INSTRUCTIONS
PART I : PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1	§2	§3	§4	§5
Problems: (Circle your seven choices)	**1 2**	**3 4**	**5 6**	**7 8**	**9 10**

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PHYSICAL CONSTANTS

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Boltzmann's constant	$k = 1.381 \times 10^{-23}$ JK^{-1}
	$= 8.617 \times 10^{-5}$ eVK^{-1}
Avogadro's number	$N_A = 6.022 \times 10^{26}$ kilomole^{-1}
Volume of one kilomole of gas at STP	22.42 m^3(STP = 0°C, 1atm)
Standard atmosphere	1.013×10^5 Pa
Mechanical equivalent of heat	4184 J kilocalorie^{-1}
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Atomic mass unit	1 amu = 1.661×10^{-27} kg
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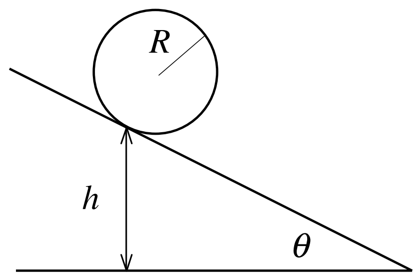
Stefan constant	$\sigma = 5.670 \times 10^{-8}$ Wm^{-2} K^{-4}

#1 : UNDERGRADUATE CLASSICAL MECHANICS

`PROBLEM:` As shown in the figure, a uniform thin rod of weight W is supported horizontally by two supports, one at each end. At $t = 0$, one of these supports is removed. Find the force on the remaining support immediately thereafter.

#2 : UNDERGRADUATE CLASSICAL MECHANICS

`PROBLEM:` A circular hoop of mass M and radius R rolls down a plane inclined at an angle θ in the Earth's gravitational field. For simplicity, assume the hoop to be infinitely thin (i.e., all the mass is at distance R from the geometric center of the hoop), and rolls without slipping.

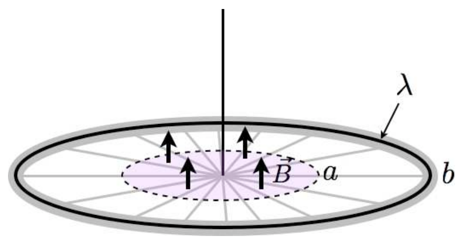


Answer the following questions if the hoop starts at rest at height h as shown in the figure:

`(a)` how fast is it moving when it reaches the bottom of the ramp?
`(b)` what is the hoops angular velocity at the bottom of the ramp?
`(c)` how long does it take to reach the bottom of the ramp?

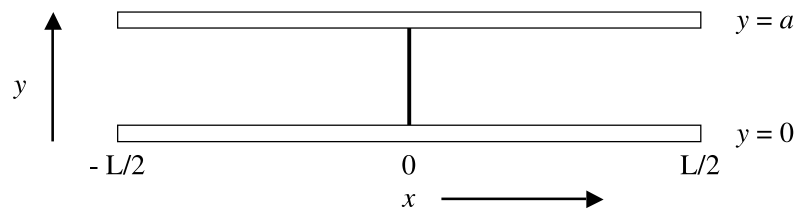
#3 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` The rim of a wheel of radius b is charged with a linear charge density λ. The wheel is suspended horizontally and is free to rotate. The spokes are made of some non-conducting material. In the central region out to a radius $a < b$ is a uniform magnetic field B pointing up; see Figure. Explain qualitatively what happens to the wheel when somebody turns the B-field off, and compute the resulting angular momentum given to the wheel.



#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: As shown in the figure, two parallel conducting plates of dimension $L \times L$ are separated by a distance $a \ll L \to \infty$ and are at electrical potential $V = 0$. A thin charged membrane of height a and length L is inserted perpendicular to the plates at $x = 0$. The potential on this membrane is $V(0, y) = V_0 \sin(\pi y/a)$. The plates and the membrane extend a distance L in the direction perpendicular to the plane of the figure.



(a) Find the electrical potential, $V(x, y)$, in the region between the plates to the right of the membrane (i.e., for $x > 0$). (You may ignore values of $x \geq L/2$.)

(b) Find the sign and magnitude of the charge density, $\sigma(x)$, on the conducting plates at $y = 0$ and $y = a$ to the right of the membrane, $x > 0$.

(c) Find the magnitudes and directions of the forces on the entire upper and lower plates.

#5 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a quantum mechanical system which only has two available states, i.e. the ket $|\psi\rangle$ is a vector in a 2-dimensional, complex vector space. This space has a complete, orthonormal basis of kets $|e_i\rangle$, for $i = 1, 2$. The Hamiltonian of this system is $H = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2|$, where $E_1 < E_2$. There is another observable, B, with $B = b|e_1\rangle\langle e_2| + b|e_2\rangle\langle e_1|$, where b is a positive real number.

The following experiments are performed in sequence:

- At time $t_1 = 0$, the observable B is measured.
- At time t_2 (with $t_2 > t_1$), the energy is measured.
- At time t_3 (with $t_3 > t_2$), the observable B is measured.
- At time t_4 (with $t_4 > t_3$), the observable B is measured again.

Answer the following questions about the outcomes of these experiments:

`(a)` Suppose that the outcome of experiment at time t_1 is the larger of the two possible values. Write the wavefunction for general time t in the range $t_1 < t < t_2$.

`(b)` Suppose that at time t_2 the larger possible energy is measured. Write the wavefunction for general time t in the range $t_2 < t < t_3$.

`(c)` Suppose that at time t_3 the *larger* possible value of the observable B is measured. For what values (list all of them) of $\Delta t \equiv t_4 - t_3$ is there 100% probability that the *smaller* value of B will be measured at time t_4?

#6 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Find the tightest upper bound on the ground state energy of the one-dimensional harmonic oscillator by using a trial wave function of the form

$$\psi(x) = \frac{D}{x^2 + a^2}$$

where D is to be determined by normalization and a is an adjustable parameter.

(You may express your answers in terms of dimensionless integrals.)

#7 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The internal energy of a non-relativistic Fermi gas at low temperatures is given by the expression

$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right],$$

where ε_F is the Fermi energy of the gas. Note that $\varepsilon_F \propto (N/V)^{2/3}$.

`(a)` Using the expression for U (and your knowledge of Thermodynamics), derive the corresponding expressions for the pressure P and the Helmholtz free energy F of the gas.

`(b)` Using these expressions for P and F, show that the corresponding expression for the chemical potential μ of the gas is

$$\mu = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

#8 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The “surface waves” in a low-temperature Bose liquid, upon quantization, behave like a two-dimensional gas of non-interacting excitations called “ripplons”. Like potons or phonons, these excitations are *indefinite* in number and obey Bose-Einstein statistics. Their energy-momentum relation, however, is $\varepsilon = a \cdot p^{3/2}$, where a is a constant.

Find the temperature dependence of the total energy per unit area of these excitations.

#9 : UNDERGRADUATE PHYSICAL ESTIMATES

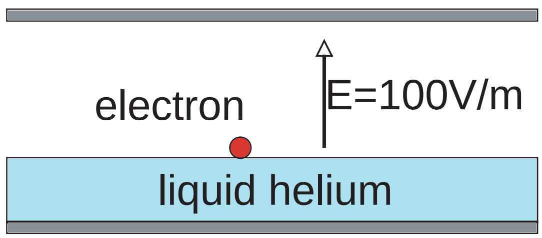
PROBLEM:

(a) The Sun may be regarded as a black-body radiator, of radius $R_\odot \approx 7 \times 10^8$ m, at a temperature $6,000\ K$. Calculate the "solar radiative flux" per unit area per unit time as observed on the surface of the Earth. The distance between the Sun and the Earth is $D = 1.5 \times 10^{11}$ m.

(b) Next, suppose that the Earth maintains itself in a steady state by continually radiating away into space as much energy as it receives from the Sun. If the Earth too were regarded a blackbody radiator, what would its steady-state temperature be?

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

`PROBLEM:` As shown in the figure, a capacitor has a thin layer of liquid helium on one of the metal plates. The electric field in the space between the liquid layer and the other plate is $E = 100$ volts/meter, pointing upwards. An electron is trapped on the upper liquid surface by the electric field. The helium surface may be regarded as impenetrable by the electron. Estimate the order of magnitude of the uncertainty in the vertical position of the electron in Angstroms without explicitly solving the Schrödinger equation. [You may ignore any possible image charge effects.]



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`SOLUTION:`

Downward acceleration $\ddot{x}$ of the center of mass of the rod is given by

$$m\ddot{x} = W - F,$$

where F is the force on the support. The angular momentum equation gives

$$W\frac{L}{2} = I\ddot{\theta},$$

where $I = \frac{1}{3}mL^2$ is the moment of inertia of the rod about an end, and L is the length of the rod.

For small θ, i.e., short times, $\ddot{x} = \frac{L}{2}\ddot{\theta}$. Combining these equations, we find

$$F = W - m\ddot{x} = W - m\tfrac{L}{2}\ddot{\theta} = W - m\frac{L}{2}\frac{WL}{2I} = W - m\frac{L}{2}\frac{3WL}{2mL^2} = \frac{W}{4}.$$

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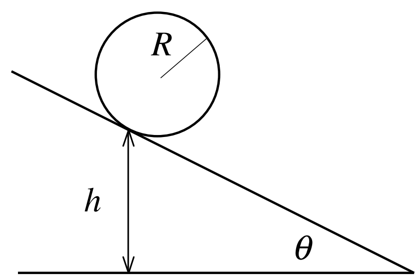


Answer the following questions if the hoop starts at rest at height h as shown in the figure:

(a) how fast is it moving when it reaches the bottom of the ramp?
(b) what is the hoops angular velocity at the bottom of the ramp?
(c) how long does it take to reach the bottom of the ramp?

