SP07 Sol'ns

Part 1

#1 : UNDERGRADUATE MECHANICS

PROBLEM: Consider an Atwood machine with a massless pulley and two masses, m and M, which are attached at opposite ends to a string of fixed length that is hung over the pulley. For this Atwood machine, the center of the pulley is supported by a spring of spring constant k.

(a) Find the Lagrangian and the resulting equations of motion.

(b) Find the equilibrium position of the pulley and its frequency of oscillations. Consider your result in the limits $m = M$ and discuss. *Hint:* using the method of Langrange multipliers significantly reduces the complexity of this problem.

#1 : UNDERGRADUATE MECHANICS

SOLUTION:

(a) We will assume that the center of the pulley is a distance z measured in the downward direction from equilibrium. The distance from the center of the pulley for M is x again measured in the downward direction. For mass m the distance from the center of the pulley is $L - x$ where L is a constant related to the length of the string. The kinetic energy of the two masses is

$$T = \frac{1}{2}M\left(\dot{z} + \dot{x}\right)^2 + \frac{1}{2}m\left(\dot{z} - \dot{x}\right)^2.$$

The potential energy of the two masses and the spring is

$$U = -Mg\left(z + x\right) - mg\left(z - x\right) + \frac{1}{2}kz^2,$$

so that the Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\left(\dot{z} + \dot{x}\right)^2 + \frac{1}{2}m\left(\dot{z} - \dot{x}\right)^2 + Mg\left(z + x\right) + mg\left(z - x\right) - \frac{1}{2}kz^2.$$

The x equation of motion is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \\
\left(M - m\right)g &= M\left(\ddot{z} + \ddot{x}\right) + m\left(\ddot{x} - \ddot{z}\right) = \left(M + m\right)\ddot{x} + \left(M - m\right)\ddot{z}
\end{aligned}$$

The z equation of motion is

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \\
(M+m)\,g - kz &= M\left(\ddot{z}+\ddot{x}\right) + m\left(\ddot{z}-\ddot{x}\right) = (M+m)\,\ddot{z} + (M-m)\,\ddot{x}
\end{aligned}$$

(b) Solving for $\ddot{x}$ from the x equation

$$\begin{aligned}
(M+m)\,\ddot{x} &= (M-m)\left(g-\ddot{z}\right) \\
\ddot{x} &= \frac{M-m}{M+m}\left(g-\ddot{z}\right).
\end{aligned}$$

Substituting this into the z equation

$$\begin{aligned}
(M+m)\,g - kz &= (M+m)\,\ddot{z} + \frac{(M-m)^2}{M+m}\left(g-\ddot{z}\right) \\
(M+m)^2 g - (M-m)^2 g - (M+m)\,kz &= (M+m)^2\,\ddot{z} - (M-m)^2\,\ddot{z} \\
4Mmg - (M+m)\,kz &= 4Mm\ddot{z} \\
\frac{4Mmg}{M+m} - kz &= \frac{4Mm}{M+m}\ddot{z}.
\end{aligned}$$

The new position of equilibrium occurs when $\ddot{z} = 0$ and is

$$z_o = \frac{4Mm}{M+m}\frac{g}{k}.$$

The frequency of oscillation is

$$\omega^2 = \frac{k\,(M+m)}{4Mm} = \frac{k}{4\mu}, \quad \text{where} \quad \mu = \frac{Mm}{M+m}.$$

In the limit $M = m$ the frequency of oscillation reduces to $\omega^2 = k/2m$, which is exactly what you would expect as neither mass is accelerating due to the gravitational field.

Alternate Solution

(a) The solution to this problem is much easier to obtain with the use of Lagrange multipliers. Let the distance to each mass be given by x_M and x_m respectively while the distance to the center of the suspended pulley is y. All of these distances are measured positive in the downward direction relative to some stationary position, e.g the center of the pulley when there are no masses attached to the string. The distance along the string from the from the center of the pulley to mass M is simply $x_M - y$ and correspondingly the equilivalent distance to m is $x_m - y$. Since the length of the string is unchanged, the constraint equation is

$$x_M - y + x_m - y = x_M + x_m - 2y = L = const.$$

Hence the modified Lagrangian is simply

$$\mathcal{L}' = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 - \frac{1}{2}ky^2 + \lambda\left(x_M + x_m - 2y\right),$$

where λ is a Lagrange mulitiplier. The three Lagrange equations of motion take the simple form

$$M\ddot{x}_M = Mg + \lambda, \quad m\ddot{x}_m = mg + \lambda, \quad \text{and} \quad -ky - 2\lambda = 0.$$

These are subject, of course, to the constraint $x_M + x_m - 2y = L \rightarrow \ddot{x}_M + \ddot{x}_m = 2\ddot{y}$.

(b) Summing the equations for x_M and x_m we find

$$\begin{aligned} \ddot{x}_M + \ddot{x}_m &= 2g + \lambda/M + \lambda/m \\ 2\ddot{y} &= 2g - \frac{k}{2}y\left(\frac{1}{M} + \frac{1}{m}\right) \\ \ddot{y} &= g - \frac{k}{4\mu}y, \end{aligned}$$

where again $\mu = Mm/\left(M + m\right)$. The equilibrium position is found when $\ddot{y} = 0$, or $y_o = 4\mu g/k$. The frequency of oscillations about this point is $\omega^2 = k/4\mu$. When $M = m$ then the frequency of oscillation is found from $\omega^2 = k/2m$ which is what you would expect as neither mass is accelerating due to the gravitational field.

