INSTRUCTIONS
PART I : SPRING 2006 PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1		§2		§3		§4		§5	
Problems: (Circle your seven choices)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: A grandfather clock has a pendulum length of 1.0 m and a bob of mass $m = 0.5\,\mathrm{kg}$. A mass of 2 kg falls 0.7 m in seven days to keep the amplitude of the pendulum oscillations steady at 0.03 rad.

(a) The quality factor Q of a damped oscillator is defined as

$$Q = 2\pi \times \frac{\text{average energy}}{\text{energy lost per cycle}} \ .$$

What is the Q of the system?

(b) With no energy input from the falling mass, the pendulum obeys the small angle equation of motion

$$\ddot{\theta} + 2\beta\,\dot{\theta} + \omega_0^2\,\theta = 0 \ .$$

Find ω_0 and β.

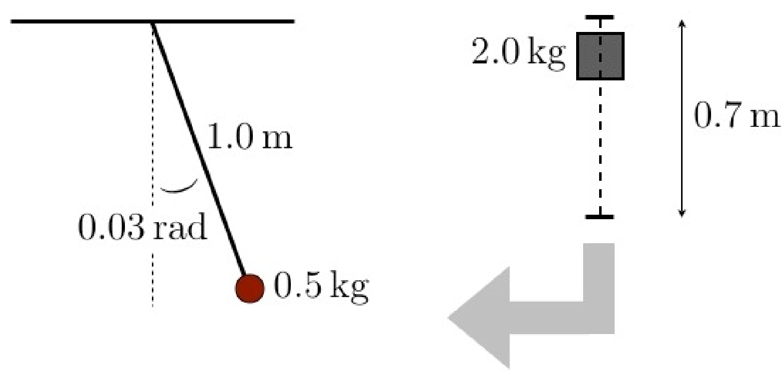


Figure 1: Schematic of the grandfather clock, showing energy flow from falling mass to the damped pendulum. The falling mass must be reset weekly.

#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: A mechanical system consists of two particles, one of which moves in three dimensions, the other of which is confined to a plane. The particle masses are m_1 and m_2, respectively. The potential energy of the system is

$$U(x, y, z, \rho, \phi) = V(u, v) \ ,$$

where

$$u \equiv \alpha x + \beta y + \gamma z \quad , \quad v \equiv y + a\phi \ .$$

Thus, the potential U depends only on two linear combinations of the five degrees of freedom.

(a) Write down the Lagrangian and the equations of motion.

(b) Noether's theorem says that to each continuous one-parameter family of transformations of the generalized coordinates, there corresponds an associated conserved quantity. For this system, identify all such one-parameter families and conserved quantities.

(c) Is anything else conserved by the dynamics?

#3 : UNDERGRADUATE ELECTROMAGNETISM

In unbounded vacuum, the initial electric and magnetic fields are given by

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t=0) &= \hat{\mathbf{y}} f(x) \\ \mathbf{B}(\mathbf{r}, t=0) &= 0.\end{aligned}$$

Find $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ for $t \geq 0$.

#4 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` Compute the force of attraction between a neutral metallic sphere of radius a and a point charge q positioned a distance r from the center of the sphere, where $r > a$.

#5 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A proton (spin-$\frac{1}{2}$) is in the spin state

$$\chi = \frac{1}{4}\begin{pmatrix} 2-3i \\ i\sqrt{3} \end{pmatrix}, \tag{1}$$

where the spinor corresponds to the representation where, *e.g.,* the state with spin up along the z-axis is

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2}$$

`(a)` What is the expectation value for the angular momentum operator in the z-direction?

`(b)` What is the expectation value for the angular momentum operator in the x-direction?

`(c)` What is the expectation value for the angular momentum operator in the y-direction?

#6 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider a two state system governed by Hamiltonian H and with energy eigenstates $|E_1\rangle$ and $|E_2\rangle$ where $H|E_1\rangle = E_1|E_1\rangle$ and $H|E_2\rangle = E_2|E_2\rangle$.

Two other states are:

$$|x\rangle = \frac{|E_1\rangle + |E_2\rangle}{\sqrt{2}} \ ;$$

$$|y\rangle = \frac{|E_1\rangle - |E_2\rangle}{\sqrt{2}} \ .$$

At time $t = 0$ the system is in state $|x\rangle$. At what subsequent times is the probability to find the system in state $|y\rangle$ biggest and what is this probability?

#7 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` A horizontal insulated cylinder is partitioned by a frictionless, insulating piston (which prevents heat exchange between the two sides). On each side of the piston, we have 30 liters of an ideal monatomic gas ($C_V = \frac{3}{2}nR$ and $\gamma = \frac{5}{3}$) at a pressure of 1 atmosphere and a temperature $T = 300\,\mathrm{K}$. Heat is slowly injected into the gas on the left, causing the piston to move until the gas on the right is compressed to a pressure of $P = 2$ atmospheres.

`(a)` What are the final volume V and temperature T on the right side?

`(b)` What are the final values of P, V, and T on the left side?

`(c)` What is the change in the internal energy ΔU for the gas on the right side? How much heat Q_{if} did it absorb, and how much work W_{if} is done to the right chamber?

(d.) What are the values of ΔU, W_{if}, and Q_{if} for the gas on the left side?

Hint #1: For an ideal gas with adiabatic changes, PV^γ is constant.

Hint #2: The internal energy of an ideal gas is a function of temperature only.

#8 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` Compute the mean energy fluctuation for a quantum mechanical harmonic oscillator with mass m and frequency ω at temperature T.

#9 : UNDERGRADUATE OTHER

`PROBLEM:` In a high vacuum system, a spherical chamber of radius $R = 1$ m is pumped through a small hole of radius $a = 3$ cm. You may assume that the pump is perfect in that no atoms pass back through the hole into the chamber. The stainless steel chamber has been "baked" so that the room temperature rate of outgassing from the surface is 10^{12} molecules/m^2·sec (primarily H_2). Determine the equilibrium pressure in the chamber at room temperature.

Hint: The mean free path between collisions is large compared to R. You may need the values $k_B = 1.4 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ and $m_{H_2} = 3.4 \times 10^{-27}$kg.

#10 : UNDERGRADUATE OTHER

`PROBLEM:` Stars in mechanical equilibrium can sustain a net outward-directed photon flux up to a limiting value, beyond which the rate of momentum transfer from photons to electrons and, hence, to protons via the Coulomb interaction, exceeds the local force of gravity. For a star of mass $M = 1\,\mathrm{M}_\odot \approx 2 \times 10^{33}\,\mathrm{g}$ what is the luminosity (total energy per unit time radiated from the stellar surface) required to produce this critical flux? Express your result in units of $\mathrm{erg\,s^{-1}}$; an order of magnitude estimate is fine. You may find it useful to know that the Thomson cross section for photon-electron scattering is $\sigma_\mathrm{T} \approx 6.65 \times 10^{-25}\,\mathrm{cm}^2$, the gravitational constant is $G \approx 6.67 \times 10^{-8}\,\mathrm{erg\,cm\,g^{-2}}$, and the rest mass of a proton is $m_\mathrm{p} \approx 938.26\,\mathrm{MeV} \approx 1\,\mathrm{amu}$.

INSTRUCTIONS
PART II : SPRING 2006 PHYSICS DEPARTMENT EXAM

Please take a few minutes to read through all problems before starting the exam. Ask the proctor if you are uncertain about the meaning of any part of any problem. You are to do seven (7) of the ten (10) problems.

