PHYSICS DEPARTMENT EXAM
FALL 2005. SOLUTIONS

#1 : UNDERGRADUATE CLASSICAL MECHANICS

`PROBLEM:` A meteorite of mass M_1 is incident with the relative velocity v_0 and impact parameter s on a planet of radius R and mass M_2. Determine the largest s for the collision to occur assuming that the two bodies attract each other according to Newton's law of gravitation.

Hint: Use polar coordinates.

4. A particle of mass M_1 is incident with velocity v_0 and impact parameter s on a sphere of radius R and mass M_2 that is initially at rest. Determine the cross section for collision assuming that the two masses attract each other according to Newton's law of gravitation.

$$\text{const} = H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} - \frac{GM_1M_2}{r}$$

where $m = \dfrac{M_1M_2}{M_1+M_2}$, $P_\theta = m v_0 s$

set $P_r = 0$ at $r = R$

$$\frac{m v_0^2}{2} + \frac{GM_1M_2}{R} = \frac{m^2 v_0^2 s^2}{2m R^2}$$

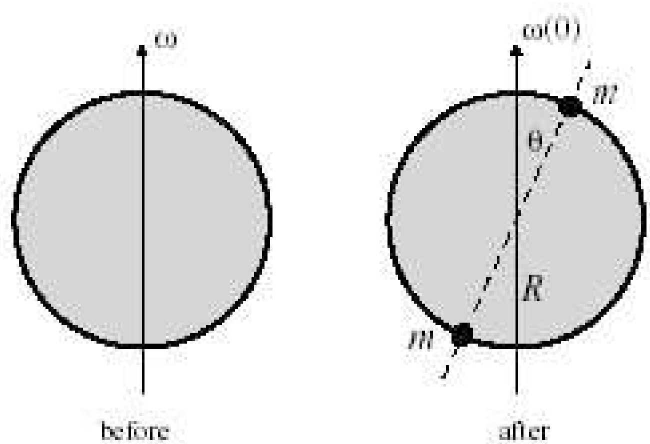
$$\sigma = \pi s^2 = \pi R^2\left[1 + \frac{GM_1M_2/R}{\left(\frac{M_1M_2}{M_1+M_2}\right)v_0^2/2}\right]$$

#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: Two comets of identical mass m simultaneously smash into the earth at diametrically opposite points. The axis along which the comets strike is at an angle θ with respect to the rotational axis of the earth. The velocities of the comets are such that immediately after the collision the earth's angular velocity $\boldsymbol{\omega}_e$ is unchanged (no "impact torque" transmitted).

(a) Assuming that before the collision the earth could be considered a sphere with the moment of inertia I_0, find the new inertia tensor of the system, along its principal axes.

(b) Solve for the subsequent rotation of the system. Give the answer in terms of the components of $\boldsymbol{\omega}_e(t)$ along the (body-fixed) principal axes.



Hint: A quick way to rederive Euler's equations is to use the following relation valid for any vector $\mathbf{A}$:

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{A}.$$

Solution

(a) The new inertia tensor is

$$I_{\alpha\beta} = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_0 \end{pmatrix}$$

where $I_\perp = I_0 + \Delta I$ and $\Delta I = 2mR^2$ is the contribution from the comets.

(b) Euler's equations are derived from $d\mathbf{L}/dt = 0$, which entails $(d\mathbf{L}/dt)_{body} = \mathbf{L} \times \boldsymbol{\omega}$. Here $\mathbf{L}$ is the angular momentum. Along the pricipal axes, $L_\alpha = I_{\alpha\alpha}\omega_\alpha$, so that the equations of motion read

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_1\dot{\omega}_2 = (I_2 - I_3)\omega_3\omega_1, \quad I_1\dot{\omega}_3 = (I_2 - I_3)\omega_1\omega_2,$$

where $I_1 = I_2 = I_\perp$ and $I_3 = I_0$. Therefore, $\omega_3(t) = \text{const}$, while $\dot{\omega}_1 = \nu\omega_2$, $\dot{\omega}_2 = -\nu\omega_1$, with

$$\nu = \omega_3 \Delta I / I_0 = (2mR^2/I_0)\cos\theta.$$

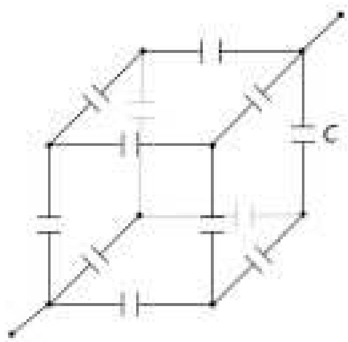
We define the 1 and 2 axes such that $\omega_1(0) = -\omega_e \sin\theta$ and $\omega_2(0) = 0$. Solving the above equations of motion, we obtain

$$\omega_1(t) = -\omega_e \sin\theta\cos(\nu t), \quad \omega_2(t) = \omega_e \sin\theta\sin(\nu t), \quad \omega_3(t) = \omega_e\cos\theta.$$

#3 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` Twelve capacitors of capacitance C each are assembled into a circuit that have the topology of a regular cube. Find the capacitance C_{tot} that would be measured between diagonally opposite nodes of such a circuit.

Hint: Consider a node not directly connected to the measuring device. If the electrostatic potential of this node is ϕ, can you determine any other potentials in the circuit from symmetry arguments?



Solution

Let the electrostatic potentials of two diagonally opposite nodes be $V/2$ and $-V/2$. Let the sum of the electric charges on the three capacitor plates connected to the $V/2$-node be Q. Then the corresponding quantity for the diagonally opposite corner is $-Q$, while the capacitance in question is $C_{tot} = Q/V$. By symmetry, the three nodes adjacent to $V/2$-node have the same potential ϕ. The potentials of the remaining nodes are all equal to $-\phi$. Therefore, $Q = 3q_1$ where $q_1 = C(V/2 - \phi)$. Due to electroneutrality of the $\pm\phi$ nodes, $-q_1 + 2q_2 = 0$, where $q_2 = 2C\phi$. From these equations we get $\phi = V/10$, $q_1 = (2/5)CV$, and $q_2 = (1/5)CV$. Finally, $C_{tot} = (6/5)C$.