#2 : UNDERGRADUATE MECHANICS

`PROBLEM`: Two particles (with reduced mass μ) that are orbiting each other interact via a potential energy $U = \frac{1}{2}kr^2$, where $k > 0$ and r is the distance between them.

(a) Find the equilibrium distance r_0 at which the two particles can circle each other at a constant distance of separation as a function of the angular momentum L.

(b) Determine if this is a position of stable equilibrium.

(c) Without solving an orbit equation, determine if those orbits are closed.

#2 : UNDERGRADUATE MECHANICS `SOLUTION`:

(a) The effective potential energy is

$$U_{eff}(r) = \frac{1}{2}kr^2 + \frac{L^2}{2\mu r^2}.$$

The position(s) of equilibrium are found from

$$\begin{aligned} \frac{dU_{eff}(r_o)}{dr} &= kr_o - \frac{L^2}{\mu r_o^3} = 0, \\ r_o^4 &= \frac{L^2}{\mu k}. \end{aligned}$$

(b) The second derivative of the effective potential at equilibrium is

$$\frac{d^2U_{eff}(r_o)}{dr^2} = k + 3\frac{L^2}{\mu r_o^4} = k + 3\frac{\mu k r_o^4}{\mu r_o^4} = 4k.$$

Since $k > 0$, the second derivative of the potential at equilibrium is greater than zero and this is a position of stable equilibrium.

(c) From the equation of motion

$$\mu\ddot{r} = -\frac{dU_{eff}(r)}{dr}.$$

For small displacements from equilibrium, $r = r_o + \delta r$, the equation of motion becomes

$$\begin{aligned} \mu\ddot{\delta r} &= -\frac{dU_{eff}(r_o)}{dr} - \frac{d^2U_{eff}(r_o)}{dr^2}\delta r = -4k\delta r, \\ \mu\ddot{\delta r} + 4k\delta r &= 0 \end{aligned}$$

so that the frequency of small radial oscillations about $r = r_o$ is $\omega^2 = 4k/\mu$.

(d) The angular frequency for a circular orbit is

$$\dot{\phi} = \frac{L}{\mu r_o^2}.$$

From the solution for r_o in part (a), $r_o^4 = L^2/\mu k$, therefore

$$\dot{\phi} = \frac{L\sqrt{\mu k}}{\mu L} = \sqrt{k/\mu}.$$

This is to be compared with the radial frequency (for a near circular orbit) of

$$\omega = 2\sqrt{k/\mu}.$$

We see that in this limit the relative distance between the particles goes through exactly two cycles for each full orbit. Hence the orbit is closed.

#3 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM`: A thin wire carries constant current I into one plate of a charging capacitor, and another thin wire carries constant current I out of the other plate. The capacitor plates are disks of radius a and separation $w \ll a$ (so edge effects can safely be neglected). The region between the plates has $\epsilon \approx \epsilon_0$, $\mu \approx \mu_0$, but does have a non-negligible, constant conductivity σ. *Note:* The capacitor is not an ideal capacitor, since the material between the plates is not a perfect insulator.

(a) Supposing that the charges are uniformly distributed on the plates, find a differential equation for the charge $Q(t)$ on the plates, and solve it for $Q(t)$, taking $Q(0) = 0$.

(b) Find the electric and magnetic fields in the gap. Approximate the electric field as just that due to the charged plates. When computing the magnetic field, include all sourcing contributions.

#3 : UNDERGRADUATE ELECTROMAGNETISM

`SOLUTION`: Choose coordinates so that $\vec{I} = I\hat{z}$ in the wire. With charge $Q(t)$, the electric field inside is $\vec{E} = \frac{Q(t)}{\pi a^2 \epsilon_0}\hat{z}$. The current density inside is then $\vec{J} = \sigma\vec{E} = \frac{\sigma Q(t)}{\pi a^2 \epsilon_0}\hat{z}$. Integrating this over an arbitrary surface of constant z, between the plates, gives the current flowing from one plate to another: $\vec{I}_{flow} = \int \vec{J} da = \hat{z}\sigma Q(t)/\epsilon_0$. Charge conservation implies that the charge on the plate satisfies

$$\frac{dQ(t)}{dt} = I - I_{flow} = I - \frac{\sigma}{\epsilon_0} Q,$$

with I the constant current flowing into the plate from the wire. We can solve this differential equation:

$$Q(t) = \frac{I\epsilon_0}{\sigma}\left(1 - e^{-\sigma t/\epsilon_0}\right).$$

The magnetic field inside satisfies $\vec{B} = B(s)\hat{\phi}$, and Ampere gives

$$B(s)2\pi s = \mu_0 \int da(J_z + \epsilon_0 \frac{\partial E_z}{\partial t}) = \frac{\mu_0}{\epsilon_0}(Q(t)\sigma + \epsilon_0 \dot{Q}(t))\left(\frac{s^2}{a^2}\right) = \mu_0 I \frac{s^2}{a^2},$$

so inside the plates, $s \leq a$, we have

$$\vec{B}(s) = \frac{\mu_0 I s}{2\pi a^2}\widehat{\phi}.$$

Outside the plates we have $\vec{B} = \mu_0 I / 2\pi s$, as usual for a wire.

#4 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM`: A sphere of radius R has volume charge density $\rho = Kr^n$, for some constants K and n. The region $r > R$ is filled with a conductor (all the way to infinity).

(a) Find the volume charge density ρ in the region $r > R$, inside the conductor, and the surface charge density at $r = R$.

(b) Find the electric field $\mathbf{E}$ everywhere, i.e. for $r < R$ and for $r > R$.

