_y(\hat{y})$$

$$\vec{F}_g = M g (-\hat{y})$$

$$\vec{N} = l_s \sin\beta \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & 0 \\ -T_x & T_y & 0 \end{vmatrix} - M g (l_s \sin\beta + l_A)\underbrace{(\hat{x} \times \hat{y})}_{\hat{z}}$$

$$= \hat{z}\left\{ M g l_s \sin\beta - M g (l_s \sin\beta + l_A) \right\}$$

$$= -M g l_A \hat{z}$$

Thus, the precession direction is counter clockwise (as viewed from above). We know $\vec{\Omega} = \Omega(+\hat{y})$.

Now use $\vec{N} = \frac{d\vec{L}_{tot}}{dt}$

$$\vec{L}_{tot} = \vec{L}_{cm} + \vec{L}'$$

$\vec{L}' = I\omega\,\hat{x}$, ang mom. due to spin about axle

$\vec{L}_{cm} = M r^2 \Omega\,\hat{y}$, ang mom. due to precession of center of mass of the disk.

Note: $\hat{x}, \hat{y}$ and $\hat{z}$ are body fixed coordinates but only

$$\frac{d}{dt}\hat{y} = 0 \quad \text{(always pointed up)}.$$

That is $\hat{x}$ and $\hat{z}$ change directions as a function of time.

$$\Rightarrow \quad \frac{d\vec{L}_{tot}}{dt} = \cancel{\frac{d\vec{L}_{cm}}{dt}}^{\,0} + \frac{d\vec{L}'}{dt}$$

since $\vec{L}_{cm}$ is constant

$$= \cancel{\left(\frac{d\vec{L}'}{dt}\right)_{Body}}^{\,0} + \vec{\Omega} \times \vec{L}'$$

since $\vec{L}'$ has a constant magnitude

$$= I\omega\Omega\,(\hat{y} \times \hat{x})$$

$$= I\omega\Omega\,(-\hat{z})$$

Thus, $\quad M g l_A = I \omega \Omega$

$$\Omega = \frac{M g l_A}{I \omega} = \frac{2 g l_A}{\omega R^2}$$

Our equation for β becomes:

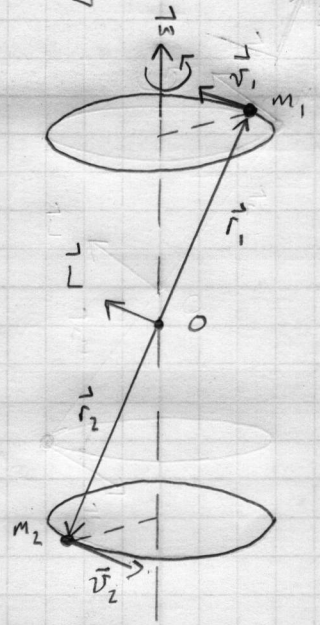
$$\boxed{\tan\beta = (l_s \sin\beta + l_A)\,\frac{4 g l_A}{\omega^2 R^4}}$$

This is trancendental so this cannot be solved analytically.

Reference: Classical Dynamics, Marion & Thorton

page 421 and 447 (ex 11.10).

Consider the rotating dumbell (connecting rods are massless)



$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \qquad \text{(since } m_i \text{ traces out a circle)}$$

$$\vec{L} = \sum_{i=1}^{2} m_i \vec{r}_i \times \vec{v}_i$$

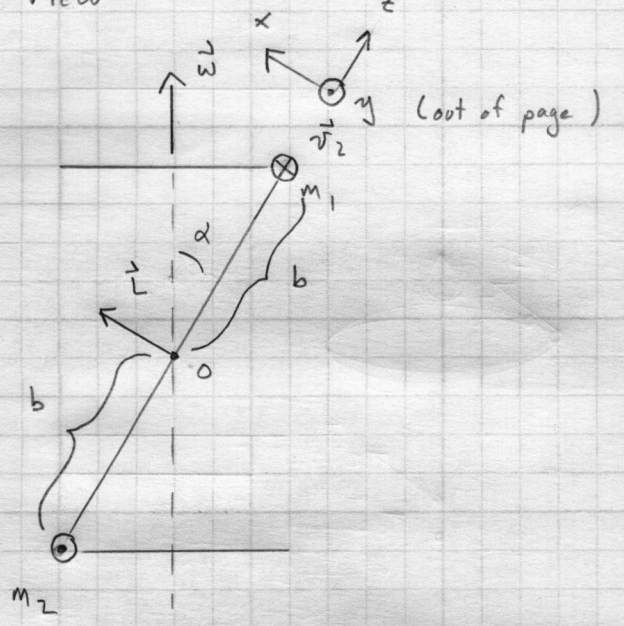
$$= m_i \sum_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

→ so $\vec{\omega} \times \vec{r}_i$ is $\perp$ to $\vec{\omega}$ and $\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$ is not in general $\parallel$ to $\vec{\omega}$. This is clearly a case when $\hat{L} \neq \hat{\omega}$.