#4 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` A total charge Q is uniformly distributed on the surface of a sphere of radius R. Find the force exerted by the lower hemisphere on the upper hemisphere.

Hint: One possible method of solution employs Maxwell's electric stress tensor given by (in CGS units)

$$T_{ij} = \frac{1}{8\pi}\delta_{ij}E^2 - \frac{1}{4\pi}E_iE_j.$$

Solution

`Method 1`: The electric field of the sphere is equal to $\mathbf{E}(\mathbf{r}) = (Q/r^2)\hat{\mathbf{r}}$ for $r > R$ and $\mathbf{E} = 0$ for $r < R$. By symmetry, the force in question must be along the z-direction. This force can be computed integrating the stress tensor over any surface that encloses the lower hemisphere. It is convenient to choose such a surface to be another hemisphere of a very large radius. In this case only its flat part contributes:

$$F_z = -\int T_{zi}dS_i = -\int \frac{E^2}{8\pi}dxdy = -\int_R^\infty \frac{E^2(r)}{8\pi}2\pi r dr = -\frac{Q^2}{8R^2}.$$

`Method 2:` The lower hemisphere does not exert a net force on itself, so the force in question F can be computed according to

$$F = \int_{z<0} E_z(\mathbf{r})\rho(r)d^3r,$$

where E_z is the z-component of the *total* electric field. The charge density $\rho(r)$ is concentrated in a thin layer near the surface of the sphere. However, we cannot immediately take the width a of such a distribution to be infinitesimally small. If we do so, $E_z(\mathbf{r})$ would be discontinuous and the integral would be ambiguous. The correct procedure is to assume that a is small but finite and take the limit $a \to 0$ at the end of the calculation. In spherical polar coordinates we find

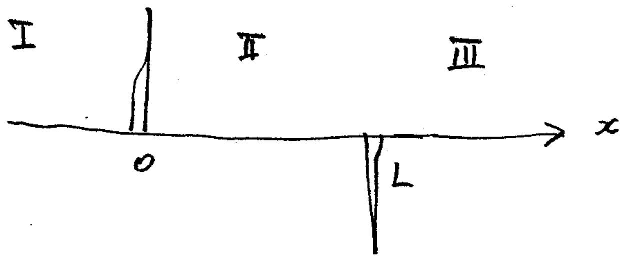
$$F = \int_{\pi/2}^{\pi} \sin\theta\cos\theta d\theta \int_0^R 2\pi r^2 E(r)\rho(r)dr = -\frac{1}{4}\int_0^R E(r)\frac{d}{dr}[E(r)r^2]dr,$$

The last equation follows from the Gauss law. In the limit $a \to 0$, we can replace r^2 by R^2 in the integrand, which yields

$$F = -R^2E^2(R+0)/8 = -Q^2/8R^2.$$

#5 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` A one-dimensional quantum particle with mass m and momentum $p = \hbar k$ is incident from $x = -\infty$ on a potential $V = \lambda\delta(x) - \lambda\delta(x - L)$. Find the transmission coefficient $T(k)$ and the values of k for which $|T| = 1$.



$$E = \frac{\hbar^2 k^2}{2m}$$

I : $\psi = e^{ikx} + R e^{-ikx}$

II : $\psi = A e^{ikx} + B e^{-ikx}$

III : $\psi = T e^{ikx}$

boundary condition if $V = A\,\delta(x - x_0)$

ψ continuous at x_0

$$-\frac{\hbar^2}{2m}\left[\psi'(x^0 + \epsilon) - \psi'(x^0 - \epsilon)\right] + A\psi(x_0) = 0$$

$$\therefore \quad \Delta\psi'(x_0) = \frac{2mA}{\hbar^2}\psi(x_0)$$

so at $x = 0$: $\quad 1 + R = A + B$

$$ik(A - B) - ik(1 - R) = \frac{2m\lambda}{\hbar^2}(1 + R)$$

at $x = L$: $\quad A e^{ikL} + B e^{-ikL} = T e^{ikL}$

~~$ikT = ik$~~

$$ik T e^{ikL} - ik(A e^{ikL} - B e^{-ikL}) = -\frac{2m\lambda}{\hbar^2} T e^{ikL}$$

so we have 4 equations in 4 unknowns:

solving gives

$$T = \frac{\hbar^4 k^2}{\hbar^4 k^2 + m^2\lambda^2(1 - e^{2ikL})}$$

$|T| = 1 \Rightarrow e^{2ikL} = 1.$

since otherwise |denominator| > |numerator|

so $k = \dfrac{n\pi}{L}$

#6 : UNDERGRADUATE QUANTUM MECHANICS

`PROBLEM:` Two particles of mass m are fixed on the ends of a massless rigid rod of length b. The location of the center of mass of the system is fixed in space but otherwise the entire assembly is free to rotate about the center of mass. Find the energy eigenvalues and eigenfunctions for this system.

Solution

The Hamiltonian of the system $H = \mathbf{L}^2/2I$ where $I = mb^2/2$ is the moment of inertia of the rod about an axis going through the center of mass perpendicular to the rod axis. The energy eigenvalues are $E_L = (\hbar^2/2I)L(L+1)$, where $L = 0, 1, \ldots$ Each eigenvalue is $2L+1$-fold degenerate. The eigenfunctions are the usual spherical harmonics $Y_{LM}(\theta, \phi)$.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The second law of thermodynamics prohibits a transfer of heat from 'cold' to 'hot' in a two-body system. There is no such a restriction in a closed system of several bodies. Show that explicitly by following these steps:

(a) Consider three thermal reservoirs with initial temperatures $T_1^0 < T_2^0 < T_3^0$. In simple physical terms, explain why it is possible to lower the temperature of reservoir No. 1 without violating the second law.

(b) Assuming the heat capacity of each reservoir is equal to a temperature-independent constant C, show that the entropy of i-th reservoir is equal to $S_i = C \ln T_i$, where T_i is its temperature.

