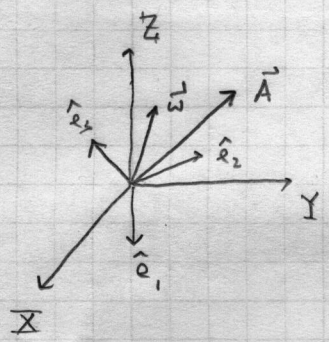
Non-Inertial Systems

Vector derivatives in rotating coordinates



$$\vec{A} = \sum A_i \hat{e}_i$$

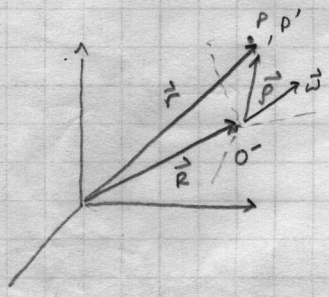
$$\dot{\vec{A}} = \sum \dot{A}_i \hat{e}_i + \sum A_i \dot{\hat{e}}_i$$

$\dot{\hat{e}} = \vec{\omega} \times \hat{e}$

$$= \sum \dot{A}_i \hat{e}_i + \sum A_i \vec{\omega} \times \hat{e}_i$$

$$= (\dot{\vec{A}})_r + \vec{\omega} \times \vec{A}$$

motion in moving coordinate system (P is at object P' coincident w/ P (at $t = t_0$) but fixed in O' frame)



$$\vec{r} = \vec{R} + \vec{\rho}$$

$$\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{\rho}}$$ ($\dot{\vec{\rho}}$: velocity of P w/r O')

$$= \dot{\vec{R}} + (\dot{\vec{\rho}})_r + \vec{\omega} \times \vec{\rho}$$

($\dot{\vec{R}}$: vel of P' w/r O; $(\dot{\vec{\rho}})_r$: vel of P w/r O')

$$\ddot{\vec{r}} = \ddot{\vec{R}} + (\dot{\vec{\rho}})_r^{\cdot} + (\vec{\omega} \times \vec{\rho})^{\cdot}$$

$$\frac{d}{dt}(\dot{\vec{\rho}})_r = (\ddot{\vec{\rho}})_r + \vec{\omega} \times (\dot{\vec{\rho}})_r$$

$$\frac{d}{dt}(\vec{\omega} \times \vec{\rho}) = \dot{\vec{\omega}} + \cancel{\vec{\omega} \times \vec{\omega}} \times \vec{\rho} + \vec{\omega} \times (\dot{\vec{\rho}}_r + \vec{\omega} \times \vec{\rho})$$

$$= \dot{\vec{\omega}} \times \vec{\rho} + \vec{\omega} \times \dot{\vec{\rho}}_r + \vec{\omega} \times \vec{\omega} \times \vec{\rho}$$

$\ddot{\vec{R}}$ absolute accederation of O'

$\dot{\vec{\omega}} \times \vec{\rho}$ $\vec{\omega} \times \vec{\omega} \times \vec{\rho}$ absolute acceleration P' w/r O'

$$\ddot{\vec{r}} = \underbrace{\ddot{\vec{R}} + \dot{\vec{\omega}} \times \vec{\rho} + \vec{\omega} \times \vec{\omega} \times \vec{\rho}}_{\text{absolute acceleration of } P'} + (\ddot{\vec{\rho}})_r + 2\omega \times (\dot{\vec{\rho}})_r$$

($(\ddot{\vec{\rho}})_r$: acceleration of P w/r O' frame)

Coriolis: (1) ~~rate~~ direction change of velocity w/r O' system

(2) rate of change of $\vec{\omega} \times \vec{\rho}$ dire to changing direction / magnitude of $\vec{\rho}$ w/r moving coordinates

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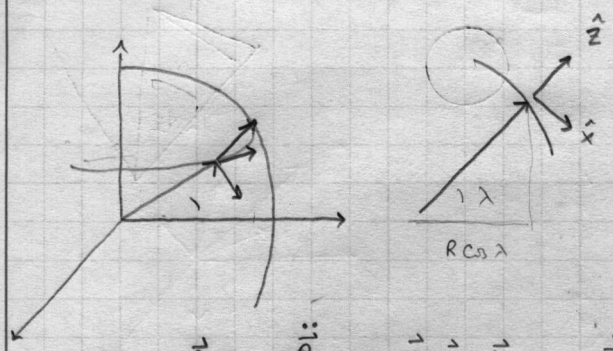
Coriolis Effect

