INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks,) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 : UNDERGRADUATE MECHANICS

`PROBLEM:` A 1 lb (~ 0.45 kg) fish is swimming at 1 mph ($\sim 0.45 m/s$) North and experiences a fluid drag force proportional to its velocity. If its drag coefficient is 0.01 $N/m/s$ and the fish suddenly stops swimming, how far North does it coast before coming to a stop? Assume the water is deep enough so the fish doesn't sink to the bottom.

`SOLUTION:`

Newton's 2nd law: $F = m\dot{v} = -\gamma v$, so $\dot{x} = v = v_0 \exp(-\gamma t/m)$ and thus $x = \frac{mv_0}{\gamma}(1 - \exp(-\gamma t/m))$. Taking $t \to \infty$, $x \to mv_0/\gamma \approx 20$ meters.

#2 : UNDERGRADUATE MECHANICS

`PROBLEM:` A pencil floats before you in a weightless environment. You flick the very tip of the pencil, imparting an impulse force perpendicular to the pencil axis. How much of the energy transferred to the pencil is in rotational motion vs. translational motion? Also, would the path of the tip be described as forming closed loops, executing a perfect no-slip cycloid, or as a wavy pattern? Approximate the pencil as a thin, uniform rod.

`SOLUTION:`

Answers: The energy is 75% rotational, 25% translational. The path of the tip forms closed loops.

Imagine that a very small particle impacts the end of the pencil at its tip, a distance $L/2$ from the pencil's center of mass. Independent of the degree of elasticity in the collision, some momentum, p, is imparted to the pencil, with angular momentum $\ell = pL/2$. [If the particle arrives with v_i and leaves with v_f it will deposit momentum $p = m(v_i - v_f)$ and the corresponding ℓ.]

The pencil, of mass M and length L, will therefore acquire a speed of $V = p/M$ and an angular velocity, ω, according to $\ell = I\omega = pL/2$, or $\omega = MVL/2I$. The moment of inertia, I, is:

$$I = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} r^2 \, dr = \frac{1}{12} ML^2$$

We can therefore find that $\omega = 6V/L$. At the tip, the velocity is $v_{\text{tip}} = \omega L/2 = 3V$. Thus the tip speed is three times that of the speed of the pencil center of mass.

The translational kinetic energy of the pencil is $T_{\rm trans} = \frac{1}{2}MV^2$, and the rotational energy is $T_{\rm rot} = \frac{1}{2}I\omega^2 = \frac{3}{2}MV^2$. Therefore, of the total energy, $T_{\rm tot} = 2MV^2$, three quarters is rotational, and one quarter is translational.

Because the tip speed exceeds the center-of-mass speed, the path of the tip will describe a series of closed loops in space.

#3 : UNDERGRADUATE E&M

`PROBLEM:` What is the critical angle for total external reflection for photons of wavelength λ and frequency $\omega = 2\pi c/\lambda$ in the vacuum, falling on a metal plate with electron density n_e? Assume that electrons in a metal are essentially free.

`SOLUTION:`

The critical angle is determined from Shell's law $n_1 \cos\theta_1 = n_2 \cos\theta_2$, where angles are measured with respect to the surface. $\theta_2 = 0$ for the critical angle. Therefore,

$$\cos\theta_c = n. \tag{1}$$

Now we calculate the index of refraction $n(\omega)$ in a metal.

Equation of motion for a free electron $m\frac{d^2x}{dt^2} = -eE$. For AC electric field $E = E_0 e^{-i\omega t}$ and $x = x_0 e^{-i\omega t}$, we obtain $x = \frac{eE}{m\omega^2}$.

The polarization of the metal as the dipole moment per unit volume is $P = -exn = -\frac{ne^2 E}{m\omega^2}$, so that the polarizability $\alpha = P/E = -\frac{n_e e^2}{m\omega^2}$. For the dielectric function $\epsilon = 1 + 4\pi\alpha$ we obtain $\epsilon = 1 - 4\pi\frac{n_e e^2}{m\omega^2}$, therefore

$$n^2 = \epsilon = 1 - 4\pi\frac{n_e e^2}{m\omega^2}. \tag{2}$$

Combining (1) and (2) we obtain

$$\cos^2\theta_c = 1 - \frac{4\pi n_e e^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \tag{3}$$

or

$$\sin\theta_c = \frac{\omega_p}{\omega}, \tag{4}$$

where $\omega_p = \left(\frac{4\pi n_e e^2}{m}\right)^{1/2}$ is the plasma frequency. For $\omega < \omega_p$ one obtains total reflection at all angles.

#4 : UNDERGRADUATE E&M

`PROBLEM:` Consider two electric charges in the $x - y$ plane: a charge $+q$ at $y = d/2$ and a charge $-q$ located at $-d/2$; a dipole separated by a distance d.

(i) Write down the electric field at a point located at a general point r, θ in the $x - y$ plane. The angle θ is measured clockwise from the y-axis and $r = 0$ corresponds to $x = y = 0$. The solution can be left as a function of the cartesian coordinates or in terms of r, θ.

(ii) Simplify this expression for the electric field when $r \gg d$; that is, at large distances from $x = y = 0$.

(iii) If the dipole is then embedded in a constant electric field, E, with field lines parallel to the $x = y$ axis (i.e., at $\theta = 45°$), what is the torque on the dipole?

(iv) How much energy must be *supplied* to rotate the dipole orthogonal to the electric field, E?

`SOLUTION:`

(i) The potential, V, can be used to get the electric field. The potential $V = kq(1/r_+ - 1/r_-)$, where $r_{+,-}$ is the distance from the positive or negative charge. The solution can be left as a function of the cartesian coordinates or r, θ.

(ii) When $r \gg d$ we get $V = kqd\cos\theta/r^2$ and $E = -\nabla V = ...$

(iii) The torque is $\vec{\tau} = \vec{r} \times \vec{F} = \vec{d} \times \vec{E}$, so $\tau = qEd\sin\theta = \frac{\sqrt{2}}{2}qEd$.