(c) Consider the specific case $T_1^0 = 4(\sqrt{10} - 3) \approx 0.65$, $T_2^0 = 1$, and $T_3^0 = 5$. Taking advantage of energy and entropy conservation, demonstrate that the lowest achievable T_1 is $T_1 = T_1^0/2 = 2(\sqrt{10} - 3)$.

Hint: Show that the final temperatures of Nos. 2 and 3 must be equal.

Solution

(a) To cool down reservoir No. 1 one can employ an ideal heat engine as a refrigerator that receives power from another ideal heat engine that runs between reservoirs No. 2 and 3.

(b) The entropy is computed as follows:

$$S_i = \int \delta Q_i / T_i = C \int dT_i / T_i = C \ln T_i.$$

(c) The cooling scheme described in part (a) works until reservoirs Nos. 2 and 3 equilibrate, as suggested in the Hint. To proceed formally, one uses conservation of the total energy and entropy, which yields two constrains $\sum T_i = \text{const}$ and $\sum \ln T_i = \text{const}$, respectively. The lowest T_1 corresponds to the condition of minimum of function $T_1 - \lambda_E \sum T_i - \lambda_S \sum \ln T_i$, where λ_E and λ_S are the Lagrange multipliers. One immediately obtains that T_2 and T_3 must be equal, confirming the above physical argument. This entails $T_2 = T_3 = (T_1^0 + T_2^0 + T_3^0 - T_1)/2$. Rewriting the entropy conservation in the form $T_1 T_2 T_3 = \text{const}$, one arrives at the cubic equation:

$$(1/4)(T_1^0 + T_2^0 + T_3^0 - T_1)^2 T_1 = T_1^0 T_2^0 T_3^0.$$

For the numerical values given, it reads

$$(1/4)T_1^3 - (2\sqrt{10} - 3)T_1^2 + (2\sqrt{10} - 3)^2 T_1 = 20(\sqrt{10} - 3).$$

By direct substitution one verifies that $T_1 = 2(\sqrt{10} - 3) \approx 0.32$ is the solution. (The other two solutions $T_1 \approx 5.0$ and 7.9 are higher than T_1^0.)

#8 : UNDERGRADUATE STATISTICAL MECHANICS

PROBLEM:

1. An ideal monatomic gas undergoes a quasi-static, isothermal expansion from (P_A, V_A, T) to (P_B, V_B, T).

(a) Use classical thermodynamics to compute the heat Q and the work W input to the gas and also the change in its entropy S in such a process.

(b) Derive the formula for the entropy change from a *statistical mechanics* argument and show that it agrees with your result in part (a).

(c) Compute the change in the Gibbs free energy $G = U - TS + PV$.

2. Find the enthalpy $H = U + PV$ of n moles of an ideal gas whose molecules have $f > 3$ degrees of freedom each.

Solution

1. The work done by the surroundings is $W = -\int_{V_A}^{V_B} p dV = -p_A V_A \ln(V_B/V_A)$. (Equation of state $PV =$ const was used.)

(a) By the first law, $Q + W = \Delta U = 0$, so that $Q = p_A V_A \ln(V_B/V_A)$. The change in entropy is $\Delta S = Q/T = (p_A V_A/T)\ln(V_B/V_A)$.

(b) The number of ways to place N indistinguishable particles in a volume V is $\Omega \propto V^N/N!$ Hence, the entropy is given by $S \simeq k_B N \ln(V/N) + \text{const}$ in the limit of large N. (The constant term is ultimately fixed by quantum mechanics.) The change in entropy is $\Delta S = k_B N \ln(V_B/V_A)$. Using $N = P_A V_A / k_B T$, the result of part (a) is recovered.

(c) The change in the Gibbs free energy is given by

$$\Delta G = \Delta U - T\Delta S + \Delta(PV) = -p_A V_A \ln(V_B/V_A).$$

2. From the equipartition theorem, the internal energy is given by $(f/2)nRT$. Using the ideal gas equation $PV = nRT$, we then have $H = [1 + (f/2)]nRT$.

#9 : UNDERGRADUATE GENERAL

`PROBLEM:` Underwater explosions usually emit multiple shock waves because the bubble of gaseous detonation products that forms immediately thereafter undergoes a few cycles of oscillations. The period of such oscillations was measured to be $T = 3\,\text{ms}$ when a lump of a common explosive TNT was detonated at the depth of $h = 3.8\,\text{km}$.

(a) Estimate the energy E released in the explosion using only dimensional arguments, i.e., by constructing the quantity of dimension of time from relevant physical parameters.

(b) The mass of the TNT charge in the explosion was $2.27\,\text{kg}$ (5 lb). Predict what oscillation period will result for a 10 Kton ($= 10^7\,\text{kg}$ of TNT) nuclear blast at the same depth.

Solution

(a) The only relevant physical parameters of a deep point-source explosion are the total energy released E, the mass density of the water ρ, and the equilibrium water pressure $p = \rho g h$. There is only one combination of these parameters that has the dimension of time:

$$T = AE^{1/3}\rho^{1/2}p^{-5/6} = A(E/\rho)^{1/3}/(gh)^{5/6},$$

where A is an unknown numerical coefficient. Solving for E, we obtain

$$E = A^{-3}\rho T^3 (gh)^{5/2}.$$

If $A \sim 1$, we obtain an estimate $E = 0.72 \times 10^7\,\text{J}$. (Actually, $2.27\,\text{kg}$ of TNT produces $0.95 \times 10^7\,\text{J}$, so that $A \approx 0.9$.)

(b) For the nuclear blast we find $T = 3\,\text{ms} \times (10^7\,\text{kg}/2.27\,\text{kg})^{1/3} = 0.5\,\text{s}$.

