

**PHYSICS DEPARTMENT EXAM
FALL 2007. PART I**

INSTRUCTIONS

- Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. The questions are grouped in five Sections: Mechanics, E&M, Quantum, StatMech, and General. You must attempt at least one problem from each Section. Credit will be assigned for seven (7) questions only.
- You should not have anything close to you other than your pens, pencils, calculator, and food items. Please deposit your belongings (books, notes, backpacks, etc.) in a corner of the exam room.
- Departmental examination paper is provided. Colored scratch paper is also provided. Please make sure you:
 - a. Write the problem number and your ID number on each sheet;
 - b. Write only on one side of the paper;
 - c. Start each problem on the attached examination sheets;
 - d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.
- At the conclusion of the examination period, please staple sheets from each problem together. **Circle the seven problems you wish to be graded:**

Mechanics	E & M	Quantum	Stat Mech	General					
1	2	3	4	5	6	7	8	9	10

- Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: A rope of mass M and length L is suspended in the earth's gravitational field, g , with the bottom end of the rope touching a surface. The rope is then released from rest and falls limply on the surface (i.e., without the elements bouncing upwards). Find the force $F(t)$ on the surface as a function of time, $0 < t < \infty$, and make a sketch of it. At what time does $F(t)$ reach its maximum? What is the value of this maximum force?

SOLUTION: All the elements of the rope are in free fall. It takes time $T = \sqrt{2L/g}$ for the last element to reach the surface. Hence, at $t > T$ we have

$$F(t) = Mg, \quad t > T.$$

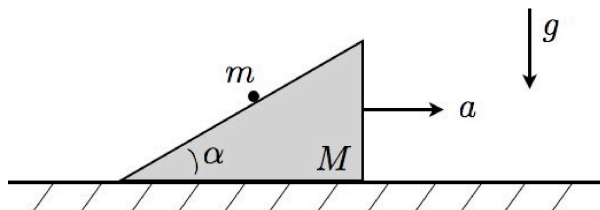
Consider now $0 < t < T$. Here F is a sum of two terms, $F = F_1 + F_2$. The first one is the weight $F_1 = \mu l g$ of the part that has already fallen. Here $l(t) = gt^2/2$ is the length of the fallen piece and $\mu = M/L$ is the mass per unit length. The second term is the transfer of momentum $F_2 = dP/dt$ from the element of length $dl = v dt$ that comes to rest during the time $(t, t + dt)$. The velocity of this segment is $v = gt$, and so $F_2 = \mu v(v dt)/dt = \mu g^2 t^2$. Accordingly,

$$F(t) = \frac{1}{2} \mu g^2 t^2 + \mu g^2 t^2 = \frac{3M}{2L} g^2 t^2 = 3Mg \frac{t^2}{T^2}, \quad 0 < t < T.$$

We see that F reaches a maximum $F = 3Mg$ at $t = T$, then experiences a sudden drop to the three times smaller value, after which it remains constant.

#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: A wedge of mass M rests on a horizontal frictionless surface. A point mass m is placed on the wedge, whose surface is also frictionless. Find the horizontal acceleration a of the wedge.



SOLUTION: Let N be the normal reaction force, then the Newton equation for the wedge projected on the horizontal axis gives

$$Ma = N \sin \alpha \quad \therefore \quad N = Ma / \sin \alpha .$$

Consider now the motion of the small mass. Its acceleration is $\mathbf{a} + \mathbf{a}_{\parallel}$. The second term \mathbf{a}_{\parallel} is along the incline, and so it vanishes if projected on the direction normal to the incline. The corresponding Newton law reads

$$-ma \sin \alpha = N - mg \cos \alpha .$$

Substituting N from the first equation and solving for a , we get

$$a = g \frac{\sin \alpha \cos \alpha}{(M/m) + \sin^2 \alpha} .$$

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: A circular loop of wire is of radius R and carries current I . The wire lies in the plane $z = 0$ with its center at the origin of coordinates. Let (ρ, θ, z) be the cylindrical coordinates. Determine:

- (a) Magnetic field \mathbf{B} at a given point $(0, 0, z)$ on the z -axis.
 (b) The radial component $B_\rho(\rho, \theta, z)$ of \mathbf{B} at a distance $\rho \ll R$ off the z -axis.

Hint: For an arbitrary vector \mathbf{X}

$$\operatorname{div} \mathbf{X} = \frac{1}{\rho} \partial_\rho(\rho X_\rho) + \frac{1}{\rho} \partial_\theta X_\theta + \partial_z X_z.$$

SOLUTION: (a) By symmetry, $\mathbf{B} = B_z \hat{z}$ at $\rho = 0$. Let $I dl$ be the differential current element along the ring, then the Biot-Savart law yields

$$dB_z = \frac{I dl}{c} \frac{\sin \phi}{R^2 + z^2}, \quad \sin \phi = \frac{R}{\sqrt{R^2 + r^2}},$$

where ϕ is the angle between the vector connecting this element to the observation point and the vertical. Integration over the ring leads to $dl \rightarrow 2\pi R$, and so

$$B_z = \frac{2\pi I}{c} \frac{R^2}{(R^2 + z^2)^{3/2}}.$$

(b) By Taylor expansion

$$B_\rho \simeq \rho \partial_\rho B_\rho(0, 0, z) \equiv \rho C, \quad \rho \ll R.$$

Next,

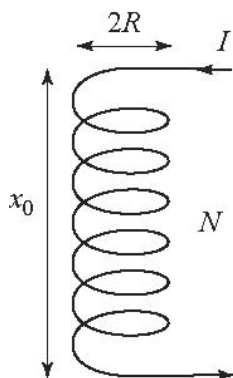
$$0 = \operatorname{div} \mathbf{B}|_{\rho=0} = 2C + \partial_z B_z \quad \therefore \quad C = -\partial_z B_z / 2.$$

Computing the last derivative, we finally get

$$B_\rho \simeq \frac{3\pi I}{c} \frac{R^2 \rho z}{(R^2 + z^2)^{5/2}}.$$

#4 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: A spiral spring has N turns and initial length x_0 . How does its length change if a small current I is made to flow through it? The spring has an elastic constant k for longitudinal deformations. Assume that the spring can be treated as a perfect solenoid and that its radius R remains fixed.