The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. (E.g. Section 1: one or both of problem 1 and problem 2.) Credit will be assigned for seven (7) questions only. **Circle the seven problems you wish to be graded:**

Section:	§1		§2		§3		§4		§5	
Problems: (Circle your seven choices)	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**

SPECIAL INSTRUCTIONS DURING EXAM

1. You should not have anything close to you other than your pens & pencils, calculator and food items. Please deposit your belongings (books, notes, backpacks, *etc.*) in a corner of the exam room.
2. Departmental examination paper is provided. Please make sure you:
   a. Write the problem number and your ID number on each sheet;
   b. Write only on one side of the paper;
   c. Start each problem on the attached examination sheets;
   d. If multiple sheets are used for a problem, please make sure you staple the sheets together and make sure your ID number is written on each of your exam sheets.

Colored scratch paper is provided and may be discarded when the examination is over. At the conclusion of the examination period, please staple sheets from each problem together. Submit this top sheet to one of the proctors, who will check that you have circled the correct problem numbers above. Then submit your completed exam, separated into stacks according to problem number.

#11 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` A point particle of mass m slides frictionlessly along a hoop of radius R and radius M. The hoop rolls without slipping along a horizontal surface.

`(a)` What quantities are conserved by the motion?

`(b)` What is the frequency of small oscillations of the point mass, when it is close to the bottom of the hoop?

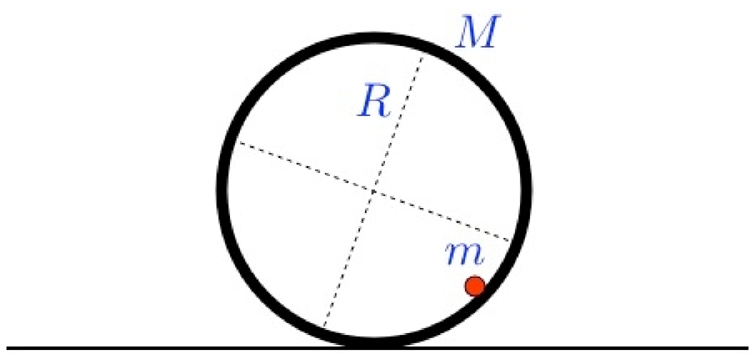


Figure 2: A point mass m sliding inside a rolling hoop of radius R.

#12 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` Consider a "mechanical mirror," as shown in the figure below (the horizontal axis is the x-axis, the vertical axis is the y-axis). The walls are hard and curved with a surface $\pm y\,(x)$. Take all motion to be in the x-y plane.

`(a)` Without writing equations describe what will happen to a perfectly elastic rubber ball injected at x_0 with critical velocity v_{x0}, v_{y0}. Supply a few sketches.

`(b)` Assuming that $|y|_{\max} \ll L$, use adiabatic theory to determine the conditions for a particle to be confined in this system.

`(c)` More generally, "characterize" what class of particles will be confined. State your answer it terms of a phase space plot.

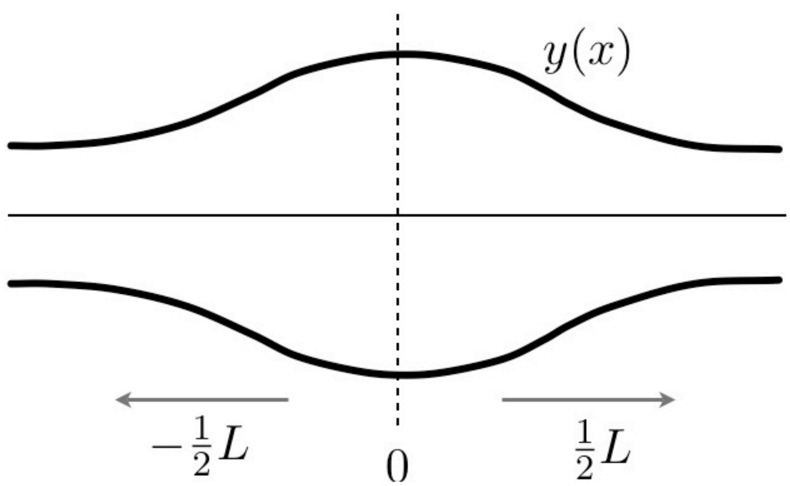


Figure 3: The mechanical mirror.

#13 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` In a first collision, an electron with velocity $\boldsymbol{v}_0$ is incident with impact parameter b on an electron that is initially at rest. The initial conditions for a second collision are the same as those for the first except that the electron at rest is replaced by a positron at rest. In both collisions, parameters are ordered as $e^2/b \ll mv_0^2 \ll mc^2$, where m is the electron mass. Since the motion is non-relativistic, the radiated energy from the two charge system in each collision can be treated in the multipole approximation.

`(a)` For which of the two collisions is the total radiated energy larger? Be sure to explain your answer carefully.

`(b)` Calculate the radiated energy for the collision with largest radiated energy.

#14 : GRADUATE ELECTROMAGNETISM

A thin spherical shell of radius R carries the uniform surface charge σ and rotates about an axis through its center with angular frequency ω, where $R\omega \ll c$. In this problem you will be asked to calculate the magnetic field inside the shell ($r < R$) and outside the shell ($r > R$). Since $\boldsymbol{\nabla} \times \boldsymbol{B} = 0$ in both these regions, one can set $\boldsymbol{B} = -\boldsymbol{\nabla}\Phi_1$, for $r < R$ and $\boldsymbol{B} = -\boldsymbol{\nabla}\Phi_2$ for $r > R$, where Φ_1 and Φ_2 are magnetic potentials.

(a) From Maxwell equations, write down the partial differential equations satisfied by Φ_1 (for $r < R$) and Φ_2 (for $r > R$). Also, determine the boundary conditions satisfied by Φ_1 and Φ_2 at $r = R$. Here, (r, θ, ϕ) is a spherical coordinate system centered on the center of the sphere and with the polar axis directed along the rotation vector ω.

(b) Determine $\boldsymbol{B}_1$ for $r < R$ and $\boldsymbol{B}_2$ for $r > R$.

#15 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Nucleons in nuclei can be regarded as point-like spin-$\frac{1}{2}$ fermions moving in a collective spherical square well potential some 50 MeV deep but otherwise *noninteracting*. This "independent particle" model works remarkably well despite (really because of) the strong nature of the nucleon-nucleon force. The spherical square well potential can be regarded as having radius $R \approx 1.2\,\mathrm{fm} \cdot A^{1/3}$, where the nuclear mass number is $A = Z + N$ with Z protons and N neutrons. Take $N = Z$, treat neutrons and protons independently, and estimate the *average* kinetic energy and speed of nucleons. What is the rough minimum binding energy of a nucleon in this model?

Hints: Take the neutron and proton rest masses to be the same as that of the proton, $m_{\mathrm{p}}c^2 \approx 938.3\,\mathrm{MeV}$. Remember that $1\,\mathrm{fm} = 10^{-13}\,\mathrm{cm}$ and $\hbar c \approx 197.33\,\mathrm{MeV} \cdot \mathrm{fm}$. Treat the nucleons as having plane waves as wavefunctions, *i.e.* treat the neutrons and protons as independent Fermi gases.

#16 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` Consider two linear oscillators each with spring constant k. These oscillators are coupled by an interaction term $H_{\rm int} = a\, x_1\, x_2$, where x_1 and x_2 are the oscillator (displacement) variables, and a is a constant. Find the energy levels for this system.

Hint: transform to new coordinates.

