

INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	1 2	3 4	5 6	7 8	9 10

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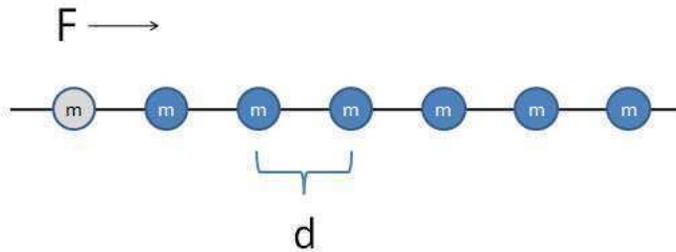
PHYSICAL CONSTANTS

Gas constant	$R = 8.314 \times 10^3 \text{ J kilomole}^{-1} \text{ K}^{-1} (R = N_A k)$
Boltzmann's constant	$k = 1.381 \times 10^{-23} \text{ JK}^{-1}$ $= 8.617 \times 10^{-5} \text{ eVK}^{-1}$
Avogadro's number	$N_A = 6.022 \times 10^{26} \text{ kilomole}^{-1}$
Volume of one kilomole of gas at STP	$22.42 \text{ m}^3 (\text{STP} = 0^\circ\text{C}, 1\text{atm})$
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Mechanical equivalent of heat	$4184 \text{ J kilocalorie}^{-1}$
Temperature of triple point of H_2O	$273.16 \text{ K} = 0.01^\circ\text{C}$
Atomic mass unit	$1 \text{ amu} = 1.661 \times 10^{-27} \text{ kg}$
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Permeability of free space	$\mu_0 = 4\pi \times 10^{-7} \text{ NA}^{-2}$
Stefan constant	$\sigma = 5.670 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$

#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: Beads of equal mass m are strung at equal original distances d on a long horizontal wire. The beads are initially at rest but can move along the wire without friction. The leftmost bead is continuously accelerated (towards the right) by a constant force F . The other beads do not feel F , but do undergo collisions with the leftmost bead and each other. As a result of the collisions, a compression wave propagates to the right down the wire. What are the speeds of the leftmost bead and the front of the 'shock wave' after a long time, if the collisions of the beads are:

- (a) completely inelastic,
- (b) perfectly elastic?



#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: What is the minimum proton energy needed in an accelerator to produce antiprotons \bar{p} by the reaction

$$p + p \rightarrow p + p + (p + \bar{p})$$

The rest energy of a proton and antiproton is 938 MeV. What is the minimum kinetic energy for each particle to produce this reaction?

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: Find the distribution of bulk and surface currents that will produce a magnetic field $\vec{B} = B_0(\rho/b)^2\hat{\phi}$ for $\rho \leq b$ and $\vec{B} = 0$, $\rho \geq b$, where B_0 and b are constants, and this problem is referred to a cylindrical coordinate system (ρ, ϕ, z) with $\hat{\rho} \times \hat{\phi} = \hat{z}$.

#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: Consider a “leaky” spherical capacitor, namely a sphere with radius a surrounded by a larger sphere with radius b . The region between the spheres is filled with a uniform medium with conductivity σ and dielectric permittivity ϵ .

- (a) Find the characteristic time for a charge Q_0 on the inner sphere to decay to $1/e$ of its original value.
- (b) Show that this result is true for any charge density–fluctuation in a uniform medium with conductivity σ and dielectric constant ϵ ; namely that any charge-density fluctuation will decay in this same time. [Hint: use the continuity equation for electric charge.]

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Consider a particle of mass m and charge q moving in the presence of constant, uniform, crossed \vec{E} and \vec{B} fields. For concreteness orient your coordinate system so that \vec{E} points along the z axis and \vec{B} points along the y axis. Write the Schrodinger equation. Reduce it to a one dimensional problem by separating variables. Find the expectation value of the z -coordinate of the particle and use this to calculate the expectation value of the x -component of the particle's velocity.

CODE NUMBER: _____

SCORE: _____

6

#6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Find the transmission probability as a function of energy of the incident particle for scattering in one dimension off the potential $V(x) = \kappa(\delta(x + a) + \delta(x - a))$, where κ and a are positive constants.

CODE NUMBER: _____

SCORE: _____

7

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: Consider an electron gas with particle density n . Determine the numerical value of n for which the Fermi energy ϵ_F of the gas is equal to the rest energy, $m_e c^2$, of the electron. What is the corresponding value of the Fermi velocity v_F of the system?

#8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice, at the ice point (absolute temperature T_0). It is desired to use a cyclic refrigerator to freeze an additional mass m of water in the bucket. The refrigerator also rejects heat Q_R , which all goes into warming up a body of constant heat capacity C and, initially, also at temperature T_0 .

(a) What is the change in the entropies in the bucket, the refrigerator, and the body in this process?

(b) What is the minimum work required to run the refrigerator for this process?

#9 : UNDERGRADUATE MATH METHODS

PROBLEM: Consider the differential equation

$$\alpha \frac{dX}{dt} + X = f(t), \quad \text{where} \quad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{f}(\omega) e^{i\omega t},$$

where α is a positive, real constant.

(a) Find the general solution for $X(t)$ in terms of $\tilde{f}(\omega)$ and $X(t=0) = X_0$.

(b) Find the solution for the case $\tilde{f}(\omega) = Ae^{-i\omega t_1}$, where A and t_1 are real constants, and $t_1 > 0$. Evaluate the integral by contour integration, for all values of t . **Please draw** the appropriate closed contour for all values of t , noting that the appropriate contour differs depending on whether t is below or above a certain value. Also draw the location of all poles.

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

PROBLEM: Consider a self-gravitating, uniform sphere of mass M , radius R , and temperature T radiating as a black body. Consider the possibility that the energy source for the radiation is supplied by the sphere's gravitational contraction (this was the prevailing theory about the energy source for the Sun and stars in the 19th century.)

- (a) Derive the formula for the time required for the sphere to radiate away all its gravitational potential energy. This is called the Kelvin-Helmholtz time.
- (b) Evaluate the Kelvin-Helmholtz time for the Sun. (assume the Sun's mass, radius, and surface temperature are $M_{\odot} = 2 \times 10^{30}$ kg, $R_{\odot} = 7 \times 10^8$ m, $T_{\odot} = 5800$ K).
- (c) Compare the Sun's Kelvin-Helmholtz time to the age of the oldest rocks on Earth (about 4 billion years old). Is gravitational contraction a viable energy source for the Sun? Briefly discuss.

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

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#11 : GRADUATE CLASSICAL MECHANICS

PROBLEM: A system has the following Hamiltonian:

$$H(x, p_x, y, p_y) = \frac{1}{2}p_x^2 + xp_y^2 + xy^2.$$

Solve for $x(t)$ and $y(t)$, assuming $x(0) = y(0) = 0$.

(Hint 1: $\int^y \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\left(\frac{y}{a}\right)$.)

(Hint 2: use Hamilton-Jacobi theory)

#12 : GRADUATE CLASSICAL MECHANICS

PROBLEM: Consider an infinite, one dimensional “diatomic” chain of oscillators shown below. Each mass is separated by a distance ℓ from its neighbor, by a spring of strength k . The masses m_1, m_2 alternate.

- (a) Show that this chain supports two collective modes for purely longitudinal oscillations. Calculate their dispersion relations.
- (b) Characterize the modes in the limit $\ell\alpha < 1$, where α is the wave number



#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: A steady state current density $\vec{J}(\vec{r})$ is non-zero only in the sphere $|\vec{r}| < R$. Let $\vec{B}(\vec{r})$ be the magnetic field induced by this current. Determine a numerical value for the flux through the plane $x = 0$; that is, determine the value of

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz B_x(x, y, z)$$

Prove your answer.

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: A plane electromagnetic wave of angular frequency $\omega = 10^{10} \text{ sec}^{-1}$ is normally incident on a planar slab of copper. Obtain an expression for the reflection coefficient of the wave (i.e., for \mathbf{R}) in terms of ω and the conductivity of copper σ . What is an approximate numerical value for $(1 - \mathbf{R})$? (assume $\sigma = 5 \times 10^{17}$).

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: Consider a spin-1/2 particle with magnetic moment μ in a uniform, non-constant magnetic field $\vec{B} = \vec{B}(t)$. The particle wave function at time $t = 0$ is

$$\begin{pmatrix} e^{-i\alpha \cos \delta} \\ e^{i\alpha \sin \delta} \end{pmatrix} \quad (1)$$

Find the expectation value of the spin of the particle, \vec{s} , at a later time t .