SOLUTION:

(a) By conservation of energy, we have

$$T + U = T_{kin} + T_{rot} + U = const.$$
$$T_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}MR^2\omega^2, T_{kin} = \frac{1}{2}MV^2$$

Rolling without slipping implies $V = \omega R$. Hence,

$$T = MV^2, \quad U = Mgh$$
$$\therefore V = \sqrt{gh}$$

(b) $\omega = V/R = \sqrt{gh}/R$

(c) Let x be the vertical distance hoop has dropped measured from its starting elevation, and s be the distance the hoop has rolled along the inclined plane. From above, we have

$$V(x) = \sqrt{gx} = \frac{ds}{dt} = \frac{1}{\sin\theta}\frac{dx}{dt}$$

Solving for dx as a function of dt:

$$x^{-1/2}dx = \sqrt{g}\sin\theta \cdot dt$$

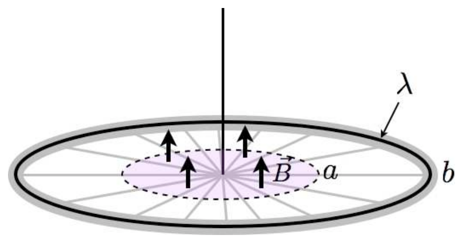
$$\int_0^h 2d(x^{1/2}) = 2h^{1/2} = \sqrt{g}\sin\theta \int_0^t dt = \sqrt{g}\sin\theta \cdot t$$

$$\therefore t = 2\sqrt{\frac{h}{g}} \Big/ \sin\theta$$

Notice that for $\theta = 90°$, this is twice the time for a point mass to fall a distance h in the Earth's gravitational field.

#3 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` The rim of a wheel of radius b is charged with a linear charge density λ. The wheel is suspended horizontally and is free to rotate. The spokes are made of some non-conducting material. In the central region out to a radius $a < b$ is a uniform magnetic field B pointing up; see Figure. Explain qualitatively what happens to the wheel when somebody turns the B-field off, and compute the resulting angular momentum given to the wheel.



`SOLUTION:` Qualitatively, the changing magnetic field will induce an electric field, curling around the axis of the wheel. This will exert a force on the charges on the rim, and the wheel starts spinning. According to Lenz's law, it will spin in the direction to maintain the upward flux. The rotation is thus counterclockwise when viewed from above.

Quantitatively, Faraday's law allows us to relate the E field to the change in flux as follows:

$$\oint E dl = -\frac{d\phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

The torque, $\vec{T}$, on a line segment $d\vec{l}$ is given by $d\vec{T} = \vec{r} \times \vec{F} = b\lambda E dl \hat{\boldsymbol{z}}$. The total torque is thus calculated as

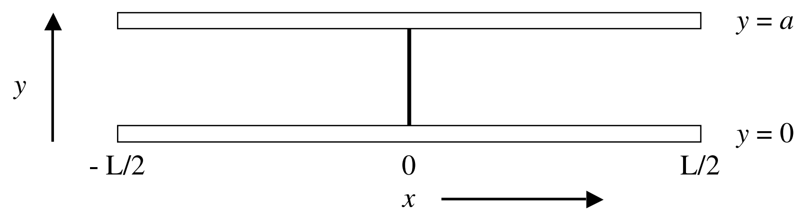
$$|\vec{T}| = b\lambda \oint E\, dl = -b\lambda\pi a^2 \frac{dB}{dt}$$

From this, we can find the total angular momentum L the wheel reaches:

$$L = \int |\vec{T}| dt = -b\lambda\pi a^2 \int_B^0 \frac{dB}{dt} dt = b\lambda\pi a^2 B$$

#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: As shown in the figure, two parallel conducting plates of dimension $L \times L$ are separated by a distance $a \ll L \to \infty$ and are at electrical potential $V = 0$. A thin charged membrane of height a and length L is inserted perpendicular to the plates at $x = 0$. The potential on this membrane is $V(0, y) = V_0 \sin(\pi y/a)$. The plates and the membrane extend a distance L in the direction perpendicular to the plane of the figure.



(a) Find the electrical potential, $V(x, y)$, in the region between the plates to the right of the membrane (i.e., for $x > 0$). (You may ignore values of $x \geq L/2$.)

(b) Find the sign and magnitude of the charge density, $\sigma(x)$, on the conducting plates at $y = 0$ and $y = a$ to the right of the membrane, $x > 0$.

(c) Find the magnitudes and directions of the forces on the entire upper and lower plates.

SOLUTION:

(a) The electric potential V satisfies the Laplace equation, $\nabla^2 V = 0$. Given the boundary conditions

$$V(x, y = 0) = 0 = V(x, y = a), \qquad \text{and} \qquad V(x = 0, y) = V_0,$$

the solution is of the form

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{ikx}.$$

Inserting this solution into the Laplace equation, we have

$$-\left(\frac{\pi}{a}\right)^2 - k^2 = 0,$$

or $k = \pm i\pi/a$. Thus, the solution (for $x \geq 0$) is

$$V(x, y) = V_0 \sin\left(\frac{\pi y}{a}\right) e^{-\pi x/a}.$$

(We can ignore $x \geq L/2$ since $e^{-piL/2a} \ll 1$ for $L/a \gg 1$.)

(b) To find the charge density σ at the surface of the conductors, we need the electric field $\vec{E}$ at the surface. The latter can be obtained from the potential $V(x, y)$ as

$$\vec{E} = -\vec{\nabla} V = \frac{\pi V_0}{a} \left[\sin\left(\frac{\pi y}{a}\right) \hat{\boldsymbol{x}} - \cos\left(\frac{\pi y}{a}\right) \hat{\boldsymbol{y}} \right] e^{-\pi x/a}.$$

At the surfaces of the conducting plates at $y = 0$ and $y = a$, the induced charge densities are the same, with

$$\sigma(x, y = 0) = \sigma(x, y = a) = \epsilon_0 \vec{E} \cdot \hat{\boldsymbol{n}} = -\frac{\epsilon_0 \pi V_0}{a} e^{-\pi x/a}, \qquad x \geq 0$$

for both plates.

(c) Force exerted on a conductor is given by

$$\vec{F} = \int \sigma \, \vec{E}_{\text{ext}} \, dA,$$

integrated over the surface area of the conductor, with $E_{\text{ext}} = E_{\text{self}} = E/2$.

On the upper plate (and $x \geq 0$),

$$\begin{aligned}
\vec{F} &= L \int_0^{L/2 \to \infty} dx \, \sigma(x, y = a) \cdot \frac{1}{2} \vec{E}(x, y = a) \\
&= -\frac{\epsilon_0 \pi^2 V_0^2 L}{2a^2} \left[\int_0^\infty e^{-2\pi x/a} dx \right] \hat{\boldsymbol{y}} \\
&= -\frac{\pi}{4} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}}
\end{aligned}$$

Including also the part from $x \leq 0$, the total force exerted on the top plate is

$$\vec{F}_{\text{upper}} = -\frac{\pi}{2} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}},$$

i.e., the top plate is attracted towards the lower plate.

By symmetry, the lower plate is attracted towards the upper plate with force of the same magnitude, i.e.,

$$\vec{F}_{\text{lower}} = +\frac{\pi}{2} \epsilon_0 V_0^2 L \, \hat{\boldsymbol{y}}.$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Consider a quantum mechanical system which only has two available states, i.e. the ket $|\psi\rangle$ is a vector in a 2-dimensional, complex vector space. This space has a complete, orthonormal basis of kets $|e_i\rangle$, for $i = 1, 2$. The Hamiltonian of this system is $H = E_1|e_1\rangle\langle e_1| + E_2|e_2\rangle\langle e_2|$, where $E_1 < E_2$. There is another observable, B, with $B = b|e_1\rangle\langle e_2| + b|e_2\rangle\langle e_1|$, where b is a positive real number.

The following experiments are performed in sequence:

- At time $t_1 = 0$, the observable B is measured.
- At time t_2 (with $t_2 > t_1$), the energy is measured.
- At time t_3 (with $t_3 > t_2$), the observable B is measured.
- At time t_4 (with $t_4 > t_3$), the observable B is measured again.

Answer the following questions about the outcomes of these experiments:

(a) Suppose that the outcome of experiment at time t_1 is the larger of the two possible values. Write the wavefunction for general time t in the range $t_1 < t < t_2$.

(b) Suppose that at time t_2 the larger possible energy is measured. Write the wavefunction for general time t in the range $t_2 < t < t_3$.

(c) Suppose that at time t_3 the *larger* possible value of the observable B is measured. For what values (list all of them) of $\Delta t \equiv t_4 - t_3$ is there 100% probability that the *smaller* value of B will be measured at time t_4?