#120

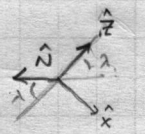


$$\frac{\vec{F}}{m} = -g\hat{z}$$

$$\vec{a} = \ddot{\vec{R}} + \vec{\omega}\times\vec{\omega}\times\vec{\rho} + \dot{\vec{\omega}}\times\vec{\rho}^{\,0} + \ddot{\vec{\rho}}_r + 2\vec{\omega}\times\dot{\vec{\rho}}$$

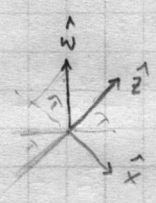
$$\ddot{\vec{R}} = -R\cos\lambda\,\omega^2\,\hat{s}$$

$$= -R\cos\lambda\,\omega^2\left(\hat{z}\cos\lambda + \hat{x}\sin\lambda\right)$$



$$\hat{s} = -\hat{z}\cos\lambda - \hat{x}\sin\lambda$$

cont: $\vec{\omega} = \omega\,\hat{\omega}$,



$$\hat{\omega} = -\hat{x}\cos\lambda + \hat{z}\sin\lambda$$

$$\vec{\rho} = \vec{\rho}_{(0)} + \omega\vec{\rho}_1 + \omega^2\vec{\rho}_2$$

$\mathcal{O}(\omega^0)$:

$$-g\hat{z} = (\ddot{\vec{\rho}}_0)_r \rightarrow \begin{cases} \ddot{z} = -g \\ \ddot{y} = 0 \\ \ddot{x} = 0 \end{cases} \quad \begin{cases} z = h - gt^2/2 \\ y = 0 \\ x = 0 \end{cases}$$

$\mathcal{O}(\omega^1)$:

$$0 = \ddot{\vec{R}} + \vec{\omega}\times\vec{\omega}\times\vec{\rho} + \ddot{\vec{\rho}}_r + 2\vec{\omega}\times\dot{\vec{\rho}} + g\hat{z}$$

$$\omega\ddot{\vec{\rho}}_1 + 2\vec{\omega}\times\dot{\vec{\rho}}_0 = 0$$

$$\hat{\omega}\times\dot{\vec{\rho}}_0 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & -gt \end{vmatrix} = -\hat{y}\left(gt\cos\lambda\right)$$

y_1 : $\ddot{y}_1 + 2gt\cos\lambda = 0$

$$y_1 = \frac{gt^3}{3}\cos\lambda + \dot{x}_{1_0}t + x_{1_0}^{\,0}$$

x_1 : $\ddot{x}_1 = 0$ ... $\hat{z}$: $\ddot{z}_1 = 0$

$\mathcal{O}(\omega^2)$:

$$-R\cos\lambda\,\omega^2\left(\hat{z}\cos\lambda + \hat{x}\sin\lambda\right) + \omega^2\left(\hat{\omega}\times\hat{\omega}\times\vec{\rho}_0\right) + \omega\ddot{\vec{\rho}}_2 + \omega 2\vec{\omega}\times\dot{\vec{\rho}}_1 = 0$$

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$$\hat{\omega} \times \hat{\omega} \times \vec{\rho}_0 = \hat{\omega} \times \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & z_0 \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & z_0\cos\lambda & 0 \end{vmatrix} = -z_0\cos\lambda\sin\lambda\,\hat{x} - z_0\cos^2\lambda\,\hat{z}$$

$$\hat{\omega} \times \dot{\vec{\rho}}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & gt^2\cos\lambda & 0 \end{vmatrix} = -gt^2\cos\lambda\sin\lambda\,\hat{x}$$

$\hat{x}$: $\quad -R\cos\lambda\sin\lambda = \left(h - \frac{gt^2}{2}\right)\cos\lambda\sin\lambda + \ddot{x}_2 = 2gt^2\cos\lambda\sin\lambda = 0$

$$\ddot{x}_2 = \left(R + h + \frac{3}{2}gt^2\right)\cos\lambda\sin\lambda$$

$$x = \left(\frac{(R+h)t^2}{2} + \frac{gt^4}{8}\right)\cos\lambda\sin\lambda$$

$\hat{y}$: $\quad \ddot{y}_2 = 0 \qquad y_2 = 0$

$\hat{z}$: $\quad -R\cos^2\lambda + z_0\cos^2\lambda + \ddot{z}_2 = 0$

$$\ddot{z}_2 = \left(R + h - \frac{gt^2}{2}\right)\cos^2\lambda$$

$$z_2 = \left[\frac{(R+h)t^2}{3} - \frac{gt^4}{24}\right]\cos^2\lambda$$

$$x = \omega^2\left[\frac{(R+h)t^2}{2} - \frac{gt^4}{6}\right]\cos\lambda\sin\lambda$$

$$y = \frac{\omega g t^3}{3}\cos\lambda$$

$$z = \left(h - \frac{gt^2}{2}\right) + \left(\omega^2\left[\frac{(R+h)t^2}{3} - \frac{gt^4}{24}\right]\cos^2\lambda\right)$$