(iv) How much energy must be *supplied* to rotate the dipole orthogonal to the electric field, E? This is the work, which equals the torque multiplied by the angle through which the rotation takes place: $W = qEd\cos\theta = \frac{\sqrt{2}}{2}qEd$.

#5 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Neutrally buoyant plastic spheres having 1-micron radii are immersed in still water (viscosity= 0.001 N-s/m^2) at 30 degrees Celsius. Under these conditions, a sphere diffuses a distance of about 30 microns, on average, from its starting position in 10 minutes due to Brownian motion

(i) What distance, on average would a sphere diffuse if the radius were doubled to 2 microns? (Assuming all other parameters are the same)?

(ii) What distance, on average, would the sphere (radius = 1 micron) diffuse if the temperature was increased to 60 degrees Celsius?

(iii) What distance, on average, would the sphere (radius = 1 micron) diffuse if the time were doubled to 20 minutes? (With temperature = 30 degrees Celsius.)

(iv) What distance, on average, would the sphere (radius = 1 micron) diffuse in 10 minutes if the viscosity was doubled to $0.002N - s/m^2$? (With temperature = 30 degrees Celsius.)

`SOLUTION:` According to Einstein's theory of Brownian motion the mean square displacement after time t is $\langle(\Delta r)^2\rangle \propto Dt$, where D is the diffusion coefficient and t is the elapsed time. The diffusion coefficient is $D = kT/\gamma$, γ is the drag coefficient, k is Boltzmann's constant and $T$is the absolute temperature (303K). According to Stokes' law, $\gamma \propto \eta r$ for a small sphere moving slowly through a fluid, where η=viscosity and r=radius of the sphere.

Thus average distance $= \sqrt{\langle \Delta r^2 \rangle} \propto \sqrt{Dt} \propto \sqrt{\frac{kT}{\gamma} t} \propto \sqrt{\frac{kT}{\eta r} t}$.

(i) Doubling r, distance is multiplied by $1/\sqrt{2} = 0.707$.

(ii) Raising temperature from 30 to 60C, absolute temperature is raised from 30+273=303 to 333K, so distance is multiplied by $\sqrt{333/303} =$ 1.05.

(iii) Doubling t, distance is multiplied by $\sqrt{2} = 1.41$.

(iv) Doubling η, distance is multiplied by $1/\sqrt{2} = 0.707$.

#6 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The Rayleigh-Jeans law expresses the low-frequency behavior of the Planck Distribution. This problem involves the derivation of the Rayleigh-Jeans law. First, consider a cubical metallic cavity with length L. The wave equation leads to electric field solutions of the form:

$$E = E_o \sin(\tfrac{n_x \pi x}{L}) \sin(\tfrac{n_y \pi y}{L}) \sin(\tfrac{n_z \pi z}{L}) \sin(\tfrac{2\pi c t}{\lambda}) \tag{5}$$

Where $n_{x,y,z}$ are the mode numbers and λ is the electromagnetic wavelength.

(i) Substitute this equation into the wave equation to obtain a relationship between $n_x^2 + n_y^2 + n_z^2$, L, and λ.

(ii) Including both polarization modes, the total number of modes, N, in the cavity as a function of wavelength, is found to be $N = \frac{8\pi L^3}{3\lambda^3}$. Compute the *density* of modes as a function of wavelength. Remember that N is a function of L. Express your answer as the density of modes divided by the cavity volume.

(iii) If the cavity is at temperature T, the energy density per unit wavelength in the cavity ($du/d\lambda$) is proportional to the density of modes (as found in the previous part) times the energy per mode. Using this, together with the classical (equipartition) result for the energy per mode, compute $\frac{du}{d\lambda}$. Also, find the the energy density as a function of frequency, ν : $\frac{du}{d\nu}$.

(iv) Recall that expression for $du/d\nu$ obtained in the previous part is approximately valid for large kT/ν, but obviously breaks down for large frequencies. Planck fixed this (by replacing equipartition with something else). Using these facts, write down the Planck distribution for $\frac{du}{d\nu}$, with the correct numerical factors and low frequency behavior.

`SOLUTION:`

(i) The wave equation is: $\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$. Substituting the ansatz given in the problem, we arrive at $n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{\lambda^2}$.

(ii) We need to simply calculate $\frac{dN}{d\lambda} = -8\pi L^3/\lambda^4$. The number of modes decreases with increasing wavelength so there is a negative sign. The number of modes per unit volume per unit wavelength is obtained by dividing by the volume of the cubical cavity: L^3.

(iii) Classically, the energy per mode is k_bT, where k_b is Boltzmann's constant. This leads to $\frac{du}{d\lambda} = 8\pi k_b T/\lambda^4$. To get this in terms of frequency, $\frac{du}{d\nu}$ we need to convert using the chain rule of differentiation: $\frac{du}{d\nu} = \frac{du}{d\lambda}\frac{d\lambda}{d\nu} = \frac{du}{d\lambda}\frac{d(c/\nu)}{d\nu} = -\frac{du}{d\lambda}\frac{c}{\nu^2}$. Using the original equation for $du/d\lambda$, and converting λ to c/ν we get: $\frac{du}{d\nu} = k_bT\frac{8\pi\nu^2}{c^3}$.

(iv) The Planck distribution for the energy density of radiation is $du/d\nu = \frac{8\pi\hbar\nu^3}{c^3}[e^{\hbar\nu/k_bT} - 1]^{-1}$. Check the normalization: allowing ν to go to zero, and expanding the exponential, we get $\frac{du}{d\nu} = k_bT\frac{8\pi\nu^2}{c^3}$.

#7 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A particle of mass m moves in a potential $V(r) = -V_0$ for $r < a$ and $V(r) = 0$ for $r > a$. Find the smallest V_0 for which there is a bound state of zero angular momentum in the potential. Recall $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi}) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}$.

