DEPARTMENT OF PHYSICS

DEPARTMENTAL WRITTEN EXAMINATION

FALL, 2004

UPDATED SOLUTION SET : 9/28/04

#1 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: A cannonball is dropped from the top of a tower of height h located at a northerly latitude of λ. Assuming the cannonball is initially at rest with respect to the tower, and neglecting air resistance, calculate its deflection (magnitude and direction) due to (a) centrifugal and (b) Coriolis forces by the time it hits the ground. Evaluate for the case $h = 100$ m, $\lambda = 45°$. The radius of the Earth is $R_{\rm e} = 6.4 \times 10^6$ m.

#1 : UNDERGRADUATE CLASSICAL MECHANICS

SOLUTION: The equation of motion for a particle near the Earth's surface is

$$\ddot{\boldsymbol{r}} = -2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} - g_0 \hat{\boldsymbol{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \ ,$$

where $\omega = \omega\,\hat{\boldsymbol{z}}$, with $\omega = 2\pi/(24\,\text{hrs}) = 7.3 \times 10^{-5}\,\text{rad/s}$. Here, $g_0 = GM_{\text{e}}/R_{\text{e}}^2 = 980\,\text{cm/s}^2$. We use a locally orthonormal coordinate system $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}\}$ and write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\varphi}} + (R_{\text{e}} + z)\,\hat{\boldsymbol{r}} \ ,$$

where $R_{\text{e}} = 6.4 \times 10^6\,\text{m}$ is the radius of the Earth. Expressing $\hat{\boldsymbol{z}}$ in terms of our chosen orthonormal triad,

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \ ,$$

where $\theta = \frac{\pi}{2} - \lambda$ is the polar angle, or 'colatitude'. Since the height of the tower and the deflections are all very small on the scale of R_{e}, we may regard the orthonormal triad as fixed and time-independent. (In general, these unit vectors change as a function of $\boldsymbol{r}$.) Thus, we have $\dot{\boldsymbol{r}} \simeq \dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\varphi}} + \dot{z}\,\hat{\boldsymbol{r}}$, and we find

$$\begin{aligned}
\hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} &= -\dot{y}\,\cos\theta\,\hat{\boldsymbol{\theta}} + (\dot{x}\,\cos\theta + \dot{z}\,\sin\theta)\,\hat{\boldsymbol{\varphi}} - \dot{y}\sin\theta\,\hat{\boldsymbol{r}} \\
\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) &= -\omega^2 R_{\text{e}} \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} - \omega^2 R_{\text{e}} \sin^2\!\theta\,\hat{\boldsymbol{r}} \ ,
\end{aligned}$$

where we neglect the $\mathcal{O}(z)$ term in the second equation, since $z \ll R_{\text{e}}$.

The equation of motion, written in components, is then

$$\begin{aligned}
\ddot{x} &= 2\omega\cos\theta\,\dot{y} + \omega^2 R_{\text{e}} \sin\theta\cos\theta \\
\ddot{y} &= -2\omega\cos\theta\,\dot{x} - 2\omega\sin\theta\,\dot{z} \\
\ddot{z} &= -g_0 + 2\omega\sin\theta\,\dot{y} + \omega^2 R_{\text{e}} \sin\theta\cos\theta \ .
\end{aligned}$$

While these (inhomogeneous) equations are linear, they also are coupled, so an exact analytical solution is not trivial to obtain. Fortunately, the deflections are small, so we can solve this perturbatively. We write $x = x^{(0)} + \delta x$, *etc.*, and solve to lowest order by including only the g_0 term on the RHS. This gives $z^{(0)}(t) = z_0 - \frac{1}{2} g_0\, t^2$, along with $x^{(0)}(t) = y^{(0)}(t) = 0$. We then substitute this solution on the RHS and solve for the deflections, obtaining

$$\delta x(t) = \tfrac{1}{2}\omega^2 R_{\mathrm{e}}\, \sin\theta \cos\theta\, t^2$$
$$\delta y(t) = \tfrac{1}{3}\omega g_0\, \sin\theta\, t^3$$
$$\delta z(t) = \tfrac{1}{2}\omega^2 R_{\mathrm{e}}\, \sin^2\!\theta\, t^2\ .$$