SOLUTION:

(a) First solve for the eigenvalues and eigenvectors of the operator B:

$$\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} \cdot \begin{pmatrix} |e_1\rangle \\ |e_2\rangle \end{pmatrix} = \lambda \cdot \begin{pmatrix} |e_1\rangle \\ |e_2\rangle \end{pmatrix}.$$

The eigenvalues are $\lambda = \pm b$, and the eigenvectors are

$$\begin{aligned} |+b\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle) \\ |-b\rangle &= \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle). \end{aligned}$$

Since the eigenvalue $+b$ is measured at time t_1, we have

$$|\psi(t_1)\rangle = |+b\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle).$$

Then the time evolution of the wavefunction for $t_1 < t < t_2$ is given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-iE_1 t/\hbar}|e_1\rangle + e^{-iE_2 t/\hbar}|e_2\rangle).$$

(b) $|\psi(t_2)\rangle = |e_2\rangle$ since the energy E_2 is measured at time t_2. The time evolution afterwards is then simply

$$|\psi(t)\rangle = e^{-iE_2 t/\hbar}|e_2\rangle.$$

(where the overall phase is immaterial).

(c) Much as in part (a), the wavefunction for $t_3 < t < t_4$ is

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(e^{-iE_1(t-t_3)/\hbar}|e_1\rangle + e^{-iE_2(t-t_3)/\hbar}|e_2\rangle\right) \\ &= \frac{1}{\sqrt{2}}e^{-iE_1(t-t_3)/\hbar}\left(|e_1\rangle + e^{-i(E_2-E_1)(t-t_3)/\hbar}|e_2\rangle\right) \end{aligned}$$

Clear, for $|\psi(t = t_4)\rangle \sim |-b\rangle$, we must have $e^{-i\Delta t(E_2-E_1)/\hbar} = -1$, i.e. for $\Delta t = \pi\hbar(2n+1)/(E_2 - E_1)$, for integer n.

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`PROBLEM:` Find the tightest upper bound on the ground state energy of the one-dimensional harmonic oscillator by using a trial wave function of the form

$$\psi(x) = \frac{D}{x^2 + a^2}$$

where D is to be determined by normalization and a is an adjustable parameter.

(You may express your answers in terms of dimensionless integrals.)

`SOLUTION:`

To solve this problem, we use the variational principle that the ground state energy is given by

$$E_0 \leq \int_{-\infty}^{\infty} \psi^*(x)\hat{\mathcal{H}}\,\psi(x)$$

for the trial wave function $\psi(x)$ given.

First, fix $D(a)$ by normalization:

$$\begin{aligned}
\int_{-\infty}^{\infty} dx|\psi(x)|^2 = \int_{-\infty}^{\infty} dx\frac{D^2}{(x^2+a^2)^2} &= 1 \\
\frac{D^2}{a^3}\int_{-\infty}^{\infty} dy\frac{1}{(1+y^2)^2} &= 1.
\end{aligned}$$

$$\Rightarrow \qquad D^2 = a^3/I_1, \quad \text{where} \quad I_1 = \int_{-\infty}^{\infty} dy\frac{1}{(1+y^2)^2}.$$

Next, compute the expectation value of $\hat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2$ using ψ:

$$\begin{aligned}
E_0 &\leq D^2(a)\left\{\int_{-\infty}^{\infty} dx\frac{1}{x^2+a^2}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\frac{1}{x^2+a^2}\right) + \int_{-\infty}^{\infty} dx\frac{\frac{1}{2}kx^2}{(x^2+a^2)^2}\right\} \\
&= D^2(a)\left\{\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} dx\left(\frac{-2x}{(x^2+a^2)^2}\right)^2 + \frac{k}{2}\int_{-\infty}^{\infty} dx\frac{x^2}{(x^2+a^2)^2}\right\} \\
&= D^2(a)\left\{\frac{\hbar^2}{2ma^5}\int_{-\infty}^{\infty} dy\frac{4y^2}{(1+y^2)^4} + \frac{k}{2a}\int_{-\infty}^{\infty} dy\frac{y^2}{(1+y^2)^2}\right\}.
\end{aligned}$$

Using the result for $D^2(a)$, we have

$$E_0 \leq I_1^{-2} \cdot \left\{ \frac{\hbar^2}{2ma^2} \, I_2 + \frac{ka^2}{2} \, I_3 \right\} ,$$

where

$$I_2 = \int_{-\infty}^{\infty} dy \frac{4y^2}{(1+y^2)^4} \qquad \text{and} \qquad I_3 = \int_{-\infty}^{\infty} dy \frac{y^2}{(1+y^2)^2} .$$

Minimizing the expression for E_0 with respect to a, we have

$$0 = \left. \frac{\delta E_0}{\delta a} \right|_{a=a^*} \propto -\frac{\hbar^2}{m(a^*)^3} \, I_2 + ka^* \, I_3$$

or $a^* = \left(\frac{\hbar^2}{mk} \frac{I_2}{I_3} \right)^{1/4}$.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: The internal energy of a non-relativistic Fermi gas at low temperatures is given by the expression

$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right],$$

where ε_F is the Fermi energy of the gas. Note that $\varepsilon_F \propto (N/V)^{2/3}$.

(a) Using the expression for U (and your knowledge of Thermodynamics), derive the corresponding expressions for the pressure P and the Helmholtz free energy F of the gas.

(b) Using these expressions for P and F, show that the corresponding expression for the chemical potential μ of the gas is

$$\mu = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right],$$

SOLUTION:

(a) To determine F, we need S which can be obtained from C_V.

$$C_V \equiv \left(\frac{\partial U}{\partial T} \right)_V = \left(\frac{\partial U}{\partial T} \right)_{\varepsilon_F} \qquad \text{because} \quad \varepsilon = f(N/V).$$

$$\therefore C_V = \frac{3}{5} N \, \varepsilon_F \cdot \frac{5\pi^2}{12} \cdot \frac{2k^2 T}{\varepsilon_F^2} + \cdots = \frac{\pi^2}{2} N k \, \frac{kT}{\varepsilon_F},$$

$$\text{hence} \qquad S = \int_0^T \frac{C_V \, dT}{T} = \frac{\pi^2}{2} \, N k \, \frac{kT}{\varepsilon_F} + \cdots$$

$$\text{Thus} \qquad F = U - TS = \frac{3}{5} N \varepsilon_F \left[1 - \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right]$$

P can be determined from $P = -(\partial F/\partial V)_T$ or $P = -(\partial U/\partial V)_S$. In either case, we need to know how ε_F depends on V. With $\varepsilon_F \propto V^{-2/3}$, we get

$$P = \frac{2}{3} \frac{U}{V} = \frac{2}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

(b) Finally, $\mu = \dfrac{G}{N} = \dfrac{F + PV}{N} = \varepsilon_F \left[1 - \dfrac{\pi^2}{12} \left(\dfrac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$

#8 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The "surface waves" in a low-temperature Bose liquid, upon quantization, behave like a two-dimensional gas of non-interacting excitations called "ripplons". Like potons or phonons, these excitations are *indefinite* in number and obey Bose-Einstein statistics. Their energy-momentum relation, however, is $\varepsilon = a \cdot p^{3/2}$, where a is a constant.

Find the temperature dependence of the total energy per unit area of these excitations.

`SOLUTION:`

An indefinite N implies that the chemical potential of this gas is zero.

$$\therefore U = \int_0^\infty \frac{\varepsilon\, g(\epsilon) d\epsilon}{e^{\varepsilon/kT} - 1}.$$

$g(\varepsilon)$ may be obtained by using the fact that $p = (\varepsilon/a)^{2/3}$ and employing the phase-space expression

$$\begin{aligned}
\int \frac{dx\, dy\, dp_x\, dp_y}{h^2} f(\epsilon(p)) &= A \int_0^\infty \frac{2\pi p dp}{h^2} f(\epsilon(p)) \\
&= \frac{2\pi A}{h^2} \int_0^\infty \left(\frac{\varepsilon}{a}\right)^{2/3} \cdot \left(\frac{1}{a}\right)^{2/3} \frac{2}{3}\, \varepsilon^{-1/3} f(\epsilon) d\varepsilon \\
&= \frac{4\pi A}{3h^2 a^{4/3}} \varepsilon^{1/3}\, d\varepsilon.
\end{aligned}$$

Thus, $\dfrac{U}{A} = \dfrac{4\pi}{3h^2 a^{4/3}} \displaystyle\int_0^\infty \frac{\varepsilon^{4/3}\, d\varepsilon}{e^{\varepsilon/kT} - 1} \propto T^{7/3}.$

#9 : UNDERGRADUATE PHYSICAL ESTIMATES

PROBLEM:

(a) The Sun may be regarded as a black-body radiator, of radius $R_\odot \approx 7 \times 10^8$ m, at a temperature $6,000\ K$. Calculate the "solar radiative flux" per unit area per unit time as observed on the surface of the Earth. The distance between the Sun and the Earth is $D = 1.5 \times 10^{11}$ m.

(b) Next, suppose that the Earth maintains itself in a steady state by continually radiating away into space as much energy as it receives from the Sun. If the Earth too were regarded a blackbody radiator, what would its steady-state temperature be?