`SOLUTION:`

For zero angular momentum the radial Schrödinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\psi = E\psi \qquad (r > a), \tag{6}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\psi = (V_0 + E)\psi \qquad (0 \le r \le a). \tag{7}$$

Introducing $U = r\psi$, we obtain

$$\frac{\partial^2 U}{\partial r^2} - \alpha^2 U = 0 \qquad (r > a), \tag{8}$$

$$\frac{\partial^2 U}{\partial r^2} + \beta^2 U = 0 \qquad (0 \le r \le a), \tag{9}$$

where $\alpha = \left(\frac{-2mE}{\hbar^2}\right)^{1/2}$ and $\beta = \left(\frac{2m(V_0+E)}{\hbar^2}\right)^{1/2}$.

We look for solutions in the limit $E \to 0^-$. We obtain

$$U = Ae^{-\alpha r} \qquad (r > a), \tag{10}$$

$$U = B\sin\beta r \qquad (0 \le r \le a), \tag{11}$$

where we have eliminated the singular solutions for ψ. Continuity of U and its derivative at $r = a$ requires that $\beta \cot(\beta a) = -\alpha$. For $E \to 0$, $\alpha \to 0$ and $\cot(\beta a) \to 0$. This happens when $\beta a = \pi/2$, or $V_0 = \pi^2\hbar^2/(8ma^2)$.

#8 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:`

A 3-dimensional isotropic harmonic oscillator has energy eigenvalues $E_n = \hbar\omega(n + \frac{3}{2})$, where $n = 0, 1, 2, \cdots$.

(a) Find the degree of degeneracy of quantum state $n = 2$.

(b) Find the degree of degeneracy of the quantum state n for general n.

`SOLUTION:`

The 3-D quantum oscillator state is the sum of the 3 1-D quantum oscillators with eigenvalues $\hbar\omega(n_x + \frac{1}{2})$, etc, each with quantum number: n_x, n_y, and n_z, where each quantum number is an integer $0, 1, 2, \cdots$, and so $n = n_x + n_y + n_z$. We want the number of combinations (n_x, n_y, n_z) of these three numbers that add up to n.

(a) Consider first $n = 2$. We can write $2 = 1 + 1 + 0$ or $2 = 0 + 0 + 2$, and each has 3 permutations, so $D_2 = 6$.

(b) Now consider general n. Start by fixing n and n_z. The number of of (n_x, n_y) pairs is given by those combos that have $n_x + n_y = n - n_z$. These are $(n_x, n_y) = (0, n - n_z), (1, n - n_z - 1), (2, n - n_z - 2)$, up to $(n - n_z, 0)$. There are $n - n_z + 1$ of these combos. This is for fixed n_z, so we sum over n_z. Total degeneracy is $D_n = \sum_{n_z=0}^{n}(n - n_z + 1) = (n + 1)n - n(n + 1)/2 + 1(n + 1)$. This can be simplified to $D_n = \frac{(n+2)(n+1)}{2}$.

#9 : UNDERGRADUATE MATH

`PROBLEM:`

Consider the 5x5 matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Find its eigenvalues.

(b) Find the eigenvector corresponding to eigenvalue 0.

`SOLUTION:`

The matrix is of the general form

$$A_{ij} = \delta_{i,j\pm 1} \tag{12}$$

for $i = 2, 3, ..., N-1$, and $A_{1,2} = A_{N,N-1} = 1$, $A_{11} = A_{NN} = 0$, with $N = 5$. The eigenvectors c_j and eigenvalues λ satisfy

$$c_{j-1} + c_{j+1} = \lambda c_j \qquad j = 2, 3, ..., N-1 \tag{13a}$$

$$c_2 = \lambda c_1 \tag{13b}$$

$$c_{N-1} = \lambda c_N \tag{13c}$$

so defining $c_0 = c_{N+1} = 0$ we can use Eq. (13a) with $j = 1, 2, ...N$. The solution is $c_j = e^{ikj}$ or $c_j = e^{-ikj}$, and eigenvalues

$$\lambda = e^{ik} + e^{-ik} = 2cos(k) \quad . \tag{14}$$

The general solution is

$$c_j = ae^{ikj} + be^{-ikj} \tag{15}$$

and the coefficients a, b as well as the parameter k are determined from the boundary conditions $c_0 = c_{N+1} = 0$ (and normalization). From $c_0 = 0$ we find $b = -a$, hence

$$c_j \propto sin(kj) \tag{16}$$

and from $c_{N+1} = 0$ we find

$$k = \frac{\pi}{N+1}\nu \tag{17}$$

with $\nu = 1, 2, ..., N$.

(a) For our case, $N = 5$ and Eq. (14) yields

$$\lambda_\nu = 2cos(\tfrac{\pi}{6}\nu) \qquad \nu = 1, 2, 3, 4, 5 \tag{18}$$

so the eigenvalues are 0, ± 1, $\pm\sqrt{3}$.

(b) The eigenvalue $\lambda = 0$ corresponds to $\nu = 3$. From Eqs. (16) and (17), the (normalized) eigenvector is

$$c_j = \tfrac{1}{\sqrt{3}}(1, 0, -1, 0, 1) \tag{19}$$

#10 : UNDERGRADUATE GENERAL PHYSICS

`PROBLEM:`

An accidental bullet rips through the side of a 737 passenger airplane in flight at high-altitude, perhaps carrying passengers home from a gun show. Assuming that the structural damage is limited to the bullet hole itself (no big rips result), and that the external environment is practically a vacuum, use kinetic theory to estimate the timescale over which the air would be evacuated from the airplane? For reference, one atomic mass unit is about 930 MeV, and air is mostly comprised of N_2, with O_2 making up almost all the rest.

`SOLUTION:`

We first estimate some relevant dimensions. The airplane fuselage has a diameter of at least 3 m and a length of at least 20 m, making a volume of approximately $V \sim 150$ m^3. We can estimate the area of the bullet hole to be approximately 1 cm^2, or 0.0001 m^2.

How fast will air particles escape through the hole? Each air particle has approximately kT of energy, so that the 3-D particle velocity is $v \approx \sqrt{2kT/m}$. Here we have several options for evaluation:

- Use the fact that $kT \approx \frac{1}{40}$ eV at room temperature, and mc^2 for an atomic mass unit is 930 MeV, and air—composed of about three-quarters N_2 at 28 mass units per molecule, and one-quarter O_2 at 32 mass units per molecule—is about 29 proton masses per molecule. Also armed with $c \approx 3 \times 10^8$ m/s, we find that $v \sim 410$ m/s.