The deflection along $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ is due to the centrifugal term, while that along $\hat{\boldsymbol{\varphi}}$ is due to the Coriolis term. (At higher order, the two terms interact and the deflection in any given direction can't uniquely be associated to a single fictitious force.) To find the deflection of an object dropped from a height h, solve $z^{(0)}(t^*) = 0$ to obtain $t^* = \sqrt{2h/g_0}$ for the drop time, and substitute. For $h = 100\,\mathrm{m}$ and $\lambda = \frac{\pi}{2}$, find $\delta x(t^*) = 17\,\mathrm{cm}$ south (centrifugal) and $\delta y(t^*) = 1.6\,\mathrm{cm}$ east (Coriolis).

#2 : UNDERGRADUATE CLASSICAL MECHANICS

PROBLEM: Consider the reaction $\pi^+ + n \rightarrow K^+ + \Lambda^0$. The rest masses of the particles are $m_\pi = 140\,\mathrm{MeV}/c^2$, $m_n = 940\,\mathrm{MeV}/c^2$, $m_K = 494\,\mathrm{MeV}/c^2$, and $m_\Lambda = 1115\,\mathrm{MeV}/c^2$. What is the threshold kinetic energy of the pion to create a kaon at an angle of 90° in the lab frame, in which the neutron is at rest?

#2 : UNDERGRADUATE CLASSICAL MECHANICS

`SOLUTION:` We conserve 4-momentum in the lab frame:

$$p^\mu_\pi + p^\mu_n = p^\mu_K + p^\mu_\Lambda \ ,$$

where $p = (p^0, p^1, p^2, p^3) = (E/c, \boldsymbol{p})$ is the 4−momentum. We use a $(-,+,+,+)$ metric, in which case the scalar product of two 4-vectors is $a \cdot b \equiv a_\mu \, b^\mu = -a^0 \, b^0 + \boldsymbol{a} \cdot \boldsymbol{p}$, and is (inertial) frame-independent. For a particle of mass m, then, $p \cdot p = m^2 c^2$ (evaluate in rest frame). Thus,

$$\begin{aligned} p_\Lambda \cdot p_\Lambda &= (p_\pi + p_n - p_K) \cdot (p_\pi + p_n - p_K) \\ &= -m^2_\pi c^2 - m^2_n c^2 - m^2_K c^2 + 2\, p_\pi \cdot p_n - 2\, p_\pi \cdot p_K - 2\, p_n \cdot p_K \\ &= -m^2_\Lambda c^2 \ . \end{aligned}$$

In the lab frame,

$$p_\pi = (E_\pi/c, \boldsymbol{p}_\pi) \quad , \quad p_n = (m_n c, 0) \quad , \quad p_K = (E_K/c, \boldsymbol{p}_K) \ ,$$

so $p_\pi \cdot p_n = -E_\pi m_n$, $p_\pi \cdot p_K = -E_\pi E_K/c^2 + \boldsymbol{p}_\pi \cdot \boldsymbol{p}_K$, and $p_n \cdot p_K = -E_K m_n$. Substituting these dot products into our earlier formula, we obtain

$$-m^2_\Lambda c^2 = -m^2_\pi c^2 - m^2_n c^2 - m^2_K c^2 - 2m_n E_\pi + 2m_n E_K + 2E_\pi E_K/c^2 - 2\boldsymbol{p}_\pi \cdot \boldsymbol{p}_K \ .$$

We're told the lab frame angle between pion and kaon is 90°, so we set $\boldsymbol{p}_\pi \cdot \boldsymbol{p}_K = 0$ and obtain

$$E_\pi = \frac{m^2_\Lambda c^4 - m^2_\pi c^4 - m^2_n c^4 - m^2_K c^4 + 2E_K m_n c^2}{2(m_n c^2 - E_K)}$$

The minimum of E_π is achieved when E_K takes the smallest possible value, which is $E_K = m_K c^2$. Finally, then,