SOLUTION: It is convenient to think that the current I was created by some external source while the spring was kept at the original length x_0 . The coil was then short-circuited leaving the current flowing. Finally, the spring was allowed to expand or contract freely. In this formulation, the magnetic flux $\Phi = LI$ remains constant since the circuit has zero resistance, which simplifies the derivation.

Since the current flows in the same direction in the adjacent coils of the spring, these coils attract. Hence, the spring would shorten. Let x be the new length. To find the contraction $\Delta x = x - x_0$ we can minimize the sum of the magnetic and the elastic energy,

$$E = \frac{\Phi^2}{2c^2L} + \frac{k}{2}(\Delta x)^2.$$

where

$$L = 4\pi^2 N^2 R^2 / c^2 x$$

is the inductance of the spring (the derivation of this known formula is elementary). Since $\Phi = \text{const}$, it is easy to take the derivative of E with respect

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to x . Equating it to zero, one finds that the minimum energy is reached at

$$\Delta x \simeq -\frac{2\pi^2}{c^2} \frac{N^2 R^2 I^2}{kx_0^2}.$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: Consider a one-dimensional quantum particle with the Hamiltonian

$$H = T + V(x), \quad T = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}.$$

Suppose that m suddenly changes from m_0 to $m_1 = m_0/\lambda$ at $t = 0$. Assuming the particle was in the ground state at $t < 0$, find: (i) probability to remain in the ground state at $t > 0$ and (ii) change in the energy expectation value $\langle H \rangle$. Consider two cases:

(a) Infinite square well, i.e., $V(x) = 0$ for $0 < x < L$ and infinite otherwise.

(b) Parabolic well, $V(x) = Cx^2/2$. *Hint:* For any $a > 0$,

$$\int_{-\infty}^{\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}.$$

SOLUTION: In a sudden perturbation the wavefunction does not change; hence, the probability in question is the squared overlap of the two ground-state wavefunctions, $\langle \psi_1 | \psi_0 \rangle^2$. The potential energy $\langle V \rangle$ also does not change. Therefore,

$$\langle H \rangle_{t=-0}^{t=+0} = \langle T \rangle_{t=-0}^{t=+0} = (\lambda - 1) \langle T \rangle_{t=-0}.$$

(a) In this case the ground-state wavefunction $\psi_0 = \sin(\pi x/L)$ at $t < 0$ is also the ground state ψ_1 at $t > 0$; hence the probability to remain in the ground state is 1. The change in energy is

$$\langle H \rangle_{t=-0}^{t=+0} = \frac{\lambda - 1}{2} \frac{\pi^2 \hbar^2}{m_0 L^2}.$$

(b) The normalized ground-state wavefunction is $\psi = \pi^{-1/4} l^{-1/2} \exp(-x^2/2l^2)$, where $l = (\hbar^2/mC)^{1/4}$. The overlap is

$$\langle \psi_1 | \psi_0 \rangle = \frac{1}{\sqrt{\pi}} \frac{1}{(l_0 l_1)^{1/2}} \int_{-\infty}^{\infty} dx \exp \left[-\frac{x^2}{2} \left(\frac{1}{l_0^2} + \frac{1}{l_1^2} \right) \right] = \sqrt{\frac{2l_0 l_1}{l_0^2 + l_1^2}}.$$

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The probability in question is

$$\langle \psi_1 | \psi_0 \rangle^2 = \frac{2l_0l_1}{l_0^2 + l_1^2} = \frac{2(l_1/l_0)}{1 + (l_1^2/l_0^2)} = \frac{2\lambda^{1/4}}{1 + \lambda^{1/2}} < 1.$$

The change in energy is

$$\langle H \rangle_{t=-0}^{t=+0} = \frac{1}{4} \hbar \omega_0 (\lambda - 1) = \frac{\lambda - 1}{4} \hbar \sqrt{\frac{C}{m_0}}.$$

#6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: A particle of mass m is placed above a rigid horizontal surface. In the presence of a gravitational field g its vertical motion is quantized. Find the asymptotic expression for the n th energy level for $n \gg 1$, with $n = 0$ being the ground state.

SOLUTION: To get the requested accuracy, it is sufficient to use the standard WKB approximation,

$$\psi_n(z) = \frac{\cos[\phi(z) - \pi/4]}{\sqrt{k(z)}}, \quad \phi(z) = \int_z^H k(\zeta) d\zeta, \quad k(z) = \sqrt{\frac{2m}{\hbar^2} (E_n - mgz)},$$

where E_n is the WKB value of the energy and $H = E_n/mg$ is the classical turning point. The phase shift $\pi/4$ in the argument of the cosine is important: it ensures that ψ_n matches onto an exponentially decaying solution at $z > H$. The boundary condition $\psi_n(0) = 0$ is met if $\phi(0) = (n + 3/4)\pi$, i.e.,

$$\int_0^H d\zeta \sqrt{\frac{2m}{\hbar^2} (E_n - mg\zeta)} = \frac{2}{3} \frac{\sqrt{2m}}{mg\hbar} E_n^{3/2} = (n + 3/4)\pi, \quad n = 0, 1, \dots,$$

so that

$$E_n \simeq \left[\frac{9}{8} \left(n + \frac{3}{4} \right)^2 \pi^2 \hbar^2 g^2 m \right]^{1/3}, \quad n \gg 1.$$

This expression is accurate to $o(1/n)$, and so keeping the term $3/4$ on the background of n is still legitimate.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: Molecules of an ideal gas have internal energy levels that are equidistant, $E_n = n\varepsilon$, where $n = 0, 1, \dots$ and ε is the level spacing. The degeneracy of n th level is $n + 1$. Find the contribution of these internal states to the energy of the gas of N molecules at temperature T .