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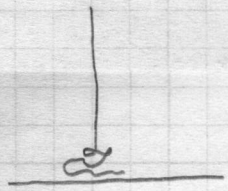
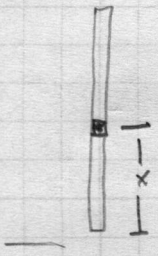
Problem 2



μ ~ linear mass density

$\Delta p = I \rightarrow$ assume free fall



$$V(x) = \mu dx\, g\, x$$

$$\mu \frac{v^2}{2} = \mu g x \rightarrow v = \sqrt{2gx}$$

$$\Delta v = v - 0 = \sqrt{2gx}$$

$$\Delta p = \mu \Delta V \Delta x$$

$$\frac{\Delta p}{\Delta x} = \mu \Delta V = \sqrt{2gx}$$

$$I = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} = \mu\left(\sqrt{2gx}\right)^2 = 2\mu g x$$

$$F = I + W = 2\mu g x + \mu g x$$

$$F = 3\mu g x$$

example in Marion & Thorton

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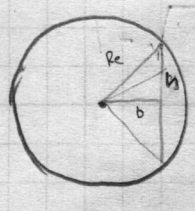
Fall 97



$$b = \sqrt{R_e^2 - h^2}$$

$$2h = 500 \text{ KM}$$

$$R_e = 6400 \text{ KM}$$

$$\vec{r} = b\hat{x} + y\hat{y} \qquad \frac{\vec{r}}{r^2} = \frac{b\hat{x} + y\hat{x}}{\sqrt{b^2+y^2}}$$

$$r = \sqrt{b^2+y^2}$$

$$\nabla^2\varphi = 4\pi G\rho$$

$$\int \nabla\varphi \cdot dA = 4\pi G \rho \frac{4\pi}{3} r^3$$

$$\nabla\varphi_r \, 4\pi r^2 = 4\pi G\rho \frac{4}{3}\pi r^3$$

$$\nabla\varphi_r = \frac{4\pi G\rho}{3} r$$

$$g = -\nabla\varphi_r$$

$$\vec{g} = -\frac{4\pi G\rho}{3} r\,\hat{r}$$

$$= (-\hat{r})\,\frac{4\pi G}{3}\,\frac{M_e}{\frac{4\pi R^3}{3}} \cdot r$$

$$\vec{g} = -\hat{r}\,\frac{GM_e}{R^2}\,\frac{r}{R} = -\hat{r}\, g\left(\frac{r}{R}\right)$$

$$\varphi(r) = -\int_{R_e}^{r} \vec{g}\cdot dr = \left(\frac{g}{R}\right)\left(\frac{r^2 - R_e^2}{2}\right) = -\left(\frac{g}{R}\right)\frac{R_e^2 - r^2}{2}$$

$$E_i = 0 + m\varphi(R_e) = 0$$

$$E(y) = \frac{1}{2}mv^2 + m\varphi(r) = E_i = 0$$

$$v^2 = -2\varphi(r)$$

$$= +\left(\frac{g}{R}\right)\left(R_e^2 - r^2\right) = \frac{g}{R_e}\left(R_e^2 - b^2 - y^2\right)$$

$$= \frac{g}{R_e}\left(R_e^2 - R_e^2 + h^2 - y^2\right)$$

$$v = \sqrt{\frac{g}{R_e}\left(h^2 - y^2\right)}$$

$$\vec{v} = \cancel{x}^{0}\hat{x} + v_y\hat{y} \rightarrow t = -\sqrt{\frac{R_e}{Mg}}\int_{-h}^{h}\frac{dy}{\sqrt{h^2-y^2}} = -2\sqrt{\frac{R_e}{g}}\int_0^h \frac{dy}{\sqrt{h^2-y^2}}$$

$$= -2\sqrt{\frac{R_e}{g}}\int_0^h \frac{(dy/h)}{\sqrt{1-(y/h)^2}} = -2\sqrt{\frac{R_e}{g}}\int_0^{\tilde{h}}\frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}}$$

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$$u = 1 - (y/h)^2 = 1 - \tilde{y}^2 \quad \rightarrow \quad \tilde{y} = \sqrt{1-u}$$