(c) Find the potential ϕ everywhere, taking ϕ to vanish at infinity.

(d) How much energy is stored in this system?

#4 : UNDERGRADUATE ELECTROMAGNETISM

`SOLUTION`: a) $\rho = 0$ in a conductor, and since $\vec{E} = 0$, the total charge enclosed by a gaussian surface in a conductor is zero, so the surface charge at $r = R$ cancels that of the charged sphere:

$$\sigma = -\frac{1}{4\pi R^2}\int_0^R Kr^n 4r^2 dr d\Omega = -\frac{KR^{n+1}}{n+3},$$

which is uniformly (by the spherical symmetry) distributed on the inner surface of the conductor.

b) By symmetry $\vec{E}(\vec{r}) = E(r)\widehat{r}$. Gauss' law applied to a gaussian surface sphere of radius $r < R$ gives

$$\vec{E}(r) = \frac{Q_{encl}(r)}{4\pi r^2 \epsilon_0}\widehat{r} = \frac{Kr^{n+1}}{(n+3)\epsilon_0}\widehat{r} \qquad \text{for} \quad r < R.$$

Again $\vec{E} = 0$ for $r > R$, inside the conductor.

c) Writing the above $\vec{E}$ as $-\vec{\nabla}\phi$, we obtain

$$\phi = -K\frac{(r^{n+2} - R^{n+2})}{\epsilon_0(n+2)(n+3)} \quad \text{for} \quad r < R$$

and $\phi = 0$ for $r \geq R$.

d)

$$\mathcal{U} = \frac{1}{2}\int \rho\phi dV = \frac{\epsilon_0}{2}\int \vec{E}^2 dV = \frac{2\pi K^2 R^{2n+5}}{\epsilon_0(2n+5)(n+3)^2}$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: A particle of mass m is in the ground state of a harmonic oscillator with spring constant $k = m\omega^2$. At $t = 0$, the spring constant changes suddenly to $k' = \lambda^2 m\omega^2$, where λ is a constant. Find the probability that the oscillator remains in its ground state.

#5 : UNDERGRADUATE QUANTUM MECHANICS

SOLUTION: **Solution:** The oscillator changes from frequency ω to $\lambda\omega$. In terms of wavefunctions, the initial wavefunction is

$$\psi_i = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \tag{1}$$

The new ground state wavefunction is

$$\psi_0 = \left(\frac{m\lambda\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\lambda\omega x^2}{2\hbar}} \tag{2}$$

The probability is

$$p = |\langle\psi_0|\psi\rangle|^2 = \left|\int_{-\infty}^{\infty} \mathrm{d}x\ \psi_0^*(x)\psi_i(x)\right|^2 = \frac{2\sqrt{\lambda}}{1+\lambda} \tag{3}$$

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: AConsider two identical blocks of material, both with heat capacity C, which is independent of temperature. The "hot" one is initially at a temperature T_H. The "cold" one is initially at temperature T_C, with $T_C < T_H$. The two blocks act as the hot and cold "reservoirs" used to run a cyclic engine. The engine runs until both blocks reach a common temperature T_f. The entire process is reversible.

(a) What is the temperature T_f?

(b) How much work does the engine do?

(c) Compute the entropy change ΔS for each of the two blocks, and for the engine, in this process.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

SOLUTION: The entropy change of each block is $\Delta S = \int dQ/T = C\ln(T_f/T_i)$. The entropy change of the engine is zero (cyclic) and the total entropy change is zero (reversible). Implies

(a) $T_f = \sqrt{T_H T_C}$.

(b) $W = C(T_H - T_f) - C(T_f - T_C) = C(T_H + T_C - 2\sqrt{T_H T_C})$.

(c) $\Delta S_C = \frac{1}{2}C\ln(T_H/T_C) = -\Delta S_H$, $\Delta S_{eng} = 0$.

#8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: Suppose that a certain system of N particles has for its number of available states a function $\Omega(N, U, V) = \Omega(N, UV^b)$, i.e. the number of states available to the system depends on the internal energy U, and the volume V, only via the combined variable UV^b, with b some constant.

(a) The system initially has volume $V_i = 2m^3$ and energy $U_i = 100J$. It undergoes an isentropic expansion to volume $V_f = 4m^3$. What is the final energy U_f?

(b) Derive the expression for the pressure of this gas, as a function of U and V (and the constant b).

#8 : UNDERGRADUATE STATISTICAL MECHANICS

SOLUTION: (a) Isentropic means $\Delta S = 0$, so $\Delta\Omega = 0$, so $U_f V_f^b = U_i V_i^b$, so $U_f = 2^{-b}100J$.

(b) $dU = TdS - PdV + \mu dN$, so $P = -(\partial U/\partial V)_{S,N}$. Constant S means UV^b is a constant, so $dUV^b + bUV^{b-1}dV = 0$, so $P = bU/V$.

#9 : UNDERGRADUATE OTHER

PROBLEM: A helicopter can hover when the power output of its engine is P. A second helicopter is an exact copy of the first one but its linear dimensions are twice larger. What power output is needed to enable this second helicopter to hover? *Hint:* Use your intuition to decide which physical parameters (e.g., density of air, density of helicopter, etc.) of the system are important and then apply a dimensional analysis.

#9 : UNDERGRADUATE OTHER

SOLUTION: It is reasonable to assume that P depends on the weight of the helicopter mg, its linear size L, and the density of air ρ, as a certain power-law:

$$P \propto (mg)^{\alpha} \times L^{\beta} \times \rho^{\gamma}.$$

The combination of exponents that gives the correct dimension of P is unique:

$$P \propto (mg)^{3/2} L^{-1} \rho^{-1/2}.$$

Since $mg \propto L^3$, this entails

$$P \propto L^{9/2} L^{-1} = L^{7/2}.$$

Therefore, the required power is $2^{7/2} P = 8\sqrt{2}\, P$.