#17 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` An interacting gas consists of hard-core spheres with a peculiar three-body attraction. The equation of state is a modified version of the van der Waals equation:

$$p = \frac{k_{\mathrm{B}}T}{v - v_0} - \frac{\alpha}{3v^3} \ ,$$

where v is the volume per particle, and where v_0 and α are positive.

Find expressions for the critical values T_{c}, p_{c}, and v_{c}.

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` Consider a generalization of the Ising model in which the spin variable s_i on each site i may take the value 0 as well as ± 1. For convenience, let the spins be arranged on a 3D simple cubic lattice. The total energy of a given configuration $\{s_i\}$ is

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - \mu \sum_i s_i^2, \tag{3}$$

where $\langle i, j \rangle$ denotes nearest neighbor sites.

`(a)` Apply the Weiss mean-field approximation (*i.e.,* replace the value of the spin of neighboring sites with the appropriate mean field value) to obtain a self-consistent equation for the mean magnetization σ. It will be convenient to use the effective parameters $t \equiv k_{\mathrm{B}}T/(6J)$ and $\delta \equiv e^{-\mu/k_{\mathrm{B}}T}$.

For the following it will be useful to assume that $|\sigma| \ll 1$.

`(b)` For small δ, find the critical temperature $t_c(\delta)$ below which spontaneous magnetization occurs. Show that $|\sigma| \propto (t_c - t)^{1/2}$

`(c)` Show that something goes wrong with the solution in part (b.) for sufficiently large δ.

#19 : GRADUATE OTHER

`PROBLEM:` Consider a uniform flux of neutral heavy particles (rest mass $mc^2 \sim 1000\,\mathrm{MeV}$) with speeds $\sim 10^{-3}\,c$ incident on a spherically symmetric atomic nucleus with radius 5 fm. Whenever these particles are inside the nucleus they feel a weak binding potential $V_0 = -2 \times 10^{-9}\,\mathrm{MeV}$. Outside the nucleus the interaction potential is zero. Estimate the total scattering cross section for these "Weakly Interacting Massive Particles," or WIMPS.

If there was a number density $10^{-2}\,\mathrm{cm}^{-3}$ of these particles moving at the indicated speeds what mass detector would you need (assume that you are using germanium nuclei – 72 amu – in the detector) to see one scattering event per day?

#20 : GRADUATE OTHER

`PROBLEM:` The index of refraction in the atmosphere of a planet varies according to the law $n(h) = n_0 - \alpha h$, where h is the height and both n_0 and α are constant. Find the height h at which electromagnetic waves propagate along a circular orbit around the planet. The planet's radius is R.

Solution #1 U.G. Classical mechanics

(a) The average energy dissipation is

$$\langle \dot{E} \rangle = -\frac{(2\,\mathrm{kg}) \times (9.8\,\mathrm{m/s^2}) \times 0.7\,\mathrm{m}}{7 \times 24 \times 3600\,\mathrm{s}} = -2.3 \times 10^{-5}\,\mathrm{J/s}\,.$$

The average energy is

$$\langle E \rangle = \tfrac{1}{2} \times (0.5\,\mathrm{kg}) \times (9.8\,\mathrm{m/s^2}) \times (1.0\,\mathrm{m}) \times (0.03\,\mathrm{rad})^2 = 2.2 \times 10^{-3}\,\mathrm{J}\,.$$

Assuming the dissipation is small, the natural period of the pendulum is

$$T = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\left(\frac{1.0\,\mathrm{m}}{9.8\,\mathrm{m/s^2}}\right)^{1/2} = 2.0\,\mathrm{s}$$

Thus,

$$Q = 2\pi \times \frac{2.21 \times 10^{-3}\,\mathrm{J}}{(2.3 \times 10^{-5}\,\mathrm{J/s}) \times (2.0\,\mathrm{s})} = 48$$

(b) Clearly

$$\omega_0 = \sqrt{\frac{g}{\ell}} = \left(\frac{9.8\,\mathrm{m/s^2}}{1.0\,\mathrm{m}}\right)^{1/2} = 3.1\,\mathrm{s^{-1}}$$

As for the energy, the equation of motion says that

$$\dot{E} = \frac{d}{dt}\left(\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\omega_0^2 x^2\right) = -2\beta m\dot{x}^2\,,$$

hence, taking the time average,

$$\langle \dot{E} \rangle = -2\beta\,\langle m\dot{x}^2 \rangle = -2\beta\,\langle E \rangle$$

Thus,

$$\beta = -\frac{\langle \dot{E} \rangle}{2\langle E \rangle} = \frac{\omega_0}{2Q} = 3.2 \times 10^{-2}\,\mathrm{s^{-1}}\,.$$

Nota bene: The frequency of the oscillation is shifted by the damping from ω_0 to $\omega = -i\beta + \sqrt{\omega_0^2 - \beta^2}$. The real part of ω determines the period of the damped oscillations, with $T = 2\pi/\sqrt{\omega_0^2 - \beta^2}$. For our system, $\beta^2/\omega_0^2 = 1.1 \times 10^{-4} \ll 1$, and we can approximate $T \simeq 2\pi/\omega_0$ to good accuracy.

Solution # 2 U. C. classical mechanics

(a) The Lagrangian is

$$L = \tfrac{1}{2}m_1\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}m_2\,(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2) - V(\alpha x + \beta y + \gamma z, y + a\phi).$$

(b) The Lagrangian is rendered invariant under time-independent transformations satisfying $dV = 0$ and $d\rho = 0$, i.e.

$$\alpha\,dx + \beta\,dy + \gamma\,dz = 0 \quad , \quad dy + a\,d\phi = 0 \quad , \quad d\rho = 0\ .$$

The two-parameter family may be written as

$$\begin{aligned}
x(\zeta_1, \zeta_2) &= x - \frac{\gamma}{\alpha}\,\zeta_1 + \frac{\beta}{\alpha}\,\zeta_2 \\
y(\zeta_1, \zeta_2) &= y - \zeta_2 \\
z(\zeta_1, \zeta_2) &= z + \zeta_1 \\
\phi(\zeta_1, \zeta_2) &= \phi + a^{-1}\,\zeta_2 \\
\rho(\zeta_1, \zeta_2) &= \rho\ .
\end{aligned}$$

The associated conserved quantities are

$$\Lambda_i = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial q_\sigma}{\partial \zeta_i}\bigg|_{\zeta_i}\ ,$$

whence

$$\Lambda_1 = -\frac{\gamma}{\alpha}\,m_1\,\dot{x} + m_1\,\dot{z} \quad , \quad \Lambda_2 = \frac{\beta}{\alpha}\,m_1\,\dot{x} - m_1\,\dot{y} + a^{-1}\,m_2\,\rho^2\,\dot{\phi}\ .$$

(c) Since $\partial L/\partial t = 0$, the total energy is also conserved:

$$E = \tfrac{1}{2}m_1\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \tfrac{1}{2}m_2\,(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2) + V(\alpha x + \beta y + \gamma z, y + a\phi).$$

#3 Solution U G. E&M problem 2

T. M. O'Neil
Phys. 100C
Spring 1997

T. M. O'Neil
Your Name

MIDTERM EXAM

Each problem worth 10 points.