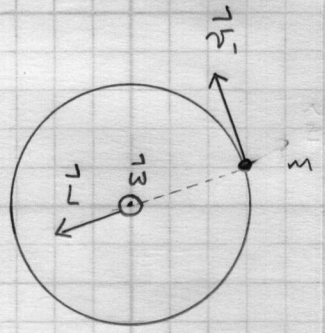
→ Now lets solve this problem for an explicit case:

- let $|\vec{r}_1| = |\vec{r}_2| = b$
- Also, let z-axis be along the rod as in diagram below. Also let x-axis be along $\vec{L}$.

Side View



Top View



The coordinate system (the body fixed frame) has non-constant unit vectors. In this frame:

$$\vec{\omega} = \omega(\sin\alpha\,\hat{x} + \cos\alpha\,\hat{z}) = \omega_1\hat{x} + \omega_3\hat{z}$$

$$I = \begin{pmatrix} (m_1+m_2)b^2 & & \\ & (m_1+m_2)b^2 & \\ & & 0 \end{pmatrix}$$

$$\vec{L} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & 0 \end{pmatrix}\begin{pmatrix} \omega_1 \\ 0 \\ \omega_3 \end{pmatrix} = I_1\omega_1\hat{x}$$

$$= (m_1+m_2)b^2\,\omega\sin\alpha$$

Using Euler Equations:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = N_2$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

CAMPAD 22-141 50 SHEETS 22-142 100 SHEETS 22-144 200 SHEETS

since $\dot{\omega} = \omega_2 = I_3 = 0$

This reduces to: $N_1 = N_3 = 0$

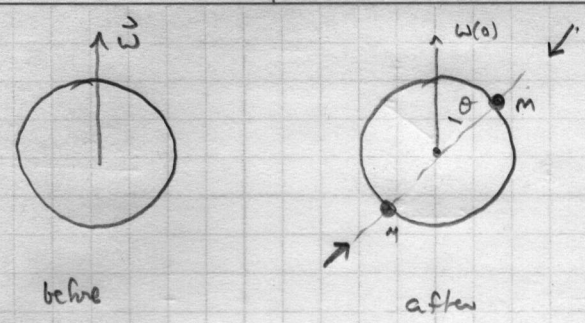
$$N_2 = I_1 \omega_1 \omega_3$$
$$= (m_1 + m_2) b^2 \omega^2 \sin\alpha \cos\alpha$$

So for a constant $\vec{\omega}$, we require a non-zero torque. This is why we most balance our tires because the friction in the Bearings provides this torque; thus, wearing out your bearings.

22-141 50 SHEETS
22-142 100 SHEETS
22-144 200 SHEETS
CAMPAD

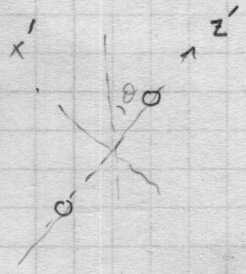
fall 05

2



a) find new I, $\quad I_{old} = \frac{2}{5} m R^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

→ body fixed frame ! $\quad I_{new} = \begin{pmatrix} 2mR^2 & & \\ & 2mR^2 & \\ & & 0 \end{pmatrix}$



b) find subsequent rotation:

find $\vec{\omega}(0)$ in body frame!