SP07 Sol'ns

Part 2

#11 : GRADUATE MECHANICS

PROBLEM: A thin hoop of radius R whose mass M is distributed uniformly around the perimeter of the hoop is free to roll along a horizontal track without slipping. Attached to the inside of this hoop is a bead of mass m which is free to slide without friction around the hoop. The whole system is in a uniform gravitational acceleration g.

(a) Find the Lagrangian for this system.

(b) Find the Lagrangian equations of motion and any possible equilibrium positions for the bead. Determine if these positions are stable or unstable.

(c) Find the frequency of small amplitude oscillations of the bead about all possible positions of stable equilibrium. Consider your results in the limit $M \gg m$ and discuss.

#11 : GRADUATE MECHANICS

SOLUTION:

(a) The generalized coordinate that we will use for the hoop is X, the distance to the center of the hoop. Since it is a horizontal track there are no potential energy considerations for the hoop. The kinetic energy of the hoop is

$$T_{hoop} = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\omega^2.$$

The momentum of inertia for this hoop is $I = MR^2$. from the no slip condition $\dot{X} = R\omega$. Hence the kinetic energy for the hoop is

$$T_{hoop} = \frac{1}{2}(2M)\dot{X}^2 = M\dot{X}^2.$$

The generalized coordinate for the bead is the angle ϕ which is measured from a horizontal axis passing through the center of the hoop. The x and y coordinates of the bead are

$$x = X + R\sin\phi \quad \text{and} \quad y = R(1-\cos\phi).$$

The velocities are

$$\dot{x} = \dot{X} + R\cos\phi\dot{\phi} \quad \text{and} \quad \dot{y} = R\sin\phi\dot{\phi}.$$

Hence the kinetic energy of the bead is

$$T_{bead} = \frac{1}{2}m\left(\dot{X}^2 + 2R\cos\phi\dot{X}\dot{\phi} + R^2\dot{\phi}^2\right).$$

The potential energy of the bead is

$$U = mgy = mgR\left(1 - \cos\phi\right)$$

Combining these terms to find the Lagrangian yields

$$\mathcal{L} = \frac{1}{2}\left(2M + m\right)\dot{X}^2 + \frac{1}{2}m\left(2R\cos\phi\dot{X}\dot{\phi} + R^2\dot{\phi}^2\right) - mgR\left(1 - \cos\phi\right).$$

(b) The EOM for X is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} &= \frac{\partial\mathcal{L}}{\partial X} \\ \frac{d}{dt}\left[\left(2M + m\right)\dot{X} + mR\cos\phi\dot{\phi}\right] &= \left(2M + m\right)\ddot{X} - mR\sin\phi\dot{\phi}^2 + mR\cos\phi\ddot{\phi} = 0. \end{aligned}$$

Note that this expression is basically the conservation of momentum in the x direction.

The EOM for ϕ is

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} &= \frac{\partial\mathcal{L}}{\partial\phi} \\ \frac{d}{dt}\left[mR\cos\phi\dot{X} + mR^2\dot{\phi}\right] &= -mgR\sin\phi - mR\sin\phi\dot{X}\dot{\phi} \\ -mR\sin\phi\dot{\phi}\dot{X} + mR\cos\phi\ddot{X} + mR^2\ddot{\phi} &= -mgR\sin\phi - mR\sin\phi\dot{X}\dot{\phi} \\ mR\cos\phi\ddot{X} + mR^2\ddot{\phi} &= -mgR\sin\phi. \end{aligned}$$

Substituting for $\ddot{X}$ from the X equation yields

$$R\cos\phi\left(\frac{mR\sin\phi\dot{\phi}^2 - mR\cos\phi\ddot{\phi}}{2M+m}\right) + R^2\ddot{\phi} = -gR\sin\phi.$$

At equilibrium we have $\dot{\phi} = \ddot{\phi} = 0$ and

$$-gR\sin\phi = 0 \rightarrow \phi = 0 \text{ or } \pi$$

are positions of equilibrium. First we will consider $\phi = \pi$. To do this we consider small deviations from $\phi = \pi$, i.e. $\phi = \pi + \epsilon$ and only keep linear terms in ϵ. With these restrictions the EOM for ϕ becomes

$$-\frac{m}{2M+m}R^2\ddot{\epsilon} + R^2\ddot{\epsilon} = \frac{2M}{2M+m}R^2\ddot{\epsilon} = gR\epsilon.$$

This EOM is not stable as the force is not a restoring force. Now we will consider small deviations from $\phi = 0$, i.e. $\phi = \epsilon$ and only keep linear terms in ϵ. With these restrictions the EOM for ϕ becomes

$$-\left(\frac{m}{2M+m}\right)R^2\ddot{\epsilon} + R^2\ddot{\epsilon} = \frac{2M}{2M+m}R^2\ddot{\epsilon} = -gR\epsilon.$$

Now the gravitational forces act as restoring forces and $\phi = 0$ is a position of stable equilibrium.

(c) Rewriting the equation for small deviations from stable equilibrium we find

$$\ddot{\epsilon} = -\frac{2M+m}{2M}\frac{g}{R}\epsilon,$$

so that the frequency of small amplitude oscillations is

$$\omega^2 = \frac{2M+m}{2M}\frac{g}{R}.$$

In the limit $M >> m$ we find that

$$\omega^2 = \frac{g}{R},$$

which is the frequency for a simple pendulum. That is what you would expect in that limit as the oscillations of the bead would not impact the extremely massive hoop.