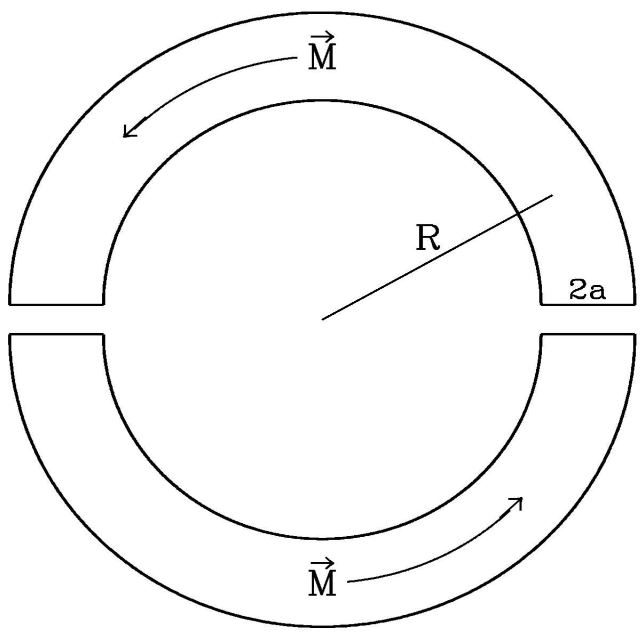
$$E_\pi = \frac{m^2_\Lambda - m^2_\pi - m^2_n - m^2_K + 2m_n m_K}{2(m_n - m_K)}\, c^2 = 1149\,\text{MeV} \ ,$$

and so the threshold kinetic energy of the pion is

$$T_\pi = E_\pi - m_\pi c^2 = 1009\,\text{MeV} \ .$$

#3 : UNDERGRADUATE ELECTROMAGNETISM

PROBLEM: The figure below shows a toroidal permanent magnet that has been cut in half, forming two horse shoe shaped magnets. The magnets are characterized by a magnetization M, major radius R, and minor radius a, where $a \ll R$. Determine the force of attraction between the two halves.



#3 : UNDERGRADUATE ELECTROMAGNETISM

`SOLUTION:` We have $\boldsymbol{\nabla} \times \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{J}_{\text{free}} = 0$, as well as

$$\boldsymbol{\nabla} \cdot \boldsymbol{H} = \boldsymbol{\nabla} \cdot \boldsymbol{B} - 4\pi \boldsymbol{\nabla} \cdot \boldsymbol{M} \simeq 0 \ .$$

That $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is from Maxwell; the result $\boldsymbol{\nabla} \cdot \boldsymbol{M} \simeq 0$ follows from $a \ll R$, since the magnetization is uniform in magnitude and very slowly varying in direction. Thus, $\boldsymbol{H}$ is a constant. By symmetry, $\boldsymbol{H}$ should point along $\boldsymbol{M}$, *i.e.* in the azimuthal direction. But with $\boldsymbol{j} = \partial \boldsymbol{E}/\partial t = 0$ we have $\oint \boldsymbol{H} \cdot d\boldsymbol{\ell} = 0$, and hence $\boldsymbol{H} = 0$. Therefore, $\boldsymbol{B} = 4\pi \boldsymbol{M}$.

We have that $B_n \equiv \hat{\boldsymbol{n}} \cdot \boldsymbol{B}$ is continuous across interfaces (with no surface currents), hence B_n is continuous in the gap. The energy density in the electromagnetic field is $B^2/8\pi$, hence for a separation x the field energy in the gap regions is

$$U(x) = 2 \cdot \frac{B^2}{8\pi} \cdot \pi a^2 x \ ,$$

where $\pi a^2 x$ is the volume of each gap region. The force is then

$$\begin{aligned} F = -\frac{\partial U}{\partial x} &= -\tfrac{1}{4}\, B^2 a^2 \\ &= -4\pi^2 a^2 M^2 \ , \end{aligned}$$