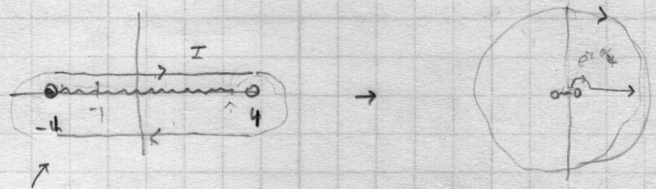
$$du = -2\tilde{y}\, d\tilde{y} = -2\sqrt{1-u}\, d\tilde{y}$$

$$t = 2\sqrt{\frac{Re}{g}} \frac{1}{+2} \cdot \int_0^1 \frac{du}{\sqrt{u(1-u)}}$$

$$= +\pi\sqrt{\frac{Re}{g}}$$

$h > 1$

$$\int_{-h}^{h} \frac{dy}{\sqrt{(h-y)(h+y)}}$$



$y/h = u$

$$= \int_{-h}^{h} \frac{(dy/h)}{\sqrt{1-(y/h)^2}}$$

$$2I = \text{residue} \frac{1}{\sqrt{1-y^2}} = \frac{1}{\sqrt{-1}} \frac{1}{\sqrt{y_1^2 - 1}}$$

which branch?

$$= \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \arcsin(1) - \arcsin(-1) = \frac{-1}{-i} \frac{1}{y} \left(\frac{1}{\sqrt{1-(1/y)^2}} \right)$$

$$= \pi$$

$$\approx -i \frac{1}{y} \left(1 - \frac{1}{2}\left(-\frac{1}{y^2}\right) + \cdots \right)$$

$$\text{residue} = -i$$

$$2I = (2\pi i)(i)(-1)$$

$$2I = +2\pi$$

$$I = +\pi$$

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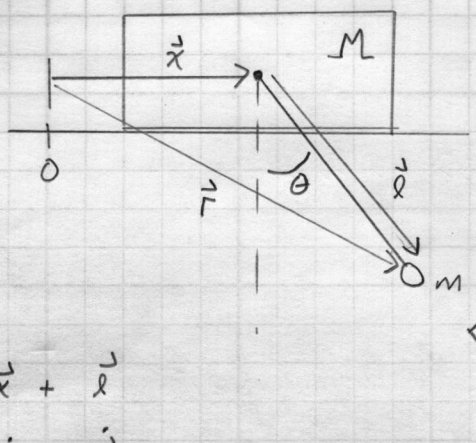
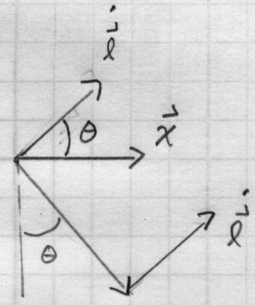
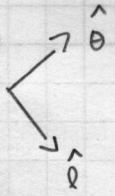
Simple Legrangian







$$\vec{r} = \vec{x} + \vec{l}$$

$$\dot{\vec{r}} = \dot{\vec{x}} + \dot{\vec{l}}$$

$$\dot{\vec{r}}^2 = \dot{\vec{x}}^2 + \dot{\vec{l}}^2 + 2\dot{\vec{x}}\cdot\dot{\vec{l}}$$

$$= \dot{x}^2 + \dot{l}^2 + 2\dot{l}\dot{x}\cos\theta$$

$$\vec{l} = l\hat{l}$$

$$\dot{\vec{l}} = \cancel{\dot{l}\hat{l}}^{\,0} + \vec{\omega}\times\vec{l}$$

$$= \dot{\theta} l \hat{\theta}$$

$$T = \frac{m}{2}|\dot{\vec{r}}|^2 + \frac{M}{2}\dot{x}^2 = \frac{1}{2}\left\{(M+m)\dot{x}^2 + m(l\dot{\theta})^2 + 2ml\dot{x}\dot{\theta}\cos\theta\right\}$$

$$U = -mgl(1-\cos\theta)$$

$$L = T - U$$

x] $\dfrac{\partial L}{\partial x} = 0 \Rightarrow \dfrac{\partial L}{\partial \dot{x}} = P_x = \text{const}$

$$= 2(M+m)\dot{x} + 2ml\dot{\theta}\cos\theta$$

$$\dot{x} = \frac{P_x}{2(M+m)} - \frac{ml\dot{\theta}\cos\theta}{M+m}$$

$$\dot{x}' = -\left(\frac{m}{M+m}\right) l\left\{\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right\}$$