SOLUTION: For a non-interacting ideal gas,

$$E = -\frac{\partial}{\partial \beta} N \ln \zeta,$$

where ζ is the single-molecule partition function

$$\zeta = \sum_{n=0}^{\infty} (n+1) \exp(-\beta n\varepsilon).$$

This partition function can be evaluated as follows ($x \equiv \beta\varepsilon$):

$$\zeta = -e^x \frac{d}{dx} \sum_{n=0}^{\infty} \exp(-(n+1)x) = -e^x \frac{d}{dx} \frac{e^{-x}}{1 - e^{-x}} = [1 - \exp(-\beta\varepsilon)]^{-2}.$$

Hence, the sought contribution to the energy is

$$E = \frac{2N\varepsilon}{\exp(\varepsilon/kT) - 1}.$$

Alternatively, one can reproduce this result as follows. One can imagine that every molecule has two independent internal degrees of freedom of harmonic oscillator type, with energy spacing ε each. It is easy to see that this model gives the same spectrum and degeneracies if the energy is counted from the ground state. With this convention, the average energy of a single harmonic oscillator is $\varepsilon n_B(\varepsilon)$, where $n_B(\varepsilon)$ is the Bose-Einstein occupation number. Therefore, for the entire gas we get $E = 2N\varepsilon n_B(\varepsilon)$, in agreement with the first derivation.

#8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM: What is the change of entropy that occurs when two moles of an ideal gas A and three moles of an ideal gas B, both at standard temperature and pressure are allowed to mix? What if the gases are the same, e.g., A and A?

SOLUTION: As we will show, the entropy increases if the gases being mixed are not identical. According to the general principles of statistical mechanics, the entropy is $S = k \ln W$, where W is the number of microstates that correspond to a given macrostate. For an ideal gas, we have $W = W_{\text{tr}} W_{\text{int}}$ where W_{tr} and W_{int} are number of microstates due to translational and internal degrees of freedom, respectively. If this gas is non-degenerate, then

$$W_{\text{tr}} = \frac{1}{N!} \left(\frac{V}{\lambda_T^3} \right)^N \sim \left(\frac{e}{\lambda_T^3} \frac{V}{N} \right)^N, \quad N \gg 1,$$

where N is the number of molecules, V is the volume, and λ_T^3 is the cube of the thermal wavelength (effectively, the “volume” occupied by a molecule at temperature T). The important factorial term $N!$ eliminates the overcounting of states for indistinguishable particles.

The only parameters that change as a result of the mixing are V and the number of moles n . Therefore, we can write

$$S(n) = nR \ln(V/n) + \text{const},$$

where R is the universal gas constant. Note that at standard temperature and pressure V and n are directly proportional, $V = (22.41) \times n$. Using this fact, the increase in entropy due to the mixing can be written as

$$\begin{aligned} \Delta S &= \Delta S_A + \Delta S_B = n_1 R \ln \left(\frac{V_1 + V_2}{V_1} \right) + n_2 R \ln \left(\frac{V_1 + V_2}{V_2} \right) \\ &= n_1 R \ln \left(\frac{n_1 + n_2}{n_1} \right) + n_2 R \ln \left(\frac{n_1 + n_2}{n_2} \right) = R \ln \left(\frac{3125}{108} \right) \approx 3.4R. \end{aligned}$$

On the other hand, if the gases were identical, e.g., A and A, then

$$\Delta S = (n_1 + n_2) R \ln \left(\frac{V_1 + V_2}{n_1 + n_2} \right) - n_1 R \ln \left(\frac{V_1}{n_1} \right) - n_2 R \ln \left(\frac{V_2}{n_2} \right) = 0,$$

as expected.

#9 : UNDERGRADUATE GENERAL

PROBLEM: Lightning is known to release a large amount of energy in a form of a short burst. Let W be the energy output per unit length of the lightning and f be the dominant acoustic frequency of the thunder it emits.

(a) Use dimensional analysis to express W in terms of f and physical parameters of the surrounding air, e.g., the speed of sound, density, *etc.*

(b) Under typical conditions, the thunder is heard at $f = 100$ Hz, the speed of sound in air is $v = 343$ m/s, and the length of the lightning is ~ 1 km. Estimate the total energy produced by such a lightning and compare it with the energy release of one ton of TNT explosive, 4.6×10^9 J.