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: A half-silvered but perfectly plane mirror is set in the $z = 0$ plane. An incident beam of light will be transmitted and reflected with equal intensity by the mirror from either side of the mirror. The mirror has such perfect symmetry that a photon incident on one side of the mirror will have exactly the same transmission and reflection *amplitudes* as a photon incident in the mirror symmetric direction on the other side of the mirror.

(a) Find the transformation matrix (known as the scattering or S matrix) which relate the two incoming states of a photon with wave vector $\vec{k} = (k \sin \theta, 0, k \cos \theta)$ and its mirror image $\vec{k}_m = (k \sin \theta, 0, k \cos(\pi - \theta))$ to the two outgoing states of the same pair of vectors.

(b) If the incoming state of a photon is either a symmetric or an antisymmetric combination of the two incoming states described in part (a), find the outgoing states in both cases and, hence or otherwise, find the symmetric and antisymmetric phase shifts due to the scattering by the mirror.

(c) If two photons, one incident in the \vec{k} direction and the other in the \vec{k}_m direction, are timed to arrive at the same region of the mirror with maximum wave function overlap, find the probabilities of finding the outgoing photons in the all possible cases of n photons along \vec{k} , m photons along \vec{k}_m , namely $(n, m) = (2, 0), (1, 1), (0, 2)$.

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The potential energy of a one-dimensional, *anharmonic* oscillator is given by

$$V(q) = cq^2 - gq^3,$$

where c and g are positive constants. Using classical statistics and treating the anharmonic term as a perturbation, evaluate the leading contribution of this term to the heat capacity of the oscillator and to the mean value of the position coordinate q .

#18 : GRADUATE STATISTICAL MECHANICS

PROBLEM: A classical N -particle system has the following density of states,

$$\Omega(E, N) \propto \exp \left\{ N \left[\frac{\epsilon}{2} + \epsilon^2 - \frac{\epsilon^3}{3} \right] \right\}$$

where $\epsilon \equiv E/N \geq 0$ is the energy per particle expressed in dimensionless energy unit. You can assume the particles to be distinguishable and $N \gg 1$.

(a) What is the entropy of the system $S(E, N)$? At a finite temperature T , show that the Helmholtz free energy per particle $f(T) \equiv F(T, N)/N$ can be obtained by minimizing a function $\Psi(\epsilon; T)$ for $N \rightarrow \infty$.

(b) How can the average thermal energy per particle, $\bar{\epsilon}(T)$ be obtained from $\Psi(\epsilon; T)$? Sketch $\Psi(\epsilon; T)$ first for $T_2 \equiv 2/k_B$ and then for $T_1 \equiv 1/k_B$, where k_B is Boltzmann's constant. Indicate on your graph the location of $\bar{\epsilon}$ at T_1 and T_2 .

Hint: At T_1 you should decide between two possibilities, one with $\psi > 0$ and another with $\psi < 0$.

(c) Sketch $\Psi(\epsilon; T)$ as T is reduced further from T_1 . Show that a phase transition occurs at temperature $0 < T_c < T_1$ and find the value of T_c . What is the order of the phase transition?

#19 : GRADUATE MATHEMATICAL PHYSICS

PROBLEM: Using contour integration, evaluate the integrals

$$\int_0^{\infty} dx x^{\alpha-1} \cos x$$

and

$$\int_0^{\infty} dx x^{\alpha-1} \sin x$$

where $0 < \alpha < 1$.

#20 : GRADUATE OTHER

PROBLEM: A sphere of mass M , radius R is dropped into a fluid of density ρ_0 , and sinks. Assuming the fluid has kinematic viscosity ν , determine the parameter scaling of the sphere's terminal velocity. You can assume the flow around the sphere is laminar.

- (a) Use dimensional analysis to determine the drag force on the sphere, for laminar flow. "Laminar flow" suggests that the drag force should be proportional to friction and velocity.
- (b) Use your result from (a) to estimate the terminal velocity of the sphere.

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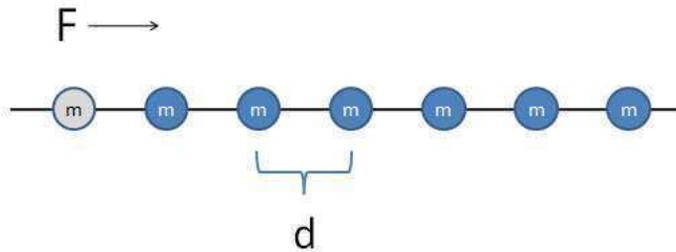
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#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: Beads of equal mass m are strung at equal original distances d on a long horizontal wire. The beads are initially at rest but can move along the wire without friction. The leftmost bead is continuously accelerated (towards the right) by a constant force F . The other beads do not feel F , but do undergo collisions with the leftmost bead and each other. As a result of the collisions, a compression wave propagates to the right down the wire. What are the speeds of the leftmost bead and the front of the 'shock wave' after a long time, if the collisions of the beads are:

- (a) completely inelastic,
 (b) perfectly elastic?



SOLUTION:

- (a) Let v_0 denote the asymptotic common speed.

In a given time interval Δt , the cluster collides with $v_0\Delta t/d$ further beads, which increases its mass by $\Delta m = mv_0\Delta t/d$ and its momentum by $\Delta p = v_0\Delta m = mv_0^2\Delta t/d$. According to Newton's law of motion,

$$F = \frac{\Delta p}{\Delta t} = \frac{mv_0^2}{d},$$

which yields $v_0 = \sqrt{Fd/m}$ for the ultimate speed in the case of inelastic collisions.

(b) In an elastic collision between two equal mass bodies with one of them initially at rest, their velocities are exchanged. The body initially moving with velocity v stops, while the second one moves away with velocity v .

The leftmost bead accelerates uniformly and reaches a speed of

$$v_1 = \sqrt{\frac{2Fd}{m}} = \sqrt{2}v_0$$

before the first (elastic) collision takes place. It then transfers its speed to the second bead and stops, after which it starts accelerating again as a result of the external force. The second bead moves at a constant speed v_1 , collides with the third bead and stops. The third and subsequent beads behave similarly, and a 'shock wave' propagates forward at speed v_1 .

Meanwhile, the leftmost bead is again accelerated to speed v_1 , collides with the second bead, which is now at rest, and the process is repeated, thus starting a new 'shock wave'. The speed of the leftmost bead varies uniformly from zero to v_1 , its average value is $v_1/2 = v_0/\sqrt{2} = \sqrt{Fd/(2m)}$.

#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: What is the minimum proton energy needed in an accelerator to produce antiprotons \bar{p} by the reaction

$$p + p \rightarrow p + p + (p + \bar{p})$$

The rest energy of a proton and antiproton is 938 MeV. What is the minimum kinetic energy for each particle to produce this reaction?

SOLUTION: The minimum energy will occur when the four particles are all at rest in the center of mass system after collision. Conservation of energy in the CM system gives

$$2E_{p,CM} = 4m_p c^2$$

or

$$E_{p,CM} = 2m_p c^2 = 2E_0$$

which implies $\gamma = 2$ or $\beta = \sqrt{3}/2$.

To find the energy required in the lab system (one proton initially at rest), we transform back to the lab

$$E_{lab} = \gamma(E' + vp'_1) \tag{1}$$

The velocity of the CM frame with respect to the Lab frame is just the velocity of the proton in the CM frame. So $u = v$. Then

$$vp'_1 = v(p_{CM}) = v(\gamma mu) = \gamma mv^2 = \gamma mc^2 \beta^2$$

Since $\gamma = 2, \beta = \sqrt{3}/2$,

$$vp'_1 = \frac{3}{2}E_0$$

Substituting into Eq. 1 above,

$$E_{lab} = \gamma(2E_0 + \frac{3}{2}E_0) = 2(\frac{7}{2}E_0) = 7E_0$$

Therefore the minimum proton energy in the lab system is $7m_p c^2$, of which $6m_p c^2$ is kinetic energy.

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: Find the distribution of bulk and surface currents that will produce a magnetic field $\vec{B} = B_0(\rho/b)^2\hat{\phi}$ for $\rho \leq b$ and $\vec{B} = 0$, $\rho \geq b$, where B_0 and b are constants, and this problem is referred to a cylindrical coordinate system (ρ, ϕ, z) with $\hat{\rho} \times \hat{\phi} = \hat{z}$.