#12 : GRADUATE MECHANICS

PROBLEM: Functions $A(\mathbf{p}, \mathbf{q})$ of coordinates $\mathbf{q}$ and their conjugate momentum $\mathbf{p}$ generate motions through phase space $(\mathbf{q}, \mathbf{p})$ via Poisson bracket, by the equation

$$\frac{dB(\mathbf{p}, \mathbf{q})}{d\eta} = \{B(\mathbf{p}, \mathbf{q}), A(\mathbf{p}, \mathbf{q})\}_{PB},$$

where η is a scalar label along such a motion, $B(\mathbf{p}, \mathbf{q})$ is an arbitrary function on phase space, and $\{\}_{PB}$ denotes the Poisson bracket.

Show explicitly that the components of angular momentum $\mathbf{L}(\mathbf{p}, \mathbf{q}) = \mathbf{q} \times \mathbf{p}$ generate rotations in phase space. Show this for $\mathbf{p}$ and for $\mathbf{q}$. Show that scalar functions in phase space are unchanged by the Poisson bracket operation with $\mathbf{L}$.

#12 : GRADUATE MECHANICS

SOLUTION:

Angular momentum is to be used as the generator of motions in phase space. Let's answer all the questions using the third component of angular momentum. The answer is similar for L_1 and for L_2.

$L_3 = q_1p_2 - q_2 - p_1$, and this generates motion in p,q space via

$$\begin{aligned} \frac{dB(\mathbf{p},\mathbf{q})}{d\eta} = \{L_3, B(\mathbf{p},\mathbf{q})\} &= \{q_2p_1 - q_1p_2, B(\mathbf{p},\mathbf{q})\}, \\ &= p_2\frac{\partial B(\mathbf{p}\,q)}{\partial p_1} - p_1\frac{\partial B(\mathbf{p}\,q)}{\partial p_2} - q_1\frac{\partial B(\mathbf{p}\,q)}{\partial q_2} + q_2\frac{\partial B(\mathbf{p}\,q)}{\partial q_1} \end{aligned} \tag{1}$$

Now we take $B(\mathbf{q},\mathbf{p})$ separately as the components of $\mathbf{q} = (q_1, q_2, q_3)$:

$$\begin{aligned} \frac{dq_1}{d\eta} &= +q_2 \\ \frac{dq_2}{d\eta} &= -q_1 \\ \frac{dq_3}{d\eta} &= 0. \end{aligned} \tag{2}$$

The solution to these is

$$\begin{aligned} q_1(\eta) &= q_1\cos(\eta) + q_2\sin(\eta) \\ q_2(\eta &= q_2\cos(\eta) - q_1\sin(\eta) \\ q_3(\eta &= q_3, \end{aligned} \tag{3}$$

where the values q_1, q_2, q_3 on the right hand side are those at $\eta = 0$.

This is clearly a rotation of the vector $\mathbf{q}$ around the z-axis by an angle η.

The same operations show that the components of $\mathbf{p}$ are also rotated by L_3 and that the other components of L_1, L_2, L_3 rotate vectors.

Now take $B(\mathbf{q},\mathbf{p}) = \mathbf{q}\cdot\mathbf{p} = q_1p_1 + q_2p_2 + q_3p_3$, and rotate this using L_3 to find 0.

Any vector made out of $\mathbf{p}$ and $\mathbf{q}$ only can be written as $\mathbf{p}(A(\mathbf{q}^2, \mathbf{p}^2, \mathbf{p}\cdot\mathbf{q}) + \mathbf{q}(B(\mathbf{q}^2, \mathbf{p}^2, \mathbf{p}\cdot\mathbf{q})$, and A and B are functions of the rotational scalars. This acts like a vector under rotations and these are generated by the components of L.

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: In a region of space there is a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{x}}$. An uncharged solid copper sphere of radius R moves with a velocity $\mathbf{v} = v_0\hat{\mathbf{y}}$ through the magnetic field. Compute the electric field everywhere (inside and outside the sphere), as seen in the lab frame. Also, find the surface charge density on the surface of the sphere. Take $v_0 \ll c$.

#13 : GRADUATE ELECTROMAGNETISM

SOLUTION:

<u>#13 GRADUATE ELECTROMAGNETISM</u>

SOLUTION:

(a)

Inside sphere, $\vec{F} = q(E + \frac{\vec{v}\times\vec{B}}{c}) = 0.$

$\therefore \vec{E}_{in} = -\frac{1}{c}\vec{v}\times\vec{B} = -\frac{1}{c}v_0 B_0 \hat{y}\times\hat{x} = \frac{1}{c}v_0 B_0 \hat{z} = \frac{1}{c}v_0 B_0 \cos\theta\cdot\hat{r} - \frac{1}{c}v_0 B_0 \sin\theta\cdot\hat{\theta}.$

$\Phi_{in} = -\int \vec{E}_{in}\cdot d\vec{r} = -\frac{1}{c}v_0 B_0 r\cos\theta.$

Outside sphere, $\Phi_{out}(r)$ satisfies $\nabla^2\Phi_{out} = 0$. Expand potential in Legendre polynomials:

$$\Phi_{out}(r,\theta) = \sum_{\ell}\left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$= a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta$$

Potential cannot diverge as $r\to\infty$, therefore $a_i = 0$. Matching potentials at r=R, we have