SOLUTION:

(a) Radiative energy emitted by the Sun per unit time is σT^4, where is the Stefan-Boltzmann constant $\sigma \approx 5.67 \times 10^{-8}\,\mathrm{W/m^2 \cdot K^4}$ and T the temperature of the Sun. So the *total* rate of the solar emission is $\sigma T^4 \cdot 4\pi R_\odot^2$. The "solar radiation flux" observed on the Earth is then given by $\sigma T^4 \cdot 4\pi R_\odot^2/(4\pi D^2)$, where D is the Sun-Earth distance. We thus get

$$(5.67\times10^{-8}\,\mathrm{W\ m^{-2}K^{-4}})\times(6,000\,\mathrm{K})^4\times(7\times10^8\ \mathrm{m})^2/(1.5\times10^{11}\ \mathrm{m})^2 \approx 1,600\,\mathrm{W/m^2}.$$

(b) The *total* power received by the Earht is $1,600\ \mathrm{W/m^2}\ \times$ the corss-sectional area of the Earth, πr_E^2, r_E being the Earth's radius. If the Earth emits into space the same amount of power as it receives from the Sun, then its steady-state temperature T_{ss} would be given by the condition

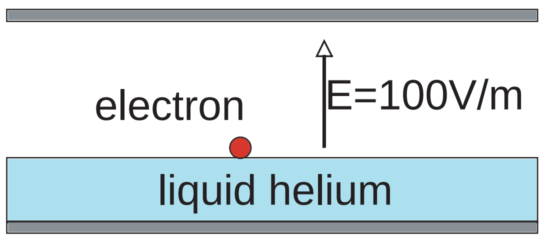
$$\sigma T_{\mathrm{ss}} \times 4\pi r_E^2 = 1,600\ \mathrm{W/m^2} \times \pi r_E^2.$$

Thus,

$$T_{\mathrm{ss}} = \left(\frac{1,600\ \mathrm{W/m^2}}{4 \times 5.67 \times 10^{-8}\,\mathrm{W\ m^{-2}K^{-4}}}\right)^{1/4} = 290\ \mathrm{K}.$$

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

`PROBLEM:` As shown in the figure, a capacitor has a thin layer of liquid helium on one of the metal plates. The electric field in the space between the liquid layer and the other plate is $E = 100$ volts/meter, pointing upwards. An electron is trapped on the upper liquid surface by the electric field. The helium surface may be regarded as impenetrable by the electron. Estimate the order of magnitude of the uncertainty in the vertical position of the electron in Angstroms without explicitly solving the Schrödinger equation. [You may ignore any possible image charge effects.]



`SOLUTION:` Let the vertical distance from the liquid surface be z. The potential is given

$$\begin{aligned} V(z) &= +\infty, \quad \text{if} \quad z < 0, \\ &= eEz, \quad \text{if} \quad z > 0. \end{aligned}$$

The ground state energy is

$$\begin{aligned} \mathcal{E} &\sim \frac{(\Delta p)^2}{2m} + eE\Delta z \\ &\sim \frac{\hbar^2}{2m(\Delta z)^2} + eE\Delta z, \end{aligned}$$

where the momenutm uncertainty Δp is $\sim \hbar/\Delta z$, the uncertainty in position.

Minimization of the ground state energy leads to

$$\Delta z = (\hbar^2/meE)^{1/3}$$

The numerical estimate is

$$\Delta z = a_B \left(\frac{\hbar^2}{ma_B^2} \; \frac{1}{eEa_B} \right)^{1/3} = 0.53\mathring{A} \left(\frac{27.2 \text{ eV}}{10^{-9} \text{ eV/Å } 0.53\mathring{A}} \right)^{1/3} = 10^3 \mathring{A}.$$

INSTRUCTIONS
PART II : PHYSICS DEPARTMENT EXAM

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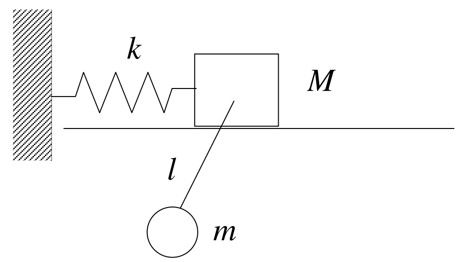
#11 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` Find the frequencies of the two normal modes of *small* oscillations for the coupled mass system shown below, consisting of a simple pendulum of length l and mass m attached to a block of mass M that slides in one dimension on a frictionless table and is attached to a wall by a spring of constant k.



#12 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` A one-dimensional mechanical continuum is described by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\phi_t^2 + r\,\phi_x^2 - \tfrac{1}{2}\,\phi_{xx}^2 - \tfrac{1}{2}\,\Delta\,\phi^2 \ ,$$

where $\phi_t = \partial\phi/\partial t$, $\phi_{xx} = \partial^2\phi/\partial x^2$, *etc.* Here the field, space, and time values are made dimensionless, with r and Δ being dimensionless parameters.

`(a)` Find the equation of motion for $\phi(x, t)$.

`(b)` Show that wave solutions exist, and find each branch of the dispersion relation $\omega(k)$.

`(c)` What conditions on r and Δ guarantee stability? It may be useful to make a sketch of ω^2 *versus* k^2.

`(d)` When (r, Δ) lie in a regime of instability, what is the wavelength for the maximally unstable (*i.e.* largest growth rate) wave solution?

#13 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` A solid sphere of radius R and conductivity σ is immersed in a uniform external magnetic field, $\boldsymbol{B} = \hat{\boldsymbol{z}} B_0$. At $t = 0$, the external magnetic field is suddenly switched off. Initially the magnetic field remains "frozen" into the conducting sphere, but then slowly decays.

`(a)` Estimate the time scale for the decay in terms of the parameters given and universal constants. (You do not need to solve any equation explicitly.)

`(b)` Derive the B-field inside and outside the sphere immediately after the external field is switched off. (For convenience, you may express your answer in term of a potential function.)

`(c)` Using your result to part `(b)`, calculate the heat produced during the decay process.

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Two halves of a thin, conducting, spherical shell are separated by a thin insulating layer. An oscillating potential is applied to the two halves, so that the halves have potential $+V\ \cos\omega t$ and $-V\ \cos\omega t$. The radius of the shell is R, where $R \ll c/\omega$. Calculate the time-averaged power radiated by this system.

#15 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` In a heavy atom, the radius of the innermost ("K" shell) electron orbital is only about 100 times larger than the nuclear radius R. Its energy therefore exhibits a detectable shift $\varDelta E$ due to the finite size of the nucleus. The shift $\varDelta E$ is expected to depend on both the atomic number Z and the atomic weight A, with the latter arising from the dependence of the nuclear radius R on A:

$$R = r_0\, A^{1/3} \quad , \quad r_0 = 2.3 \times 10^{-5}\, a_{\rm B} \ ,$$

where $a_{\rm B} = \hbar^2/me^2$ is the Bohr radius.

In this problem, you are asked to compute the "isotope shift" $\varDelta E(A) - \varDelta E(A')$ for two different isotopes of thallium ($Z = 81$) with atomic weights $A = 203$ and $A' = 205$. (This isotope shift can be readily measured as a shift in the K edge of X-ray energy.) You may compute $\varDelta E$ using first order perturbation theory, assuming that the nucleus is a sphere of uniform charge density, and neglecting screening effects due to other electrons. The exponential envelope of the hydrogenic wavefunctions can also be neglected on the scale of the nucleus.

#16 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` The identical plane rotators with coordinates θ_1 and θ_2 are governed by the Hamiltonian

$$\mathcal{H} = \frac{A}{\hbar^2}\,(p_{\theta_1}^2 + p_{\theta_2}^2) - B\,\cos(\theta_1 - \theta_2)$$

where A, B are positive constants. Note that θ_1, θ_2 are angular variables, i.e., $\theta_i = \theta_i + 2\pi$. Determine the energy eigenvalues and eigenfunctions to linear order in B for $B \ll A$. To this end,

`(a)` Make a linear change of coordinates so that the problem separates (that is, neither the kinetic term nor the perturbation (B-term) couple the two variables).

`(b)` Find the eigenvalues and eigenstates of the unperturbed Hamiltonian ($B = 0$); list the degeneracies of this solution.

`(c)` For any two unperturbed states ψ and χ, compute $\langle\chi|V|\psi\rangle$ where $V = -B\,\cos(\theta_1 - \theta_2)$. (*Hint: Use the eigenstates expressed in terms of the variables introduced in part* (a), *but switch back to the original variables to carry out the integrals*).

`(d)` Use the above result to determine the shift in energy eigenvalues. Be mindful of diagonalizing the matrix of energy shifts for degenerate unperturbed levels, as needed.

#17 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` The rotational states of a diatomic molecule are characterized by energy eigenvalues

$$\epsilon_\ell = \ell(\ell+1)\frac{\hbar^2}{2I}, \qquad \ell = 0, 1, 2, ...$$

I is the moment of inertia of the molecule.

Set up the partition function of this molecule. At high temperatures, compute the specific heat (per such molecule), $C_{\rm rot}$, to leading order correction from the classical result $C_{\rm rot} = k_B$, where k_B is the Boltzmann constant.

Hint: It will be useful to know the following formula (known as the Euler-MacLaurin formula),

$$\sum_{\ell=0}^{\infty} f(\ell + \tfrac{1}{2}) = \int_0^\infty f(x)dx + \frac{1}{24}f'(0) - \frac{7}{5760}f'''(0) + \ldots,$$

where $f(x) = x\,e^{-\alpha x^2}$ for a positive constant α.

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider N classical spins of $s = \{\pm 1\}$ on a one-dimensional lattice. The interaction of spins on site m and n (denoted by s_m and s_n respectively) is given by the exchange term J_{mn}, so that the total energy of a given configuration of spins $\{s_i\}$ is

$$\mathcal{H} = -\frac{1}{2} \sum_{m,n=1}^{N} J_{mn}\, s_m s_n .$$

`(a)` For nearest-neighbor coupling, i.e., $J_{mn} = J\, \delta_{m,n\pm 1}$, show that there cannot be spontaneous magnetization in the thermodynamic limit at any finite temperature $T > 0$.

Hint: You do not need to solve the full problem to reach this conclusion; it suffices to consider the energy and entropy of low energy, large scale excitations that may destabilize the ground state.