θ] $\dfrac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\dot{x}\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + ml\dot{x}'\cos\theta - ml\dot{x}\dot{\theta}\sin\theta$$

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EOM => $m\ell^2\ddot{\theta} + m\ell\ddot{x}\cos\theta + mg\ell\sin\theta = 0$

↑ substitute for $\ddot{x}$

$$m\ell^2\ddot{\theta} - \left(\frac{m^2}{M+m}\right)\ell^2\cos\theta\left\{\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right\} + mg\ell\sin\theta = 0$$

$$\boxed{\left(m - \frac{m^2}{M+m}\cos^2\theta\right)\ell^2\ddot{\theta} + \left(\frac{m^2}{M+m}\right)\ell^2\dot{\theta}^2\sin\theta\cos\theta + mg\ell\sin\theta = 0}$$

small angle : $\theta \rightarrow$ small

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$\left(m - \frac{m^2}{M+m}\right)\ell^2\ddot{\theta} + mg\ell\theta = 0$$

$$\left(\frac{mM + m^2 - m^2}{M+m}\right)\ell^2\ddot{\theta} = -mg\ell\theta$$

$$\ddot{\theta} = -\underbrace{\frac{g}{\ell}\left(\frac{m+M}{M}\right)}_{\omega^2}\theta$$

$$\omega = \sqrt{\frac{g}{\ell}\left(\frac{m+M}{M}\right)}$$

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A note corrections A

Solution 2

#2 : UNDERGRADUATE MECHANICS

PROBLEM: A bead of mass m is constrained to move along a rigid, frictionless wire attached to a rotating disk at an angle $\theta \leq \frac{\pi}{2}$ to the plane of the disk, and at a radius R from the origin. The wire rotates with the disk with angular velocity ω. At the origin of the disk is a mass M which exerts a gravitational force on m. Ignore the gravitational force of the Earth.

(a) If x is the position of the bead along the wire, write down the Lagrangian function for the bead.

(b) Derive the equation of motion for the bead using Lagrange's equation.

#2 : UNDERGRADUATE MECHANICS

SOLUTION: a)

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m[\omega(R + x\cos\theta)]^2$$

$$U = -\frac{GMm}{[(R + x\cos\theta)^2 + (xsin\theta)^2]^{\frac{1}{2}}} = -\frac{GMm}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{1}{2}}}$$

Therefore

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m[\omega(R + x\cos\theta)]^2 + \frac{GMm}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{1}{2}}}.$$

b)

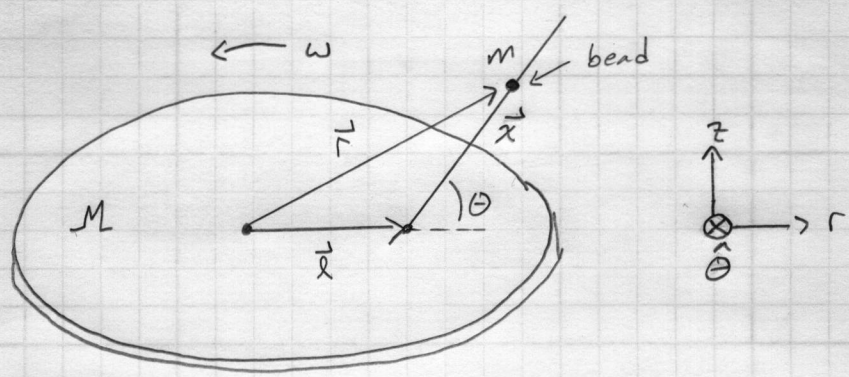
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}.$$

$$\frac{\partial L}{\partial x} = m\omega^2(R + x\cos\theta)\cos\theta - \frac{GM(R\cos\theta + x)m}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{3}{2}}}.$$

Therefore

$$m\ddot{x} = m\omega^2(R + x\cos\theta)\cos\theta - \frac{GM(R\cos\theta + x)m}{[R^2 + 2Rx\cos\theta + x^2]^{\frac{3}{2}}}.$$



let $\vec{r}$ be the vector from the center of mass of the disk. The disk is spining w/ const. angular speed, ω.

Note: R, θ, ω are const.