1. In unbounded vacuum, the initial electric and magnetic fields are given by

$$\mathbf{E}(\mathbf{r}, t=0) = \hat{y} f(x)$$

$$\mathbf{B}(\mathbf{r}, t=0) = 0 \ .$$

Find $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ for $t \geq 0$.

$$\underset{\sim}{E}(\underset{-}{r}, t) = \hat{y}\frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

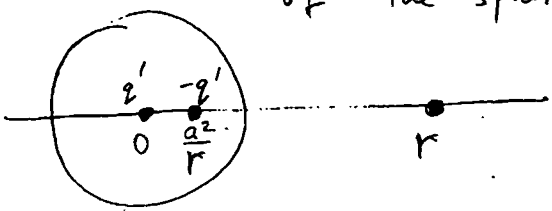
$$\underset{\sim}{B}(\underset{-}{r}, t) = \hat{z}\frac{1}{2c}\left[f(x-ct) - f(x+ct)\right]$$

4 U.G. E&M

Undergraduate EM

Compute the force of attraction between a neutral metallic sphere of radius a and a point charge positioned a distance r from the center of a sphere.

Solution: According to the method of images, the field of the sphere coincides with that of two charges:



A charge q' at its center and a charge $-q'$ at $\left(\frac{a^2}{r}, 0, 0\right)$

Here $q' = \frac{a}{r}$.

Therefore, the attraction force is:

$$F = \frac{q'}{\left(r - \frac{a^2}{r}\right)^2} - \frac{q'}{r^2} = \frac{(2r^2 - a^2)\,a^3}{(r^2 - a^2)^2\, r^3}.$$

For example, at $r \gg a$ we have $F \simeq \frac{2a^3}{r^5}$.

#5 U. G. Q. Solution

$$\chi = \frac{1}{4}\begin{pmatrix} 2-3i \\ i\sqrt{3} \end{pmatrix}$$

(a.) ~~$\langle S_z \rangle$~~ $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in this representation

$$\langle S_z \rangle = \left[\frac{1}{4}\left(\overline{2+3i \quad -i\sqrt{3}}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{4}\begin{pmatrix} 2-3i \\ i\sqrt{3} \end{pmatrix}\right]\frac{\hbar}{2}$$

$$= \frac{1}{16}\cdot\frac{\hbar}{2}\left(\overline{2+3i \quad -i\sqrt{3}}\right)\begin{pmatrix} 2-3i \\ -i\sqrt{3} \end{pmatrix} = \frac{\hbar}{2\cdot 16}\{4+9-3\}$$

$$= \frac{5}{16}\hbar$$

(b.) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle S_x \rangle = \frac{1}{16}\cdot\frac{\hbar}{2}\cdot\left[\left(\overline{2+3i \quad -i\sqrt{3}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2-3i \\ i\sqrt{3} \end{pmatrix}\right]$$

$$= \frac{1}{16}\cdot\frac{\hbar}{2}\cdot\left(-6\sqrt{3}\right) = \frac{-3\sqrt{3}}{16}\hbar$$

(c.) $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\langle S_y \rangle = \frac{1}{16}\cdot\frac{\hbar}{2}\cdot\left[\left(\overline{2+3i \quad -i\sqrt{3}}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 2-3i \\ \sqrt{3}\,i \end{pmatrix}\right]$$

$$= \frac{1}{16}\cdot\frac{\hbar}{2}\cdot 4\sqrt{3} = \frac{\sqrt{3}}{8}\hbar$$

#6 UG Quantum Solution

$$|\psi(t=0)\rangle = |x\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = \tfrac{1}{\sqrt{2}}\left(|E_1\rangle e^{-iE_1 t/\hbar} + |E_2\rangle e^{-iE_2 t/\hbar}\right)$$

$$\Rightarrow \quad \langle y|\psi(t)\rangle = \tfrac{1}{\sqrt{2}}\left(\langle E_1| - \langle E_2|\right)|\psi(t)\rangle$$

$$= \tfrac{1}{2}\left(e^{-iE_1 t/\hbar} - e^{-iE_2 t/\hbar}\right)$$

$$\text{Prob} = |\langle y|\psi(t)\rangle|^2 = \tfrac{1}{4}\left(1 - e^{-i(E_2-E_1)t/\hbar} - e^{+i(E_2-E_1)t/\hbar} + 1\right)$$

$$= \tfrac{1}{4}\left[2 - 2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right]$$

$$= \tfrac{1}{2}\left(1 - \cos(\quad)\right)$$

$$= \sin^2\left[\frac{(E_2-E_1)t}{2\hbar}\right]$$

maximum probability to be in $|y\rangle$ is 1

and that occurs @ times

$$\left(\frac{E_2-E_1}{2\hbar}\right)t = \frac{\pi}{2}n \qquad \text{for } n = 1, 3, 5, \ldots$$

$$t = \frac{n\pi\hbar}{E_2 - E_1}$$

#7 Solution

2006 Spring Exam – UG Stat Mech

A horizontal insulated cylinder is partitioned by a frictionless, insulating piston (which prevents heat exchange between the two sides). On each side of the piston, we have 30 liters of an ideal monatomic gas ($C_V = \frac{3}{2}nR$ and $\gamma = \frac{5}{3}$) at a pressure of 1 atm and a temperature of 300K. Heat is slowly injected into the gas on the left, causing the piston to move until the gas on the right is compressed to a pressure of 2 atm.

(a) What are the final V and T on the right side?

(b) What are the final P, V, and T on the left side?

(c) What is the change in internal energy ΔU for the gas on the right side? How much heat Q_{if} did it absorb, and how much work W_{if} is done to the right chamber?

(d) What are the values of ΔU, W_{if}, and Q_{if} for the gas on the left side?

To simplify the arithmetics, you may express all energies in unit of atm-liter.

[Hint 1: For an ideal gas, under adiabatic equation, PV^γ is constant.

Hint 2: The the internal energy of an ideal gas is a function of temperature only.]

Solution:

Let the initial pressure, temperature, volume of each chamber be $P_0 = 1$ atm, $T_0 = 300K$, $V_0 = 30$ liters respectively, and the final pressure of the right chamber be $P_r = 2$ atm.

(a) Since the compression of the right chamber is adiabatic, we have $P_0V_0^\gamma = P_rV_r^\gamma$. Hence

$$V_r = V_0\left(P_0 / P_r\right)^{1/\gamma} \approx 20 \text{ liters}.$$

From the ideal gas law then, $T_r = T_0\left(P_r / P_0\right)^{\frac{\gamma-1}{\gamma}} \approx 396K$.

(b) From mechanical equilibrium, we have $P_l = P_r = 2$ atm. From volume conservation we have $V_l = 2V_0 - V_r \approx 40$ liters. Then, $T_l = T_0\left(P_lV_l / P_0V_0\right) \approx 800K$.

(c) For the right chamber, we have $\Delta U_r = C_V \cdot (T_r - T_0)$ since U only depends on the temperature for ideal gas, and C_V is temperature-independent. In terms of the initial parameters, $nR = P_0V_0 / T_0$, and $C_V = \frac{3}{2}P_0V_0 / T_0$. Hence $\Delta U_r = \frac{3}{2}P_0V_0(T_r - T_0)/T_0 \approx 14.4$ atm-liter.
As the process is adiabatic, $Q_{if} = 0$, it then follows from the first law that the work done on the right chamber is $W_{if} = \Delta U_r = 14.4$ atm-liter.

(d) For the left chamber, $\Delta U_l = C_V(T_l - T_0) = \frac{3}{2}P_0V_0(T_l - T_0)/T_0 = 75$ atm-liter. Since the left and right chamber are always in mechanical equilibrium, then the work done *by* the left chamber is the work done *on* the right chamber, i.e., $W_{if} = -\Delta U_r \approx -14.4$ atm-liter. From the first law, it then follows that $Q_{if} = \Delta U_l - W_{if} \approx 89.4$ atm-liter as expected based on energy conservation.

#8 Solution

UG Statistical Mechanics ~~Problem~~ 1

Compute the mean energy fluctuation for a quantum mechanical harmonic oscillator with mass m and frequency ω at temperature T.