`(b)` Suppose the exchange term in $\mathcal{H}$ above has the form $J_{m \neq n} = J/|m-n|^{\sigma}$ where σ is a positive real number. Show that spontaneous magnetization may occur at some finite temperature $T_c > 0$ only if σ is smaller than a threshold value σ_c. Find the value of σ_c. (You do not need to determine the value of T_c.)

#19 : GRADUATE MATHEMATICAL PHYSICS

`PROBLEM:` Find an analytic expression for the sum

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + x^2)^2} \ ,$$

where x is real. *Hint*: It will be useful to consider the pole structure of the function

$$f(\omega) = \frac{1}{e^{2\pi\omega} - 1} \ .$$

#20 : GRADUATE OTHER

`PROBLEM:` Consider a random walk of N steps in one-dimension. Suppose that the steps are completely uncorrelated, and that the probability distribution of the displacement x of a single step is given by

$$\rho(x) \propto \frac{1}{1+|x|^\alpha}, \qquad -\infty < x < \infty$$

where $\alpha > 0$ is a constant. We want to know the probability distribution $P(R; N)$ for the total displacement $R = \sum_{n=1}^{N} x_n$ for $N \gg 1$, with x_n being the displacement of the n^{th} step.

`(a)` Write down the functional form of $P(R; N)$ expected from the Central Limit Theorem up to a normalization constant. Express parameter(s) of P in terms of integral(s) of ρ. (You do not need to derive the Central Limit Theorem.) For what range of α is the result (and hence the Central Limit Theorem) valid?

`(b)` Compute the distribution $P(R; N)$ directly from $\rho(x)$ for $\alpha = 2$, and find its explicit form including normalization constant. How far does a walker typically go after N steps? Compare your result to that of part `(a)` and comment.

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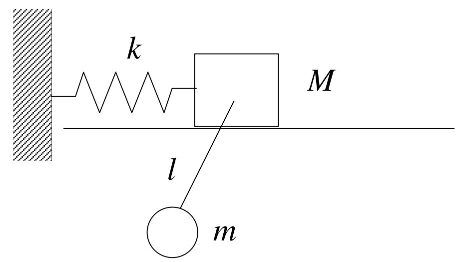

`SOLUTION:`

- The Kinetic energy of the block is $M\dot{x}^2/2$.
- Kinetic energy of the pendulum is $m(\dot{x} - l\dot{\theta})^2/2$.
- Potential energy of the spring-mass system is $kx^2/2$.
- Potential energy of pendulum, for $\theta \ll 1$ is $mgl\theta^2/2$.

So the Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + m(\dot{x} - \frac{1}{2}l\dot{\theta})^2 - \frac{1}{2}kx^2 - \frac{1}{2}mgl\theta^2.$$

We want to make a linear change of variables into new variables y_1 and y_2 that puts the above Lagrangian into the form

$$L = \frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2\right) - \frac{1}{2}\left(\omega_1^2 y_1^2 + \omega_2^2 y_2^2\right),$$

from which we read the frequencies of the normal modes ω_1 and ω_2.

To accomplish this we make a sequence of transformations. First let

$$z_1 = \sqrt{M}x \qquad \text{and} \qquad z_2 = \sqrt{m}(x - l\theta).$$

In terms of these

$$L = \tfrac{1}{2}(\dot{z}_1^2 + \dot{z}_2^2) - \tfrac{k}{2M} z_1^2 - \tfrac{mg}{2l} \left(\tfrac{z_1}{\sqrt{M}} - \tfrac{z_2}{\sqrt{m}} \right)^2 .$$

Note that the potential term is, in matrix notation

$$V = \tfrac{1}{2} \begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} \begin{pmatrix} \frac{k}{M} + \frac{mg}{Ml} & \frac{g}{l}\sqrt{\frac{m}{M}} \\ \\ \frac{g}{l}\sqrt{\frac{m}{M}} & \frac{g}{l} \end{pmatrix} \begin{pmatrix} z_1 & z_2 \end{pmatrix}$$

We need to diagonalize this by making one last coordinate transformation, a rotation from the (z_1, z_2) plane to the (y_1, y_2) plane. But since we are only asked to compute the frequencies of the normal modes we need only find the eigenvalues (and not the eigenvectors) of the 2×2 matrix. Computing the characteristic equation gives:

$$\left(\tfrac{k}{M} + \tfrac{mg}{Ml} - \lambda\right) \left(\tfrac{g}{l} - \lambda \right) - \tfrac{g^2}{l^2} \tfrac{m}{M} = 0$$

The solutions are elementary,

$$\boxed{\omega_{1,2}^2 = \tfrac{1}{2} \left(\tfrac{k}{M} + \left(1 + \tfrac{m}{M}\right) \tfrac{g}{l} \pm \sqrt{ \left(\tfrac{k}{M} + \left(1 + \tfrac{m}{M}\right) \tfrac{g}{l} \right)^2 - 4 \tfrac{gk}{lM} } \right)}$$

Some simple checks (not required). For a very stiff spring, $k \to \infty$, we have solutions

$$\omega_1^2 \approx \tfrac{k}{M} \qquad \text{and} \qquad \omega_2^2 \approx \tfrac{g}{l}$$

These just correspond to uncoupling the system! Note also that this is also the limit $g \to 0$. For $g = 0$ one mode (ω_2) vanishes. This just corresponds to the pendulum executing uniform circular motion. Another check is the opposite limit, that of a weak spring, $k \to 0$:

$$\omega_1^2 \approx \tfrac{k}{m+M} \qquad \text{and} \qquad \omega_2^2 \approx \left(1 + \tfrac{m}{M}\right) \tfrac{g}{l}$$

For $k = 0$, no spring at all, you expect a normal mode of zero frequency, corresponding to translation of the whole system (and you see from above that $\omega_1 = 0$ at $k = 0$). If k is small but not zero then this mode becomes an oscillation of the block and pendulum together.

#12 : GRADUATE CLASSICAL MECHANICS

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$$\mathcal{L} = \tfrac{1}{2}\phi_t^2 + r\,\phi_x^2 - \tfrac{1}{2}\,\phi_{xx}^2 - \tfrac{1}{2}\,\Delta\,\phi^2 \ ,$$

where $\phi_t = \partial\phi/\partial t$, $\phi_{xx} = \partial^2\phi/\partial x^2$, *etc.* Here the field, space, and time values are made dimensionless, with r and Δ being dimensionless parameters.

(a) Find the equation of motion for $\phi(x,t)$.

(b) Show that wave solutions exist, and find each branch of the dispersion relation $\omega(k)$.

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(d) When (r,Δ) lie in a regime of instability, what is the wavelength for the maximally unstable (*i.e.* largest growth rate) wave solution?

SOLUTION:

(a) The Euler-Lagrange equation is:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\,\frac{\partial\mathcal{L}}{\partial\phi_t} - \frac{\partial}{\partial x}\,\frac{\partial\mathcal{L}}{\partial\phi_x} + \frac{\partial^2}{\partial x^2}\,\frac{\partial\mathcal{L}}{\partial\phi_{xx}} = 0 \ .$$

Thus,

$$\Delta\,\phi + \phi_{tt} + 2r\,\phi_{xx} + \phi_{xxxx} = 0 \ .$$

(b) The wave equation is linear, hence we can superpose solutions to get another solution. We try a wave solution of the form

$$\phi(x,t) = A\,e^{i(kx-\omega t)} \ .$$

The equation of motion then yields

$$\omega^2 = k^4 - 2rk^2 + \Delta \ .$$

There are two branches to the dispersion, corresponding to taking the positive or negative square root for ω^2:

$$\omega_\pm(k) = \pm\sqrt{k^4 - 2rk^2 + \Delta} \ .$$

(c) We can factorize and write

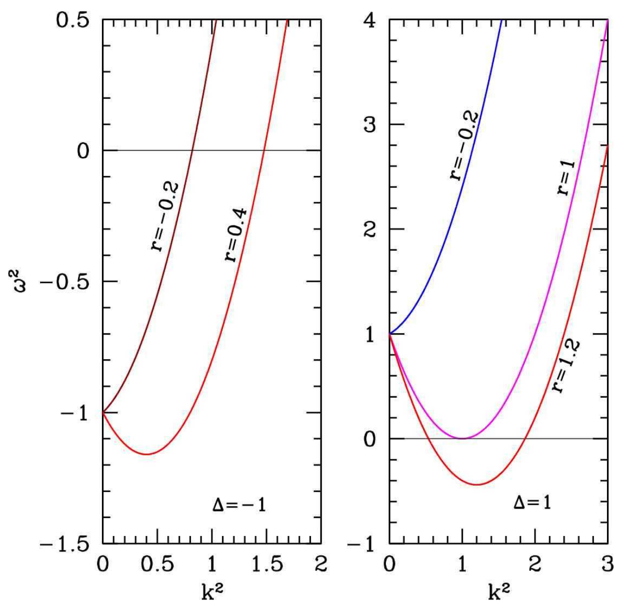
$$\omega^2 = (k^2 - k_-^2)(k^2 - k_+^2) \ ,$$

where

$$k_\pm^2 = r \pm \sqrt{r^2 - \Delta} \ .$$

The system is stable provided that $\mathrm{Im}\,\omega_\pm(k) < 0$ for both branches. Note that $\omega_\pm^2(0) = \Delta$, so there is always instability when $\Delta < 0$. The condition for stability is:

$$\text{stability} \quad \Longleftrightarrow \quad \Delta > 0 \ , \ r < \sqrt{\Delta} \ .$$



(d) The most unstable wavevector k is the one which makes $\mathrm{Im}\,\omega(k)$ the most negative. Thus, we want ω^2 to be negative and a minimum with respect to k^2. We then differentiate,

$$\frac{\partial \omega^2}{\partial (k^2)} = -2r + 2k^2 \ .$$

If $\Delta < 0$, then the most unstable wavevector occurs at $k^* = 0$ if $r < 0$ and at $k^* = \sqrt{r}$ if $r > 0$ (see figure). For $\Delta > 0$, instability requires $r > \sqrt{\Delta}$, and the maximally unstable wavevector occurs at $k^* = \sqrt{r}$. Note $k^* = 0$ means $\lambda = \infty$ and $k^* = \sqrt{r}$ means $\lambda = 2\pi/\sqrt{r}$.