Write out vectors in terms of our one generalized coordinate: $|\vec{x}| = x$.

$$\vec{\omega} = \omega \hat{z}$$

$$\vec{R} = R \hat{r}$$

$$\vec{x} = x(\cos\theta \hat{r} + \sin\theta \hat{z})$$

$$\vec{r} = \vec{R} + \vec{x}$$

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{\vec{R}} + \dot{\vec{x}}$$

$$\Rightarrow \quad \dot{\vec{R}} = \cancel{(\dot{\vec{R}})_{rot}}^{\,0} + \vec{\omega} \times \vec{R} = \omega R (\hat{z} \times \hat{r}) = \omega R \hat{\theta}$$

$$\dot{\vec{x}} = (\dot{\vec{x}})_{rot} + \vec{\omega} \times \vec{x}$$

$$= \dot{x} \hat{x} + \omega x (\hat{z} \times \hat{x}) = \dot{x} \hat{x} + \omega x \cos\theta \hat{\theta}$$

$$\dot{\vec{R}} \cdot \dot{\vec{x}} = \omega R \hat{\theta} \cdot \left(\dot{x} \hat{x} + \omega x (\hat{z} \times \hat{x}) \right)$$

$$= \omega^2 R x \hat{\theta} \cdot \left[\hat{z} \times (\cos\theta \hat{r} + \sin\theta \hat{z}) \right]$$

$$= \omega^2 R x \cos\theta$$

$$\Rightarrow \quad |\dot{\vec{r}}|^2 = |\dot{\vec{R}}^2 + \dot{\vec{x}}|^2$$

$$= \omega^2 R^2 + \dot{x}^2 + \omega^2 x^2 \cos^2\theta \quad 2\omega^2 R x \cos\theta$$

$$= \dot{x}^2 + \omega^2 \left(R + x\cos\theta \right)^2$$

Kinetic energy :

$$T = \frac{1}{2} m |\dot{\vec{r}}|^2$$

$$= \frac{1}{2} m \left\{ \dot{x} + (R + x\cos\theta)^2 \right\}$$

Potential energy :

$$U = - \frac{GMm}{|\vec{r}|}$$

$$= - GMm \cdot \frac{1}{(R^2 + x^2 + 2\vec{R}\cdot\vec{x})^{1/2}}$$

$$= \frac{- GMm}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}}$$

a) $L = T - U$

$$= \frac{1}{2} m \left\{ \dot{x}^2 + \omega^2 (R + x\cos\theta)^2 \right\} + \frac{GMm}{\sqrt{R^2 + x^2 + 2Rx\cos\theta}}$$

b) $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$

$$\frac{\partial L}{\partial x} = m\omega^2 (R + x\cos\theta)\cos\theta - \frac{GMm\,(R\cos\theta + x)}{(R^2 + 2Rx\cos\theta + x^2)^{3/2}}$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

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Solution 12

#12 : GRADUATE MECHANICS

PROBLEM: A spherical marble of mass m and radius a rolls without slipping near the bottom of a perfectly rough spherical bowl of radius b, which is fixed in position. Calculate the angular frequency for small oscillations.

#12 : GRADUATE MECHANICS

SOLUTION: The no slip condition is $b\dot{\theta} = a\dot{\phi}$. The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}I\dot{\phi}^2 + mg(b-a)\cos\theta \\ &= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{5}mb^2\dot{\theta}^2 + mg(b-a)\cos\theta \end{aligned}$$

Expanding for small θ gives

$$L = \frac{1}{2}\left[m(b-a)^2 + \frac{2}{5}mb^2\right]\dot{\theta}^2 + mg(b-a) - \frac{1}{2}mg(b-a)\theta^2$$

so that

$$\omega^2 = \frac{mg(b-a)}{m(b-a)^2 + \frac{2}{5}mb^2} = \frac{5g(b-a)}{5(b-a)^2 + 2b^2}$$

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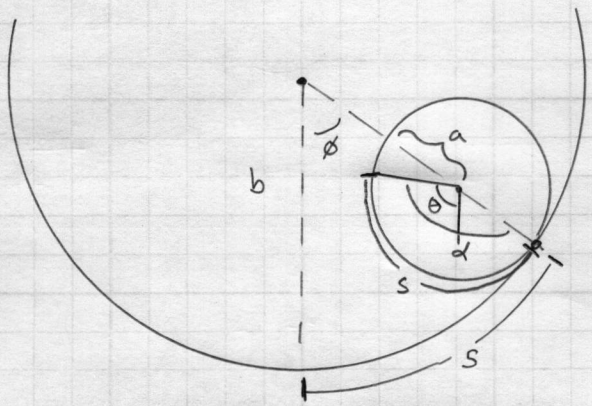

rolling condition: the amount of arc traversed for point of constant must match up!