SOLUTION: (a) A sudden release of a large energy along the track of the lightning creates an initially rapidly expanding cylinder of a superhot gas. Remembering that the thermal velocity of molecules coincides with the speed of sound v up to a coefficient, we conclude that the expansion of a very hot gas is necessarily supersonic. Hence, it creates a shock wave. Since the energy is delivered as a short burst rather than continuously, the expanding gas cools down, slows down, and the shock eventually becomes subsonic. At that moment the sound waves can run ahead of it and be heard as thunder. This description suggests that the most important here are the inertial and sound propagation characteristics of air, i.e., mass density ρ and the speed of sound v . Temperature, diffusion coefficient, viscosity, *etc.* are irrelevant because the process is far from equilibrium. Ambient air pressure P does not add anything either because $P \sim \rho v^2$. Hence, we expect $W = W(f, \rho, v)$. We try the scaling form

$$W = cf^\alpha \rho^\beta v^\gamma,$$

where c , α , β , and γ are some dimensionless numbers. The requirement of W to have correct units fixes the last three as follows:

$$W = c\rho v^4/f^2 \quad \therefore \quad W [\text{J/m}] = c \frac{\rho [\text{kg/m}^3] \times (v [\text{m/s}])^4}{(f [\text{Hz}])^2}.$$

Hence, the lower the frequency of the thunder, the higher must be the energy of the lightning. Although it may seem counterintuitive, it can be understood based on the argument that the wavelength of the thunder is set by the radius of the supersonic core around the lightning. Obviously, this radius increases with W .

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(b) For a crude estimate, we can drop the unknown numerical coefficient c . Using the suggested numbers of $f = 100$ Hz, $v = 343$ m/s, and also $\rho = 1.2$ kg/m³, we get $W = 1.7 \times 10^6$ J/m. Hence, the total energy released by the lightning is $\sim 1.7 \times 10^9$ J or ~ 0.4 ton of TNT.

#10 : UNDERGRADUATE GENERAL

PROBLEM: Ideal gas has density n , molecular mass m , initial temperature T_0 , and collisional cross-section σ . At time $t = 0$ a small amount of heat Q is released in a neighbourhood of a point inside the gas. Determine how the temperature difference $T - T_0$ at that point decays at large time t . No detailed calculations are necessary; rather, use estimates and dimensional analysis to derive the scaling trend.

SOLUTION: Small heat disturbances spread by thermal conductivity. At large time the size of the heated region is $R(t) \sim \sqrt{D_T t}$, where D is the thermal diffusion coefficient. In an ideal gas D_T is of the same order as the regular diffusion coefficient $D = v_T l / 3$, where $v_T = \sqrt{3kT_0/m}$ is the thermal velocity and $l = 1/\sigma n$ is the mean-free path. The conservation of energy requires

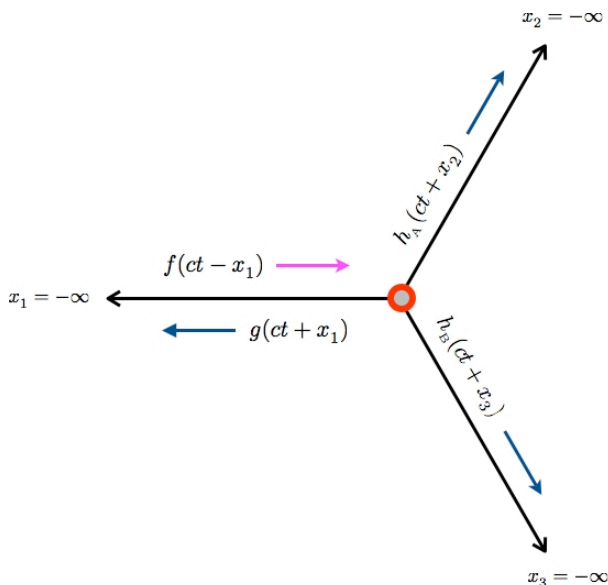
$$cnR^3(t)[T(t) - T_0] \sim Q,$$

where $c \sim k$ is the specific heat per molecule. Therefore,

$$T(t) - T_0 \sim \frac{Q}{kn} \left(\frac{\sigma^2 n^2 m}{kT_0} \right)^{3/4} \frac{1}{t^{3/2}}.$$

#11 : GRADUATE CLASSICAL MECHANICS

PROBLEM: Three identical strings are connected to a ring of mass m that can slide frictionlessly along a vertical pole. Each string has tension τ and the linear mass density σ . In equilibrium, all strings are in the same horizontal plane. The motion of the strings is in the vertical z -direction.



We can choose coordinates x_1 , x_2 , and x_3 for the three strings, with $-\infty < x_i \leq 0$ and the ring position being $x_i = 0$. When a plane wave of a given momentum k is incident on the ring from the first string, it creates transmitted waves down the other two strings and a reflected wave on the first string:

$$z_1 = \hat{f}(k) \exp(ikx_1 - i\omega t) + \hat{g}(k) \exp(-ikx_1 - i\omega t),$$

$$z_2 = \hat{h}_A(k) \exp(-ikx_2 - i\omega t), \quad z_3 = \hat{h}_B(k) \exp(-ikx_3 - i\omega t).$$

- (a) Write the set of equations of motion for the problem. Define all coefficients, e.g., $c \equiv \omega/k$.
- (b) Find the reflection coefficient $\hat{g}(k)/\hat{f}(k)$.
- (c) Test the correctness of your formula by considering two limits, $k \rightarrow 0$ (a very long and slow pulse) and $k \rightarrow \infty$ (a very short and fast one).