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: A solid sphere of radius R and conductivity σ is immersed in a uniform external magnetic field, $\boldsymbol{B} = \hat{\boldsymbol{z}}B_0$. At $t = 0$, the external magnetic field is suddenly switched off. Initially the magnetic field remains "frozen" into the conducting sphere, but then slowly decays.

(a) Estimate the time scale for the decay in terms of the parameters given and universal constants. (You do not need to solve any equation explicitly.)

(b) Derive the B-field inside and outside the sphere immediately after the external field is switched off. (For convenience, you may express your answer in term of a potential function.)

(c) Using your result to part (b), calculate the heat produced during the decay process.

SOLUTION:

(a) For a slow decay, we can use the quasi-static approximation,

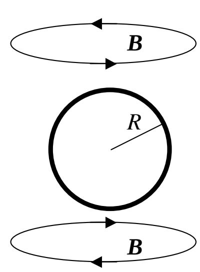
$$\nabla^2 \boldsymbol{B} = \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{B}}{\partial t}.$$

Taking the estimates $\nabla^2 \boldsymbol{B} \sim \dfrac{B_0}{R^2}$ and $\dfrac{\partial \boldsymbol{B}}{\partial t} \sim \dfrac{B_0}{\tau}$, we have $\tau \sim \dfrac{4\pi\sigma}{c^2}R^2$.

(b) The field inside the sphere is $\boldsymbol{B} = \hat{\boldsymbol{z}}B_0$

To find dipole field outside the sphere, note that $\nabla \times \boldsymbol{B} = 0$ so $\boldsymbol{B} = \nabla\phi$.

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \Rightarrow \qquad \nabla^2\phi = 0.$$



Try the dipole solution $\phi(r, \theta) = \dfrac{A}{r^2}\cos\theta$ for $r > R$.

At surface, require that normal component of $\boldsymbol{B}$ is continuous, i.e.,

$$\underbrace{\left.\frac{\partial\phi}{\partial r}\right|_R}_{\frac{-2A\cos\theta}{R^3}} = B_0\,\hat{z}\cdot\hat{r} = B_0\cos\theta \qquad \Rightarrow \qquad A = -\frac{B_0\,R^3}{2}$$

`(c)` The heat produced is equal to the magnetic field intensity for the sphere with frozen in magnetic field.

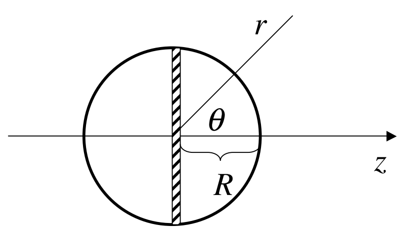
$$\text{Heat} = W_M = \underbrace{\frac{B_0^2}{8\pi}\cdot\frac{4}{3}\pi R^3}_{\text{inside}} + \underbrace{\int_R^\infty r^2 dr\int_0^\pi 2\pi\sin\theta d\theta\frac{(\nabla\phi)^2}{8\pi}}_{\text{outside}} = \frac{B_0^2R^3}{4}$$

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` Two halves of a thin, conducting, spherical shell are separated by a thin insulating layer. An oscillating potential is applied to the two halves, so that the halves have potential $+V\cos\omega t$ and $-V\cos\omega t$. The radius of the shell is R, where $R \ll c/\omega$. Calculate the time-averaged power radiated by this system.

`SOLUTION:` Since dipole radiation dominates here, it is necessary to calculate the dipole moment for the configuration of electric potential, specified at the conducting shell,

$$\varphi(R,\theta) = \begin{cases} +V\cos\omega t & 0 \leqslant \theta \leqslant \pi/2 \\ -V\cos\omega t & \pi/2 \leqslant \theta \leqslant \pi \end{cases}$$



We can use electrostatics for this purpose since $R \ll c/\omega$.

For $r > R$, the electric potential can be generally written as

$$\phi(r,\theta) = \sum_\ell A_\ell \frac{P_\ell[\cos\theta]}{r^{\ell+1}}$$

in spherical coordinate system. Compared to the potential of an ideal dipole,

$$\phi_{\text{dipole}} = \frac{\vec{r}\cdot\vec{P}}{r^3} = \frac{P\cos\theta}{r^2},$$

the dipole moment is given by $P = A_1$ since $P_1[\cos\theta] = \cos\theta$.

To find the dipole moment, we note

$$\begin{aligned}
\int_0^\pi \sin\theta d\theta P_1[\cos\theta]\cdot\varphi(R,\theta) &= \int_0^\pi \sin\theta d\theta P_1[\cos\theta]\cdot\phi(R,\theta) \\
V\cos\omega t\left\{\int_0^{\pi/2}\sin\theta d\theta\cos\theta - \int_{\pi/2}^{\pi}\sin\theta d\theta\cos\theta\right\} &= \frac{A_1}{R^2}\int_0^\pi d\theta\sin\theta\cos^2\theta \\
V\cos\omega t\left\{\left[-\frac{\cos^2\theta}{2}\right]_0^{\pi/2} - \left[-\frac{\cos^2\theta}{2}\right]_{\pi/2}^{\pi}\right\} &= \frac{A_1}{R^2}\left[-\frac{\cos^3\theta}{3}\right]_0^\pi \\
V\cos\omega t\left\{+\frac{1}{2}+\frac{1}{2}\right\} &= +\frac{A_1}{R^2}\frac{2}{3}
\end{aligned}$$

So the dipole moment is

$$P = \frac{3}{2}R^2\,V\ \cos\omega t.$$

Average power irradiated by a dipole is then

$$\begin{aligned}
\langle\text{Power Radiated}\rangle_{\text{time}} &= \left\langle\frac{2}{3}\frac{|\ddot{\vec{P}}|^2}{c^3}\right\rangle \\
&= \frac{2}{3}\frac{9}{4}\frac{R^4V^2\omega^4}{c^3}\langle\cos^2\omega t\rangle \\
&= \frac{3}{4}\frac{R^4V^2\omega^4}{c^3}
\end{aligned}$$

#15 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` In a heavy atom, the radius of the innermost ("K" shell) electron orbital is only about 100 times larger than the nuclear radius R. Its energy therefore exhibits a detectable shift ΔE due to the finite size of the nucleus. The shift ΔE is expected to depend on both the atomic number Z and the atomic weight A, with the latter arising from the dependence of the nuclear radius R on A:

$$R = r_0\, A^{1/3} \quad , \quad r_0 = 2.3 \times 10^{-5}\, a_{\rm B} \ ,$$

where $a_{\rm B} = \hbar^2/me^2$ is the Bohr radius.

In this problem, you are asked to compute the "isotope shift" $\Delta E(A) - \Delta E(A')$ for two different isotopes of thallium ($Z = 81$) with atomic weights $A = 203$ and $A' = 205$. (This isotope shift can be readily measured as a shift in the K edge of X-ray energy.) You may compute ΔE using first order perturbation theory, assuming that the nucleus is a sphere of uniform charge density, and neglecting screening effects due to other electrons. The exponential envelope of the hydrogenic wavefunctions can also be neglected on the scale of the nucleus.

`SOLUTION:` The first order of business is computing the electrostatic potential due to a uniform charge distribution. From $\nabla^2\phi = -4\pi\rho$ we have $\phi(\boldsymbol{r}) = -\frac{2\pi}{3}\rho r^2 + \phi(0)$. This is valid up for $0 \le r \le R$, where R is the radius of the uniform spherical charge distribution. For $r > R$, we have $\phi(\boldsymbol{r}) = Ze/r$, where $Ze = \frac{4\pi}{3}\rho R^3$ is the total nuclear charge. Matching at $r = R$ then gives

$$\phi(\boldsymbol{r}) = \begin{cases} \frac{Ze}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) & \text{for } 0 \le r \le R \\ \\ \frac{Ze}{r} & \text{for } r \ge R \ . \end{cases}$$

The potential energy for the K shell electron, neglecting screening (since the K shell is the innermost) is $V(\boldsymbol{r}) = -e\phi(\boldsymbol{r})$.