SOLUTION:

from handwritten solution

$$2. \quad \vec{B} = B_0 \left(\frac{s}{b}\right)^2 \hat{\phi} \quad s \leq b$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int_0^s \vec{J}(s') 2\pi s' ds' = 2\pi s B_0 \left(\frac{s}{b}\right)^2$$

$$2\pi s \mu_0 \vec{J}(s) = 2\pi \frac{s^3}{b^2} B_0$$

$$\underline{\vec{J}(s) = \frac{3 B_0 s}{\mu_0 b^2} \hat{z} \quad s \leq b}$$

$s \geq b \quad \vec{B} = 0$ so need return surface current

$$\underline{I} = 2\pi \int_0^b \vec{J}(s) ds = \frac{6\pi B_0 b^3}{\mu_0 b^2 3}$$

current / unit circumference

$$\underline{\vec{K} = -\frac{B_0}{\mu_0} \hat{z} \quad @ s = b}$$

current / unit circumference

#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: Consider a “leaky” spherical capacitor, namely a sphere with radius a surrounded by a larger sphere with radius b . The region between the spheres is filled with a uniform medium with conductivity σ and dielectric permittivity ϵ .

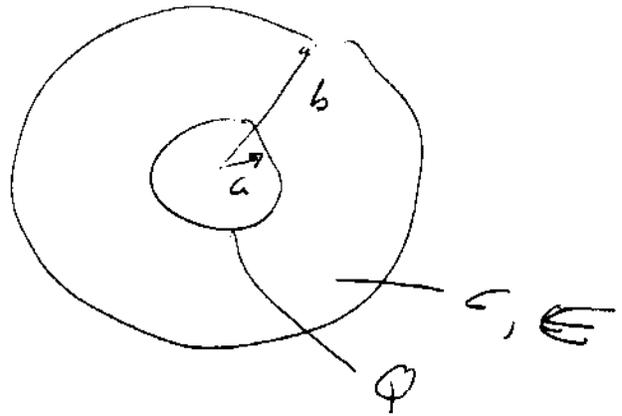
- (a) Find the characteristic time for a charge Q_0 on the inner sphere to decay to $1/e$ of its original value.
- (b) Show that this result is true for any charge density–fluctuation in a uniform medium with conductivity σ and dielectric constant ϵ ; namely that any charge-density fluctuation will decay in this same time. [Hint: use the continuity equation for electric charge.]

SOLUTION:

see handwritten solution

3.

(a)



$$4\pi r^2 E = \frac{Q}{\epsilon}$$

$$J = \sigma E = \frac{\sigma Q}{4\pi r^2 \epsilon}$$

$$I = 4\pi r^2 J = \frac{\sigma Q}{\epsilon} = -\frac{dQ}{dt}$$

$$Q = Q_0 e^{-t/(\epsilon/\sigma)}$$

$$\tau_{1/e} = \frac{\epsilon}{\sigma}$$

(b) Continuity eq. for charge

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho_f}{\partial t} = 0$$

ρ_f is free charge

$$\nabla \cdot \mathbf{J} = \sigma \nabla \cdot \mathbf{E} = \frac{\sigma \rho_f}{\epsilon}$$

↑
Poisson eq.

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\epsilon} \rho_f$$

$$\rho_f = \rho_{f0} e^{-t/(\epsilon/\sigma)}$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Consider a particle of mass m and charge q moving in the presence of constant, uniform, crossed \vec{E} and \vec{B} fields. For concreteness orient your coordinate system so that \vec{E} points along the z axis and \vec{B} points along the y axis. Write the Schrodinger equation. Reduce it to a one dimensional problem by separating variables. Find the expectation value of the z -coordinate of the particle and use this to calculate the expectation value of the x -component of the particle's velocity.

SOLUTION: To write a Schrödinger equation with magnetic field we need the vector potential, with $\vec{B} = \vec{\nabla} \times \vec{A}$. This is gauge dependent, so choose a convenient gauge. For example, I will use here $\vec{A} = (Bz, 0, 0)$ with B the magnitude of the resulting magnetic field in the z -direction. Since the fields are constant we can use a time-independent Schrödinger equation. The potential is that given by a constant uniform electric field, $V = -qEz$. The magnetic field enters through minimal substitution: $\vec{p} \rightarrow \vec{p} - (q/c)\vec{A}$. So we have the Schrödinger equation:

$$H\psi = \left[\frac{1}{2m} \left\{ \left(p_x - \frac{q}{c}Bz \right)^2 + p_y^2 + p_z^2 \right\} - qEz \right] \psi = \mathcal{E}\psi$$

I used \mathcal{E} for the energy to avoid confusion with the magnitude E of the electric field. Of course, it is understood that \vec{p} is a derivative operator.

Now, to separate this notice that the coordinates x and y do not enter explicitly. So write

$$\psi(x, y, z) = e^{ik_x x + ik_y y} \phi(z)$$

which gives

$$\left[\frac{1}{2m} \left\{ \left(\hbar k_x - \frac{q}{c}Bz \right)^2 + \hbar^2 k_y^2 + p_z^2 \right\} - qEz \right] \phi = \mathcal{E}\phi$$

This is just the simple harmonic oscillator with the minimum of the potential shifted to some non-zero z :

$$\frac{1}{2m} \left(\frac{qB}{c} \right)^2 z^2 - \left(\frac{1}{m} \frac{qB\hbar k_x}{c} + qE \right) z = \frac{1}{2m} \left(\frac{qB}{c} \right)^2 \left((z - z_0)^2 + z_0^2 \right)$$

Since the expectation value of the coordinate in the simple harmonic oscillator is zero, the shifted case has expectation value at z_0 . Solving we have

$$\langle z \rangle = z_0 = \frac{c}{qB} \left(\hbar k_x + \frac{mcE}{B} \right)$$

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The expectation value of the velocity, $\vec{v} = (1/m)(\vec{p} - (q/c)\vec{A})$ follows:

$$\langle v_x \rangle = \frac{1}{m} \left(\langle p_x \rangle - \frac{qB}{c} \langle z \rangle \right) = -\frac{Ec}{B}$$

Note that this is the same as the classical result for crossed E - B fields.

#6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Find the transmission probability as a function of energy of the incident particle for scattering in one dimension off the potential $V(x) = \kappa(\delta(x+a) + \delta(x-a))$, where κ and a are positive constants.

SOLUTION: Let $\psi(x)$ be the wave-function. In the regions where the potential vanishes the wave function is a solution to the free Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi = E\psi \quad (2)$$

The general solution is of the form $ae^{ikx} + be^{-ikx}$ with $k > 0$ giving an energy $E = \hbar^2 k^2 / 2m$ and a and b arbitrary complex coefficients. A plane wave approaching from the far left has $\psi \sim \exp(ikx)$. The general solution of the Schrodinger equation is then

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx} & \text{for } x < -a, \\ Ae^{ikx} + Be^{-ikx} & \text{for } -a < x < a, \\ Te^{ikx} & \text{for } x > a. \end{cases} \quad (3)$$

The interpretation is that T is the probability amplitude that a particle of energy E is transmitted, and R that it is reflected.

Integrating the Schrodinger equation over a region of size 2ϵ centered about $x = a$ and letting $\epsilon \rightarrow 0$ we have equations for the discontinuity in the derivative of the wave function:

$$\int_{a-\epsilon}^{a+\epsilon} dx \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + V(x)\psi(x) - E\psi \right] = 0 \quad (4)$$

$$-\frac{\hbar^2}{2m} (\psi'(a+) - \psi'(a-)) + \kappa\psi(a) = 0 \quad (5)$$

where $\psi'(x) = d\psi/dx$ and $\psi'(a\pm) = \lim_{x \rightarrow a\pm} \psi'(x)$. A similar expression can be written at $x = -a$. The wavefunction must be continuous so we have, in addition, the conditions $\psi(\pm a-) = \psi(\pm a+)$.