SOLUTION: (a) The wave equations read

$$\frac{\partial^2 z_i}{\partial x_i^2} = \frac{1}{c^2} \frac{\partial^2 z_i}{\partial t^2}, \quad c = \sqrt{\frac{\tau}{\sigma}}.$$

Next, if $Z \exp(-i\omega t)$ is the vertical coordinate of the ring, then Newton's second law implies $F = -m\omega^2 Z \exp(-i\omega t)$. Here the force F on the ring is the sum of the vertical components of the tension in the three strings at $x_i = 0$:

$$F = -\tau \sum_{i=1}^3 \left. \frac{\partial z_i}{\partial x_i} \right|_{x_i=0} = -i\tau k e^{-i\omega t} (\hat{f} - \hat{g} - \hat{h}_A - \hat{h}_B).$$

(b) The continuity at the ring demands

$$Z = \hat{f} + \hat{g} = \hat{h}_A = \hat{h}_B.$$

Eliminating \hat{h}_A and \hat{h}_B from the Newton's law for the ring, we readily obtain

$$\hat{g}(k) = -\left(\frac{k + iQ}{k + 3iQ} \right) \hat{f}(k),$$

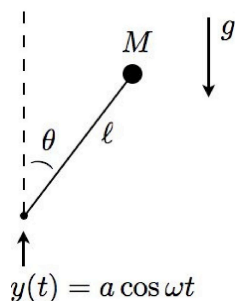
where $Q \equiv \tau/mc^2$ has dimensions of inverse length. Substituting this into formulas for \hat{h}_A and \hat{h}_B , we have

$$\hat{h}_A(k) = \hat{h}_B(k) = \left(\frac{2iQ}{k + 3iQ} \right) \hat{f}(k).$$

(c) For a very long wavelength pulse, composed of plane waves for which $|k| \ll Q$, we have $\hat{g}(k) \simeq -\frac{1}{3} \hat{f}(k)$. Thus, the reflected pulse is inverted, and is reduced by a factor of 3 in amplitude. The other outgoing pulses have amplitudes $(2/3)\hat{f}$. This is consistent with the energy conservation: $1^2 = (1/3)^2 + (2/3)^2 + (2/3)^2$ (for $\omega \rightarrow 0$, the ring oscillates very slowly, and so it has no appreciable kinetic energy). Conversely, for a very *short* wavelength pulse, $k \gg Q$, we have perfect reflection with inversion, and no transmission. This is due to the inertia of the ring.

#12 : GRADUATE CLASSICAL MECHANICS

PROBLEM: The pivot of an inverted simple pendulum is rapidly oscillated vertically with amplitude a and frequency ω (see diagram).



Find a condition on ω such that $\theta = 0$ is a point of *stable* equilibrium.

Hint: Separate the equation of motion into “fast” and “slow” parts. Eliminate the former. The remaining equation for the slow part determines whether the system is stable.

SOLUTION: In the oscillating frame of the pivot,

$$g_{\text{eff}} = g + \frac{d^2y}{dt^2} = g - \omega^2 a \cos \omega t.$$

Therefore,

$$\ddot{\theta} = \frac{g}{\ell} \left(1 - \frac{\omega^2 a}{g} \cos \omega t\right) \theta.$$

Let us decompose θ into a “slow” $\bar{\theta}$ and a “fast” θ_1 parts:

$$\theta = \bar{\theta}(t) + \theta_1(t),$$

then

$$\frac{d^2\theta_1}{dt^2} + \frac{d^2\bar{\theta}}{dt^2} = \frac{g}{\ell} \left(1 - \frac{\omega^2 a}{g} \cos \omega t\right) (\bar{\theta} + \theta_1).$$

The “fast” equation is

$$\frac{d^2\theta_1}{dt^2} = -\frac{g}{\ell} \frac{\omega^2 a}{g} \cos \omega t \bar{\theta} \quad \therefore \quad \theta_1 = \frac{a}{\ell} \bar{\theta} \cos \omega t.$$

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The slow equation is

$$\frac{d^2\bar{\theta}}{dt^2} = \frac{g}{\ell}\bar{\theta} - \frac{g}{\ell}\langle\theta_1 \cos \omega t\rangle \frac{\omega^2 a}{g} = \frac{g}{\ell}\left(1 - \frac{\omega^2 a}{g} \frac{a}{2\ell}\right)\bar{\theta}$$

The stability is achieved if

$$\frac{\omega^2 a^2}{2g\ell} > 1.$$

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: A relativistic electron radiates while executing a *nearly circular* cyclotron motion in a uniform magnetic field \mathbf{B} . Find how the function $\gamma(t) \equiv E(t)/mc^2$ decreases with time t starting from some initial value $\gamma(0)$. Assume that $\gamma(t) \gg 1$ and that the energy radiated during one period of cyclotron motion is small compared to the electron energy $E(t)$.

Hint: the power W radiated by a relativistic electron can be written as

$$W = -\frac{2}{3} \frac{e^2}{m^2 c^3} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau},$$

where $p^\mu = (E/c, -\mathbf{P})$ and $p_\mu = (E/c, \mathbf{P})$ are contravariant and covariant 4-momenta, respectively, and τ is the proper time.

SOLUTION: The relation between the proper and lab time is

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma(t)}.$$

Therefore,

$$\frac{dp^\mu}{d\tau} = \gamma(t) \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\mathbf{P}}{dt} \right), \quad \frac{dp_\mu}{d\tau} = \gamma(t) \left(\frac{1}{c} \frac{dE}{dt}, -\frac{d\mathbf{P}}{dt} \right).$$

For motion in magnetic field

$$\frac{dE}{dt} = 0, \quad \frac{d\mathbf{P}}{dt} = \frac{e}{c} [\mathbf{v} \times \mathbf{B}].$$

Hence,

$$W = \frac{2}{3} \frac{e^2}{m^2 c^3} \gamma^2(t) \frac{v^2}{c^2} e^2 B^2 \simeq \frac{2}{3} \frac{e^4}{m^2 c^3} \gamma^2(t) B^2.$$

The energy balance equation becomes

$$\frac{d}{dt}(\gamma mc^2) = -W = -\frac{e^4}{m^2 c^3} \gamma^2(t) B^2,$$

which has the solution

$$\frac{1}{\gamma(t)} = \frac{1}{\gamma(0)} + \frac{2}{3} \frac{e^4 B^2}{m^3 c^5} t.$$

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: A small amount of water is being warmed in a microwave oven.