For the potential $V_0(\boldsymbol{r}) = -Ze^2/r$ (for all r), the ground state eigenfunction and energy are given by

$$\psi_0(\boldsymbol{r}) = \frac{(Z/a_{\rm B})^{3/2}}{\sqrt{\pi}}\, e^{-Zr/a_{\rm B}} \qquad , \qquad E_0 = -\frac{(Ze)^2}{2a_{\rm B}} \ .$$

We now define the perturbation potential

$$\Delta V(\boldsymbol{r}) = V(\boldsymbol{r}) - V_0(\boldsymbol{r}) = \begin{cases} \frac{Ze^2}{R}\left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) & \text{for } 0 \le r \le R \\ \\ 0 & \text{for } r > R\ . \end{cases}$$

The energy shift, to first order in perturbation theory, is then

$$\begin{aligned} \Delta E &= \langle \psi_0 \big| \Delta V \big| \psi_0 \rangle \\ &= 4\pi \int\limits_0^R dr\, r^2\, \frac{Z^3}{\pi a_{\rm B}^3}\, e^{-2Zr/a_{\rm B}}\, \frac{Ze^2}{R} \left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r} \right) \\ &\approx \tfrac{4}{5} Z^4 \left(\frac{R}{a_{\rm B}} \right)^2 \cdot \left(\frac{e^2}{2a_{\rm B}} \right) \ , \end{aligned}$$

where the last line follows if we approximate $e^{-2Zr/a_{\rm B}} \approx 1$ inside the integrand. Recall $e^2/2a_{\rm B} = 13.6\,\text{eV}$. Setting $Z = 81$ and $A = 203$, and invoking $R = r_0\, A^{1/3}$, we find

$$\Delta E(A) = 8.25\,\text{eV}\ .$$

The isotopic energy shift in the X-ray K edge is then

$$\begin{aligned} \Delta E(A) - \Delta E(A') &\simeq \frac{d\Delta E}{dR}\,\frac{dR}{dA}\,\Delta A \\ &= \frac{2}{3}\,\frac{\Delta A}{A}\,\Delta E \\ &= -54.2\,\text{meV}\ , \end{aligned}$$

since $\Delta A = A' - A = -2$.

#16 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` The identical plane rotators with coordinates θ_1 and θ_2 are governed by the Hamiltonian

$$\mathcal{H} = \frac{A}{\hbar^2}\,(p_{\theta_1}^2 + p_{\theta_2}^2) - B\,\cos(\theta_1 - \theta_2)$$

where A, B are positive constants. Note that θ_1, θ_2 are angular variables, i.e., $\theta_i = \theta_i + 2\pi$. Determine the energy eigenvalues and eigenfunctions to linear order in B for $B \ll A$. To this end,

`(a)` Make a linear change of coordinates so that the problem separates (that is, neither the kinetic term nor the perturbation (B-term) couple the two variables).

`(b)` Find the eigenvalues and eigenstates of the unperturbed Hamiltonian ($B = 0$); list the degeneracies of this solution.

`(c)` For any two unperturbed states ψ and χ, compute $\langle\chi|V|\psi\rangle$ where $V = -B\,\cos(\theta_1 - \theta_2)$. (*Hint: Use the eigenstates expressed in terms of the variables introduced in part* (a), *but switch back to the original variables to carry out the integrals*).

`(d)` Use the above result to determine the shift in energy eigenvalues. Be mindful of diagonalizing the matrix of energy shifts for degenerate unperturbed levels, as needed.

`SOLUTION:`

`(a)` The Schrödinger equation is

$$\left[-A\left(\frac{\partial^2}{\partial\theta_1^2} + \frac{\partial^2}{\partial\theta_2^2}\right) - B\,\cos(\theta_1 - \theta_2)\right]\,\psi = E\,\psi.$$

It is convenient to change variables to $x = \theta_1 + \theta_2$ and $y = \theta_1 - \theta_2$. Then

$$\left[-2A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - B\,\cos y\right]\,\psi = E\,\psi.$$

Note that with the new variables, the two coordinates are no longer coupled.

`(b)` For $B = 0$, we have $\left[-2A\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right]\,\psi = E\,\psi.$

This is the Schrödinger equation for a "free particle", with

$$\begin{aligned}\psi &= C\, e^{i\alpha x}\, e^{i\beta y} \\ &= C\, e^{i(\alpha+\beta)\theta_1}\, e^{i(\alpha-\beta)\theta_2}.\end{aligned}$$

Since θ_i is periodic, $\alpha\pm\beta$ must be integers; hence 2α and 2β must be integers and both even or both odd. So we have

$$\psi = C\, e^{i\frac{m}{2}x}\, e^{i\frac{n}{2}y}, \qquad m, n \in \mathbb{Z}, \text{ and } m - n \in 2\mathbb{Z}.$$

Normalization condition $\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 |\psi(\theta_1, \theta_2)|^2 = 1$ yields $C = (2\pi)^{-1}$.

Inserting the solution back into the Schrödinger equation, we find

$$E^{(0)}_{m,n} = \frac{A}{2}\,(m^2 + n^2) \qquad \text{and} \qquad \psi^{(0)}_{m,n} = \tfrac{1}{2\pi} e^{i\frac{m}{2}x}\, e^{i\frac{n}{2}y}$$

Here the superscript (0) reminds us that these are unperturbed eigenvalues and eigenstates. Note the degeneracies:

$$E^{(0)}_{m,n} = E^{(0)}_{-m,n} = E^{(0)}_{m,-n} = E^{(0)}_{-m,-n}.$$

`(c)` As instructed we compute the exepctation value of the perturbation V. Use the braket notation $\psi_{m,n} = |m, n\rangle$.

$$\langle m', n'|V|m, n\rangle = \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \left(-\frac{B}{(2\pi)^2}\right) \cos(\theta_1 - \theta_2)\, e^{i\frac{\Delta m}{2}(\theta_1+\theta_2)} e^{i\frac{\Delta n}{2}(\theta_1-\theta_2)}$$

where $\Delta m = m - m'$ and $\Delta n = n - n'$.

The integral is trivially done by noting that $\cos(\theta_1 - \theta_2) = \frac{1}{2}(e^{i(\theta_1+\theta_2)} + e^{-i(\theta_1+\theta_2)})$, and gives zero unless $\Delta m = 0$ and $\Delta n = \pm 1$. In the latter case, the integral is

$$-\frac{B}{(2\pi)^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2\, \frac{1}{2} = -\frac{B}{2}.$$

So, $\langle m', n'|V|m, n\rangle = -\dfrac{B}{2}\delta_{m'm}\left(\delta_{\frac{\Delta n}{2},-1} + \delta_{\frac{\Delta n}{2},+1}\right)$.

`(d)` There is no diagonal shift in energy, ΔE. But for fixed m and $n = \pm 1$, we have a 2×2 system to diagonalize.

$$\langle m, \pm 1|V|m, \pm 1\rangle = -\frac{B}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are

$$\det \begin{pmatrix} -\lambda & 1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 - 1 = 0 \qquad \rightarrow \qquad \lambda = \pm 1,$$

and eigenvectors

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \pm \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\Rightarrow \quad v = \pm u \quad \Rightarrow \frac{1}{\sqrt{2}} \left(|m, 1\rangle \pm |m, -1\rangle \right),$$

with energy shifts $\Delta E_{m,\pm 1} = -\frac{B}{2} \cdot (\pm 1)$.

#17 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` The rotational states of a diatomic molecule are characterized by energy eigenvalues

$$\epsilon_\ell = \ell(\ell+1)\frac{\hbar^2}{2I}, \qquad \ell = 0, 1, 2, ...$$

I is the moment of inertia of the molecule.

Set up the partition function of this molecule. At high temperatures, compute the specific heat (per such molecule), $C_{\rm rot}$, to leading order correction from the classical result $C_{\rm rot} = k_B$, where k_B is the Boltzmann constant.

Hint: It will be useful to know the following formula (known as the Euler-MacLaurin formula),

$$\sum_{\ell=0}^{\infty} f(\ell + \tfrac{1}{2}) = \int_0^\infty f(x)dx + \frac{1}{24}\, f'(0) - \frac{7}{5760}\, f'''(0) + \ldots,$$

where $f(x) = x\, e^{-\alpha x^2}$ for a positive constant α.

`SOLUTION:` The partition function is

$$\mathcal{Q} = \sum_{\ell=0}^{\infty} g_\ell e^{-\epsilon_\ell/k_B T}$$

where $g_\ell = 2\ell{+}1$ is the multiplicity of the state ℓ. In term of the temperature scale $\Theta = \hbar^2/(2Ik_B)$, the partition function $\mathcal{Q}$ can be expressed as

$$\begin{aligned}\mathcal{Q} &= \sum_{\ell=0}^{\infty}(2\ell+1)e^{-\ell(\ell+1)\Theta/T} \\ &= 2\, e^{\frac{1}{4}\Theta/T} \sum_{\ell=0}^{\infty} \left(\ell + \frac{1}{2}\right) e^{-(\ell+\frac{1}{2})^2\Theta/T}.\end{aligned}$$

Using Euler-Maclaurin formula as given with $\alpha = \Theta/T$, we get

$$\begin{aligned}
\mathcal{Q} &= 2\,e^{\alpha/4}\left[\frac{1}{2\alpha} + \frac{1}{24} + \frac{7\alpha}{960} + \ldots\right] \\
&= \frac{1}{\alpha}\,e^{\alpha/4}\left[1 + \frac{\alpha}{12} + \frac{7\alpha^2}{480} + \ldots\right] \\
\therefore \ln\mathcal{Q} &= -\ln\alpha + \frac{\alpha}{4} + \ln\left[1 + \frac{\alpha}{12} + \frac{7\alpha^2}{480} + \ldots\right] \\
&= -\ln\alpha + \frac{\alpha}{4} + \left[\frac{\alpha}{12} + \frac{7\alpha^2}{480} + \ldots - \frac{\alpha^2}{288} - \ldots\right] \\
&= -\ln\alpha + \frac{\alpha}{3} + \frac{\alpha^2}{90} + \ldots
\end{aligned}$$

Using $\alpha = \Theta/T$, we can find the free energy A and its derivatives as:

$$\begin{aligned}
A &= -k_B T\ln\mathcal{Q} = -k_B T\ln(T/\Theta) - \frac{1}{3}k_B\Theta - \frac{1}{90}k_B\frac{\Theta}{T} + \ldots, \\
\frac{\partial A}{\partial T} &= -k_B\ln(T/\Theta) - k + \frac{1}{90}k_B\left(\frac{\Theta}{T}\right)^2 + \ldots \\
\frac{\partial^2 A}{\partial T^2} &= -\frac{k_B}{T} - \frac{1}{45}\frac{k\Theta^2}{T^3} + \ldots
\end{aligned}$$

Finally, $C_{\mathrm{rot}} = -T\dfrac{\partial^2 A}{\partial T^2} = k_B\left(1 + \dfrac{1}{45}\dfrac{\Theta^2}{T^2} + \ldots\right)$.