Using these conditions in our wavefunction we have

$$e^{-ika} + Re^{ika} = Ae^{-ika} + Be^{ika} \quad \text{continuity at } x = -a \quad (6)$$

$$Ae^{ika} + Be^{-ika} = Te^{ika} \quad \text{continuity at } x = a \quad (7)$$

And from discontinuity of ψ' at $x = -a$ and $x = a$ respectively, we have:

$$-\frac{i\hbar^2 k}{2m} \left[(Ae^{-ika} - Be^{ika}) - (e^{-ika} - Re^{ika}) \right] = -\kappa \left[Ae^{-ika} + Be^{ika} \right] \quad (8)$$

$$-\frac{i\hbar^2 k}{2m} \left[Te^{ika} - (Ae^{ika} - Be^{-ika}) \right] = -\kappa Te^{ika} \quad (9)$$

Solving for A and B in terms of T in (7) and (9) we have

$$A = \left(1 + \frac{im\kappa}{\hbar^2 k}\right) T \quad (10)$$

$$B = -\frac{im\kappa}{\hbar^2 k} e^{2ika} T \quad (11)$$

and eliminating R from (6) and (8) we find

$$\left(1 + \frac{im\kappa}{\hbar^2 k}\right) A + \frac{im\kappa}{\hbar^2 k} e^{2ika} B = 1 \quad (12)$$

and using (10) and (11) in this we have

$$\left(1 + \frac{im\kappa}{\hbar^2 k}\right)^2 T + \left(\frac{m\kappa}{\hbar^2 k}\right)^2 e^{4ika} T = 1 \quad (13)$$

or

$$T = \frac{1}{1 + \frac{2im\kappa}{\hbar^2 k} + \left(\frac{m\kappa}{\hbar^2 k}\right)^2 (e^{4ika} - 1)} \quad (14)$$

The probability of transmission is just the square of this:

$$|T|^2 = \frac{1}{1 + 4\left(\frac{m\kappa}{\hbar^2 k}\right)^2 \cos^2(2ka) + 4\left(\frac{m\kappa}{\hbar^2 k}\right)^3 \sin(4ka) + 4\left(\frac{m\kappa}{\hbar^2 k}\right)^4 \sin^2(2ka)} \quad (15)$$

This is a function of incident particle's energy E , with $k = k(E) = \sqrt{2mE}/\hbar$.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: Consider an electron gas with particle density n . Determine the numerical value of n for which the Fermi energy ϵ_F of the gas is equal to the rest energy, $m_e c^2$, of the electron. What is the corresponding value of the Fermi velocity v_F of the system?

SOLUTION: see hand written solution

Solution

If one uses the nonrelativistic formula for ϵ_F ,
then

$$\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{3n}{8\pi} \right)^{2/3} = mc^2,$$

so that

$$\begin{aligned} n &= \frac{8\pi}{3} \left(\frac{2m^2 c^2}{\hbar^2} \right)^{3/2} = \frac{16\sqrt{2}\pi}{3} \left(\frac{mc}{\hbar} \right)^3 \\ &= 1.659 \times 10^{36} \text{ m}^{-3}. \end{aligned}$$

However, v_F (using nonrelativistic formula
to be
 $\epsilon_F = \frac{1}{2} m v_F^2$) turns out $\sqrt{2} c$, which is clearly
wrong. So we must use relativistic formulae.

Using the relation

$$\epsilon = c \sqrt{p^2 + m^2 c^2} - mc^2,$$

we see that $\epsilon_F = mc^2$ implies that $p_F = \sqrt{3} mc$.

However, p_F is given by the general formula

$$p_F = \hbar \left(\frac{3n}{8\pi} \right)^{1/3}.$$

Equating this expression with $\sqrt{3} mc$, we get

$$n = 8\sqrt{3}\pi \left(\frac{mc}{h}\right)^3 = \underline{3.047 \times 10^{36} \text{ m}^{-3}} \quad \checkmark$$

For v_F , we get

$$\begin{aligned} v_F &= \frac{p_F}{m_{\text{tot}}} = \frac{p_F c^2}{E_{\text{total}}} = \frac{(\sqrt{3} mc) c^2}{2 mc^2} = \frac{\sqrt{3}}{2} c \\ &= 0.866 c \quad \checkmark \\ &= \underline{2.596 \times 10^8 \text{ m/s}} \end{aligned}$$

#8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: The latent heat of melting ice is L per unit mass. A bucket contains a mixture of water and ice, at the ice point (absolute temperature T_0). It is desired to use a cyclic refrigerator to freeze an additional mass m of water in the bucket. The refrigerator also rejects heat Q_R , which all goes into warming up a body of constant heat capacity C and, initially, also at temperature T_0 .

(a) What is the change in the entropies in the bucket, the refrigerator, and the body in this process?

(b) What is the minimum work required to run the refrigerator for this process?

SOLUTION:

(a) The heat removed from bucket is Lm , and the temperature stays at T_0 .

$$\Delta S_{\text{bucket}} = -Lm/T_0, \Delta S_{\text{fridge}} = 0,$$

$$\Delta S_{\text{body}} = \int dQ/T = C \ln(T_f/T_0) = C \ln(1 + Q_R/CT_0),$$

where $T_f = T_0 + Q_R/C$.

(b) Since $\Delta S_{\text{universe}} = \Delta S_{\text{bucket}} + \Delta S_{\text{fridge}} + \Delta S_{\text{body}} \geq 0$, we have

$$Q_R \geq CT_0 (\exp(Lm/CT_0) - 1).$$

Because $\Delta U_{\text{fridge}} = 0$, the work is $Q_R - Lm$. So the minimum is

$$W \geq CT_0 (\exp(Lm/CT_0) - 1) - Lm.$$

#9 : UNDERGRADUATE MATH METHODS

PROBLEM: Consider the differential equation

$$\alpha \frac{dX}{dt} + X = f(t), \quad \text{where} \quad f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{f}(\omega) e^{i\omega t},$$

where α is a positive, real constant.

(a) Find the general solution for $X(t)$ in terms of $\tilde{f}(\omega)$ and $X(t=0) = X_0$.

(b) Find the solution for the case $\tilde{f}(\omega) = Ae^{-i\omega t_1}$, where A and t_1 are real constants, and $t_1 > 0$. Evaluate the integral by contour integration, for all values of t . **Please draw** the appropriate closed contour for all values of t , noting that the appropriate contour differs depending on whether t is below or above a certain value. Also draw the location of all poles.

SOLUTION:

(a) $X_h(t) = X_0 e^{-t/\alpha}$.

(b) For $t < t_1$, close the contour in the lower half plane. For $t > t_1$ close the contour in the upper half plane. The only pole is at $\omega = i/\alpha$, in the upper half plane. So:

$$X_p(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \frac{2\pi A}{\alpha} e^{-(t-t_1)/\alpha} & \text{if } t > t_1 \end{cases}$$

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

PROBLEM: Consider a self-gravitating, uniform sphere of mass M , radius R , and temperature T radiating as a black body. Consider the possibility that the energy source for the radiation is supplied by the sphere's gravitational contraction (this was the prevailing theory about the energy source for the Sun and stars in the 19th century.)

- (a) Derive the formula for the time required for the sphere to radiate away all its gravitational potential energy. This is called the Kelvin-Helmholtz time.
- (b) Evaluate the Kelvin-Helmholtz time for the Sun. (assume the Sun's mass, radius, and surface temperature are $M_{\odot} = 2 \times 10^{30}$ kg, $R_{\odot} = 7 \times 10^8$ m, $T_{\odot} = 5800$ K).
- (c) Compare the Sun's Kelvin-Helmholtz time to the age of the oldest rocks on Earth (about 4 billion years old). Is gravitational contraction a viable energy source for the Sun? Briefly discuss.

SOLUTION:

- (a) Because of the virial theorem, the total mechanical energy of the Sun is

$$E = -1/2E_g$$

where E_g is the Sun's gravitational potential energy. Approximating the Sun as a uniform sphere,

$$E_g = -\frac{3}{5} \frac{GM_{\odot}^2}{R_{\odot}}$$

The Kelvin-Helmholtz time is the ratio of the Sun's mechanical energy E and its luminosity L :

$$t_{KH} = \frac{E}{L} = \frac{3}{10} \frac{GM_{\odot}^2}{R_{\odot}} / 4\pi R_{\odot}^2 \sigma T_{\odot}^4 = \frac{3GM_{\odot}^2}{40\pi R_{\odot}^3 \sigma T_{\odot}^4}$$

- (b) Inserting numbers, we find $t_{KH} \sim 10^7$ yr.

(c) No, Kelvin-Helmholtz contraction is not a viable energy source for the Sun, otherwise the Earth would be far older than the Sun, forcing a complete revision of our theory of the origin of the solar system.

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	11 12	13 14	15 16	17 18	19 20

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
 - a. Write the problem number and your ID number on each sheet;
 - b. Write only on one side of the paper;
 - c. Start each problem on the attached examination sheets;
 - d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 : GRADUATE CLASSICAL MECHANICS

PROBLEM: A system has the following Hamiltonian:

$$H(x, p_x, y, p_y) = \frac{1}{2}p_x^2 + xp_y^2 + xy^2.$$

Solve for $x(t)$ and $y(t)$, assuming $x(0) = y(0) = 0$.

(Hint 1: $\int^y \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}(\frac{y}{a})$.)

(Hint 2: use Hamilton-Jacobi theory)

SOLUTION: Using Hamilton-Jacobi theory this Hamiltonian is *separable*.

H-J equation is

$$\frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 + x \left[\left(\frac{\partial S}{\partial y} \right)^2 + y^2 \right] + \frac{\partial S}{\partial t} = 0$$

Take $S = -Et + W_x(x) + W_y(y)$.