Solution:

The energy levels are

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right). \tag{1}$$

The probability that the oscillator is in state n is

$$\begin{aligned} p_n &= \frac{e^{-E_n/(k_B T)}}{Z} \\ Z &= \sum_{n=0}^{\infty} e^{-E_n/(k_B T)} \end{aligned} \tag{2}$$

The mean values $\langle E\rangle$ and $\langle E^2\rangle$ are

$$\begin{aligned} \langle E\rangle &= \sum_{n=0}^{\infty} p_n E_n = \hbar\omega \sum_{n=0}^{\infty} p_n\left(n+\tfrac{1}{2}\right) = \hbar\omega\left(\langle n\rangle+\tfrac{1}{2}\right) \\ \langle E^2\rangle &= \sum_{n=0}^{\infty} p_n E_n^2 = \hbar^2\omega^2 \sum_{n=0}^{\infty} p_n\left(n+\tfrac{1}{2}\right)^2 = \hbar^2\omega^2\left(\langle n^2\rangle+\langle n\rangle+\tfrac{1}{4}\right) \end{aligned} \tag{3}$$

The energy fluctuation is

$$(\Delta E)^2 \equiv \langle E^2\rangle - \langle E\rangle^2 = \hbar^2\omega^2\left(\langle n^2\rangle - \langle n\rangle^2\right) \tag{4}$$

The mean value of n^r is

$$\begin{aligned} \langle n^r\rangle &= \sum_{n=0}^{\infty} p_n n^r = \frac{\sum_{n=0}^{\infty} n^r e^{-(n+1/2)\hbar\omega/(k_B T)}}{\sum_{n=0}^{\infty} e^{-(n+1/2)\hbar\omega/(k_B T)}} \\ &= \frac{\sum_{n=0}^{\infty} n^r e^{-n\hbar\omega/(k_B T)}}{\sum_{n=0}^{\infty} e^{-n\hbar\omega/(k_B T)}} \\ &= \frac{\sum_{n=0}^{\infty} n^r \lambda^n}{\sum_{n=0}^{\infty} \lambda^n}, \qquad \lambda = e^{-\hbar\omega/(k_B T)} \end{aligned} \tag{5}$$

Now

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda} \tag{6}$$

Differentiating with respect to λ and multiplying by λ gives

$$\sum_{n=0}^{\infty} n\lambda^n = \frac{\lambda}{(1-\lambda)^2} \tag{7}$$

Repeating gives

$$\sum_{n=0}^{\infty} n^2\lambda^n = \frac{\lambda(\lambda+1)}{(1-\lambda)^3} \tag{8}$$

So

$$
\begin{aligned}
\langle n \rangle &= \frac{\lambda}{1-\lambda} \\
\langle n^2 \rangle &= \frac{\lambda(\lambda+1)}{(1-\lambda)^2} \\
\langle n^2 \rangle - \langle n \rangle^2 &= \frac{\lambda}{(1-\lambda)^2} \qquad (9)
\end{aligned}
$$

and so the mean energy fluctuation is

$$
\begin{aligned}
(\Delta E)^2 &= \hbar^2\omega^2 \frac{\lambda}{(1-\lambda)^2} \\
&= \hbar^2\omega^2 \frac{e^{-\hbar\omega/(k_B T)}}{(1-e^{-\hbar\omega/(k_B T)})^2} \\
&= \hbar^2\omega^2 \frac{e^{\hbar\omega/(k_B T)}}{(e^{\hbar\omega/(k_B T)}-1)^2} \qquad (10)
\end{aligned}
$$

Sol'n # 9 U.C. "other"

The mean free path is long compared to a meter, so the pumping rate is

$$\frac{dN}{dt} = \pi a^2 n \int_0^\infty dv_x\, v_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \frac{e^{-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}}}{\left(\frac{2\pi kT}{m}\right)^{3/2}}$$

$$= \pi a^2 n \frac{1}{\sqrt{2\pi}} \sqrt{\frac{kT}{m}} \underbrace{\int_0^\infty \underbrace{dx\, x}_{d\, x^2/2}\, e^{-x^2/2}}_{1}$$

$$\pi a^2 n \sqrt{\frac{kT}{m}} \frac{1}{\sqrt{2\pi}} = \frac{dN}{dt} = \left(10^{12} \frac{1}{m^2 sec}\right) 4\pi R^2$$

$$P = nkT = \left(\frac{R}{a}\right)^2 \frac{10^{12}}{m^2 sec} 4\sqrt{2\pi}\sqrt{m}\sqrt{kT}$$

$$P = nkT = \left(\frac{10^2}{3}\right)^2 10^{12}\, 4\sqrt{2\pi}\sqrt{3.4\cdot 10^{-27}}\sqrt{(300)(1.4\cdot 10^{-23})} \frac{Nt}{m^2}$$

$$= 4 \times 10^{-8} \left(Nt/m^2\right) \text{ Pascal}$$

10 Solution U. C. "Other" Problem

$$\text{flux of photons (energy)} = \frac{L}{4\pi r^2}$$

$$\text{momentum transfer rate} = \frac{L\,\sigma_T}{4\pi r^2 c}$$

so $$\frac{L\,\sigma_T}{4\pi \cancel{r^2} c} = \frac{G M m_p}{\cancel{r^2}}$$

$$L = \frac{4\pi G M m_p c}{\sigma_T}$$

$$L = \frac{4\pi M_\odot m_p c \hbar c}{m_{pl}^2 \sigma_T}$$

$$\simeq 10^{38} \text{ erg s}^{-1}$$

In general, the Eddington luminosity as this is called is

$$L_{edd} \simeq 10^{38} \text{ erg s}^{-1} \left(\frac{M}{M_\odot}\right)$$

Solution # 11 Grad mechanics

Let θ be the angle from a fixed point on the hoop to the point of contact with the surface, and let ϕ be the angle of the mass point relative to the vertical, as shown in Fig. ??. The Cartesian coordinates for the mass point are then

$$x = R\theta + R\sin\phi \quad , \quad y = R(1-\cos\phi) \ .$$

The Lagrangian is then

$$\begin{aligned} L &= \tfrac{1}{2}MR^2\dot\theta^2 + \tfrac{1}{2}I\dot\theta^2 + \tfrac{1}{2}m(R\dot\theta + R\dot\phi\cos\phi)^2 + \tfrac{1}{2}m(R\dot\phi\sin\phi)^2 - mgR(1-\cos\phi) \\ &= \tfrac{1}{2}\left(\tfrac{I}{R^2} + M + m\right)R^2\dot\theta^2 + \tfrac{1}{2}mR^2\dot\phi^2 + mR^2\cos\phi\,\dot\phi\,\dot\theta - mgR(1-\cos\phi) \ , \end{aligned}$$

where $I = MR^2$ for a perfect hoop. Note that L is invariant under $\theta \to \theta + \zeta$, where ζ is a constant, which means that there is an associated conserved quantity, which in this case is just the x-component of the momentum,

$$P_x = \frac{1}{R}\frac{\partial L}{\partial\dot\theta} = \left(\tfrac{I}{R^2} + M + m\right)R\dot\theta + mR\dot\phi\cos\phi \ .$$

The other conserved quantity is the total energy,

$$\begin{aligned} E &= T + V \\ &= \frac{P_x^2}{2\left(\frac{I}{R^2} + M + m\right)} + \frac{I + MR^2 + mR^2\sin^2\phi}{2\,(I + MR^2 + mR^2)}\cdot mR^2\dot\phi^2 + mgR(1-\cos\phi) \ . \end{aligned}$$

(a) The conserved quantities are P_x and E.