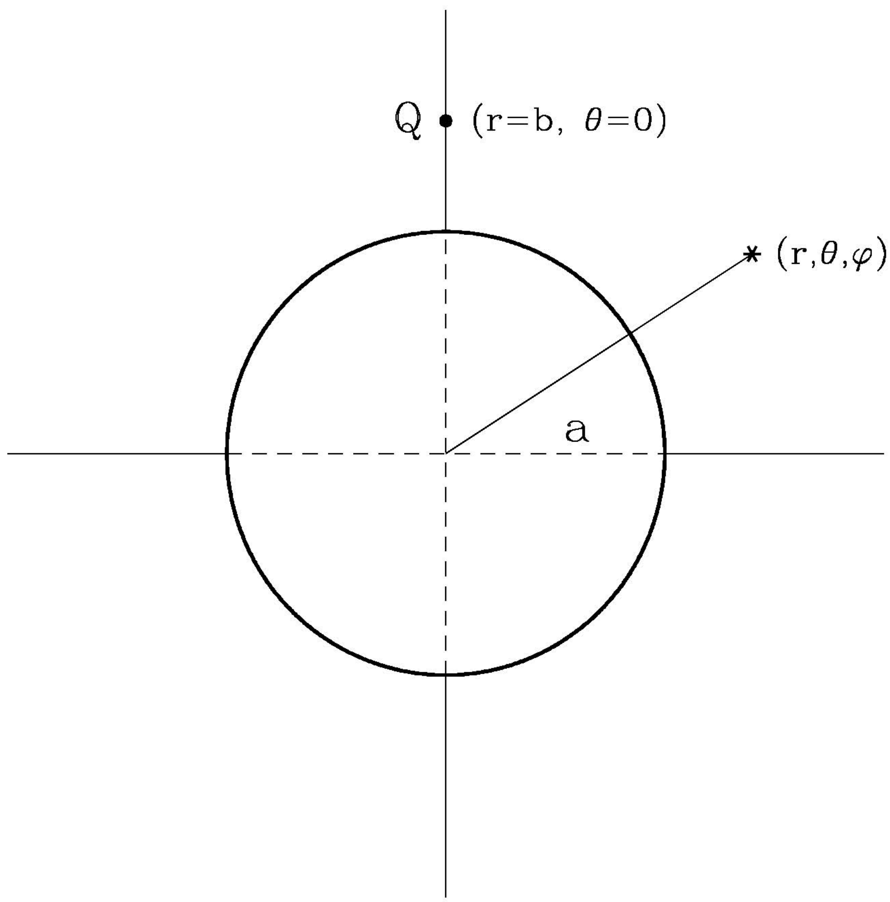
where the sign tells us that the force is attractive.

#4 : UNDERGRADUATE ELECTROMAGNETISM

`PROBLEM:` A point charge Q lies a distance b above the center of a grounded conducting sphere of radius a.

(a) Find the potential $\phi(r, \theta, \varphi)$ at an arbitrary point located outside the sphere. (Take θ to be the polar angle, with $\theta = 0$ being along $\hat{\boldsymbol{z}}$.) *Hint: Use the method of images.*

(b) How much work is required to move the point charge Q from $r = b$ to $r = \infty$?



#4 : UNDERGRADUATE ELECTROMAGNETISM

`SOLUTION:` This problem is conveniently solved using the method of images.

(a) An equipotential $\phi = 0$ is achieved over the entire sphere by placing an image charge of strength $\tilde{Q} = -(a/b)\,Q$ a distance a^2/b from the center, also at $\theta = 0$. Even if we did not remember these values, they could easily be determined by supposing the image charge lies a distance d from the center, and then demanding that the potential vanish anywhere on the surface of the sphere:

$$\phi(R,\theta,\phi) = \frac{Q}{\sqrt{a^2+b^2-2ab\,\cos\theta}} + \frac{\tilde{Q}^2}{\sqrt{a^2+d^2-2ad\,\cos\theta}} = 0 \quad \forall\,\theta\ .$$

After pushing one radical over to the other side of the equation, inverting both sides, and squaring, one then separately equates the constant terms on both sides as well as the coefficients of $\cos\theta$. This yields two equations:

$$b\,\tilde{Q}^2 = d\,Q^2$$
$$(a^2+b^2)\,\tilde{Q}^2 = (a^2+d^2)\,Q^2\ ,$$

which yield the familiar results $\tilde{Q} = -aQ/b$ and $d = a^2/b$. The potential everywhere is then

$$\phi(r,\theta,\varphi) = \frac{Q}{\sqrt{r^2+b^2-2br\,\cos\theta}} - \frac{Q}{\sqrt{\left(\frac{br}{a}\right)^2+a^2-2br\cos\theta}}\ .$$