$\Phi_{in}(R) = \Phi_{out}(R)$, therefore b_0=0 and $b_1 = -\frac{v_0 B_0 R^3}{c}$. Therefore

$$\Phi_{out} = -\frac{v_0 B_0 R^3\cos\theta}{cr^2}.$$

$$\vec{E}_{out} = -\nabla\Phi_{out} = -2\frac{v_0 B_0 R^3\cos\theta}{cr^3}\hat{r} - \frac{v_0 B_0 R^3\sin\theta}{cr^3}\hat{\theta}.$$

(b) Surface charge is given by

$$\sigma = \frac{1}{4\pi}\left(E_{\perp,out} - E_{\perp,in}\right)\Big|_{r=R} = \frac{1}{4\pi}\left(-2\frac{v_0 B_0\cos\theta}{c} - \frac{v_0 B_0\cos\theta}{c}\right) = -3\frac{v_0 B_0\cos\theta}{4\pi c}.$$

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: A point charge $-2q$ is at the origin, $\mathbf{r} = 0$, and two point charges, each $+q$, are at $\mathbf{r} = \pm a\hat{z}$. Consider the limit $a \to 0$, with $Q = qa^2$ held fixed.

(a) Find the scalar potential $\phi(\mathbf{r})$ in spherical coordinates.

(b) This system of charges is now placed inside a grounded, conducting spherical shell, of radius b (with $b \gg a$). Now find the scalar potential $\phi(\mathbf{r})$ everywhere, both inside and outside of the shell (again, in spherical coordinates).

#14 : GRADUATE ELECTROMAGNETISM

SOLUTION:

#14 GRADUATE ELECTROMAGNETISM

SOLUTION:

(a)

$$\Phi_0(\bar{r}) = \frac{q}{|\bar{r} - a\bar{z}|} + \frac{q}{|\bar{r} + a\bar{z}|} - \frac{2q}{|\bar{r}|}$$

$$= \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} + \frac{q}{\sqrt{r^2 + a^2 + 2ar\cos\theta}} - \frac{2q}{r}$$

$$= \frac{q}{r} \cdot \frac{1}{\sqrt{1 + (a/r)^2 - 2(a/r)\cos\theta}} + \frac{q}{r} \cdot \frac{1}{\sqrt{1 + (a/r)^2 + 2(a/r)\cos\theta}} - \frac{2q}{r}$$

$$\approx \frac{q}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right\}^2\right]$$

$$+ \frac{q}{r}\left[1 - \frac{1}{2}\left(\frac{a}{r}\right)^2 - \left(\frac{a}{r}\right)\cos\theta + \frac{3}{8}\left\{\left(\frac{a}{r}\right)^2 + 2\frac{a}{r}\cos\theta\right\}^2\right] - \frac{2q}{r}$$

$$\approx \frac{q}{r}\left[2 - \left(\frac{a}{r}\right)^2 + \frac{3}{4}\left\{2\frac{a}{r}\cos\theta\right\}^2 - 2\right], \text{ to second order in } a/r$$

$$\approx \frac{qa^2}{r^3}\left(3\cos^2\theta - 1\right) = \frac{Q}{r^3}\left(3\cos^2\theta - 1\right)$$

(b) On spherical shell $\Phi(r = b) = 0$. Inside shell we have $\Phi_{in}(r) = \Phi_0(r) + \Phi_1(r)$ where $\Phi_1(r)$ satisfies $\nabla^2\Phi_1 = 0$. Expand potential in Legendre polynomials:

$$\Phi_1(r,\theta) = \sum_\ell \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$= a_0 + \frac{b_0}{r} + \left(a_1 r + \frac{b_1}{r^2}\right)\cos\theta + \left(a_2 r^2 + \frac{b_2}{r^3}\right) P_2(\cos\theta)$$

Now at r=0 potential should not diverge, and hence $b_i = 0$. At r=b we have

$$0 = \frac{Q}{b^3}\left(3\cos^2\theta - 1\right) + a_0 + a_1 b\cos\theta + a_2 b^2 \frac{(3\cos^2\theta - 1)}{2}, \text{therefore } a_0 = a_1 = 0$$

and $a_2 = -\frac{2Q}{b^5}$. Therefore the interior potential is given by

$$\Phi_{in}(r) = \Phi_0(r) + \Phi_1(r) = \frac{Q}{r^3}\left(3\cos^2\theta - 1\right)\left[1 - \left(\frac{r}{b}\right)^5\right].$$

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: The Dirac Equation for a relativistic electron with no external forces is

$$(i\gamma^\mu \partial_\mu - k_c)\,\psi(x) \;=\; 0$$

where $\partial_\mu = \partial/\partial^\mu = ((1/c)\partial/\partial t, \nabla)$, $k_c = mc/\hbar$ is the Compton wave-vector, and the Dirac matrices are

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \qquad \gamma^j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}$$

where I is the unit 2×2 matrix, and σ_j are the three Pauli matrices.

(a) Find one (unnormalized) energy eigenstate with positive energy and momentum $(0, 0, p)$ along the z direction.

(b) Calculate the expectation value for the velocity for the state you have found above.

#15 : GRADUATE QUANTUM MECHANICS

SOLUTION:

2. The Dirac equation for a relativistic electron under no external force is given by

$$(i\gamma^\mu \partial_\mu - k_c)\psi(x) = 0, \tag{9}$$

where $\partial_\mu = \partial/\partial x^\mu = (\partial/c\partial t, \nabla)$, $k_C = mc/\hbar$ is the Compton wave-vector, and the Dirac γ-matrices are

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}, \tag{10}$$

I being the unit 2×2 matrix and σ_j are the three Pauli matrices.