Then W_y satisfies the ODE

$$\left(\frac{\partial W_y}{\partial y} \right)^2 + y^2 = \text{const} = p_{y0}^2 \quad \begin{array}{l} \swarrow \text{initial } p_y \\ \Rightarrow \underline{p_y^2 + y^2 = p_{y0}^2} \\ \searrow \\ p_y^2 \end{array}$$

$W_x(x)$ satisfies

$$E = \frac{1}{2} \left(\frac{\partial W_x}{\partial x} \right)^2 + x(p_y^2 + y^2) = \frac{1}{2}p_x^2 + xp_{y0}^2$$

Equations of motion: $\dot{x} = \frac{\partial H}{\partial p_x} = p_x$, $\dot{p}_x = -\frac{\partial H}{\partial x} = -p_{y0}^2$

$$\Rightarrow p_x = p_{x0} - p_{y0}^2 t$$

$$\boxed{\Rightarrow x(t) = p_{x0}t - p_{y0}^2 t^2 / 2}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = 2xp_y = 2x\sqrt{p_{y0}^2 - y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{p_{y0}^2 - y^2}} = 2x(t)dt$$

integrate:

$$\sin^{-1} \left(\frac{y}{p_{y0}} \right) = p_{x0}t^2 - p_{y0}^2 t^3/3$$

$$y(t) = p_{y0} \sin (p_{x0}t^2 - p_{y0}^2 t^3/3)$$

#12 : GRADUATE CLASSICAL MECHANICS

PROBLEM: Consider an infinite, one dimensional “diatomic” chain of oscillators shown below. Each mass is separated by a distance ℓ from its neighbor, by a spring of strength k . The masses m_1, m_2 alternate.

- (a) Show that this chain supports two collective modes for purely longitudinal oscillations. Calculate their dispersion relations.
- (b) Characterize the modes in the limit $\ell\alpha < 1$, where α is the wave number



SOLUTION:

see hand written solution

Solution - Mechanics/Grad

#12

a.) No loss of generality to associate

$$m_1 \rightarrow x_{2n}$$

$$m_2 \rightarrow x_{2n+1}$$

$$\text{so, per usual: } L = \sum_n \left[\frac{1}{2} (m_{2n} \dot{x}_{2n}^2 + m_{2n+1} \dot{x}_{2n+1}^2) \right.$$

$$\left. - \frac{1}{2} k (x_{2n-1} - x_{2n})^2 - \frac{1}{2} k (x_{2n} - x_{2n+1})^2 \right]$$

$$m_1 \ddot{x}_{2n} = -k(2x_{2n} - x_{2n-1} - x_{2n+1})$$

$$m_2 \ddot{x}_{2n+1} = -k(2x_{2n+1} - x_{2n} - x_{2n+2}).$$

Assume solution of form:

$$x_{2n} = A e^{i2n\ell\alpha} e^{-i\omega t}$$

$$x_{2n+1} = B e^{i(2n+1)\ell\alpha} e^{-i\omega t}$$

so

$$(-m_1 \omega^2 + 2k)A - k(2 \cos \ell\alpha)B = 0$$

$$-k(2 \cos \ell\alpha)A - B(-m_2 \omega^2 + k) = 0$$

$$\Rightarrow \omega^2 = k \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \pm k \left\{ \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^2 - \frac{4 \sin^2(\alpha\ell)}{m_1 m_2} \right\}^{1/2}$$

if $1/\mu = 1/m_1 + 1/m_2 \rightarrow$ reduced mass

then have dispersion relation:

$$\omega^2 = \frac{k}{\mu} \left\{ 1 \pm \left\{ 1 - \frac{4\mu^2}{m_1 m_2} \sin^2(\alpha\ell) \right\}^{1/2} \right\}$$

\Rightarrow 2 modes \pm

b.) Consider $\alpha\ell < 1 \Rightarrow$ continuum limit

$$+ \quad \omega^2 \cong 2k/\mu \quad \rightarrow \quad \text{"optical mode"}$$

$$m_1 x_n + m_2 x_{n+1} \cong 0$$

\rightarrow neighbors out of phase

$$- \quad \omega^2 \cong \frac{4k\mu}{m_1 m_2} \ell^2 \alpha^2 \rightarrow \text{"acoustic mode"}$$

\rightarrow neighbors in phase

i.e. $\omega^2 \sim k^2 c s^2$

$$c s^2_{\text{eff}} \sim \frac{4k\mu}{m_1 m_2} \ell^2, \text{ here.}$$

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: A steady state current density $\vec{J}(\vec{r})$ is non-zero only in the sphere $|\vec{r}| < R$. Let $\vec{B}(\vec{r})$ be the magnetic field induced by this current. Determine a numerical value for the flux through the plane $x = 0$; that is, determine the value of

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz B_x(x, y, z)$$

Prove your answer.

SOLUTION:

see hand written solution

solution

The flux is zero. To prove this consider the closed surface that consists of the plane $x=0$ and a hemisphere at $r \rightarrow \infty$. From

$\nabla \cdot \mathbf{B} = 0$, we find

$$0 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} B_x \, dx \, dz + \lim_{r \rightarrow \infty} \int_{\text{hemisphere}} B_r \, r^2 \, d\Omega$$

Because localized current distribution looks like dipole as $r \rightarrow \infty$

$$|B_r(r)| \propto \frac{1}{r^3} \text{ as } r \rightarrow \infty$$

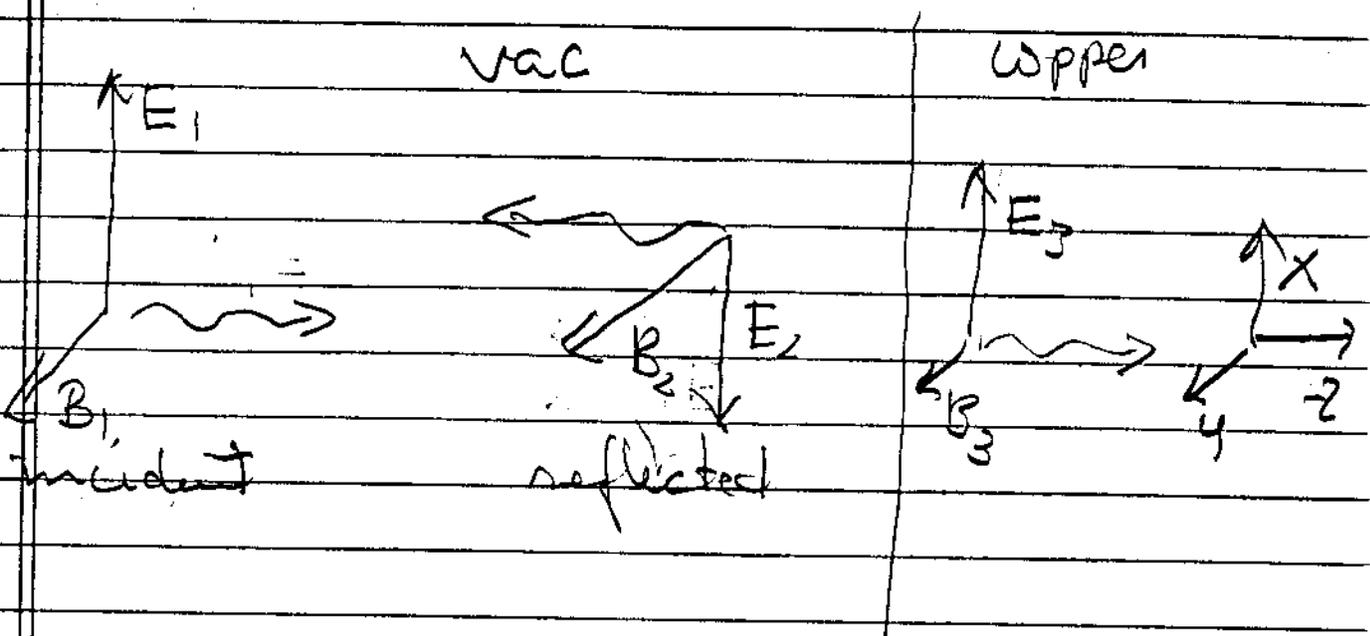
$$\therefore \lim_{r \rightarrow \infty} \int B_r \, r^2 \, d\Omega = 0$$

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: A plane electromagnetic wave of angular frequency $\omega = 10^{10} \text{ sec}^{-1}$ is normally incident on a planar slab of copper. Obtain an expression for the reflection coefficient of the wave (i.e., for \mathbf{R}) in terms of ω and the conductivity of copper σ . What is an approximate numerical value for $(1-\mathbf{R})$? (assume $\sigma = 5 \times 10^{17}$).