- Remember the physical constants in some system of units and directly evaluate. For example, $k = 1.38 \times 10^{-23}$ J/K, $m = 29 \times 1.66 \times 10^{-27}$ kg, $T \approx 293$ K, to get $v \sim 410$ m/s.
- Know that $kT \approx \frac{1}{40}$ eV at room temperature, and that 1 eV is about 1.6×10^{-19} J. Now use whatever knowledge of mass you want to turn this into a speed.
- Remember that for an ideal gas $c_{\rm s} = \sqrt{\gamma kT/m}$, where $\gamma = 7/5$ for diatomic air (or 5/3 for monatomic species), and evaluate as above. This evaluates to about 350 m/s.

The first three methods give the magnitude of air velocity, but only one component will be directed through the bullet hole, so the somewhat reduced sound speed is the most appropriate choice—though the numerical result will be virtually identical in either case compared to the crudeness of our other estimates.

If air escapes at 350 m/s through a 0.0001 m^2 hole, the volumetric loss rate is $\dot{V} = 0.035$ m^3/s. It would take a time $\tau = V/\dot{V} \sim 4500$ s, or over an hour, to evacuate the air. The actual pressure profile will be exponential in nature, with a time constant of approximately τ. Regardless, there is plenty of time to calmly stick a finger in the hole, and even get all the way back to Texas.

#11 :GRADUATE MECHANICS

PROBLEM: A quasi-particle has the following Hamiltonian for motion in two dimensions (x, y):

$$H(x, y, p_x, p_y) = p_x^2 + x^2 p_y^2 + x^2 y^2.$$

Find $x(t)$, $y(t)$ for initial conditions $x(0) = 0$, $p_x(0) = 1$, $y(0) = 0$, $p_y(0) = 1$. Hint:

$$\int \frac{dy}{\sqrt{1-y^2}} = \sin^{-1} y.$$

SOLUTION:

The Hamiltonian is *separable.* The Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} = \left(\frac{\partial S}{\partial x}\right)^2 + x^2 \left[\left(\frac{\partial S}{\partial y}\right)^2 + y^2\right],$$

with solution $S = -Et + W_x(x) + W_y(y)$

where $W_x = \int p_x dx$, $W_y = \int p_y \, dy$.

$$\Rightarrow \quad \left(\frac{\partial W_y}{\partial y}\right)^2 + y^2 = \text{const} = p_y(0)^2 + y(0)^2$$
$$\Rightarrow \quad p_y^2 + y^2 = 1.$$

Similarly $p_x^2 + x^2 p_{y_0}^2 = H \;\Rightarrow\; p_x^2 + x^2 = 1.$

Also,

$$\begin{aligned}
\dot{x} &= \tfrac{\partial H}{\partial p_x} = 2p_x\ ,\ \dot{p}_x = -\tfrac{\partial H}{\partial x} = -2x\ p_y^2(0) = -2x \\
\Rightarrow \ddot{x} &= -4x\ \Rightarrow\ x = A\sin\,2t \\
\dot{x}(0) &= 2A = 2p_x(0) = 2\ \Rightarrow\ A = 1 \\
\dot{y} &= \tfrac{\partial H}{\partial p_y} = 2x^2 p_y\,; \\
\Rightarrow \dot{y} &= 2\sin^2 2t\sqrt{1-y^2} \\
\tfrac{dy}{\sqrt{1-y^2}} &= 2dt\ \sin^2 2t = dt(1-\cos\,4t) \\
&\downarrow \\
\sin^{-1} y &= t - \tfrac{1}{4}\sin 4t \\
\Rightarrow y &= \sin\left(t - \tfrac{1}{4}\sin 4t\right) \\
x &= \sin 2t.
\end{aligned}$$

#12 :GRADUATE MECHANICS

`PROBLEM:` A finite one-dimensional system of balls has Lagrangian

$$L = \sum_{j=1}^{N} \frac{1}{2} m\ \dot{\phi}_j^2 - \sum_{j=1}^{N-1} \frac{1}{2} k \left(\phi_{j+1} - \phi_j\right)^2,$$

where ϕ_j is the displacement of the j^{th} ball.

(a) Derive the equations of motion, and show that they have solutions of the form

$$\phi_j = A\ \exp\left[i(\beta j - \omega t)\right].$$

Find the relation between β and ω.

(b) Impose periodic boundary conditions on this one-dimensional "crystal". That is, imagine that the crystal is actually infinite, but that the displacement of the $(N+j)^{\text{th}}$ ball is identical to the displacement of the j^{th} ball. Find the values of β which are consistent with this condition.

(c) In parts (a) and (b), you have found a complete set of N normal modes for the crystal. Define normal coordinates Q_n $(n = 1, \cdots, N)$ and write the Lagrangian L in terms of the Q_n. (Note: first add a term $\frac{1}{2}k\left(\phi_N - \phi_1\right)^2$ to the potential energy, coupling the first and last masses, so the one-dimensional system is periodic.)

(d) Take the continuum limit, allowing the spacing a between the masses to approach zero, the number of balls N to infinity, while $\lim_{a\to 0} m/a = \rho$, $\lim_{a\to 0} ka = \kappa$, and $\lim_{a\to 0, N\to\infty} Na = D$ remain finite, and express the Lagrangian density $\mathcal{L} \equiv \lim_{a\to 0} L/a$ in the continuum limit in terms of a field $\phi(x,t)$. Show that the equation of motion for this Lagrange density is a wave equation. What is the velocity v of the wave equation?