(a) Find one (unnormalized) energy eigenstate state with a positive energy and momentum $(0, 0, p)$ along the z direction.

Solution – The translational symmetry of the Dirac operator $\gamma^\mu \partial_\mu$ or the Hamiltonian H means that the solution can be a plane wave:

$$\psi(x) = u(p)e^{-ip\cdot x/\hbar} = u(\vec{p})e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar}. \tag{11}$$

Putting the wave function (11) in the Dirac equation gives

$$\begin{bmatrix} mc^2 & \vec{\sigma}\cdot\vec{p}c \\ \vec{\sigma}\cdot\vec{p}c & -mc^2 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = E \begin{bmatrix} u_A \\ u_B \end{bmatrix}. \tag{12}$$

These in turn give

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}c}{E - mc^2} u_B, \tag{13}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}c}{E + mc^2} u_A, \tag{14}$$

and

$$u_j = \frac{(\vec{\sigma}\cdot\vec{p}c)^2}{E^2 - (mc^2)^2} u_j, \quad j = A \text{ or } B \tag{15}$$

From the last expression, the eigenenergies are

$$E = \pm\sqrt{(mc^2)^2 + p^2c^2}. \tag{16}$$

If we take

$$u_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{17}$$

u_B is given by Eq. (14). The positive-energy state is

$$u = N \begin{bmatrix} 1 \\ 0 \\ pc/(|E| + mc^2) \\ 0 \end{bmatrix}, \tag{18}$$

where we have used $\vec{p} = (0, 0, p)$.

(b) Deduce a reasonable form for the velocity operator.

Solution – From the Heisenberg equation of motion,

$$\frac{dx_k}{dt} = \frac{i}{\hbar}[H, x_k], \tag{19}$$

and the commutation relations

$$\begin{aligned} [p_j, x_k] &= -i\hbar\delta_{jk}, \\ [H, x_k] &= -i\hbar\alpha_k c, \end{aligned} \tag{20}$$

we obtain the velocity operator

$$\frac{dx_k}{dt} = \alpha_k c. \tag{21}$$

The velocity operator is $\vec{\alpha} c$

(c) Calculate the expectation value for the velocity for the state you have found above.

Solution – The velocity is along the z direction and is

$$\langle u|\alpha_z|u\rangle = N^2 \frac{2pc^2}{E + mc^2}. \tag{22}$$

(d) Does the velocity of this state exhibits *Zitterbewegung* or Flutter?

Solution – No, *Zitterbewegung* is due to interference between a positive and a negative energy solution. A single plane wave state would yield no *Zitterbewegung*.

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: In studying the hydrogen atom one takes the proton to be a point charge with mass M. Suppose instead that the proton charge is distributed uniformly over the surface of a spherical shell of radius $r_0 = 10^{-15}\,\mathrm{m}$. Using perturbation theory, calculate the shift in energy of the $1s$ level of hydrogen to first order in the perturbation.

#16 : GRADUATE QUANTUM MECHANICS

SOLUTION:

Grad Quantum

1. ~~What is~~ Find an estimate for the effect of the finite size of the proton on the ground state energy of the hydrogen atom. Take the radius of the proton to be $1\,\text{fm} = 10^{-13}\,\text{cm}$.

Hint: simplest estimate: treat the proton as a thin, uniformly charged <u>spherical shell</u> of radius s. ~~This way, In~~ This way, the potential energy of the electron when it is inside the shell is constant.

where $a = 0.529\,\text{Å}$. The ground state energy is $E = -\frac{mc^2}{2}\alpha^2$, $\alpha^2 = \left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)^2 \approx \left(\frac{1}{137.04}\right)^2$

The unperturbed Hydrogen ground state w.f. is $\Psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$

solution: $\Rightarrow$ potential in shell $= \frac{-e^2}{4\pi\epsilon_0 s}$

$\Rightarrow$ perturbed Hamiltonian: $\cancel{\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{s}-\frac{1}{r}\right)}$

$$H' = \begin{cases} \frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{s}\right) & 0 < r < s \\ 0 & r > s \end{cases}$$

$$\Delta E = \langle \Psi | H' | \Psi \rangle$$

$$\Delta E = \frac{-e^2}{4\pi\epsilon_0} \frac{1}{\pi a^3} 4\pi \int_0^s \left(\frac{1}{s} - \frac{1}{r}\right) e^{-2r/a} r^2\, dr$$

$$= \frac{-e^2}{\pi\epsilon_0 a^3}\left(\frac{1}{s}\int_0^s r^2 e^{-2r/a}\, dr - \int_0^s r e^{-2r/a}\, dr\right)$$

(1) solution continued

$$= \frac{e^2}{4\pi\varepsilon_0 a}\left[\left(1 - a/s\right) + \left(1 + a/s\right) e^{-2s/a}\right]$$

let $\varepsilon = \frac{2s}{a} \Rightarrow$ small!

then term in square brackets is

$$\left(1 - 2/\varepsilon\right) + \left(1 + 2/\varepsilon\right)\left(1 - \varepsilon + \varepsilon^2/2 - \varepsilon^3/6 + \dots\right)$$

$$= \varepsilon^2/6 + (\ )\varepsilon^3 + (\ )\varepsilon^4 + \dots$$

to leading order $\Delta E = \left(\frac{e^2}{4\pi\varepsilon_0 a}\right)\frac{\cancel{4}2 s^2}{3\cancel{4} a^2}$