(b) For $|\phi| \ll 1$, we have

$$E = E_0 + \frac{I + MR^2}{2\,(I + MR^2 + mR^2)}\cdot mR^2\dot\phi^2 + \tfrac{1}{2}mgR\phi^2 + \mathcal{O}(\phi^4) \ .$$

Setting $\dot E = 0$ we obtain

$$\ddot\phi = -\omega^2\phi \ ,$$

with

$$\omega^2 = \left(1 + \frac{mR^2}{I + MR^2}\right)\frac{g}{R} \ .$$

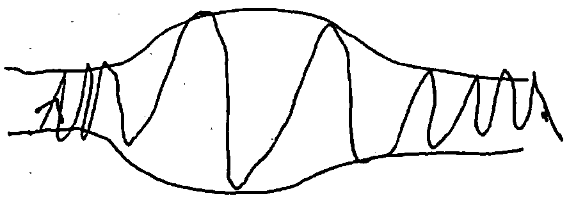
For a perfect hoop with $I = MR^2$,

$$\omega^2 = \left(1 + \frac{m}{2M}\right)\frac{g}{R} \ .$$

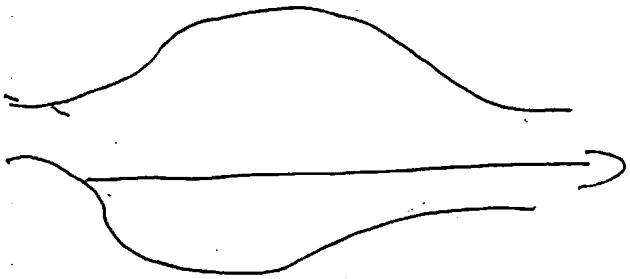
Solution   Grad mech # 12

a.) Depending on v_{y0}, v_{x0}, particle will mainly bounce in $\hat{y}$ direction and be confined, or stream out in $\hat{x}$ direction, and be lost.

i.e.



vs



b.) $y_{max} \ll L \Rightarrow \tau_{bounce\,y} \sim \left(v_\perp / 2\, y(x)\right)^{-1}$

$$\tau_{bounce\,x} \sim L/v_\parallel$$

so $\tau_{by} \ll \tau_{bx}$ $\Rightarrow$ many bounces in y direction before 1 bounce in x direction

so for adiabatic invariant:

$$2\pi I = \int_{-y(x)}^{y(x)} m v_y \, dy + \int_{y(x)}^{-y(x)} (-m v_y)\, dy$$

$$= 4m\, y(x)\, v_y$$

so have adiabatic invariant

$$I = 2y(x)\, m v_y / \pi$$

Now $$E = \frac{1}{2} m \left(v_x^2 + v_y^2\right)$$

$$= \frac{m}{2}\left(\frac{\pi^2 I^2}{4y(x)^2 m^2} + v_x^2\right)$$

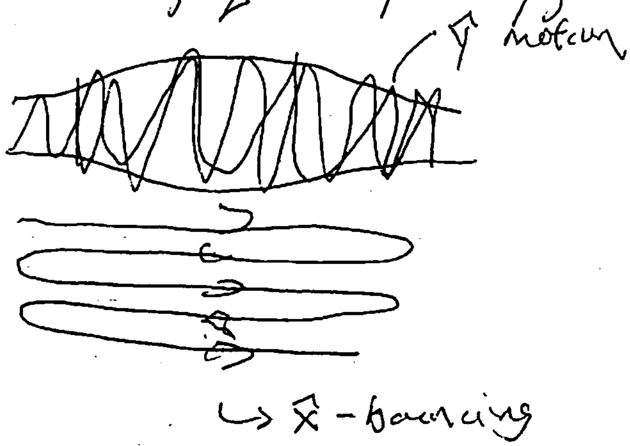
$\Rightarrow$

$$v_x^2 = \frac{2E}{m} - \frac{\pi^2}{4(y(x))^2}\frac{I^2}{m^2}$$

so, for $|x| < L/2$, if have an x such that

$$\frac{m}{2}\pi^2 I^2 / 4y(x)^2 m^2 > E$$

the particle is confined, and bounces back and forth, quickly in y, slowly in x.



c.) For particle injected at x_0, y_0

$$I = \frac{2y(x_0)}{\pi} m v_{y_0}$$

so confinement criterion is:

$$\left(\frac{y(x_0)}{y(x)}\right)^2 v_{y_0}^2 > \frac{2E}{m}$$

so for confinement:

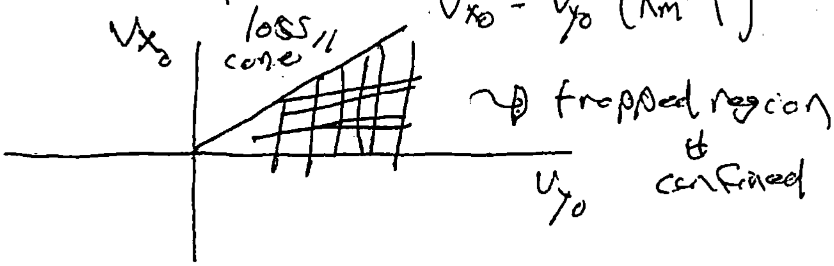
$$\left(\frac{y(x_0)}{y(x)}\right)^2 v_{y_0}^2 > v_{y_0}^2 + v_{x_0}^2$$

$$\left(\frac{v_y(x_0)}{y(x)}\right)^2 - 1 > \frac{v_{x_0}^2}{v_{y_0}^2}$$

so confinement criterion is:

$$\frac{V_{x_0}^2}{V_{y_0}^2} < R_m - 1 \qquad R_m = \left(\frac{y(x_0)}{y(x)}\right)^2$$

$$V_{x_0} = V_{y_0}(R_m - 1)^{1/2}$$



$x_0 \sim$ center

Obviously, larger $R_m \Rightarrow$ better confinement.

Sol'n #13 Grad E&M

-1-

(a) • The radiated energy is larger for the second collision (electron-positron collision). Electric dipole radiation does not vanish for this collision.

• For the first collision (electron-electron collision) electric dipole and magnetic dipole radiation vanish, so only quadrupole contributes.

For electron-electron collision

$$\underset{\sim}{p} = -e\underline{\ddot{r}_1} - e\underline{\ddot{r}_2} = -\frac{e}{m}\underline{(\dot{r}_1 + \dot{r}_2)}$$

electric

electric dipole radiation goes as

$$P = \frac{2}{3}\frac{|\ddot{p}|^2}{c^3}$$

but $\ddot{p} = -\frac{e}{m}(\ddot{r}_1 + \ddot{r}_2) = 0$

by conservation of momentum

magnetic dipole radiation goes as

$$P = \frac{2}{3}\frac{|\ddot{m}|^2}{c^3}$$

but $m = -e\, r_1 \times v_1 - e\, r_2 \times v_2$

$= -\frac{e}{m}(l_1 + l_2)$

$\therefore\ \ddot{m} = 0$ by conservation of angular momentum

(b) for electron-positron system

$$\underset{\sim}{p} = e(\underbrace{\underset{\sim}{r}_1 - \underset{\sim}{r}_2}_{\underset{\sim}{r}})$$