`SOLUTION:` (a) The Euler-Lagrange equations for an infinite one-dimensional system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_j} = \frac{\partial L}{\partial \phi_j} \quad \Rightarrow$$

$$m\ddot{\phi}_j = k\left(\phi_{j+1} - \phi_j\right) - k\left(\phi_j - \phi_{j-1}\right) = k\left(\phi_{j+1} - 2\phi_j + \phi_{j-1}\right)$$

Solutions of the form:

$$\phi_j = A\ \exp\left[i(\beta j - \omega t)\right] \quad \Rightarrow$$

$$-m\omega^2 = k\left(e^{i\beta} - 2 + e^{-i\beta}\right) = 2k\left(\cos\beta - 1\right)$$

$$\omega^2 = \frac{2k}{m}\left(1 - \cos\beta\right)$$

(b) Periodic boundary conditions : $\phi_{N+j} = \phi_j$.

$$e^{i\beta N} = 1 \quad \Rightarrow \quad \beta_n = \frac{2\pi n}{N},\ n = 0, 1, 2, 3, \cdots, N-1.$$

(c) Normal coordinates defined by discrete Fourier series

$$Q_n \equiv \sum_{j=1}^{N} \phi_j\ e^{-i\frac{2\pi}{N}nj}$$

$$\phi_j \equiv \sum_{n=1}^{N} Q_n\ e^{i\frac{2\pi}{N}nj}$$

$$L = \sum_{j=1}^{N} \frac{1}{2} m\ \dot{\phi}_j^2 - \sum_{j=1}^{N} \frac{1}{2} k\left(\phi_{j+1} - \phi_j\right)^2$$

$$L = \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{j=1}^{N} \left\{ \frac{1}{2} m \dot{Q}_n \dot{Q}_m \; e^{i\frac{2\pi}{N}(n+m)} - \frac{1}{2} k Q_n Q_m \left(e^{i\frac{2\pi}{N}n} - 1 \right) \left(e^{i\frac{2\pi}{N}m} - 1 \right) \; e^{i\frac{2\pi}{N}(n+m)} \right\}$$

$$L = \sum_{n=1}^{N} \sum_{m=1}^{N} \delta_{n,-m} \left\{ \frac{1}{2} m \dot{Q}_n \dot{Q}_m \; - \frac{1}{2} k Q_n Q_m \left(e^{i\frac{2\pi}{N}n} - 1 \right) \left(e^{i\frac{2\pi}{N}m} - 1 \right) \right\}$$

$$L = \sum_{n=1}^{N} \left\{ \frac{1}{2} m \dot{Q}_n \dot{Q}_{-n} \; - \frac{1}{2} k Q_n Q_{-n} \left(e^{i\frac{2\pi}{N}n} - 1 \right) \left(e^{-i\frac{2\pi}{N}n} - 1 \right) \right\}$$

$$L = \sum_{n=1}^{N} \left\{ \frac{1}{2} m \dot{Q}_n \dot{Q}_{-n} \; - \frac{1}{2} k Q_n Q_{-n} \left(2 - 2\cos\frac{2\pi}{N} n \right) \right\}$$

$$L = \frac{1}{2} m \sum_{n=1}^{N} \left\{ \dot{Q}_n \dot{Q}_n^* - \omega_n^2 Q_n Q_n^* \right\}$$

$$\omega_n^2 = \frac{2k}{m} \left(1 - \cos\beta_n\right) = \frac{2k}{m} \left(1 - \cos\frac{2\pi}{N} n \right)$$

where $Q_n^* = Q_{-n}$.

(d) Take the limits $\lim a \to 0$, $\lim N \to \infty$ with $\lim \frac{m}{a} \to \rho$, $\lim ka \to \kappa$, $\lim Na \to D$.

Discrete Lagrangian

$$L = \sum_{j=1}^{N} \frac{1}{2} m \; \dot{\phi}_j^2 - \sum_{j=1}^{N} \frac{1}{2} k \left(\phi_{j+1} - \phi_j\right)^2$$

with spacing $\Delta x = a$. Replace $\phi_j = A \exp\left[i(\beta j - \omega t)\right]$ by

$$\phi(x) = A \exp\left[i \left(\tilde{\beta} x - \omega t \right) \right], \qquad \tilde{\beta} \equiv \frac{\beta}{a}, \quad x \equiv ja$$

$$\mathcal{L} \equiv \lim_{a\to 0} \frac{L}{a} = \lim_{a\to 0} \left[\frac{1}{2} \frac{m}{a} \dot{\phi}^2 - \frac{1}{2} \frac{k}{a} a^2 \left(\frac{\partial\phi}{\partial x} \right)^2 \right] = \frac{1}{2} \rho \; \dot{\phi}^2 - \frac{1}{2} \kappa \left(\frac{\partial\phi}{\partial x} \right)^2$$

$$\partial^\mu \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \right) = \frac{\delta\mathcal{L}}{\delta\phi}$$

$$\rho \; \ddot{\phi} - \kappa \; \frac{\partial^2\phi}{\partial x^2} = 0$$

$$\ddot{\phi} - \frac{\kappa}{\rho}\frac{\partial^2\phi}{\partial x^2} = 0$$

Wave equation with $v^2 = \kappa/\rho$.

#13 :GRADUATE E&M

`PROBLEM:` A long rod of *paramagnetic material* with linear magnetic permeability μ, and cross-sectional area a, is inserted a distance x into a long solenoid of length d, cross-sectional area A, carrying current I with total number of turns N. The current is held fixed by an external battery. Find the force (magnitude and direction) on the rod.

`SOLUTION:`The change in magnetic energy W_m due to insertion of rod $= \frac{1}{2}\Delta L\, I^2$ where ΔL is the change in the inductance of the system due to the rod. For a closed system this would be the work done on the rod. However, since current is fixed, the external battery also does work on the system as the rod is moved. The work done by the battery is $-I\int dt\, EMF$ where $EMF = I\partial L/\partial t$, so the work by the battery on the rod is $-\Delta L\, I^2$.

Thus, the total work done on the rod is $\Delta E = -\frac{1}{2}\Delta L\, I^2$. The force F on the rod is $F = -\partial\Delta E/\partial x$. It remains only to find an expression for ΔL. This can be found from the general expression

$$W_m = \tfrac{1}{2}L\, I^2 = \int dV\, \mu H^2/8\pi.$$

Maxwell's equation implies $H = 4\pi IN/(dc)$, everywhere in the solenoid and the rod (neglecting end effects since the rod is long). This implies

$$\begin{aligned}\Delta L &= \tfrac{H^2}{4\pi I^2}(\mu a x + (A-a)x + A(d-x) - Ad)\\ &= \tfrac{H^2}{4\pi I^2}a(\mu-1)x.\end{aligned}$$

Therefore $\Delta E = -\frac{H^2}{8\pi}a(\mu-1)x$, so $F = \frac{H^2}{8\pi}a(\mu-1)$. The force is in the direction of increasing x (into the solenoid), since for a paramagnetic material $\mu > 1$.