#10 : UNDERGRADUATE GENERAL

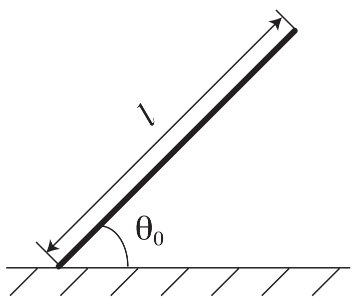
`PROBLEM:` Estimate how long it takes a typical nucleon (neutron or proton) to fly across the diameter of the nucleus of Ag (61 neutrons and 47 protons). Assume that you can model the nucleus as a gas of non-interacting fermions moving in a collective potential well. The radius of the nucleus is roughly $R \approx r_0 A^{1/3}$, where A is the total number of nucleons and $r_0 = 10^{-13}\,\mathrm{cm}$. The mass of a nucleon is $1.67 \times 10^{-24}\,\mathrm{g}$.

Solution

In a Fermi gas, particles propagate ballistically, with typical velocity $v \sim v_F$. The time to cross the nucleus is therefore $t \sim R/v_F = mR/\hbar k_F$. The Fermi momentum is of the order of the inverse proton-proton separation, $k_F \sim 1/r_0$, so $t \sim mRr_0/\hbar = mr_0^2 A^{1/3}/\hbar \sim 10^{-22}\,\mathrm{s}$.

#11 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` A thin rod of length l is supported at one end by a smooth floor (see figure). The rod is released from a configuration where it makes an angle θ_0 relative to the horizontal. Write down the Lagrangian for this system. Determine how long it takes for the rod to fall to the floor (*the answer in terms of a definite integral will be sufficient*.) Also determine how far the lower end moves during this time.



A thin rod of length l is supported at one end by a smooth floor (see figure). The rod is released from a configuration where it makes an angle θ_0 relative to the horizontal. Write down a Lagrangian for this system and determine how long it takes for the rod to fall to the floor. Also determine how far the lower end of the rod moves during this time.

$$L = \frac{m}{2}\left[\dot{x}^2 + \left(\frac{l}{2}\right)^2 \cos^2\theta\,\dot{\theta}^2\right] + \frac{ml^2\dot{\theta}^2}{24} - \frac{mgl\sin\theta}{2}$$

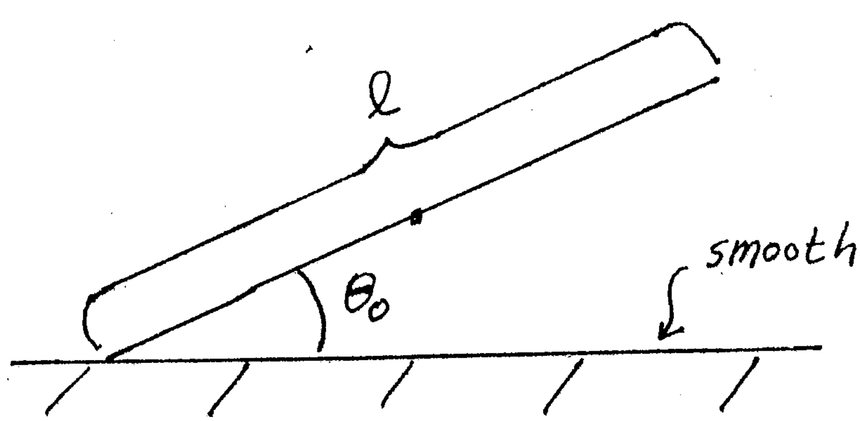
$$\dot{x} = \text{const.} = 0 \quad , \quad x = 0$$

$$H = \left(\frac{m}{2}\frac{l^2}{4}\cos^2\theta + \frac{ml^2}{24}\right)\dot{\theta}^2 + \frac{mgl\sin\theta}{2} = \frac{mgl\sin\theta_0}{2}$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{(g/l)\frac{1}{2}(\sin\theta_0 - \sin\theta)}{\frac{1}{8}\cos^2\theta + \frac{1}{24}}$$

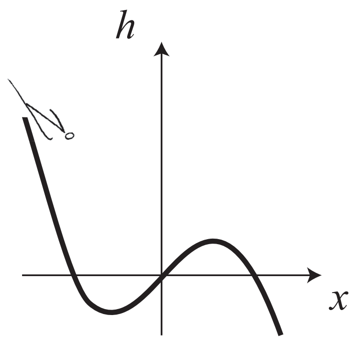
$$\sqrt{g/l}\; t = \int_0^{\theta_0} \sqrt{\frac{\frac{1}{4}\cos^2\theta + \frac{1}{12}}{\sin\theta_0 - \sin\theta}}\, d\theta$$

$$\Delta X_{end} = \frac{l}{2} - \frac{l}{2}\cos\theta_0$$



#12 : GRADUATE CLASSICAL MECHANICS

`PROBLEM:` A skier begins to slide down a ski jump shaped as a cubic parabola $h(x) = x - x^3/(3a^2)$, where h is the local height, x is the horizontal coordinate, and a is a constant. The skier flies off at the point $x = a$. The initial velocity of the skier, friction, changes in the body position or muscular forces generated by the skier are all negligible. Find the height h_0 where the motion started.



Solution

`Method 1:` The skier stays on the ski jump as long as the projection of the gravity on the local normal exceeds the necessary centripetal force:

$$mg\cos\phi(x) > \frac{mv^2(x)}{R(x)},$$

where $v(x)$ is the skier velocity, $\phi(x) = \arctan h'(x)$ is the angle between the local tangent and the horizontal, and $R(x)$ is the local radius of curvature. The point of detachment $x = x_d = a$ happens to be the point of maximum for the function $h(x)$, so that $\phi(x_d) = 0$ and $1/R(x_d) = -h''(x_d) = 2/a$. At this point the above condition turns into the equality:

$$g = \frac{v^2(x_d)}{R(x_d)} = \frac{2}{a}v^2(x_d).$$

From the energy conservation $h_0 = h(x_d) + v^2(x_d)/2g$. Therefore,

$$h_0 = h(x_d) + \frac{a}{4} = \frac{11}{12}a.$$

The Lagrangian is

$$L = \tfrac{1}{2} m\big(\dot{x}^2 + \dot{y}^2\big) - mgy$$

and there is a single constraint,

$$G(x,y) = y - h(x) = 0 \ .$$

The Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda \, \frac{\partial G}{\partial q_i} \ .$$