$$\vec{\omega}_0 = \omega(\cos\theta\,\hat{z} + \sin\theta\,\hat{x})$$

Eulers eq:

$$I_1\dot{\omega}_1 - (\cancel{I_2 - I_3})\,\omega_2\omega_3 = 0 \qquad I_2 - I_3 = 2mR^2$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\,\omega_3\omega_1 = 0$$

$$I_3\dot{\omega}_2 - (\cancel{I_1 - I_2}^{\,0})\,\omega_1\omega_3 = 0 \quad \rightarrow \quad \omega_3 = \text{const}$$

$$\left.\begin{aligned} \dot{\omega}_1 &= -\left(\frac{\Delta I\,\omega_3}{I}\right)\omega_2 \\ \dot{\omega}_2 &= -\left(\frac{\Delta I\,\omega_3}{I}\right)\omega_1 \end{aligned}\right\} \quad \Omega = \frac{2mR^2\,\omega_3}{\left(\frac{2}{5}\right)m_e R^2} = 5\left(\frac{m}{m_e}\right)\omega_3 = 5\left(\frac{m}{m_e}\cos\theta\right)$$

$$\omega_1(t) = \omega\cos\Omega t\,\sin\theta$$

$$\omega_2(t) = \omega\sin\Omega t\,\cos\theta$$

$$\omega_3(t) = \omega\cos\theta$$

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Force free symetric top $(I_1 = I_2)$

$$(I_1 - I_3)\,\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$$

$$(I_3 - I_1)\,\omega_3\omega_1 - I_1\dot{\omega}_2 = 0$$

$$I_3\dot{\omega}_3 = 0 \rightarrow \omega_3 = \text{const}$$

$$\left.\begin{aligned}\dot{\omega}_1 &= -\Omega\omega_2\\ \dot{\omega}_2 &= +\Omega\omega_1\end{aligned}\right\} \rightarrow \text{solution} \quad \omega_1(t) = A\cos\Omega t + \omega_{2_0}$$

$$\omega_2(t) = A\sin\Omega t + \omega_{2_0}$$

$$\Omega = \left(\frac{I_3 - I_1}{I}\omega_3\right) \rightarrow \text{angle by which } \omega(t) \text{ rotate about } \hat{e}_3 \text{ (symmetry axis)}$$

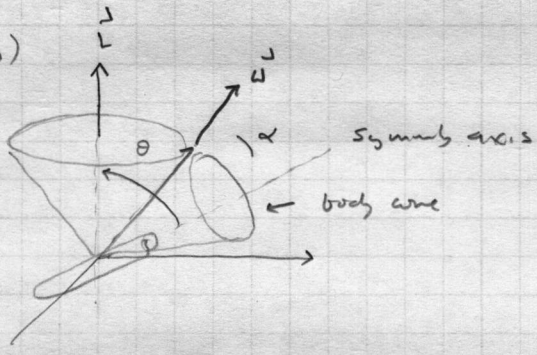
$$|\omega| = \sqrt{\omega_3^2 + A^2} = \text{const}$$

$\frac{d\vec{L}}{dt} = 0 \rightarrow \vec{L}$ is fixed in space

$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} \rightarrow$ T const implies $\vec{\omega}\cdot\vec{L} = \text{const}$

Prolate $(I_1 > I_3)$

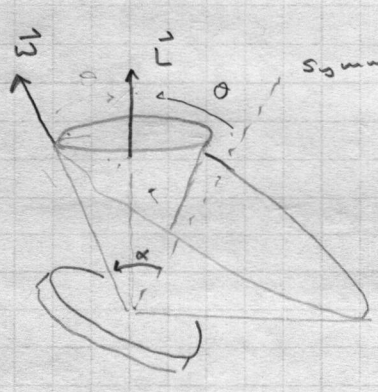
Ω & ω_3 same sign



$$\tan\theta = \frac{I_1}{I_3}\tan\alpha$$

Prolate

Oblate $(I_3 > I_1)$



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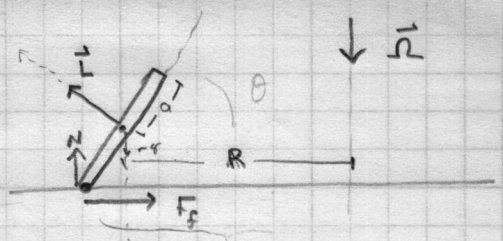
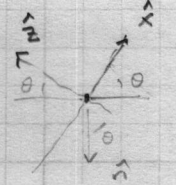
Spring 02

rolling disk :