$\underset{\sim}{r}$ ← relative position

$$\ddot{\underset{\sim}{p}} = e\ddot{\underset{\sim}{r}} = \frac{e}{\mu}\frac{e^2}{r^2}$$

$$P = \frac{2}{3}\frac{(\ddot{\underset{\sim}{p}})^2}{c^3} = \frac{2}{3}\frac{e^6}{\mu^2 c^3}\frac{1}{r^4}$$

$$W = \int_{-\infty}^{+\infty} dt\, P = \frac{2}{3}\frac{e^6}{\mu^2 c^3}\int_{-\infty}^{+\infty}\frac{dt}{r^4}$$

$$r^2 \simeq b^2 + (v_0 t)^2$$ since $mv_0^2 \gg \frac{e^2}{b}$

$$W = \frac{2}{3}\frac{e^6}{\mu^2 c^6}\int_{-\infty}^{+\infty}\frac{dt}{[(v_0 t)^2 + b^2]^2}$$

(a) $\underline{B}_1 = -\nabla \Phi_1$, $\underline{B}_2 = -\underline{\nabla} \Phi_2$

$0 = \underline{\nabla} \cdot \underline{B}_1 = -\nabla^2 \Phi_1 \qquad 0 = \underline{\nabla} \cdot \underline{B}_2 = -\nabla^2 \Phi_2$

at $r = R$ $\quad \underline{\nabla} \cdot \underline{B} = 0$ implies

$$\left.\frac{\partial \Phi_1}{\partial r}\right|_R = \left.\frac{\partial \Phi_2}{\partial r}\right|_R = 0$$

The surface current in the shell is

$$K_\phi = \sigma \omega R \sin\theta$$

From $\quad \underline{\nabla} \times \underline{B} = \frac{4\pi}{c} \underline{J}$

$$-\frac{1}{R} \left.\frac{\partial \Phi_2}{\partial \theta}\right|_R + \frac{1}{R} \frac{\partial \Phi_1}{\partial \theta} = \frac{4\pi}{c} K_\phi$$

(b) From b.c. anticipate that only $P_1[\cos\theta] = \cos\theta$ will be need in Legendre expansion

$$\Phi_1 = A r \cos\theta$$

$$\Phi_2 = \frac{D}{r^2}\cos\theta$$

$$A\cancel{\cos\theta} = \left.\frac{\partial\Phi_1}{\partial r}\right|_R = \left.\frac{\partial\Phi_2}{\partial r}\right|_R = -\frac{2D}{R^2}\cancel{\cos\theta}$$

$$A = -\frac{2D}{R^3}$$

$$\frac{D}{R^3}\cancel{\sin\theta} - A\cancel{\sin\theta} = \left.+\frac{1}{R}\frac{\partial\Phi_1}{\partial\theta}\right|_R - \frac{1}{R}\frac{\partial\Phi_2}{\partial\theta} = \frac{4\pi\sigma\omega R}{c}\cancel{\sin\theta}$$

$$\frac{3D}{R^3} = \frac{4\pi\sigma\omega R}{c}$$

D

$$D = \frac{4\pi}{3}\frac{\sigma\omega}{c}R^4 \quad , \quad A = -\frac{2}{3}\frac{4\pi\sigma\omega R}{c}$$

$$\underset{\sim}{B}_1 = -\nabla\Phi_1 = -\underset{\sim}{\nabla}\left(-\frac{2}{3}\frac{4\pi\sigma\omega R}{c}\underbrace{r\cos\theta}_{z}\right)$$

$$= \hat{z}\,\frac{2}{3}\frac{4\pi\sigma\omega R}{c}$$

$$\underset{\sim}{B}_2 = -\underset{\sim}{\nabla}\Phi_2 = -\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta}\right)\frac{4\pi\sigma\omega R^4}{3c}\frac{\cos\theta}{r^2}$$

$$\underset{\sim}{B}_2 = \frac{4\pi\sigma\omega R^4}{3c}\left[\hat{r}\left(\frac{2}{r^3}\cos\theta\right) + \frac{\hat{\theta}\sin\theta}{r^3}\right]$$

#15 solution Grad QM Problem # 15

treat nucleons as a Fermi gas
(with plane wave wave functions)

$$d(\text{density}^{\#}) = dn = \frac{g}{2\pi^2}\frac{d\Omega}{4\pi}\frac{p^2\, dp}{e^{(E/kT - \mu/kT)}+1}.$$

since # of states in vol. V is

$$dN = \frac{V d^3p}{(2\pi\hbar)^3}$$

and occupation probability i.

$$\frac{1}{e^{(E_\nu-\mu)/kT}+1}$$

note $d^3p = dp_x\, dp_y\, dp_z$

$$= p^2\, dp\, \underbrace{\sin\theta\, d\theta\, d\varphi}_{d\Omega}$$

and $E^2 = p^2c^2 + m^2c^4 \Rightarrow p\,dp = E\,dE \Rightarrow p^2\,dp = p\,E\,dE$

assume complete degeneracy so $\mu/kT \gg 1$

$$n = \frac{g}{2\pi^2}\int_0^{p_F} p^2\, dp = \frac{g\, p_F^3}{6\pi^2} = \frac{p_F^3 c^3}{3\pi^2(\hbar c)^3}$$

since $g=2$ for nucleons $g=2s+1$ and $s=1/2$

but we also know the number density from

$$n = \frac{(A/2)}{4/3\,\pi R^3} = \frac{3}{8\pi}\frac{A}{(1.2)^3 A} = \frac{3}{8\pi(1.2)^3} \simeq \frac{0.069}{(\text{fm})^{-3}} \simeq 0.07\,\text{fm}^{-3}$$

for protons and/or neutrons

$$\Rightarrow \frac{p_F^3c^3}{3\pi^2(\hbar c)^3} = 0.07\,\text{fm}^{-3} \Rightarrow p_F c = 250.48\ \text{MeV}$$

$$\Rightarrow \frac{p_F^2c^2}{2mc^2} \simeq 33\ \text{MeV}$$ = kinetic energy @ the top of either the neutron or proton Fermi sea.

$\Rightarrow$ if the potential well is 50 MeV deep the minimum binding of a nucleon would be $\simeq 50-33 = 17$ MeV

$$\Rightarrow \text{average kinetic energy} = \langle E\rangle = \frac{\int_0^{p_F}\frac{p^2}{2m}p^2\,dp}{\quad} = \frac{3}{\ }\left(\frac{p_F^2}{\ }\right)$$

#15 Solution Grad QM Problem # 15

$\Rightarrow \langle E \rangle \approx 20\,\text{MeV} \approx \frac{1}{2} m v^2$ so average speed is

$$v_{av} = \sqrt{v^2} = 0.2\,c$$

a not insignificant fraction of the speed of light!