Thus, we obtain

$$\begin{aligned} m\ddot{x} &= -\lambda \, h'(x) \\ m\ddot{y} &= \lambda - mg \ . \end{aligned}$$

The constraint is $y = h(x)$. The constraint may be differentiated to yield

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h''(x)\,\dot{x}^2 + h'(x)\,\ddot{x} \ .$$

Substituting into the second equation of motion, we obtain

$$\frac{\lambda}{m} = g + h''(x)\,\dot{x}^2 + h'(x)\,\ddot{x}$$

and thus, from the first equation of motion,

$$\big(1 + h'(x)^2\big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 = -g\,h'(x) \ .$$

NOTE: This equation of motion is also obtained by eliminating the holonomic constraint $y = h(x)$ at the outset, and writing

$$L = \tfrac{1}{2}m\big(1 + h'(x)^2\big)\,\dot{x}^2 - mg\,h(x)\ .$$

The particle flies off the curve when the vertical force of constraint λ starts to become negative, because the curve can supply only a positive normal force. To evaluate y_0, we must express λ in terms of y_0 and x. Therefore, we must eliminate $\ddot{x}$ and $\dot{x}$ from

$$\frac{\lambda}{m} = g + h''(x)\,\dot{x}^2 + h'(x)\,\ddot{x}\ .$$

To eliminate $\ddot{x}$, we can use the equation of motion. This gives

$$\ddot{x} = -\left(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\right)\ ,$$

and thus

$$\lambda = m\left(\frac{g + h''\,\dot{x}^2}{1 + h'^2}\right)\ .$$

This has a simple interpretation at points where $h' = 0$: $\lambda = mg + mv^2/R$, where R is the local radius of curvature.

To eliminate $\dot{x}$, using conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot{x}^2 + mg\,h(x)\ ,$$

yielding

$$\dot{x}^2 = 2g\,\frac{y_0 - h}{1 + h'^2}\ .$$

Putting it all together,

$$\lambda = \frac{mg}{\big(1 + h'^2\big)^2}\,\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\}\ .$$

The particle flies off when $\lambda = 0$. This means

$$1 + h'(x)^2 + 2\big(y_0 - h(x)\big)\,h''(x) = 0\ .$$

For $h(x) = x - \frac{x^3}{3a^2}$, one obtains $y_0 = \frac{11}{12}\,a$.

#13 : GRADUATE ELECTROMAGNETISM

PROBLEM: Two halves of a spherical metallic shell of radius R are separated by a small insulating gap. The alternating voltage $V\cos\omega t$ is applied to the top half and the alternating voltage $-V\cos\omega t$ to the bottom half. Suppose that $\omega \ll c/R$ where c is the speed of light. Compute the amplitude of the oscillating dipole moment of the system and the time-average power it radiates.

Sol'n

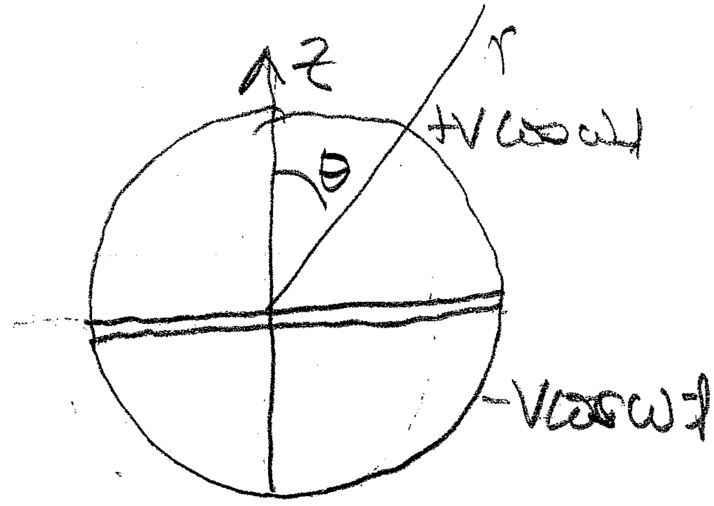
To lowest order in $\frac{\omega R}{c} << 1$, electric dipole radiation dominates

time-average power $\equiv \langle P \rangle = \frac{1}{3} \frac{|\omega^2 \underset{\sim}{p}_\omega|^2}{c^3}$

where $\underset{\sim}{p}_\omega = \int d^3\underset{\sim}{r}\, \underset{\sim}{r}\, \rho_\omega(r)$

and $\rho(t) = \mathrm{Re}\, \rho_\omega(\underset{\sim}{r}) e^{-i\omega t}$

Since $\omega R/c << 1$, can determine $\rho_\omega(\underset{\sim}{r})$ using near field (electrostatic approximation).



Potential outside shell is given by

$$\varphi(r,\theta,t) = \sum_l \frac{A_l(t)}{r^{l+1}} P_l[\cos\theta]$$

Surface charge is

$$\sigma(\theta,t) = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial r}\Big|_R = \frac{+1}{4\pi}\sum_l \frac{(l+1)A_l(t)}{R^{l+2}} P_l[\cos\theta]$$

by symmetry $\underset{\sim}{P}(t) = \hat{z} P_z(t)$

$$P_z(t) = \int_0^\pi 2\pi R^2 \sin\theta \, d\theta \; R\cos\theta \; \sigma(\theta,t)$$

$$P_z(t) = 2\pi R^3 \underbrace{\int_0^\pi \sin\theta \cos\theta \; \sigma(\theta,t) \, d\theta}_{\frac{1}{4\pi}\frac{2}{R^3} A_1(t) \underbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}_{2/3}}$$

$$P_z(t) = \frac{2}{3} A_1(t)$$

$$\frac{A_1(t)}{R^2} \int_0^{\pi} d\theta \sin\theta \cos^2\theta = \int_0^{\pi} d\theta \sin\theta \cos\theta \, \varphi(R,\theta)$$