#14 :GRADUATE E&M

`PROBLEM:` Suppose that there is a density n_Q of charge carrier particles, all with charge Q, mass m, and (non-relativisitic) velocity $\vec{v}$. A superconductor has $\vec{P} = 0$, where $\vec{P} = \partial L/\partial\vec{v}$, (recall that $S \supset \int \frac{Q}{c}\vec{A}\cdot d\vec{x}$).

(a) Show that the current is proportional to the vector potential in the superconductor, $\vec{J} = C\vec{A}$, and find the constant of proportionality C.

(b) Suppose that the superconductor has net charge density $\rho = 0$ (the charge carriers are balanced out by opposite sign, static charges). There is an applied external electric field

$$\vec{E}(\vec{x}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t),$$

with $\vec{E}_0$ some constant vector. Solve for $\vec{B}$, $\vec{J}$, and derive the dispersion relation $\omega = \omega(k)$.

SOLUTION:

(a) Since $\vec{J} = Qn_e\vec{v}$ and $\vec{P} = m\vec{v} + Q\vec{A}/c$, we have $\vec{J} = -Q^2 n_Q \vec{A}/mc$.

(b) Solve Maxwell's equations to find $\vec{B} = c\omega^{-1}\vec{k} \times \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t)$ and then solve for $\vec{J}$. Define $\lambda_L = \sqrt{mc^2/4\pi Q^2 n_Q}$. Find

$$\vec{J} = \frac{c}{4\pi}(\frac{ck^2}{\omega} - \frac{\omega}{c})\vec{E}_0 \sin(\vec{k} \cdot \vec{x} - \omega t),$$

$$\vec{B} = -(\frac{ck^2}{\omega} - \frac{\omega}{c})\lambda_L^2 \vec{k} \times \vec{E}_0 \cos(\vec{k} \cdot \vec{x} - \omega t)$$

and the dispersion relation is

$$\omega^2 - c^2k^2 = c^2\lambda_L^{-2}.$$

#15 :GRADUATE STATISTICAL MECHANICS

A spin-1 Ising model in one dimension is described by the Hamiltonian

$$H_N\{\sigma_i\} = -J\sum_{i=1}^{N} \sigma_i\sigma_{i+1}, \qquad (\sigma_i = -1, 0, +1).$$

Write down the transfer matrix (P) (where, recall, $Q_N = TrP^N$) for this interaction and show that the free energy $A_N(T)$ of this model in the thermodynamic limit is equal to

$$-NkT \ln\left(\frac{1}{2}\left[(1 + 2\cosh K) + (8 + (2\cosh K - 1)^2)^{1/2}\right]\right), \qquad (K \equiv J/kT).$$

Examine the limiting behavior of this quantity as $T \to 0$ and as $T \to \infty$, and discuss the physical interpretation of each limit.

`SOLUTION:` Assuming a closed endless structure, the partition function is

$$Q_N(T) = \sum_{\{\sigma_i\}} \exp[\beta \sum_{i=1}^{N} J\sigma_i\sigma_{i+1}] \qquad (\sigma_{N+1} = \sigma_1)$$

$$= \sum_{\{\sigma_i\}} \exp(\beta J\sigma_1\sigma_2)\exp(\beta J\sigma_2\sigma 3)\ldots\exp(\beta J\sigma_N\sigma_1)$$

$$= \sum_{\{\sigma_i\}} \langle\sigma_1|P|\sigma_2\rangle\langle\sigma_2|P|\sigma_3\rangle\ldots\langle\sigma_N|P|\sigma_1\rangle,$$

where $\langle\sigma_i|P|\sigma_{i+1}\rangle$ are the matrix elements of the transfer matrix $(P) = (e^{\beta J\sigma_i\sigma_{i+1}})$. Writing out this 3×3 matrix (with 1s in the middle row and column and $e^{\pm\beta J}$ in the corners), the eigenvalues are found to be

$$\lambda_{1,2} = \tfrac{1}{2}[(1+2\cosh K) \pm (8 + (2\cosh K - 1)^2)^{1/2}], \qquad \lambda_3 = 2\sinh K.$$

It follows that $Q_N(T) = Tr(P^N) = \lambda_1^N + \lambda_2^N + \lambda_3^N$.

In the thermodynamic limit, only the largest eigenvalue, viz. λ_1, matters – with the result that $A_N(T) = -kT\ln Q_N(T) \approx -NkT\ln\lambda_1$, which gives the stated result.

In the limit $T \to 0$, i.e. $K \to \infty$, the function $\cosh K \approx \frac{1}{2}e^K$ and hence $A \approx -NJ$; this corresponds to a state of perfect order, with $U = -NJ$ and $S = 0$.

In the limit $T \to \infty$, i.e. $K \to 0$, the function $\cosh K \to 1$ and hence $A \to -NkT\ln 3$; this corresponds to a state of complete randomness, with 3^N equally likely microstates, which entails $U = 0$ and $S = Nk\ln 3$.

#16 :GRADUATE STATISTICAL MECHANICS

`PROBLEM:` The radius of a neutron star of mass equal to a solar mass is 12.4 km. Find the radius of a neutron star of mass equal to two solar masses. Show all steps needed to derive the answer. Assume no interaction between the neutrons other than the gravitational force and treat the problem non-relativistically.

Hints:

(1) The gravitational energy of a sphere of mass M and radius R is

$$E_{grav} = -\frac{3}{5}G\frac{M^2}{R} \tag{20}$$

(2) The kinetic energy of N spin 1/2 fermions is

$$E_{kin} = \frac{3}{5}N\epsilon_F \tag{21}$$

where ϵ_F is the Fermi energy.