$\vec{L}$ fixed in body frame → $\left(\frac{d\vec{L}}{dt}\right)_{body} = 0$

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\Omega} \times \vec{L} = \vec{T}$$

o

in body frame : $\vec{\Omega} = -\Omega \cos\theta\, \hat{z} - \Omega \sin\theta\, \hat{x}$

$$\vec{L} = L\hat{z} = I_3 \omega \hat{z} = I_3 \frac{v}{a} \hat{z}$$

rolling condition → $a\omega = R\Omega$

$$\vec{\Omega} \times \vec{L} = -(I_3 \omega)(+\Omega \sin\theta)(\hat{x} \times \hat{z})$$

$$= I_3 \omega \Omega \sin\theta\, \hat{y}$$

$$= I_3 \left(\frac{v}{a}\right)\left(\frac{v}{R}\right) \sin\theta\, \hat{y}$$

$$= \frac{1}{2} M a^2 \frac{v^2}{aR} \sin\theta\, \hat{y} = \frac{1}{2} M v^2 \left(\frac{a}{R}\right) \sin\theta\, \hat{y}$$

$\vec{N} = Mg(\cos\theta\, \hat{z} + \sin\theta\, \hat{x})$ $\quad \vec{r}_N = -a\hat{x}$ $\quad \vec{T}_N = +aMg\cos\theta\, \hat{y}$

$\vec{F}_f = F(\cos\theta\, \hat{x} - \sin\theta\, \hat{z})$ $\quad \vec{r}_f = -a\hat{x}$ $\quad \vec{T}_f = +Fa\sin\theta\,(-\hat{y})$

$$\vec{T} = Mga\cos\theta\, \hat{y} - Fa\sin\theta\,(\hat{y}) \qquad F = M\frac{v^2}{R}$$
$$= \left(Mga\cos\theta - M\frac{v^2}{R}a\sin\theta\right)\hat{y}$$

$\vec{\Omega} \times \vec{L} = \vec{T}$

$$\frac{M v^2}{2} \frac{a}{R} \sin\theta = Mga\cos\theta - Mv^2\left(\frac{a}{R}\right)\sin\theta$$

$$\frac{3M}{2} v^2 \left(\frac{a}{R}\right) \sin\theta = Mga\cos\theta \rightarrow \boxed{\tan\theta = \frac{2gR}{3v^2}}$$

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Stability of Rigid Rotations $(I_3 > I_2 > I_1)$

consider I_2-body rotating around x_1 principle axis

$$\vec{\omega} = \omega_1 \hat{x} + \lambda \hat{y} + \mu \hat{z}$$

$$\left.\begin{aligned} (I_2 - I_3)\lambda\mu - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\mu\omega_1 - I_2\dot{\lambda} &= 0 \\ (I_1 - I_2)\lambda\omega_1 - I_3\dot{\mu} &= 0 \end{aligned}\right\} \text{1st order} \qquad \begin{aligned} \dot{\omega}_1 &= 0 \\ \dot{\lambda} &= \frac{I_3 - I_1}{I_2}\omega_1\mu \\ \dot{\mu} &= \frac{(I_1 - I_2)}{I_3}\omega_1\lambda \end{aligned}$$

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\right)\omega_1\dot{\mu} = \left[\left(\frac{I_3 - I_1}{I_2}\right)\left(\frac{I_1 - I_2}{I_3}\right)\omega_1^2\right]\lambda$$

$$= \Omega_{1\lambda}^2$$

$$\lambda = Ae^{i\Omega_{1\lambda}t} + Be^{-i\Omega_{1\lambda}t}$$

since $I_1 < I_3$ & $I_1 < I_2$ $\rightarrow$ $\Omega_{1\lambda}$ is real

$\rightarrow$ solution oscillates $\rightarrow$ stable solution about x_2, x_3

by permutation:

$$\Omega_1 = \omega_1\sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

$$\left.\begin{aligned} \Omega_2 &= \omega_2\sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}} \\ \Omega_3 &= \omega_3\sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}} \end{aligned}\right\} \text{imaginary}$$

rotation about x_2 $\rightarrow$ perturbation grows w/o bound!