$$\frac{A_1(t)}{R^2}\frac{2}{3} = V\cos(\omega t)\left[\int_0^{\pi/2} d\theta \sin\theta \cos\theta - \int_{\pi/2}^{\pi} d\theta \sin\theta \cos\theta\right]$$

$$= V\cos(\omega t) = \mathrm{Re}\, \frac{V}{2} e^{-i\omega t}$$

$$P_z(t) = \frac{VR^2}{2} \mathrm{Re}\, e^{-i\omega t}$$

$$p_\omega = VR^2/2$$

$$\langle P \rangle = \frac{1}{3}\frac{\omega^4}{c^3}\frac{|V|^2 R^4}{4}$$

#14 : GRADUATE ELECTROMAGNETISM

PROBLEM: A conducting spherical shell of inner radius $r = a$ rotates with angular velocity $\boldsymbol{\omega} = \omega\hat{z}$ and is immersed in the uniform magnetic field $\mathbf{B} = B\hat{z}$. Find the electric field in the vaccuum region enclosed by the shell (i.e., for $r < a$).

Hint: The first three Legendre polynomials are: $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, and $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$.

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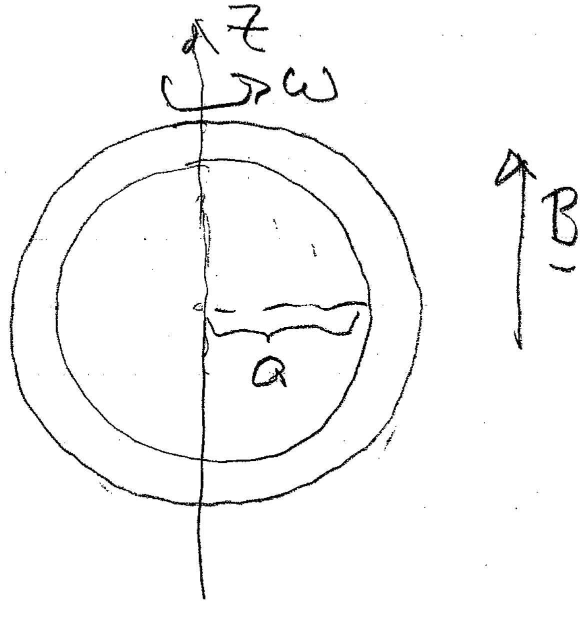
A conducting spherical shell of inner radius $r = a$ rotates with angular velocity $\underset{\sim}{\omega} = \omega\hat{z}$ and is immersed in the uniform magnetic field $\underline{B} = B\hat{z}$. Find the electric field in the vacuum region enclosed by the shell (i.e., for $r < a$).

Hint: the first four Legendre polynomials are:



$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$

$P_3 = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$

In the shell, the electric field is given by

$$0 = \underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} = \underset{\sim}{E} + \frac{(\underset{\sim}{\omega} \times \underset{\sim}{r}) \times \underset{\sim}{B}}{c}$$

$$\underset{\sim}{E} = -\underset{\sim}{r}\frac{\omega \cdot B}{c} + \underset{\sim}{B}\frac{\underset{\sim}{r} \cdot \omega}{c} = -\underset{\sim}{r_\perp}\frac{\omega B}{c}$$

$$\varphi = -\int^{r} d\underset{\sim}{r} \cdot \underset{\sim}{E} = +\frac{r_\perp^2}{2}\frac{\omega B}{c} = +\frac{B\omega}{2c} \underbrace{r^2 \sin^2\theta}_{1-\cos^2\theta}$$

for $r < a$ (region bounded by shell)

$$\varphi = \sum_l A_l r^l P_l[\cos\theta]$$

φ is continuous at $r = a$

$$\frac{B\omega}{2c} R^2 \underbrace{(1-\cos^2\theta)}_{\frac{2}{3}(1-P_2(\cos\theta))} = \sum_l A_l R^l P_l[\cos\theta]$$

$$\therefore \quad A_2 = -\frac{B\omega}{3c}, \quad A_0 = \frac{B\omega}{3c}$$

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\varphi = \underset{\sim}{\nabla}\left\{\frac{B\omega}{3c} r^2 P_2[\cos\theta]\right\}$$

$$\left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta}\right)$$

#15 : GRADUATE QUANTUM MECHANICS

PROBLEM: A particle of mass m and charge e is confined to a cubic box of side L. A weak uniform electric field E is applied parallel to one of the sides. If the electrostatic potential due to such a field is taken to be zero at the cube's center, the change in the ground state energy of the particle has the form $\Delta\varepsilon = -(\alpha/2)E^2$. Calculate the coefficient α (i.e., the polarizability).

Hint:

$$\sum_{n>0,\text{ even}}^{\infty} \frac{n^2}{(n^2-1)^5} = \frac{15\pi^2 - \pi^4}{3072}.$$

$$\psi_{n_x n_y n_z} = \left(\frac{2}{L}\right)^{3/2} \sin\frac{n_x \pi x}{L} \sin\frac{n_y \pi y}{L} \sin\frac{n_z \pi z}{L}$$

$V = -e\mathcal{E}(z - L/2)$ (convenient to choose $V=0$ at z-midpoint of box)

energy: $$E = E_0 + \langle 111|V|111\rangle + \sum_{n_x n_y n_z} \frac{|\langle n_x n_y n_z|V|111\rangle|^2}{E_{111} - E_{n_x n_y n_z}}$$

$= 0$ by parity $+ \cdots$

$\therefore$ for the energy shift, we need

$\langle n_x n_y n_z | -e\mathcal{E}(z - L/2) | 111\rangle$

by ~~othe~~ orthogonality in x, y $n_x = 1$, $n_y = 1$.