(a) Derive the formula for the amplitude E of the electric field in terms of the power P dissipated in a unit volume of water, microwave frequency f (in Hz), and the complex dielectric function of water $\varepsilon = \varepsilon_1 + i\varepsilon_2$. Assume that the field penetrates the water uniformly.

(b) Compute the voltage drop V (assuming uniform field) across the longest dimension of the oven, $L = 0.3$ m. Use the following information. Typically, it takes about a minute to heat a cup of water by 10°C , so that $P \approx 10^6 \text{ W/m}^3$. The microwave frequency is $\omega = 2\pi f$, where $f = 2.45 \text{ GHz}$. At such frequency the dielectric function of water is $\varepsilon \approx 80 + 10i$.

SOLUTION: (a) Let σ be the real part of the conductivity. In the Gaussian units $\sigma = \omega\varepsilon_2/4\pi = f\varepsilon_2/2$. The ac Joule heating is

$$P = \frac{1}{2}\sigma E^2 = \frac{1}{4}f\varepsilon_2 E^2.$$

Therefore,

$$E = 2\sqrt{P/f\varepsilon_2}.$$

(b) Substituting the numbers, we get

$$\begin{aligned} E [\text{statV/cm}] &\approx 2\sqrt{10^7 [\text{erg/cm}^3]/2.45 \times 10^9 [\text{Hz}]/10} \\ &= 0.04 [\text{statV/cm}] = 1.2 \text{ kV/m}. \end{aligned}$$

Accordingly, the voltage drop across the oven is $V = EL \approx 360 \text{ V}$.

One more consideration is in order to get the correct estimate for V . (However, failure to acknowledge it was not penalized by taking off points in grading the exam).

Strictly speaking, the field E we computed is actually the field *inside* the water. The magnitude of the field E_{out} in the rest of the oven is larger:

$$E_{\text{out}} = E[1 + (\varepsilon - 1)N]$$

where N is referred to as the depolarization factor. If water forms a very shallow puddle and the microwave field is parallel to its surface, i.e., horizontal, N is very small and $E_{\text{out}} \approx E$, so the above estimate of V stands.

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However, if the dimensions of the volume occupied by the water are comparable (e.g, water filling a common mug), then $f \sim 1/3$, so that

$$|E_{\text{out}}| \sim |\varepsilon E|/3 \approx 30|E|.$$

Therefore, the actual total voltage across the microwave oven is $\sim 360 \text{ V} \times 30 \sim 10 \text{ kV}$. This explains why safety features are necessary.

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: A particle of mass m moves in a spherically symmetric potential well $V(r) = -V_0 < 0$ at $r < a$ and $V(r) = 0$ at $r > a$. Find the smallest V_0 at which a bound state exists at zero angular momentum.

SOLUTION: The Schrödinger equation for zero angular momentum reads

$$\begin{aligned} -\frac{\hbar^2}{2m} \left[\psi''(r) + \frac{2}{r} \psi'(r) \right] &= E\psi, \quad r > a, \\ -\frac{\hbar^2}{2m} \left[\psi''(r) + \frac{2}{r} \psi'(r) \right] &= (V_0 + E)\psi, \quad r < a, \end{aligned}$$

The sought solution for $E < 0$ is

$$\begin{aligned} \psi(r) &= A \exp(-\alpha r)/r, \quad \alpha = \sqrt{-2mE}/\hbar, \quad r > a, \\ \psi(r) &= B \sin(\beta r)/r, \quad \beta = \sqrt{2m(V_0 + E)}/\hbar, \quad r < a, \end{aligned}$$

The continuity of $\psi'(r)/\psi(r)$ at $r = a$ demands

$$\beta \cot \beta a = -\alpha,$$

As $E \rightarrow 0^-$, $\alpha \rightarrow 0^+$, and so $\beta a \rightarrow \pi/2$. Thus,

$$\min V_0 = \frac{\pi^2 \hbar^2}{8ma^2}.$$

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: A system is described by the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \varepsilon\mathcal{H}_1$. \mathcal{H}_0 has a doubly-degenerate ground state of zero energy. The corresponding eigenkets are $|1\rangle$ and $|2\rangle$. \mathcal{H}_1 has the property that $\langle 1|\mathcal{H}_1|1\rangle = \langle 2|\mathcal{H}_1|2\rangle = 0$ and $\langle 1|\mathcal{H}_1|2\rangle = a$. Finally, ε is a small parameter.

(a) Find the two lowest energy states and the corresponding eigenkets of the Hamiltonian \mathcal{H} accurate to the lowest non-vanishing order in ε .

(b) Write down a general expression for the ket $|\psi(t)\rangle$ in terms of the eigenkets found above.

(c) Calculate the probability of finding the system in the state $|1\rangle$ at time t if it was in the state $|2\rangle$ at $t = 0$.