$$b\phi = a\alpha$$
$$= a(\theta + \phi)$$
$$\Rightarrow \boxed{(b-a)\phi = a\theta}$$

$$T = \frac{m}{2} v_{cm}^2 + \frac{I}{2}\dot{\theta}^2$$
$$= \frac{m}{2}(b-a)^2\dot{\phi}^2 + \frac{1}{2}\left(\frac{2}{5}ma^2\right)\left(\frac{(b-a)\cdot\dot{\phi}}{a}\right)^2$$
$$= \frac{m}{10}\dot{\phi}^2\left\{5(b-a)^2 + 2(b-a)^2\frac{a^2}{a^2}\right\}$$
$$= \frac{7}{10}m(b-a)^2\dot{\phi}^2$$

$$U = (b-a)mg(1-\cos\phi) = -(b-a)mg\cos\phi + \text{const}$$

const ← drop

$$L = \frac{7}{10} m (b-a)^2 \dot{\phi}^2 + (b-a) mg \cos\phi$$

$$\frac{\partial L}{\partial \phi} = -(b-a) mg \sin\phi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{7}{5} m (b-a)^2 \ddot{\phi}$$

$$\ddot{\phi} = -\frac{10}{7} \frac{g}{(b-a)} \sin\phi$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{10}{7} \frac{g}{(b-a)}}}$$

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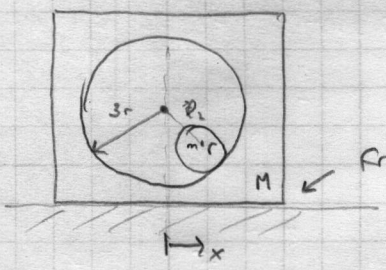
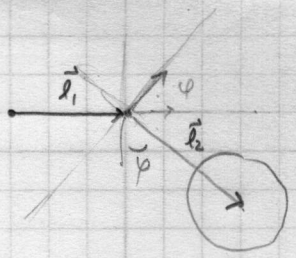


cylinder mass m
block mass M

frictionless



$$\vec{v}_{cm} = \dot{\vec{l}}_1 + \dot{\vec{l}}_2$$

$$v_{cm}^2 = \dot{l}_1^2 + \dot{l}_2^2 + 2\,\dot{\vec{l}}_1 \cdot \dot{\vec{l}}_2$$

$$= \dot{x}^2 + (2r)^2\dot{\varphi}^2 + 2\dot{x}(2r\dot{\varphi})\cos\varphi$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + 2mr^2\dot{\varphi}^2 + 2mr\dot{x}\dot{\varphi}\cos\varphi + \frac{1}{2}(I)\dot{\theta}^2$$

$$v_c = r\dot{\theta} = 2r\dot{\varphi} \rightarrow \dot{\theta} = 2\dot{\varphi}$$

$$I = \frac{M}{2}r^2$$

$$V = -2rmg(1-\cos\varphi)$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + 2mr^2\dot{\varphi}^2 + 2mr\dot{x}\dot{\varphi}\cos\varphi + \frac{mr^2}{4}\dot{\varphi}^2 + 2rmg(1-\cos\varphi)$$

$$= \frac{(M+m)\dot{x}^2}{2} + 3mr^2\dot{\varphi}^2 + 2mr\dot{x}\dot{\varphi}\cos\varphi + 2rmg(1-\cos\varphi)$$

x: $\quad \frac{\partial L}{\partial x} = 0 \rightarrow \frac{\partial L}{\partial \dot{x}} = \text{const} = (M+m)\dot{x} + 2mr\dot{\varphi}\cos\varphi$

φ: $\quad \frac{\partial L}{\partial \varphi} = -2mr\dot{x}\dot{\varphi}\sin\varphi + 2rmg\sin\varphi$

$$\frac{\partial L}{\partial \dot{\varphi}} = 2mr\dot{x}\cos\varphi + 6mr^2\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 2mr\ddot{x}\cos\varphi - 2mr\dot{x}\dot{\varphi}\sin\varphi + 6mr^2\ddot{\varphi}$$

$$(M+m)\ddot{x} + 2mr\ddot{\varphi}\cos\varphi - 2mr\dot{\varphi}^2\sin\varphi = 0$$

$$\approx (M+m)\ddot{x} + 2mr\ddot{\varphi} = 0$$

$$2mr\ddot{x}\cos\varphi - 2mr\dot{x}\dot{\varphi}\sin\varphi + 6mr^2\ddot{\varphi} + 2mr\dot{x}\dot{\varphi}\sin\varphi - 2rmg\sin\varphi = 0$$