$\therefore$ The matrix element we need is

$\langle n_z | -e\mathcal{E}(z - L/2) | 1\rangle$

$$= \int_0^L dz \left(\frac{2}{L}\right) \sin\frac{n_z \pi z}{L} \sin\frac{\pi z}{L} \left(-e\mathcal{E}(z - L/2)\right)$$

$= 0$ if n is odd by parity

$$= \frac{8e\mathcal{E}Ln}{\pi^2} \frac{1}{(n_z^2-1)^2}$$ (n_z is now called n),

so $$\Delta E = \sum_{n \text{ even}} \frac{64 e^2 \mathcal{E}^2 L^2 n^2}{\pi^4 (n^2-1)^4} \frac{1}{\frac{\hbar^2}{2m}\frac{\pi^2}{L^2}(1-n^2)}$$

$$= -\sum_{n \text{ even}} \frac{128 m e^2 \mathcal{E}^2 L^4 n^2}{\hbar^2 \pi^6 (n^2-1)^5} = -\frac{1}{2}\alpha\mathcal{E}^2.$$

$$\alpha = \sum_{n \text{ even}} \frac{256 m e^2 L^4 n^2}{\hbar^2 \pi^6 (n^2-1)^5}.$$

To 2 significant digits

$$\sum_{\substack{n \\ (n \text{ even})}} \frac{n^2}{(n^2-1)^5} \Rightarrow \quad n = 2 \text{ term} \quad \longrightarrow \quad \frac{4}{3^5}$$

$$\alpha \simeq \frac{1024}{243} \frac{me^2 L^4}{\hbar^2 \pi^6} .$$

[The exact sum is

$$\sum_{n \text{ even}} \frac{n^2}{(n^2-1)^5} = \frac{15\pi^2 - \pi^4}{3072} . \Big]$$

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: Consider two spin-1/2 particles coupled by a Heisenberg interaction $H_{int} = J\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$, where J is a constant and $\boldsymbol{\sigma}_i$ are the Pauli matrices. The spins have magnetic moments $\boldsymbol{\mu}_1 = \alpha\boldsymbol{\sigma}_1$ and $\boldsymbol{\mu}_2 = \beta\boldsymbol{\sigma}_2$, respectively. A uniform external magnetic field $\mathbf{B}$ is applied. Find the exact energy eigenvalues for this system.

Hint: Show that the projection of the total spin on the direction of the magnetic field is a good quantum number.

Solution

The Hamiltonian of the system is

$$H = J\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + B(\alpha\sigma_1^z + \beta\sigma_2^z).$$

The operator $\sigma_{tot}^z = \sigma_1^z + \sigma_2^z$ commutes with the Hamiltonian; therefore, the eigenstates can be classified according to three possible values of this quantum number: $1, 0$, and -1. The wavefunction for the first case is uniquely determined to be $|\uparrow\uparrow\rangle$. The corresponding energy is

$$E_1 = (\alpha+\beta)B + J/4.$$

Similarly, the $\sigma_{tot}^z = -1$ corresponds to the state $|\downarrow\downarrow\rangle$ with the energy

$$E_{-1} = -(\alpha+\beta)B + J/4.$$

However, $\sigma_{tot}^z = 0$ selects the two-dimensional Hilbert space $c_1|\uparrow\downarrow\rangle + c_2|\downarrow\uparrow\rangle$. In this space the Hamiltonian matrix becomes the 2×2 matrix:

$$H = \begin{bmatrix} (\alpha-\beta)B - J/4 & J/2 \\ J/2 & (\beta-\alpha)B - J/4 \end{bmatrix}$$

Diagonalizing this matrix, we get the last two eigenvalues:

$$E_{\pm} = -\frac{J}{4} \pm \sqrt{\frac{J^2}{4} + (\alpha-\beta)^2 B^2}.$$

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: In a simplified theory of the liquid-gas critical point (see figure) the pressure P of the system is related to the density $n = N/V$ and temperature T by the truncated expansion $P = nk_BT - (b/2)n^2 + (c/6)n^3$, where b and c are some positive constants.

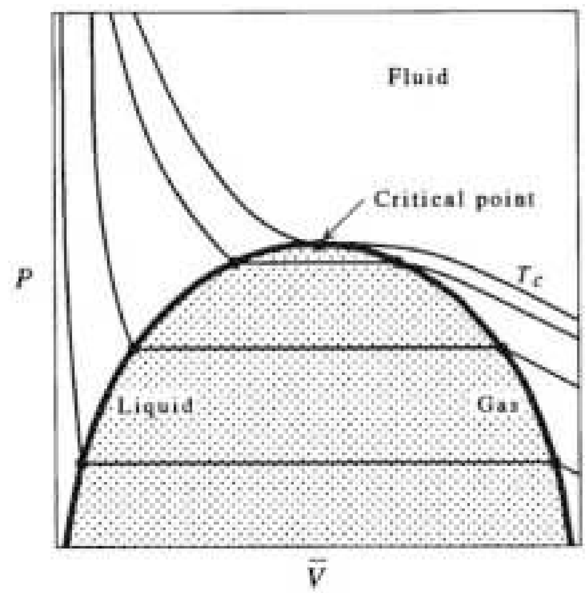


Figure 1: P–V diagram of a non-ideal gas (schematically).

(a) Find the critical temperature T_c, density n_c, and pressure P_c in terms of b and c.

(b) On the critical isotherm (i.e., for $T = T_c$), give an expression for $P - P_c$ as a function of $n - n_c$.

(c) Calculate the isothermal compressibility $\kappa_T = -(1/V)(\partial V/\partial P)_T$, and sketch its behavior as a function of T for n near n_c.

(d) Using the expression for κ_T found above, compute the temperature below which $\kappa_T < 0$. Explain why this indicates that the description of the system according to the above equation of state is unphysical below this temperature. Explain what is wrong in the formulation of the equation.

Solution

(a) The critical point is obtained from the conditions $\partial P/\partial n = \partial^2 P/\partial n^2 = 0$ at $T = T_c = \text{const}$. Starting from the cubic equation of state, we obtain $k_B T_c - bn_c + (c/2)n_c^2 = 0$ and $-b + cn_c = 0$. From the second equation, we get

$$n_c = b/c.$$

When substituted in the first equation, it gives

$$k_B T_c = b^2/(2c).$$

From the equation of state, we then find

$$P_c = b^3/(6c^2).$$

(b) Using the coordinates of the critical point computed above, we find

$$P - P_c = -\frac{b^2}{6c^2} + \frac{b^2}{2c}n - \frac{b}{2}n^2 + \frac{c}{6}n^3 = \frac{c}{6}(n - n_c)^3.$$

(c) Using $V = N/n$, we get

$$\kappa_T = \frac{1}{n}\left(\frac{\partial P}{\partial n}\right)_T^{-1} = \frac{1}{n(k_B T - bn + cn^2/2)}.$$

For $n \to n_c$, $\kappa_T \propto (T - T_c)^{-1}$, which diverges at T_c.