SOLUTION: (a) The Hamiltonian \mathcal{H} projected on the Hilbert space spanned by $|1\rangle$ and $|2\rangle$ can be written in the form of the 2×2 matrix

$$\mathcal{H} = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}.$$

Let $a = |a|e^{i\theta}$. Diagonalizing the above matrix, we find that the two lowest-energies are $E_s = |a|$ and $E_a = -|a|$. The corresponding eigenstates are

$$|s\rangle = \frac{e^{i\theta/2}}{\sqrt{2}}|1\rangle + \frac{e^{-i\theta/2}}{\sqrt{2}}|2\rangle, \quad |a\rangle = \frac{e^{i\theta/2}}{\sqrt{2}}|1\rangle - \frac{e^{-i\theta/2}}{\sqrt{2}}|2\rangle.$$

(b) The answer is ($\hbar = 1$):

$$|\psi(t)\rangle = c_s e^{-i|a|t}|s\rangle + c_a e^{i|a|t}|a\rangle, \quad c_s = \langle s|\psi(0)\rangle, \quad c_a = \langle a|\psi(0)\rangle.$$

(c) It is easy to see that

$$c_s = \langle s|2\rangle = \frac{e^{i\theta/2}}{\sqrt{2}}, \quad c_a = \langle a|2\rangle = -\frac{e^{i\theta/2}}{\sqrt{2}}.$$

In addition,

$$d_s \equiv \langle 1|s\rangle = \frac{e^{-i\theta/2}}{\sqrt{2}}, \quad d_a \equiv \langle 1|a\rangle = \frac{e^{-i\theta/2}}{\sqrt{2}},$$

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whence

$$\langle 1|\phi(t)\rangle = d_s e^{-i|a|t} c_s + d_a e^{i|a|t} c_a = \frac{1}{2} (e^{-i|a|t} - e^{i|a|t}) = -i \sin |a|t,$$

and so the probability in question is

$$|\langle 1|\phi(t)\rangle|^2 = \sin^2 |a|t.$$

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: A system is composed of N identical classical oscillators, each of mass m , defined on a one-dimensional lattice. The potential for the oscillators has the form

$$U(x) = \varepsilon |x/a|^n, \quad \varepsilon > 0, \quad n > 0.$$

(Thus, the oscillators are harmonic for $n = 2$ and anharmonic otherwise). Find the average thermal energy at temperature T .

Hint: An integral that appears in the course of evaluating the partition function cannot be computed in terms of elementary functions. Fortunately, it amounts only to an unimportant overall coefficient.

SOLUTION: Classical partition function for a single oscillator is

$$\zeta = \int_{-\infty}^{\infty} dp \exp(-\beta p^2/2m) \int_{-\infty}^{\infty} dx \exp(-\beta \varepsilon |x/a|^n).$$

By change of variables, we bring this to the form

$$\zeta = \Gamma(1/2)\Gamma(1/n) \frac{2a}{n} \left(\frac{2m}{\beta}\right)^{1/2} \left(\frac{1}{\beta\varepsilon}\right)^{1/n},$$

where

$$\Gamma(z) \equiv \int_0^{\infty} dt t^{z-1} \exp(-t).$$

A learned reader would recognize this as the Euler Gamma-function. However, knowing this is not necessary. The product $\Gamma(1/2)\Gamma(1/n)$ is just a numerical coefficient, which will disappear from the final result.

The average energy is given by

$$E = -N \frac{\partial}{\partial \beta} \ln \zeta = \left(\frac{n+2}{2n}\right) NkT.$$

This result resembles the equipartition theorem in the sense that material constants do not enter.

#18 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The Hamiltonian of N noninteracting spin-1/2 particles in magnetic field H is given by

$$\mathcal{H}_0 = -HM, \quad M = \mu \sum_{i=1}^N \sigma_i, \quad \sigma_i = \pm 1.$$

(a) Calculate the average magnetization $\langle M \rangle$, the average square of the magnetization $\langle M^2 \rangle$, and the magnetic susceptibility $\chi = (d/dH)\langle M \rangle$ at temperature T .

(b) Verify that your results obey the thermodynamic identity

$$\langle M^2 \rangle - \langle M \rangle^2 = kT\chi.$$

(c) Prove that the above identity holds even in the presence of interactions, $\mathcal{H}_0 \rightarrow \mathcal{H}_0 + \mathcal{H}_{\text{int}}$, for arbitrary $\mathcal{H}_{\text{int}}(\{\sigma_i\})$.

SOLUTION: A shorter derivation can be given if we start with part (c).

(c) The partition function is

$$Q = \sum_{\{\sigma_i\}} \exp(\beta HM(\{\sigma_i\}) - \beta E_{\text{int}}(\{\sigma_i\})),$$

whence

$$\langle M \rangle = \frac{1}{\beta Q} \frac{\partial Q}{\partial H}, \quad \langle M^2 \rangle = \frac{1}{\beta^2 Q} \frac{\partial^2 Q}{\partial H^2}.$$

Now

$$\chi = \frac{\partial}{\partial H} \langle M \rangle = \frac{1}{\beta Q} \frac{\partial^2 Q}{\partial H^2} - \frac{1}{\beta Q^2} \left(\frac{\partial Q}{\partial H} \right)^2 = \beta(\langle M^2 \rangle - \langle M \rangle^2),$$

which proves the identity.

(a) We can now apply the above formulas to the problem in hand. We have

$$Q = \prod_i \sum_{\sigma_i} \exp(\beta \mu H \sigma_i) = [2 \cosh(\beta \mu H)]^N.$$

Taking the requisite derivatives, we find

$$\begin{aligned}\langle M \rangle &= N\mu \tanh(\beta\mu H), \\ \langle M^2 \rangle &= \mu^2[N(N-1)\tanh^2(\beta\mu H) + N], \\ \chi &= \beta\mu^2 N \operatorname{sech}^2(\beta\mu H).\end{aligned}$$

(b) We have

$$\begin{aligned}\mu^2[N(N-1)\tanh^2(\beta\mu H) + N] - [N\mu \tanh(\beta\mu H)]^2 &= \mu^2 N[1 - \tanh^2(\beta\mu H)] \\ &= \mu^2 N \operatorname{sech}^2(\beta\mu H) = kT\chi.\end{aligned}$$

#19 : GRADUATE GENERAL

PROBLEM: Small transverse oscillations $u(z, t)$ of an inextensible but otherwise perfectly flexible cable suspended at one end and hanging under gravity are described by the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z} \left(gz \frac{\partial}{\partial z} u \right), \quad u(L, t) = 0,$$

where $L = \text{const}$ is the length of the cable and $0 < z < L$ is the vertical coordinate.