SOLUTION: see hand written solution

Solution (1)



$$\left. \begin{aligned} E_1 - E_2 &= E_3 \\ B_1 + B_2 &= B_3 \end{aligned} \right\} \text{b.c. at interface}$$

$\begin{matrix} E_1 & E_2 \\ B_1 & B_2 \end{matrix}$

In copper, $\sigma \gg \omega$

$$\nabla \times \underline{B}_3 \approx \frac{4\pi}{c} \underline{J}_3 = \frac{4\pi\sigma}{c} \underline{E}_3$$

$$\underbrace{\nabla \times \nabla \times \underline{B}_3}_{-\nabla^2 \underline{B}_3} = -\frac{4\pi\sigma}{c^2} \frac{\partial \underline{B}_3}{\partial t} \quad B_3 \sim e^{ik_3 z - i\omega t}$$

$$k_3^2 = \frac{i\omega 4\pi\sigma}{c^2} \quad k_3 = \sqrt{\frac{4\pi\sigma\omega}{c^2} \left(\frac{1+i}{\sqrt{2}} \right)}$$

(2)

$$\text{also } i k_3 \hat{z} \otimes \hat{y} B_3 = \frac{4\pi\sigma}{c} E_3 \hat{x}$$

$$B_3 = \frac{-4\pi\sigma}{i k_3 c} E_3$$

$$\begin{array}{l} B_1 + B_2 = B_3 = -\frac{4\pi\sigma}{i k_3 c} E_3 \\ \parallel \quad \parallel \\ E_1 \quad E_2 \end{array} \quad \underbrace{\hspace{10em}}_{(E_1 - E_2)}$$

$$E_1 \left(1 + \frac{4\pi\sigma}{i k_3 c}\right) = E_2 \left(\frac{4\pi\sigma}{i k_3 c} - 1\right)$$

$$\frac{|E_2|}{|E_1|} = \left| \frac{1 + \frac{4\pi\sigma}{i k_3 c}}{-1 + \frac{4\pi\sigma}{i k_3 c}} \right| = \left| \frac{\frac{i k_3 c}{4\pi\sigma} + 1}{1 - \frac{i k_3 c}{4\pi\sigma}} \right|$$

$$\frac{i k_3 c}{4\pi\sigma} = \frac{\omega}{\sqrt{4\pi\sigma}} \left(\frac{i-1}{\sqrt{2}}\right), \quad \frac{\omega}{\sigma} \gg 1$$

$$\left| \frac{E_2}{E_1} \right| \approx 1 - \frac{2}{\sqrt{2}} \frac{\omega}{\sqrt{4\pi\sigma}}$$

(3)

$$R = \left| \frac{F_2}{F_1} \right|^2 \approx 1 - \sqrt{\frac{2\omega}{\pi\sigma}}$$

estimate $\omega = 10^{10} \text{ sec}^{-1}$

$$\sigma \approx 5 \times 10^{17}$$

$$1 - R = \sqrt{\frac{2\omega}{\pi\sigma}} \approx \sqrt{\frac{2}{\pi} \frac{1}{5} 10^{-7}} \approx 10^{-4}$$

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: Consider a spin-1/2 particle with magnetic moment μ in a uniform, non-constant magnetic field $\vec{B} = \vec{B}(t)$. The particle wave function at time $t = 0$ is

$$\begin{pmatrix} e^{-i\alpha} \cos \delta \\ e^{i\alpha} \sin \delta \end{pmatrix} \quad (1)$$

Find the expectation value of the spin of the particle, \vec{s} , at a later time t .

SOLUTION: The time evolution of the spinor is given by Schrodinger equation,

$$i \frac{\partial \psi}{\partial t} = H(t) \psi(t) \quad (2)$$

The time dependent Hamiltonian is

$$H = -\vec{\mu} \cdot \vec{B}(t) \quad (3)$$

We choose a coordinate system with the z -axis along the direction of \vec{B} and use $\vec{\mu} = \mu \vec{s} = \frac{1}{2} \mu \vec{\sigma}$, where $\vec{\sigma}$ are the Pauli matrices. Then $H = -\frac{1}{2} \mu B(t) \sigma_z$ and recalling that $\sigma_z = \text{diag}(1, -1)$, we have

$$\frac{\partial \psi_{\pm}}{\partial t} = \pm \frac{i}{2} \mu B(t) \psi_{\pm}(t) \quad (4)$$

where ψ_+ (ψ_-) is the upper (lower) component of ψ . These are trivially integrated,

$$\psi_{\pm}(t) = e^{\pm \frac{i}{2} \mu \int_0^t B(t') dt'} \psi_{\pm}(0) \quad (5)$$

$$= \begin{pmatrix} e^{\frac{i}{2} \mu \int_0^t B(t') dt' - i\alpha} \cos \delta \\ e^{-\frac{i}{2} \mu \int_0^t B(t') dt' - i\alpha} \sin \delta \end{pmatrix} \quad (6)$$

Now we compute the expectation value

$$\langle \vec{s} \rangle = \langle \psi(t) | \frac{1}{2} \vec{\sigma} | \psi(t) \rangle \quad (7)$$

So we have

$$\langle s_z \rangle = \frac{1}{2} [\langle \psi_+(t) | \psi_+(t) \rangle - \langle \psi_-(t) | \psi_-(t) \rangle] \quad (8)$$

$$= \frac{1}{2} \cos(2\delta), \quad (9)$$

and using $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle s_x \rangle = \frac{1}{2} [\langle \psi_-(t) | \psi_+(t) \rangle + \langle \psi_+(t) | \psi_-(t) \rangle] \quad (10)$$

$$= \frac{1}{2} \sin(2\delta) \cos \left(\frac{1}{2} \mu \int_0^t B(t') dt' - \alpha \right). \quad (11)$$

Finally, with $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\langle s_y \rangle = \frac{1}{2} [i \langle \psi_-(t) | \psi_+(t) \rangle - i \langle \psi_+(t) | \psi_-(t) \rangle] \quad (12)$$

$$= -\frac{1}{2} \sin(2\delta) \sin \left(\frac{1}{2} \mu \int_0^t B(t') dt' - \alpha \right). \quad (13)$$

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: A half-silvered but perfectly plane mirror is set in the $z = 0$ plane. An incident beam of light will be transmitted and reflected with equal intensity by the mirror from either side of the mirror. The mirror has such perfect symmetry that a photon incident on one side of the mirror will have exactly the same transmission and reflection *amplitudes* as a photon incident in the mirror symmetric direction on the other side of the mirror.

(a) Find the transformation matrix (known as the scattering or S matrix) which relate the two incoming states of a photon with wave vector $\vec{k} = (k \sin \theta, 0, k \cos \theta)$ and its mirror image $\vec{k}_m = (k \sin \theta, 0, k \cos(\pi - \theta))$ to the two outgoing states of the same pair of vectors.

(b) If the incoming state of a photon is either a symmetric or an antisymmetric combination of the two incoming states described in part (a), find the outgoing states in both cases and, hence or otherwise, find the symmetric and antisymmetric phase shifts due to the scattering by the mirror.

(c) If two photons, one incident in the \vec{k} direction and the other in the \vec{k}_m direction, are timed to arrive at the same region of the mirror with maximum wave function overlap, find the probabilities of finding the outgoing photons in the all possible cases of n photons along \vec{k} , m photons along \vec{k}_m , namely $(n, m) = (2, 0), (1, 1), (0, 2)$.