(d) Clearly, κ_T becomes negative for $T < T_c$. Negative compressibility is unphysical for a thermodynamic system because such a system would not be stable to fluctuation. A small reduction in pressure exerted on the system would lead to a decrease in volume, which would lead to further decrease in pressure until the system collapses down to a very small volume. In the case of our system, the runaway collapse is eventually curbed by the formation of a liquid phase. What is wrong with the equation of state is the assumption that the system is homogeneous, as reflected by the description with a uniform density n.

#18 : GRADUATE STATISTICAL MECHANICS

PROBLEM: The energy per unit cell $e(m)$ and entropy per unit cell $s(m)$ for a magnetic phase transition are given by

$$e(m) = -\frac{1}{2}Jm^2,$$
$$s(m) = s_0 - \frac{1}{4}k_B m^2 - \frac{1}{8}k_B m^4,$$

where m is the dimensionless magnetization per cell, and J and s_0 are constants. This expansion is valid for $|m| \ll 1$.

(a) Find the critical temperature T_c.

(b) Find and sketch the specific heat $c(T)$.

(c) Suppose a magnetic field H is imposed, so that

$$e(m,H) = -\frac{1}{2}Jm^2 - \gamma H m,$$

where γ is a constant. Find $m(H,T_c)$.

Solution

(a) The free energy per cell is

$$\begin{aligned} f(m,T) &= e - Ts \\ &= \frac{1}{4}\big(k_B T - 2J\big)\, m^2 + \frac{1}{8}k_B T\, m^4 - k_B T\, s_0 \ . \end{aligned}$$

Minimizing with respect to m, we find

$$m = \begin{cases} \sqrt{\frac{J}{k_B T} - \frac{1}{2}} & \text{if } T < 2J/k_B \\ 0 & \text{if } T > 2J/k_B \ , \end{cases}$$

from which we identify $T_c = 2J/k_B$.

(b) The specific heat is

$$
\begin{aligned}
c &= -T\,\frac{\partial s}{\partial T} \\
&= -\frac{T}{4}\left(1+m^2\right)\frac{\partial m^2}{\partial T} \\
&= \begin{cases} k_B T_c (T_c + T)/16T^2 & \text{if } T < T_c \\ \\ 0 & \text{if } T > T_c\ . \end{cases}
\end{aligned}
$$

Any sketch must show a discontinuity in $c(T)$ at $T = T_c$.

(c) With the magnetic field included, we have

$$f(m,T) = -\gamma H m + \frac{1}{4}\big(k_B T - 2J\big)\, m^2 + \frac{1}{8} k_B T\, m^4 - k_B T\, s_0\ .$$

Setting $T = T_c = 2J/k_B$ eliminates the term quadratic in m. Minimizing with respect to m then gives $-\gamma H + \frac{1}{2} k_B T_c\, m^3 = 0$, or

$$m(H,T_c) = \left(\frac{\gamma H}{J}\right)^{1/3}\ .$$

#19 : GRADUATE GENERAL

`PROBLEM:` Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx.$$

Solution

The integral can be done by contour integration. Let $z = x + iy$, then $I = \int_C dz f(z)$, where

$$f(z) = \sin^2 z / z^2$$

and C runs along the real axis. Since $f(z)$ is analytic at all finite z, an arbitrary segment $(-\varepsilon, \varepsilon)$ of C can be deformed into a semicircle $z = \varepsilon e^{i\theta}$, $\pi < \theta < 2\pi$, in the lower half-plane. We call the new contour C_1. Next, we write

$$f(z) = -\frac{1}{4}(e^{2iz} + e^{-2iz} - 2)$$

to get

$$I = -\frac{1}{4}[J(2) + J(-2) - 2J(0)],$$

where

$$J(a) = \int_{C_1} \frac{dz}{z^2} e^{iaz}.$$

The contour C_1 can be closed by a large arc in the upper or lower half-plane without changing the value of the last integral if a is positive or nonnegative, respectively. If $a < 0$, no singularities are thereby enclosed, so that $J(a) = 0$. If $a \geq 0$, we get $J(a) = 2\pi i \operatorname{res}_{z=0}(e^{iaz}/z^2) = -2\pi a$. Adding all terms together, we obtain the final result

$$I = \pi.$$

#20 : GRADUATE GENERAL

PROBLEM: In an optical experiment, the transmission coefficient T of a film made of some opaque material is measured. The thickness of the film is then calculated according to the formula $w = l\ln(1/T)$, where l is a known material parameter (attenuation length). This formula is correct if the film is uniform. Consider however an imperfect film whose thickness varies in space as a random number with the distribution function

$$P(w) = Aw\exp[-(w-w_0)^4/\sigma^4].$$

If the detector can measure only the area-averaged transmission, what film thickness would be deduced from such an experiment? Assume that $l \ll \sigma \ll w_0$ and use the steepest-descent method to evaluate any integrals you need.

Solution

The area-averaged transmission is given by

$$T = A\int_0^\infty dw w e^{-f}, \quad f(w) \equiv \frac{(w-w_0)^4}{\sigma^4} + \frac{w}{l}.$$

Under the conditions specified, T can be evaluated by the steepest-descent method. The saddle-point w_* satisfies the equation $f'(w_*) = 0$, which gives

$$w_* = w_0 - (\sigma^4/4l)^{1/3}.$$

To the leading-order, $T \sim \exp[-f(w_*)]$, so that the effective width of the film is

$$w = lf(w_*) = w_0 - \frac{3}{4}\left(\frac{\sigma^4}{4l}\right)^{1/3}.$$