(a) Following Bernoulli (1732), one can seek eigenmodes of the system in the form $u(z, t) = \psi(z) \cos \omega t$, where ψ is given by Taylor series,

$$\psi(z) = \sum_{j=0}^{\infty} c_j z^j.$$

Find all the coefficients c_j assuming ω is given. Sketch the expected behavior of function $\psi(z)$ for a few first eigenmodes.

(b) A rigorous bound on the smallest eigenfrequency ω can be found from comparison with that of a perfectly rigid cable (which is a pendulum). Do so and explain whether this is a lower or an upper bound.

SOLUTION: (a) Substituting the power series into the equation of motion and equating the coefficients for equal powers, we get

$$c_{j+1} = -\frac{1}{(j+1)^2} \frac{\omega^2}{g} c_j,$$

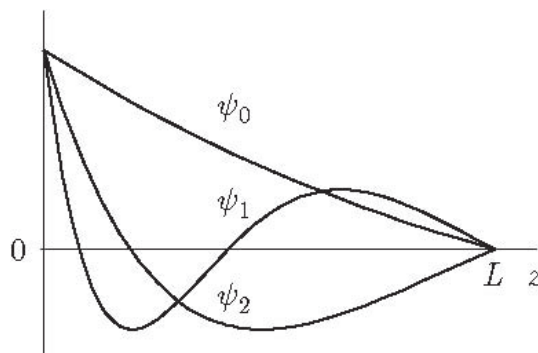
so that

$$c_j = \frac{(-1)^j}{(j!)^2} \left(\frac{\omega^2}{g} \right)^j c_0, \quad j \geq 1.$$

The arbitrary coefficient c_0 controls the overall amplitude of the harmonic oscillations. We can set it to unity. The sketches of a few eigenmodes are shown below. According to the general theory of the Sturm-Liouville problem, the lowest frequency mode ψ_0 is nodeless (except $z = L$, of course); the next one, ψ_1 , has one node, the third one, ψ_2 , has two nodes, *etc.*

(b) The eigenfrequency of a rigid cable is

$$\omega_R = \sqrt{\frac{MgL}{2I}} = \sqrt{\frac{3g}{2L}} \approx 1.22 \sqrt{\frac{g}{L}}.$$



This is a strict *upper* bound: $\omega < \omega_R$. Indeed, our Sturm-Liouville eigenvalue problem obeys a variational principle. The oscillations of the rigid cable, which are described by

$$\psi_R(z) = L - z,$$

can be considered a trial function. Hence, ω_R^2 is in fact a variational estimate of ω^2 .

Note: Those who are familiar with special functions would recognize that $\psi(z) = c_0 J_0(2\omega\sqrt{z/g})$. Accordingly, the exact result for the lowest eigenfrequency is $\omega = (r_1/2)\sqrt{g/L} \approx 1.20\sqrt{g/L}$, where $r_1 \approx 2.40$ is the first root of the Bessel function J_0 . As we can see, ω_R is within 2% of this value.

#20 : GRADUATE GENERAL

PROBLEM: In the absence of other forces, surface tension causes a liquid droplet to assume a spherical shape. Lord Rayleigh has shown that this is no longer true for an electrified droplet of a sufficiently large charge Q . (This instability has found a practical application in ink-jet printers.) Compute the corresponding critical charge Q_c for a droplet of radius R and surface tension σ .

Hint: The capacitance C of a nearly spherical object is related to its surface area S by the Aichi-Russel formula

$$C = \sqrt{S/4\pi} \quad (\text{Gaussian units})$$

SOLUTION: The total energy of the droplet is

$$E(S) = \frac{Q^2}{2C} + \sigma S = \sqrt{\frac{\pi}{S}} Q^2 + \sigma S.$$

Function $E(S)$ has the minimum at $S = S_c$,

$$S_c = \left(\frac{\sqrt{\pi} Q^2}{2 \sigma} \right)^{2/3}.$$

However, since the sphere has the minimal surface area for a given fixed volume, S cannot be smaller than $4\pi R^2$. As a result, for small Q the sphere remains the optimal shape. The critical charge is determined by the condition $S_c = 4\pi R^2$, which gives

$$Q_c = 4\sqrt{\pi\sigma R^3} \approx 7.1 \sqrt{\sigma R^3},$$

in agreement with Lord Rayleigh (1882). At Q somewhat larger than Q_c the droplet deforms into a prolate ellipsoid. This is the answer the student is expected to give for this problem.

Actually, an astute reader may realize that this answer may be incomplete. In principle, the droplet can also change its shape discontinuously, e.g., by splitting into two smaller droplets. Let us examine this “first-order transition” scenario assuming the new droplets are also spherical and equal in size.

For the droplet of charge q and radius r the energy is

$$E = \frac{q^2}{2r} + 4\pi\sigma r^2.$$

Comparing the energies of one droplet with $q = Q$ and $r = R$ with that of two droplets with $q = Q/2$ and $r = R/2^{1/3}$, we conclude that the first-order instability occurs at

$$Q > Q_m = \left(8\pi \frac{2^{1/3} - 1}{1 - 2^{-5/3}} \right)^{1/2} \sqrt{\sigma R^3} \approx 3.1 \sqrt{\sigma R^3}.$$

We see that $Q_m < Q_c$, and so the first order transition wins. More precisely, the spherical droplet with charge $Q_m < Q < Q_c$ is metastable, and so in practice it still may have a long lifetime.