SOLUTION:

(a) In terms of the transmission amplitude t and reflection amplitude r , the S matrix gives the transformation matrix converting the incoming waves denoted by (+) to the outgoing ones (-), by the given symmetry,

$$[|k(-)\rangle \quad |k_m(-)\rangle] = [|k(+)\rangle \quad |k_m(+)\rangle] \begin{bmatrix} t & r \\ r & t \end{bmatrix}.$$

Except for an overall phase factor of the S matrix, we can put.

$$t = \frac{1}{\sqrt{2}}, \quad r = \frac{1}{\sqrt{2}}e^{i\varphi}.$$

From the unitarity of the S matrix, $tr^* + rt^* = 0$ and, hence, $\varphi = \pi/2$. Thus,

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

(b) By the above basis states, the incoming state symmetric and antisymmetric states are transformed into,

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \pm \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{e^{\pm i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}.$$

Since the phase change is respectively, $2\delta_{\pm}$ for the symmetric and antisymmetric state, the phase shifts are,

$$\delta_{\pm} = \pm \frac{\pi}{8}.$$

(c) In terms of the photon creation operators, the S matrix transformation gives,

$$\begin{aligned} a_k^\dagger &\rightarrow \frac{1}{\sqrt{2}} (a_k^\dagger + ia_{k_m}^\dagger), \\ a_{k_m}^\dagger &\rightarrow \frac{1}{\sqrt{2}} (ia_k^\dagger + a_{k_m}^\dagger). \end{aligned}$$

For the two photon incoming state,

$$\begin{aligned} a_k^\dagger a_{k_m}^\dagger &\rightarrow \frac{1}{\sqrt{2}} (a_k^\dagger + ia_{k_m}^\dagger) \frac{1}{\sqrt{2}} (ia_k^\dagger + a_{k_m}^\dagger) \\ &= \frac{i}{2} (a_k^{\dagger 2} + a_{k_m}^{\dagger 2}) \\ &= \frac{i}{\sqrt{2}} (|n_k = 2, n_{k_m} = 0\rangle + |n_k = 0, n_{k_m} = 2\rangle). \end{aligned}$$

The probabilities for finding $(n, m) = (2, 0), (1, 1), (0, 2)$ are, respectively, $\frac{1}{2}, 0, \frac{1}{2}$.

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The potential energy of a one-dimensional, *anharmonic* oscillator is given by

$$V(q) = cq^2 - gq^3,$$

where c and g are positive constants. Using classical statistics and treating the anharmonic term as a perturbation, evaluate the leading contribution of this term to the heat capacity of the oscillator and to the mean value of the position coordinate q .

SOLUTION:

see hand written solution

Solution

The partition function of the anharmonic oscillator is given by

$$Q(\beta) = \frac{1}{h} \int_{-\infty}^{\infty} \int e^{-\beta H} dp dq \quad \left\{ H = \frac{p^2}{2m} + c q^2 - g q^3 \right\}.$$

The integration over p gives a factor of $\sqrt{2\pi m/\beta}$. For integration over q , we write

$$e^{-\beta c q^2 + \beta g q^3} = e^{-\beta c q^2} \left[1 + \beta g q^3 + \frac{1}{2} \beta^2 g^2 q^6 + \dots \right];$$

the integration then gives

$$\sqrt{\frac{\pi}{\beta c}} + 0 + \frac{1}{2} \beta^2 g^2 \cdot \frac{15}{8} \sqrt{\frac{\pi}{\beta^3 c^7}} + \dots$$

It follows that

$$Q(\beta) = \frac{1}{h} \sqrt{\frac{2\pi m}{\beta}} \sqrt{\frac{\pi}{\beta c}} \left[1 + \frac{15}{16} \frac{g^2}{\beta c^3} + \dots \right],$$

so that

$$\ln Q(\beta) = \text{const.} - \ln \beta + \frac{15}{16} \frac{g^2}{\beta c^3} + \dots,$$

whence

$$\begin{aligned} U &= -\frac{\partial}{\partial \beta} \ln Q = \frac{1}{\beta} + \frac{15}{16} \frac{g^2}{\beta^2 c^3} + \dots \\ &= kT + \frac{15}{16} \frac{g^2 k^2 T^2}{c^3} + \dots \end{aligned}$$

and

$$C = \frac{\partial U}{\partial T} = k + \frac{15}{8} \frac{g^2 k^2 T}{c^3} + \dots$$



Next, the mean value of the position coordinate q is given by

$$\langle q \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q \cdot e^{-\beta H} dp dq}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta H} dp dq}.$$

The integrations over p cancel out and we are left with

$$\begin{aligned} \langle q \rangle &= \frac{\int_{-\infty}^{\infty} q \cdot e^{-\beta c q^2} [1 + \beta g q^3 + \dots] dq}{\int_{-\infty}^{\infty} e^{-\beta c q^2} [1 + \beta g q^3 + \dots] dq} \\ &= \frac{0 + \beta g \cdot \frac{3}{4} \sqrt{\pi/\beta^5 c^5} + \dots}{\sqrt{\pi/\beta c} + 0 + \dots} \\ &\approx \frac{3}{4} \frac{g}{\beta c^2} = \frac{3}{4} \frac{g k T}{c^2}. \end{aligned}$$

#18 : GRADUATE STATISTICAL MECHANICS

PROBLEM: A classical N -particle system has the following density of states,

$$\Omega(E, N) \propto \exp \left\{ N \left[\frac{\epsilon}{2} + \epsilon^2 - \frac{\epsilon^3}{3} \right] \right\}$$

where $\epsilon \equiv E/N \geq 0$ is the energy per particle expressed in dimensionless energy unit. You can assume the particles to be distinguishable and $N \gg 1$.

(a) What is the entropy of the system $S(E, N)$? At a finite temperature T , show that the Helmholtz free energy per particle $f(T) \equiv F(T, N)/N$ can be obtained by minimizing a function $\Psi(\epsilon; T)$ for $N \rightarrow \infty$.

(b) How can the average thermal energy per particle, $\bar{\epsilon}(T)$ be obtained from $\Psi(\epsilon; T)$? Sketch $\Psi(\epsilon; T)$ first for $T_2 \equiv 2/k_B$ and then for $T_1 \equiv 1/k_B$, where k_B is Boltzmann's constant. Indicate on your graph the location of $\bar{\epsilon}$ at T_1 and T_2 .

Hint: At T_1 you should decide between two possibilities, one with $\psi > 0$ and another with $\psi < 0$.

(c) Sketch $\Psi(\epsilon; T)$ as T is reduced further from T_1 . Show that a phase transition occurs at temperature $0 < T_c < T_1$ and find the value of T_c . What is the order of the phase transition?

SOLUTION:

(a) The entropy of the system is given at large N by

$$S(E, N) = k_B \ln \Omega(E, N) = k_B N \cdot \left(\frac{\epsilon}{2} + \epsilon^2 - \frac{\epsilon^3}{3} \right).$$

For large N , the Helmholtz free energy is given by $F(T, N) = \min_E [E - T \cdot S(E, N)]$.

$$\begin{aligned} \therefore f(T, N) &\equiv \frac{F(T, N)}{N} \\ &= \min_{\epsilon} \left[\epsilon - k_B T \cdot \left(\frac{\epsilon}{2} + \epsilon^2 - \frac{\epsilon^3}{3} \right) \right] \\ &= k_B T \cdot \min_{\epsilon} \left[x\epsilon - \epsilon^2 + \frac{\epsilon^3}{3} \right] \end{aligned}$$

where $x \equiv \frac{1}{k_B T} - \frac{1}{2}$. Hence

$$\Psi = x\epsilon - \epsilon^2 + \frac{\epsilon^3}{3}.$$

(b) The average thermal energy \bar{E} is the energy for which $E - T \cdot S(E, N)$ is minimized. Hence $\bar{\epsilon}$ is the value of ϵ for which $\psi(\epsilon; T)$ is minimized.

At $T = T_2$, $x = 0$ and $\Psi(\epsilon; T_2) = -\epsilon^2 + \frac{\epsilon^3}{3}$. Ψ has a single minimum at $\epsilon = 2$. So $\bar{\epsilon}(T_2) = 2$.

At $T = T_1$, $x = \frac{1}{2}$ and $\Psi(\epsilon; T_1) = \frac{\epsilon}{2} - \epsilon^2 + \frac{\epsilon^3}{3}$.