If two $(I_1 = I_2)$ moments of inertia are equal $\rightarrow$ only stable about 3rd axis

Central Force

orbit eq

$$\ddot{r} - r\dot{\theta}^2 = -\frac{K}{r^2} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \rightarrow \frac{d}{dr}(r^2\dot{\theta}) = 0$$

$$l = m r^2 \dot{\theta}$$

eliminate t: $\quad \dot{\theta} = \frac{l}{m}\frac{1}{r^2}$, $u = 1/r$

$$= \frac{l}{m} u^2$$

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{l}{m}\frac{r'}{r^2} = \frac{l}{m}\left(-\frac{du}{d\theta}\right)$$

$$\ddot{r} = -\frac{l}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

$$= -\left(\frac{l}{m}\right)^2 u'' u^2$$

$$-\left(\frac{l}{\mu}\right)^2 u'' u^2 - \frac{1}{u}\left(\frac{l}{\mu}\right)^2 u^4 = -G\mu u^2$$

$$u'' + u = \frac{GM}{l^2}$$

$$u = \frac{GM}{l^2} + C\cos(\theta - \theta_0) \rightarrow \text{initial conditions}$$

determine IC. $\quad (\dot{r} = 0) \rightarrow u' = 0$

$$u = \frac{GM}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{\mu K^2}}\cos\theta\right)$$

$\theta = \theta^\circ \rightarrow \quad u' = -C\sin(\theta - \theta_0) = 0$

$$r = \left(\frac{l^2}{\mu K}\right)\frac{1}{1 + \varepsilon\cos\theta}$$

↑ choice

$$\frac{E}{\mu} = \frac{1}{2}\mu\left(\dot{r}^2 + r\dot{\theta}^2\right) - \frac{\mu}{r} = \frac{1}{2}l^2u^2 - \mu u$$

at $\theta = \theta_0$,

$$u = \frac{GM}{l^2} + C$$

$$E = \frac{1}{2}l^2C^2 - \frac{(GM)^2}{2l^2}$$

$$C = \frac{(GM)^2}{2l^2} + 2E$$

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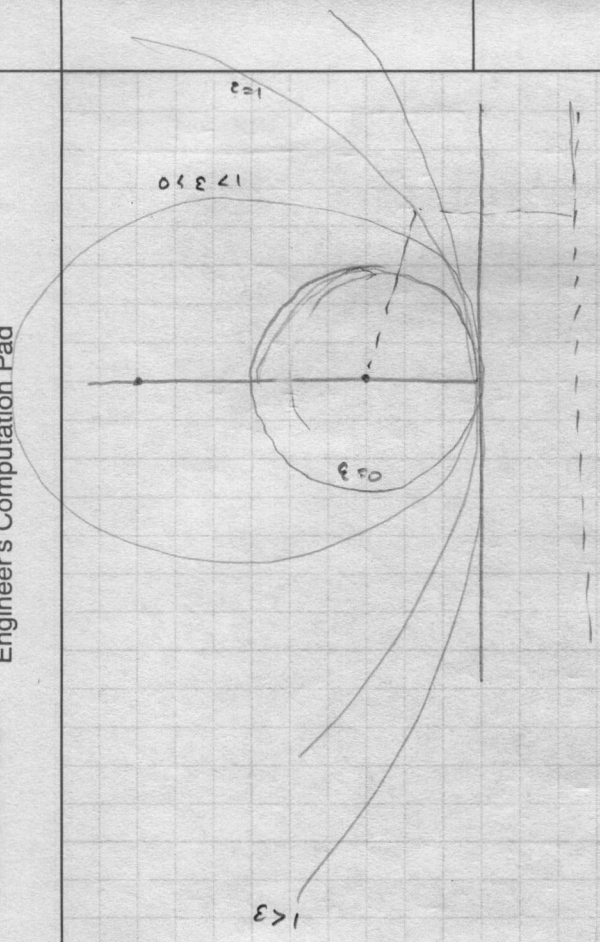
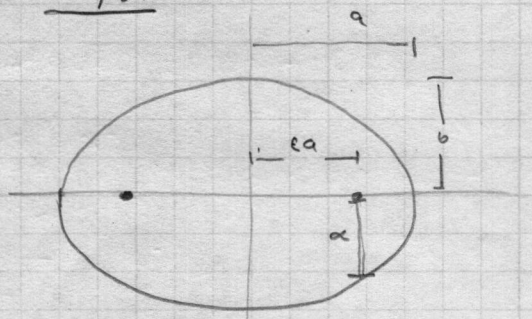


$\varepsilon > 1$	$E > 0$	Hyperbola
$\varepsilon = 1$	$E = 0$	Parabola
$0 < \varepsilon < 1$	$V_{min} < E < 0$	Ellipse
$\varepsilon = 0$	$E = V_{min}$	Circle

ellipse



$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

$$r_{min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon} \qquad a$$

$$r_{max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$

$$T^2 = \frac{4\pi^2\mu}{k} a^3$$