#16 Solution Grad Quantum #16

Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}k\left(x_1^2 + x_2^2\right) + a x_1 x_2$$

Define new variables $x_1 = \frac{\xi + \eta}{\sqrt{2}}$ $x_2 = \frac{\xi - \eta}{\sqrt{2}}$

or $\xi = \frac{x_1 + x_2}{\sqrt{2}}$ $\eta = \frac{x_1 - x_2}{\sqrt{2}}$

$$\frac{\partial}{\partial x_1} = \frac{\partial \xi}{\partial x_1}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x_1}\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial \xi}{\partial x_2}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial \eta} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right) = \left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}\right)\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial \xi} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial \eta}\right)$$

$$= \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) + \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) - \frac{\partial}{\partial \xi}\partial/\partial\eta$$

$$\Rightarrow \quad H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) + \frac{1}{2}(k+a)\xi^2 + \frac{1}{2}(k-a)\eta^2$$

So exact energy levels are

$$E = n_1 \hbar\omega_1 + n_2\hbar\omega_2 + \frac{\hbar}{2}(\omega_1 + \omega_2)$$

$n_1, n_2 \Rightarrow 0, 1, 2, \ldots$

$\omega_1^2 = \frac{k-a}{2}$ $\omega_2^2 = \frac{k+a}{2}$

Solution #17 Grad Stat mech

Multiplying the equation of state by $(v - v_0)\, v^3/p$, we have the equivalent form

$$F(v) \equiv v^4 - \left(v_0 + \frac{k_B T}{p}\right) v^3 + \frac{\alpha}{3p}\, v - \frac{\alpha}{3p}\, v_0 = 0 \ .$$

Note that $F(-\infty) = +\infty$ and $F(0^-) < 0$, and furthermore that $F''(v) > 0$ for $v < 0$. These three features mean that $F(v)$ has precisely one root along the negative v axis. At the critical point, this equation must take the form

$$(v + \Omega)\,(v - v_c)^3 = 0 \ ,$$

for some Ω. Multiplying out the factors,

$$\begin{aligned} F(v) &= v^4 - (3v_c - \Omega)\, v^3 + 3v_c\,(v_c - \Omega)\, v^2 + v_c^2\,(3\Omega - v_c)\, v - \Omega\, c_c^3 \\ &= v^4 - \left(v_0 + \frac{k_B T_c}{p_c}\right) v^3 + \frac{\alpha}{3p_c}\, v - \frac{\alpha}{3p_c}\, v_0 \ . \end{aligned}$$

Equating coefficients of the various powers of v, we obtain,

$$\begin{aligned} 3v_c - \Omega &= v_0 + \frac{k_B T_c}{p_c} \\ v_c - \Omega &= 0 \\ v_c^2\,(3\Omega - v_c) &= \frac{\alpha}{3p_c} \\ \Omega\, v_c^3 &= \frac{\alpha}{3p_c}\, v_0 \ . \end{aligned}$$

Solving, we obtain

$$\begin{aligned} \Omega = v_c &= 2\, v_0 \\ p_c &= \frac{\alpha}{48\, v_0^3} \\ T_c &= \frac{\alpha}{16\, k_B\, v_0^2} \ . \end{aligned}$$

Solution # 18 Grad Stat Mech

2006 Spring Exam – Grad Stat Mech

The s = 1 Ising Model — Consider a generalization of the Ising model in which the spin variable s_i on each site i may take the value 0 as well as ± 1. For convenience, let the spins be arranged on a 3D simple cubic lattice. The total energy of a given configuration $\{s_i\}$ is

$$H = -J\sum_{<i,j>} s_i s_j - \mu \sum_i s_i^2$$

where $\langle i,j \rangle$ denotes nearest neighbor sites.

(a) Apply the Weiss mean-field approximation to obtain a self-consistent equation for the mean magnetization σ. It will be convenient to use the effective parameters $t \equiv k_B T/(6J)$ and $\delta = e^{-\mu/k_B T}$.

For the following, it will be useful to assume $|\sigma| \ll 1$.

(b) For small δ, find the critical temperature $t_c(\delta)$ below which spontaneous magnetization occurs. Show that $|\sigma| \propto (t_c - t)^{1/2}$ immediately below the critical temperature.

(c) Show that something goes wrong with the solution in part **(b)** for sufficiently large δ.

Solution:

(a) In the Weiss mean field approach, we replace the value of the spin of the neighboring sites by the mean value, i.e., $\sum_{<i,j>} s_i \cdot s_j = 6\sum_i s_i \cdot \sigma$ for the simple cubic lattice for which there are 6 neighbors per lattice site. Then the mean spin value σ is computed self consistently as

$$\sigma = \frac{\sum_{s=\{0,\pm1\}} s\, e^{6Js\sigma/kT} \cdot e^{\mu s^2/kT}}{\sum_{s=\{0,\pm1\}} e^{6Js\sigma/kT} \cdot e^{\mu s^2/kT}} = \frac{e^{\sigma/t} - e^{-\sigma/t}}{\delta + e^{\sigma/t} - e^{-\sigma/t}}.$$

(b) Assuming that $|\sigma| \ll 1$, we expand the righ-hand side of the above mean-field equation in powers of σ and obain

$$\sigma \cdot \left[\delta + 2 + (\sigma/t)^2 + O(\sigma/t)^4\right] = 2\sigma/t + \tfrac{1}{3}(\sigma/t)^3 + O(\sigma/t)^5.$$

The critical temperature occurs where the first-order term in σ vanishes, i.e., at $t_c = \dfrac{2}{2+\delta}$. For temperature slightly below $t_c(\delta)$, we have

$$\sigma\left[\tfrac{2}{t_c} - \tfrac{2}{t}\right] = \tfrac{1}{3}(\sigma/t)^3 - \sigma(\sigma/t)^2$$

or $|\sigma| \approx t \cdot \sqrt{\left[\tfrac{2}{t_c t}(t - t_c)\right] / \left[1 - \tfrac{1}{3t}\right]} \approx \sqrt{\left[2(t - t_c)\right] / \left[1 - \tfrac{1}{3t_c}\right]}$.

(c) From the mean-field solution for σ obtained above, we see that something goes wrong when $3t_c(\delta) = 1$. Using the expression for $t_c(\delta)$ obtained above, we find that this occurs at $\delta = 4$.

#19 Solution "Other" #19 (Grad)

$\Rightarrow$ scattering amplitude in Born approx.

$$f(\theta) \approx \frac{-2m}{\hbar^2} \frac{1}{4\pi} \int d^3r \, e^{i(k-k')\cdot r} \, V(r)$$

small mom. transfer $\Rightarrow$ $(k - k')\cdot r \ll 1$

$$\Rightarrow \quad f(\theta) \approx \frac{-m}{2\pi\hbar^2} V_0 \left(\tfrac{4}{3}\pi R^3\right)$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \left(\frac{2mV_0R^3}{3\hbar^2}\right) \Rightarrow \sigma \approx 4\pi\left(\frac{2mV_0R^3}{3\hbar^2}\right)^2$$

$$\Rightarrow \quad \sigma \sim 10^{-42}\,cm^2$$

$$\text{rate} = \text{flux}\cdot\sigma = 10^{-2}\,cm^{-3} \cdot 10^7\,\frac{cm}{s} \cdot 10^{-42}\,cm^2$$

$$\approx 10^{-38}\,s^{-1}$$

1 day $\sim 10^5$ seconds $\Rightarrow$ 10^{-33} per day per atom

so for 1 particle scattered per day we need 10^{33} atoms $\Rightarrow$ $\dfrac{6\times10^{23}\,\frac{amu}{g}}{72\,\frac{amu}{atom}}[X]$

$$X = 72 \cdot 10^{33} / 6\times10^{23} \; g$$

$$= 10^{11}\,g = 10^{8}\,kg$$

#20 Graduate – Other ~~Regular~~ Solution #20

Refraction index in the atmosphere of a planet varies according to the law $n(h) = n_0 - \alpha h$, where h is the height and $n_0, \alpha = \text{const}$.

Find h at which electromagnetic waves can propagate along a circular orbit around the planet. The planet's radius is R.

Solution : Fermat's principle $\delta \int n(\vec{r})\, dl = 0$.

Assuming a circular orbit of radius $R+h$, we get

$$\frac{d}{dh}\left[n(h) \cdot 2\pi\,(R+h)\right] = 0\,, \qquad \frac{n'(h)}{n(h)} = -\frac{1}{R+h}\,,$$

$$h = \frac{1}{2}\left(\frac{n_0}{\alpha} - R\right).$$