Since $\left. \frac{\partial \Psi}{\partial \epsilon} \right|_{\epsilon=\bar{\epsilon}} = \frac{1}{2} - 2\bar{\epsilon} + \bar{\epsilon}^2 = 0$, we have two possibilities.

$$\bar{\epsilon}(T_1) = 1 \pm \sqrt{2}/2.$$

We next evaluate $\Psi(\bar{\epsilon}(T_1); T_1)$ and find which of the above two roots give a smaller value of ψ . Substituting $\bar{\epsilon}^2 = 2\bar{\epsilon} - \frac{1}{2}$ into $\Psi(\bar{\epsilon}; T_1)$, we find

$$\begin{aligned} \Psi(\bar{\epsilon}; T_1) &= \bar{\epsilon} \cdot \left(\frac{1}{2} - \bar{\epsilon} + \frac{1}{3}\bar{\epsilon}^2 \right) \\ &= \bar{\epsilon} \cdot \left(\frac{1}{2} - \bar{\epsilon} + \frac{1}{3}(2\bar{\epsilon} - \frac{1}{2}) \right) \\ &= \bar{\epsilon} \cdot \frac{1 - \bar{\epsilon}}{3} \end{aligned}$$

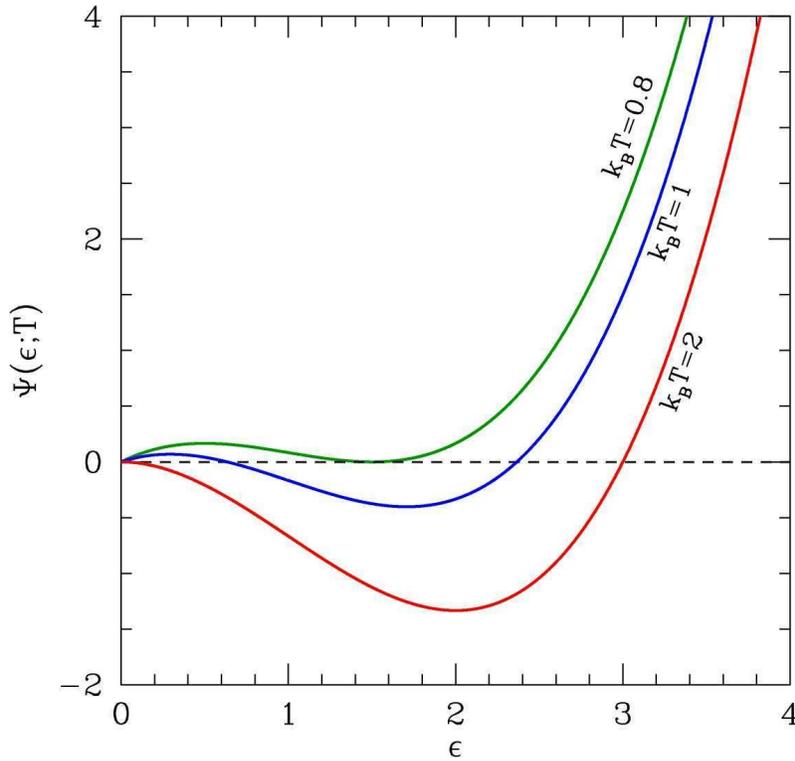
Using the solutions for $\bar{\epsilon}(T_1)$ obtained above, we have

$$\Psi(\bar{\epsilon}; T_1) = \frac{\bar{\epsilon}}{3} \cdot \left(\mp \frac{\sqrt{2}}{2} \right).$$

Since $\bar{\epsilon} > 0$, then the upper sign corresponds to the solution with $\Psi < 0$ and is the absolute minimum. Hence,

$$\bar{\epsilon}(T_1) = 1 + \frac{\sqrt{2}}{2}.$$

The other solution corresponds to a local maximum in $\Psi(\epsilon; T_1)$; see Figure.



(c) As temperature is further reduced, x increases and the value of $\Psi(\bar{\epsilon})$ will increase. At some temperature, $\Psi(\bar{\epsilon})$ crosses zero. This is the critical temperature T_c since for $T < T_c$, the value at the minimum of Ψ exceeds 0 and the true minimum moves to $\bar{\epsilon}(T) = 0$; see Figure.

Analytically, the critical point can be obtained by demanding that

$$\left. \frac{\partial \Psi}{\partial \epsilon} \right|_{\epsilon = \bar{\epsilon}(T_c)} = x(T_c) - 2\bar{\epsilon} + \bar{\epsilon}^2 = 0$$

and

$$\Psi(\bar{\epsilon}(T_c)) = \bar{\epsilon} \cdot \left(x(T_c) - \bar{\epsilon} + \frac{\bar{\epsilon}^2}{3} \right) = 0$$

Inserting the solution of the first equation, $x(T_c) = 2\bar{\epsilon} - \bar{\epsilon}^2$ into the second equation, we find

$$\bar{\epsilon}(T_c) = \frac{3}{2}, \quad \text{and} \quad x(T_c) = \frac{3}{4}.$$

Using the definition of x , we find

$$T_c = \frac{4}{5k_B}.$$

Since $\bar{\epsilon}(T)$ drops discontinuously to 0 for $T < T_c$, this is a first-order phase transition.

#19 : GRADUATE MATHEMATICAL PHYSICS

PROBLEM: Using contour integration, evaluate the integrals

$$\int_0^{\infty} dx x^{\alpha-1} \cos x$$

and

$$\int_0^{\infty} dx x^{\alpha-1} \sin x$$

where $0 < \alpha < 1$.

SOLUTION:

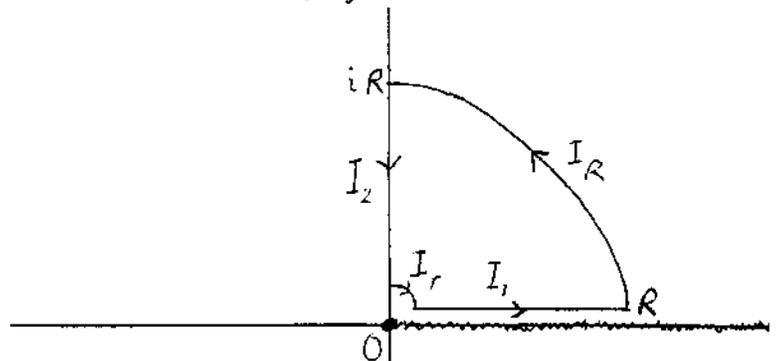
see hand written solution

Solution

We combine the two integrals into one, viz.

$$I = \int_0^{\infty} x^{\alpha-1} e^{-ix} dx, \text{ and evaluate it via } I_C = \oint_C z^{\alpha-1} e^{iz} dz,$$

where the contour C is as shown in the figure.



As $r \rightarrow 0$ and $R \rightarrow \infty$,

$$I_1 \rightarrow I, \text{ while } I_2 = \int_R^0 (r e^{i\pi/2})^{\alpha-1} e^{i \cdot ir} (e^{i\pi/2} dr) \\ = -e^{i\pi\alpha/2} \int_0^R r^{\alpha-1} e^{-r} dr \rightarrow -e^{i\pi\alpha/2} \Gamma(\alpha), \text{ for } \alpha > 0.$$

$I_R \rightarrow 0$ by virtue of Jordan's lemma and the fact that $\alpha < 1$.

I_r , being $O(r^\alpha)$, $\rightarrow 0$ because $\alpha > 0$.

With no poles enclosed, $I_C = 0$; therefore,

$$I_1 = -I_2 = e^{i\pi\alpha/2} \Gamma(\alpha).$$

Resolving into Re & Im parts, we get

$$\int_0^{\infty} x^{\alpha-1} \cos x dx = \Gamma(\alpha) \cos \frac{\pi\alpha}{2}, \text{ and}$$

$$\int_0^{\infty} x^{\alpha-1} \sin x dx = \Gamma(\alpha) \sin \frac{\pi\alpha}{2}.$$

#20 : GRADUATE OTHER

PROBLEM: A sphere of mass M , radius R is dropped into a fluid of density ρ_0 , and sinks. Assuming the fluid has kinematic viscosity ν , determine the parameter scaling of the sphere's terminal velocity. You can assume the flow around the sphere is laminar.

- (a) Use dimensional analysis to determine the drag force on the sphere, for laminar flow. "Laminar flow" suggests that the drag force should be proportional to friction and velocity.
- (b) Use your result from (a) to estimate the terminal velocity of the sphere.

SOLUTION:

see hand written solution

Solution - Graduate General

a.) For drag force, with sphere velocity U :

$$F_{drag} \sim \rho_0^\alpha v^\beta R^\gamma U^\delta$$

for dimensions

$$[\rho_0] \sim M/L^3$$

$$[v] \sim L^2/T$$

$$[R] \sim L$$

$$[U] \sim L/T$$

$$[F] \sim ML/T^2$$

so: 5 variables
3 independent dimensions

So, by Buckingham's *Pi* theorem, expect 2 dimensionless combinations, which are functions of one another.

Simplest:

$$F_d / \rho_0 R^2 U^2 \sim f(R_e), \text{ where } R_e \sim uR/v$$

$$F_d / \rho_0 R^2 U^2 \sim f(R_e)$$

Since $F_d \sim v$, $F_d \sim u$, take $f(R_e) \sim 1/R_e$ so

$$F_d \sim \rho_0 vUR \rightarrow \text{scaling of Stokes drag!}$$

General Graduate

b.) *net body force*

Body Force = Buoyancy - Gravity - Drag

$$= \rho_0 Vg - \rho_s Vg - \rho_0 vRU$$

$$V = \frac{4\pi R^3}{3}$$

$$\rho_s = M/V$$

so: $U = (1 - \rho_s/\rho_0)Vg/vR \hat{z}$

terminal velocity.

Obviously $\rho_s > \rho_0$ for sinking.