`SOLUTION:`

The Fermi energy of a system of N spin 1/2 fermions of mass m is

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m} \tag{22}$$

where k_F is the Fermi wavevector, related to the density N/V by

$$k_F = (3\pi^2\frac{N}{V})^{1/3} \tag{23}$$

The volume of a sphere of radius R is $(4\pi/3)R^3$. Hence

$$\epsilon_F = \frac{\hbar^2}{2m}(\frac{9\pi}{4})^{2/3}N^{2/3}\frac{1}{R^2} \tag{24}$$

and the kinetic energy is

$$E_{kin} = a\frac{N^{5/3}}{R^2} \tag{25}$$

with a a positive constant. The total mass $M = Nm$, hence the gravitational energy is

$$E_{grav} = -b\frac{N^2}{R} \tag{26}$$

with b a positive constant, and the total energy is

$$E_{tot} = a\frac{N^{5/3}}{R^2} - b\frac{N^2}{R} \tag{27}$$

Minimizing with respect to R yields

$$R = \frac{2a}{b}\frac{N^{5/3}}{N^2} = \frac{2a}{b}\frac{1}{N^{1/3}} \tag{28}$$

Therefore

$$R(2\ solar\ masses) = \frac{1}{2^{1/3}}R(1\ solar\ mass) = 9.84km \tag{29}$$

#17 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Two spin 1/2 particles of mass m interact via $H_{int} = C\vec{S}_1 \cdot \vec{S}_2 \delta^3(\vec{r})$, where C is a constant.

(a) Calculate the approximate differential and total scattering cross section for the spin singlet configuration in the case where the particles are distinguishable.

(b) Calculate the approximate differential and total scattering cross section for the spin triplet configuration, in the case where the two particles are distinguishable.

(c) Compute the differential scattering cross section for unpolarized scattering, again for distinguishable particles.

(d) How do the above change when the particles are indistinguishable?

(e) How would your answer to part (d) change if the interaction is a more general $H_{int} = \vec{S}_1 \cdot \vec{S}_2 f(\vec{r})$?

`SOLUTION:`

Use the Born approximation:

$$f(\vec{q}) = -\frac{2m}{4\pi\hbar^2} \int d^3\vec{r} e^{i\vec{q}\cdot\vec{r}} H_{int}(\vec{r}) = -\frac{mC}{2\pi\hbar^2} \vec{S}_1 \cdot \vec{S}_2$$

and $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2) = \frac{1}{2}(s(s+1) - \frac{3}{2})\hbar^2$, which gives $\vec{S}_1 \cdot \vec{S}_2 = -\frac{3}{4}\hbar^2$ (singlet) or $\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{4}\hbar^2$ (triplet).

The differential cross section is $\frac{d\sigma}{d\Omega} = |f(q)|^2$.

(a) For the singlet, $\frac{d\sigma}{d\Omega} = \frac{9m^2C^2}{64\pi^2}$. It's isotropic, so the total cross section is $\sigma = \frac{9m^2C^2}{16\pi}$.

(b) For the triplet, $\frac{d\sigma}{d\Omega} = \frac{m^2C^2}{64\pi^2}$ and $\sigma = \frac{m^2C^2}{16\pi}$.

(c) For unpolarized, average the one singlet with the 3 triplet to get $\frac{d\sigma}{d\Omega} = \frac{3m^2C^2}{64\pi^2}$.

(d) For indistinguishable the total wave function must be antisymmetric. The singlet scattering is unchanged, since the spin part of the singlet is already antisymmetric. The triplet scattering on the other hand vanishes, because a spatially antisymmetric wavefunction can't overlap with $\delta(\vec{r})$. So only the singlet scatters. An unpolarized beam would then scatter with $\frac{d\sigma}{d\Omega} = \frac{9}{4}\frac{m^2C^2}{64\pi^2}$.

(e) For $f(\vec{r})$ the triplet can scatter, since the antisymmetric wavefunction can then give scattering even though the particles don't overlap.

#18 : GRADUATE QUANTUM MECHANICS

Let $L_\pm = L_x \pm iL_y$ and L_z be angular momentum operators. Consider an operator V_+ which satisfies

$$[L_+, V_+] = 0, \qquad [L_z, V_+] = V_+.$$

(a) Let $|\ell, m\rangle$ be a simultaneous eigenfunction of L^2 and L_z with eigenvalues $\ell(\ell+1)$ and m, respectively. (Here, $\hbar = 1$.) Show that

$$V_+|\ell, \ell\rangle = \text{const } |\ell+1, \ell+1\rangle.$$

(b) Demonstrate, for the case of orbital angular momenta, that

$$V_+ = e^{i\phi} \ \sin\theta$$

satisfies the commutation relations of V_+ with L_+ and L_z given above. Recall that the operators L_+ and L_z are given by the differential operators

$$L_+ = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right),$$

$$L_z = -i\frac{\partial}{\partial\phi}.$$

(c) Assume that $|0,0\rangle = \text{const} \equiv 1/\sqrt{4\pi}$. Using the equations in parts (a) and (b), determine the functions $|\ell, \ell\rangle$ for arbitrary ℓ, normalized such that

$$\langle \ell, \ell|\ell, \ell\rangle = 1.$$

A useful integral is

$$\int_0^{\pi/2} d\theta \ \sin^{2\ell+1}\theta = \frac{2^\ell \ \ell!}{(2\ell+1)!!}$$

where $(2\ell+1)!! \equiv (2\ell+1)(2\ell-1)\cdots 5\cdot 3\cdot 1$.