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider N classical spins of $s = \{\pm 1\}$ on a one-dimensional lattice. The interaction of spins on site m and n (denoted by s_m and s_n respectively) is given by the exchange term J_{mn}, so that the total energy of a given configuration of spins $\{s_i\}$ is

$$\mathcal{H} = -\frac{1}{2} \sum_{m,n=1}^{N} J_{mn}\, s_m s_n .$$

`(a)` For nearest-neighbor coupling, i.e., $J_{mn} = J\,\delta_{m,n\pm 1}$, show that there cannot be spontaneous magnetization in the thermodynamic limit at any finite temperature $T > 0$.

Hint: You do not need to solve the full problem to reach this conclusion; it suffices to consider the energy and entropy of low energy, large scale excitations that may destabilize the ground state.

`(b)` Suppose the exchange term in $\mathcal{H}$ above has the form $J_{m \neq n} = J/|m-n|^{\sigma}$ where σ is a positive real number. Show that spontaneous magnetization may occur at some finite temperature $T_c > 0$ only if σ is smaller than a threshold value σ_c. Find the value of σ_c. (You do not need to determine the value of T_c.)

`SOLUTION:`

`(a)` The nearest-neighbor Ising model can be solved exactly using the transfer matrix method. However, to show the absence of spontaneous magnetization, the following consideration suffices.

- At $T = 0$, the ground state is clearly one with *all* spins taking on the same value ($+1$ or -1).
- Low energy excitations are "kink" configurations where a contiguous segment of the spins take on one value and the rest of the spins take on the opposite value, say, $s_m = +1$ for $1 < m < m^*$, and $s_m = -1$ for $m^* < m < N$. The energy cost of such an excitation is $\Delta E_{\text{kink}} = +2J$ for all kink position m^* as long as it is not at the two ends of the 1d lattice.

- Since the kink configuration has a degeneracy of the order N, the entropy gain is $\Delta S = \log N$. The total free energy cost of the kink configuration is

$$\Delta F_{\text{kink}} = 2J - k_B T \log N$$

which is negative at *any* finite temperature $T > 0$ for a sufficiently large N. Hence the ground states with uniform magnetization are unstable, replaced by kink configurations which have zero average magnetization.

(b) Continuing along the line of consideration given in part (a), we need to estimate the energy cost of a kink configuration for the non-local coupling J_{mn} —

$$\Delta E_{\text{kink}} \sim J \sum_{n=m^*+1}^{N} \sum_{m=1}^{m^*-1} \tfrac{1}{|m-n|^{\sigma}}. \tag{1}$$

For kink position not too close to the ends of the system, say $N/4 \lesssim m^* \lesssim 3N/4$, and for $N \gg 1$, the sums in Eq. (2) may be approximated by integrals, yielding the result

$$\Delta E_{\text{kink}} \sim O(J\, N^{2-\sigma})$$

to the leading order in N. This energy is larger than the entropy gain of kink formation, still of the order $\log N$, as long as $\sigma < 2$. Hence the ground state is stable and spontaneous magnetization occurs provided $\sigma < \sigma_c = 2$.

#19 : GRADUATE MATHEMATICAL PHYSICS

`PROBLEM:` Find an analytic expression for the sum

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+x^2)^2} \ ,$$

where x is real. *Hint*: It will be useful to consider the pole structure of the function

$$f(\omega) = \frac{1}{e^{2\pi\omega}-1} \ .$$

`SOLUTION:`

Consider the function

$$f(\omega) = \frac{1}{e^{2\pi\omega}-1} \ .$$

Clearly $f(\omega)$ has a simple pole at $\omega = in$ for all integer n, with residue $(2\pi)^{-1}$. Thus, for any function $H(\omega)$ which is analytic along the imaginary axis, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} H(in) &= \oint\limits_{\mathcal{C}} \frac{d\omega}{i}\, f(\omega)\, H(\omega) \\ &= -2\pi \sum_{\text{Im}(\omega)\neq 0} \text{Res}\Big[f(\omega)\, H(\omega)\Big] \ .\end{aligned}$$

In our case, we have

$$H(\omega) = \frac{1}{(\omega^2 - x^2)^2} \ ,$$

and the poles lie at $\omega = \pm x$. Since $H(\omega)$ has a double pole at these points, we must evaluate the residues according to

$$\begin{aligned}\text{Res}(+x) &= \frac{\partial}{\partial\omega}\bigg|_{\omega=x} \left[\frac{1}{(\omega+x)^2}\, \frac{1}{e^{2\pi\omega}-1}\right] \\ &= -\frac{1}{4x^3}\, \frac{1}{e^{2\pi x}-1} - \frac{1}{4x^2}\, \frac{2\pi\, e^{2\pi x}}{(e^{2\pi x}-1)^2}\end{aligned}$$

Adding to this the residue at $\omega = -x$, we have

$$F(x) = \frac{\pi}{2x^3}\, \text{ctnh}(\pi x) + \frac{\pi^2}{2x^2}\, \text{csch}^2(\pi x) \ .$$

#20 : GRADUATE OTHER

`PROBLEM:` Consider a random walk of N steps in one-dimension. Suppose that the steps are completely uncorrelated, and that the probability distribution of the displacement x of a single step is given by

$$\rho(x) \propto \frac{1}{1+|x|^{\alpha}}, \qquad -\infty < x < \infty$$

where $\alpha > 0$ is a constant. We want to know the probability distribution $P(R;N)$ for the total displacement $R = \sum_{n=1}^{N} x_n$ for $N \gg 1$, with x_n being the displacement of the n^{th} step.

`(a)` Write down the functional form of $P(R;N)$ expected from the Central Limit Theorem up to a normalization constant. Express parameter(s) of P in terms of integral(s) of ρ. (You do not need to derive the Central Limit Theorem.) For what range of α is the result (and hence the Central Limit Theorem) valid?

`(b)` Compute the distribution $P(R;N)$ directly from $\rho(x)$ for $\alpha = 2$, and find its explicit form including normalization constant. How far does a walker typically go after N steps? Compare your result to that of part `(a)` and comment.

`SOLUTION:`

`(a)` Central Limit Theorem states that the sum of a large number of random variable x_i is Gaussian distributed, with the variance of the distribution being N times the variance of the distribution of x. Hence,

$$\begin{aligned} P(R;N) &\propto e^{-R^2/(2N\sigma)} \\ \text{where} \quad \sigma &\equiv \int_{-\infty}^{\infty} dx\, x^2 \rho(x). \end{aligned}$$

In this case, we have

$$\sigma \equiv \int_{-\infty}^{\infty} dx\, \frac{x^2}{1+|x|^{\alpha}}.$$

The integral diverges for $\alpha \le 3$; thus, the Gaussian distribution is not valid for $\alpha \le 3$.

(b) Let us start from the definition,

$$\begin{aligned} P(R;N) &\equiv \int dx_1 dx_2 \cdots dx_n \, \delta\left(R - \sum_{n=1}^{N} x_n\right) \prod_{n=1}^{N} \rho(x_n) \\ &= \int \frac{dk}{2\pi} e^{ikR} \left[\int dx e^{-ikx} \rho(x)\right]^N . \end{aligned}$$

For $\alpha = 2$,

$$\int_{-\infty}^{\infty} dx e^{-ikx} \rho(x) \propto \int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{1+x^2} = \pi e^{-|k|}.$$

Hence,

$$\begin{aligned} P(R;N) &\propto \int_{-\infty}^{\infty} \frac{dk}{2\pi} \, \pi \, e^{ikR - N|k|} \\ &= 2\,\mathrm{Re}\left[\int_0^{\infty} \frac{dk}{2\pi} \, \pi \, e^{ikR - Nk}\right] \\ &= \mathrm{Re}\left[\frac{1}{N - iR}\right] \\ &= \frac{N}{N^2 + R^2}. \end{aligned}$$

Normalization: $\int dR P(R;N) = 1$. Since $\int dR \frac{N}{N^2+R^2} = \pi$, we have

$$P(R;N) = \frac{1}{\pi} \frac{N}{N^2 + R^2}.$$

Variance of this distribution is given by

$$\int dR \, R^2 \, \frac{1}{\pi} \frac{N}{N^2 + R^2} = \infty.$$

Hence the total displacement diverges.

Comparing to part (a), we see that for $\alpha \le 3$, the Gaussian distribution becomes invalid and the probability distribution function approaches new form with broad power law tail. (This is worked out here only for $\alpha = 2$ but is generally true.) The variance of the distribution diverges, reflecting the broad tail of the distribution $\rho(x)$. But the distribution itself is still well-defined as it is still normalizable.