$$\approx 2mr\ddot{x} + 6mr^2\ddot{\varphi} - 2rmg\varphi = 0$$

$$\begin{pmatrix} (M+m) & 2mr \\ 2mr & 6mr^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -2rmg \end{pmatrix}\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = -\omega^2 x \qquad \ddot{\varphi} = -\omega^2\varphi$$

$$\begin{pmatrix} (M+m)\omega^2 & 2mr\omega^2 \\ 2mr\ \omega^2 & 6mr^2\omega^2 - 2rmg \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$rm\,\omega^2\left((M+m)\left(6r\omega^2 - 2g\right) - 4mr\omega^2\right) = 0$$

$$\omega^2 = 0$$

$$(M+m)\left(6r\omega^2 - 2g\right) - 4mr\omega^2 = 0$$

$$\omega^2\left(6Mr + 2mr\right) - 2(M+m)g = 0$$

$$\omega = \sqrt{\frac{2(M+m)g}{2r(3M+m)}} = \sqrt{\frac{g}{r}\cdot\frac{M+m}{3M+m}}$$

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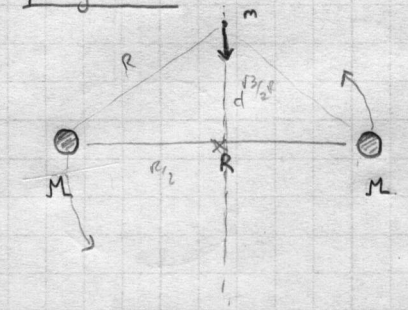


$$F_m = -GMm\left\{\frac{d\hat{y} + R/2\,\hat{x}}{\left(\sqrt{d^2+(R/2)^2}\right)^3} + \frac{d\hat{y} - R/2\,\hat{x}}{\left(\sqrt{d^2+(R/2)^2}\right)^3}\right\}$$

$$= -GMm\,\frac{2d\hat{y}}{\left(\sqrt{d^2+(R/2)^2}\right)^3} = -m\omega^2 d\,\hat{y}$$

ω? $\quad -M\left(\frac{R}{2}\right)\omega^2 = \frac{-GM^{\not 2}}{R^3}$

$$\omega^2 = \frac{2GM}{R^3}$$

$$\frac{2GM}{\left(d^2+(R/2)^2\right)^{3/2}} = \omega^2$$

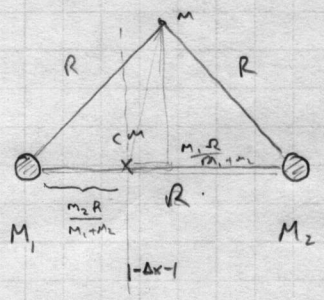
$$d^2 + (R/2)^2 = R^2$$

$$d^2 = \frac{3R^2}{4}$$

$$d = \frac{\sqrt{3}}{2}R$$

equilateral triangle

b)



cm as origin:

$$X_{cm} = \frac{M_2 R}{M_1+M_2}$$

$$R - X_{cm} = \frac{(M_1+M_2)R - M_2R}{M_1+M_2} = \frac{M_1 R}{M_1+M_2}$$

$$\frac{R}{2} - \frac{M_2 R}{M_1+M_2} = \frac{(M_1+M_2-2M_2)R}{2(M_1+M_2)}$$

$$\Delta x = \frac{(M_1-M_2)}{2(M_1+M_2)}$$

$$F_m = -GM_1m\,\frac{\left(\frac{\sqrt{3}}{2}R\,\hat{y} + \frac{MR}{M_1+M_2}\,\hat{x}\right)}{R^3} + GM_2m\,\frac{\left(\frac{\sqrt{3}}{2}R\,\hat{y} - R/2\,\hat{x}\right)}{R^3}$$

$$= -\frac{GM}{R^3}\left(\frac{\sqrt{3}}{2}(M_1+M_2)\hat{y} + \frac{1}{2}(M_1-M_2)\hat{x}\right)$$

$$= -\frac{GM}{R^3}(M_1+M_2)\left(\frac{\sqrt{3}}{2}\hat{y} + \Delta x\,\hat{x}\right)$$

$$= -\frac{Gm(M_1+M_2)}{R^3}R_m\,\hat{m} = -m\omega^2 R_m\,\hat{m} \Rightarrow \omega^2 = \frac{G(M_1+M_2)}{R^3}$$

$$M_1 \left(\frac{M_2 R}{M_1 + M_2} \right) \omega^2 = \frac{G M_1 M_2}{R^2}$$

$$\omega^2 = \frac{G (M_1 + M_2)}{R^3}$$

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