$E = E_1 = \frac{-2m e^4}{2(4\pi\varepsilon_0)^2\hbar^2}$

$$\frac{\Delta E}{E} = \frac{e^2}{4\pi\varepsilon_0}\left(-\frac{2(4\pi\varepsilon_0)}{e^2}\right)\frac{2s^2}{3a^2} = -4/3\left(\frac{s}{a}\right)^2$$

so if $a = 0.5\ \mathring{A} = 5\times10^{-9}$ cm

and $s = 1$ fm $= 10^{-13}$ cm

$$\frac{\Delta E}{E} \simeq -4/3\left(\frac{10^{-13}}{5\times10^{-9}}\right)^2 = -4/3\cdot\left(2\times10^{-5}\right)^2$$

$$= -16/3\times10^{-10} \sim -10^{-11}$$

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: Compute the chemical potential μ of an ideal gas in the Boltzmann limit, $N\lambda^3 \ll 1$, where N is the particle density, $\lambda = \sqrt{2\pi\hbar^2/(mkT)}$ is the thermal de Broglie wavelength, and m is the particle mass.

#17 : GRADUATE STATISTICAL MECHANICS

SOLUTION: The sought chemical potential is the solution of the equation

$$N = \int \frac{d^3p}{(2\pi\hbar)^3} f(\varepsilon_p),$$

where $f(\varepsilon) = \exp[(\mu - \varepsilon)/kT]$ is the Boltzmann distribution function and $\varepsilon_p = p^2/2m$ is the energy of the state with momentum p. This equation can be written as

$$N \exp(-\mu/kT) = I^3, \quad I = \int \frac{dp}{2\pi\hbar} \exp\left(-\frac{p^2}{2mkT}\right) = \frac{1}{\lambda}.$$

From here we obtain

$$\mu = kT \ln(N\lambda^3).$$

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM`: The Hamiltonian of the LC-circuit has the form analogous to that of a harmonic oscillator:

$$H = \frac{L}{2}\left(\frac{dQ}{dt}\right)^2 + \frac{1}{2C}Q^2,$$

where Q is the charge on the capacitor and $I = dQ/dt$ is the current through the inductor.

(a) Find the quantum energy levels of this system.

(b) Calculate the rms voltage noise $\langle V^2\rangle^{1/2}$ in the circuit at temperature T.

#18 : GRADUATE STATISTICAL MECHANICS

`SOLUTION`: (a) By analogy to the harmonic oscillator, we immediately find $E_n = \hbar\omega(n + 1/2)$, where $n = 0, 1, \ldots$ and $\omega = 1/\sqrt{LC}$.

(b) A quick way to compute $\langle V^2\rangle$ is to use its relation to the total energy E. Using the analogy to the harmonic oscillator once again, we have

$$\frac{E}{2} = \left\langle \frac{CV^2}{2} \right\rangle = \left\langle \frac{LI^2}{2} \right\rangle, \quad E = \hbar\omega\left(n_B + \frac{1}{2}\right), \quad n_B = \frac{1}{e^{\hbar\omega/kT} - 1}.$$

Accordingly,

$$\langle V^2\rangle = \frac{\hbar\omega}{2C}\coth\left(\frac{\hbar\omega}{2kT}\right), \quad \omega = \frac{1}{\sqrt{LC}}.$$

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left[\underbrace{\frac{e^{iax}}{x^2+b^2}}_{\text{close above}} + \underbrace{\frac{e^{-iax}}{x^2+b^2}}_{\text{close below}} \right]$$

$$= \frac{\pi}{2b} e^{-ab} + \frac{\pi}{2b} e^{-ab} = \frac{\pi}{b} e^{-ab}$$

#20 : GRADUATE MATHEMATICAL METHODS

PROBLEM: Find the asymptotic expansion as $x \to +\infty$ of the Airy integral

$$A(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(tx+t^3/3\right)}\,\mathrm{d}t$$

#20 : GRADUATE MATHEMATICAL METHODS

SOLUTION:

Find the leading term in the asymptotic expansion of

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(t^3/3 + tx)}\, dt$$

as $x \to +\infty$.

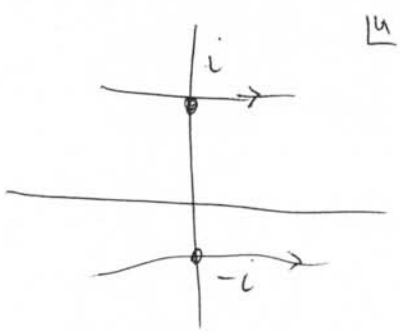
let $t = \sqrt{x}\, u$

$$Ai(x) = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{\infty} du\; e^{i x^{3/2}\left(\frac{u^3}{3} + u\right)}$$

extrema at $\left(\frac{u^3}{3} + u\right)' = u^2 + 1 = 0 \Rightarrow u = \pm i$

let $u = i + z \quad i\left(\frac{u^3}{3} + u\right) \sim -\frac{2}{3} - z^2$

$u = -i + z \qquad\qquad \sim \frac{2}{3} + z^2$



Go through the saddle point at $u = i$ which is a local maximum:

$$Ai(x) \simeq \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{\infty} dz\; e^{x^{3/2}\left(-2/3 - z^2\right)}$$

$$= \frac{1}{2\pi}\sqrt{x}\; e^{-2/3\, x^{3/2}}\sqrt{\frac{\pi}{x^{3/2}}} = \frac{1}{2\sqrt{\pi}}\,\frac{1}{x^{1/4}}\, e^{-\frac{2}{3}x^{3/2}}$$