`SOLUTION:` (a)

$$L^2\ |\ell,m\rangle = \ell\,(\ell+1)\ |\ell,m\rangle$$
$$L_z\ |\ell,m\rangle = m\,|\ell,m\rangle$$

$$L_z\,(V_+|\ell,\ell\rangle) = ([L_z,V_+] + V_+L_z)\,|\ell,\ell\rangle = (V_+ + V_+L_z)\,|\ell,\ell\rangle = (\ell+1)\,(V_+|\ell,\ell\rangle)$$

Thus, $V_+|\ell,\ell\rangle$ is a state with L_z eigenvlaue $m=\ell+1$. Explicitly,

$$V_+|\ell,\ell\rangle = \sum_{n\geq \ell+1} c_n|n,\ell+1\rangle.$$

Also,

$$L_+V_+|\ell,\ell\rangle = V_+L_+|\ell,\ell\rangle = 0$$

since $L_+|\ell,\ell\rangle = 0$ since $m=\ell$ is the state with the highest L_z eigenvalue.
This equation implies that

$$\sum_{n\geq\ell+1} c_nL_+|n,\ell{+}1\rangle = 0 = \sum_{n\geq\ell+1}\sqrt{n(n+1)-\ell(\ell+1)}|n,\ell{+}2\rangle = \sum_{n\geq\ell+2}\sqrt{n(n+1)-\ell(\ell+1)}|n,\ell{+}2\rangle,$$

which implies that

$$c_n = 0, \quad n\geq \ell+2.$$

(b)

$$[L_+,V_+] = \left[e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), e^{i\phi}\ \sin\theta\right]$$
$$= e^{i\phi}\left(\cos\theta e^{i\phi} + i\cot\theta(ie^{i\phi}\sin\theta)\right) = 0$$

$$[L_z,V_+] = \left[-i\frac{\partial}{\partial\phi}, e^{i\phi}\ \sin\theta\right] = e^{i\phi}\sin\theta = V_+$$

(c)

$$|0,0\rangle = \frac{1}{\sqrt{4\pi}}\int d\Omega = 1$$
$$|1,1\rangle = c_1\ V_+|0,0\rangle,$$

$$\langle 1,1|1,1\rangle = |c_1|^2\, \langle 0,0|V_+^\dagger V_+|0,0\rangle = |c_1|^2 \int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta)\, \sin^2\theta = \frac{8\pi}{3}|c_1|^2 = 1 \Rightarrow c_1 = \sqrt{\frac{3}{8\pi}}.$$

In general,

$$|\ell,\ell\rangle = c_\ell\ (V_+)^\ell|0,0\rangle = \frac{1}{\sqrt{4\pi}}c_\ell\ e^{i\ell\phi}\ \sin^\ell\theta$$

$$\langle \ell,\ell|\ell,\ell\rangle = 1 = |c_\ell|^2\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta)\, \sin^{2\ell}\theta = |c_\ell|^2\int_0^{\pi/2} d\theta\, \sin^{2\ell+1}\theta = |c_\ell|^2\frac{2^\ell\ \ell!}{(2\ell+1)!!}$$

$$\Rightarrow c_\ell = \sqrt{\frac{(2\ell+1)!!}{2^\ell\ \ell!}}, \qquad (2\ell+1)!! \equiv (2\ell+1)(2\ell-1)\cdots 5\cdot 3\cdot 1$$

$$|\ell,\ell\rangle = \sqrt{\frac{(2\ell+1)!!}{2^\ell\ \ell!}}\frac{1}{\sqrt{4\pi}}\ e^{i\ell\phi}\ \sin^\ell\theta$$

#19 : GRADUATE MATH METHODS

`PROBLEM:` A mathematical function has the integral representation

$$F_\nu(x) = \tfrac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty \exp(-t - \frac{x^2}{4t})t^{-\nu-1}dt,$$

where ν and x may be regarded as real, positive numbers.

With ν fixed, determine the *asymptotic* behavior of this function for $x \gg 1$.
`SOLUTION:` For large x note that $\exp(-t - \frac{x^2}{4t})$ has a maximum at $t_0 = \frac{1}{2}x$. Set $t = \frac{1}{2}x(1+u)$ and expand $(1+u)^{-1}$ in the exponent to find

$$F_\nu(x) = \tfrac{1}{2}\int_{-1}^\infty \exp[-\tfrac{1}{2}x(1+u+(1-u+u^2-u^3+\dots))](1+u)^{-\nu-1}du.$$

For large x the integrand has an effective u range which is $O(1/\sqrt{x}) \ll 1$ around zero. Therefore,

$$F_\nu(x) \approx \tfrac{1}{2}e^{-x}\int_{-\infty}^\infty e^{-\frac{1}{2}xu^2}du = \sqrt{\frac{\pi}{2x}}e^{-x},$$

independent of ν. (The function in question is the modified Bessel function $K_\nu(x)$.)

#20 : GRADUATE GENERAL

PROBLEM:

Consider a small hole in the top of a container (with outward normal $\widehat{z}$) containing a gas. Suppose the temperature of the gas is T, each molecule has mass m, and the number of molecules per unit volume is N.

(a) Note that the velocity distribution of particles exiting ("effusing") out of the hole differs from that inside the container. Considering the number of particles exiting the hole in time Δt, you'll see that the velocity distribution of particles exiting the hole is that inside the container, weighted by the flux. Faster particles are more likely to get out of the hole. Give the velocity distribution of particles effusing out of the hole, making sure it's correctly normalized.

(b) Calculate the mean upward speed $\langle v_z \rangle$ of the the molecules leaving the container. Also compute $\left\langle v_z^2 \right\rangle$ (note that it's indeed greater than $\left\langle v_z^2 \right\rangle$ inside the container, which is given by equipartition).

Hint: In doing integrals you might remember the Gamma function $\Gamma(z) = 2\int_0^\infty \exp(-t^2)t^{2z-1}dt$, and $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(z+1) = z\Gamma(z)$.

SOLUTION:

Assume inside the container that the gas molecules obey the Maxwell-Boltzmann distribution. The distribution of the effusing molecules is weighted by the flux, i.e. an additional factor of v_z for $v_z > 0$, and it vanishes for $v_z < 0$.

(a) So the distribution for $v_z > 0$ is: $(m^2/2\pi k^2T^2)v_z \exp(-mv^2/2kT)d^3v$, where the prefactor ensures that it's properly normalized to integrate to 1.

(b) Integrating v_z times the velocity distribution gives $\langle v_z \rangle = (\pi kT/2m)^{1/2}$. Doing the integral for v_z^2 gives $\left\langle v_z^2 \right\rangle = 2kT/m$, which is indeed a factor of two greater than that inside the container (it's kT/m inside the container).