(b) It is tempting to compute the potential due to the image charge at Q,

$$\phi_{\text{image}}(r)\big|_{\theta=0} = -\frac{Qa}{br-a^2} \qquad \Longrightarrow \qquad \phi_{\text{image}}(b) = -\frac{Qa}{b^2-a^2}\ ,$$

multiply by Q, and conclude that $W = aQ^2/(b^2-a^2)$ is the work required. This is wrong! The reason is that *the image charge moves with* Q. To get the right answer, integrate $F\,dr$, where F is the radial component of the force, $\boldsymbol{F} = Q\boldsymbol{E}$. The electric field due to the image at Q is

$$E(r) = -\frac{\partial\phi_{\text{image}}(r)}{\partial r}\bigg|_{b=r} = -\frac{Qar}{(r^2-a^2)^2}\ .$$

Next we multiply by Q and then integrate to get the work done *on* the charge:

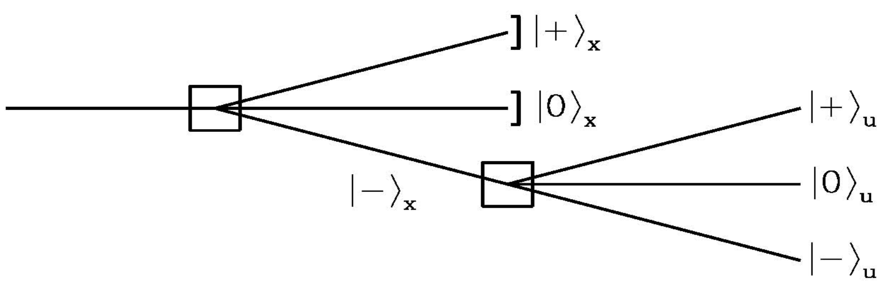
$$W = -Q \int_b^\infty dr\, E(r) = a\, Q^2 \int_b^\infty \frac{r\, dr}{(r^2 - a^2)^2} = \frac{a\, Q^2}{2(b^2 - a^2)}\ .$$

The wrong answer we obtained by the simplistic analysis is a factor of two too large.

#5 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: An unpolarized beam of $s = 0$, $\ell = 1$ atoms (never mind what kind) is passed through a Stern-Gerlach apparatus which uses a strong magnetic field gradient along the x-axis to separate the beam into three parts. Even though the beam is separated cleanly, the deflections are very small. Two of the separated beams are blocked so that only the atoms with x component of angular momentum equal to $-\hbar$ leave the apparatus. The resulting beam direction is along the z-axis.

Next, a similar apparatus with field gradient in the $\hat{\boldsymbol{u}}$ direction separates the unblocked beam into three components. The $\hat{\boldsymbol{u}}$ direction is in the $x-y$ plane at an angle θ from the x axis. (You may assume that $\hat{\boldsymbol{u}}$ is between the positive $\hat{\boldsymbol{x}}$ direction and the positive $\hat{\boldsymbol{y}}$ direction.) Detectors are set up to measure the number of atoms per second in each of the three beams. Compute these intensities in terms of $I_{\text{unpolarized}}$,the intensity of the initial beam which entered the first Stern-Gerlach apparatus, and the angle θ.



#5 : UNDERGRADUATE QUANTUM MECHANICS

`SOLUTION:` The formula for the answer can be easily written:

$$
\begin{aligned}
I_+ &= \tfrac{1}{3} I_{\text{unpolarized}} \left| \langle \chi_{u+} | \chi_{x-} \rangle \right|^2 \\
I_0 &= \tfrac{1}{3} I_{\text{unpolarized}} \left| \langle \chi_{u0} | \chi_{x-} \rangle \right|^2 \\
I_- &= \tfrac{1}{3} I_{\text{unpolarized}} \left| \langle \chi_{u-} | \chi_{x-} \rangle \right|^2
\end{aligned}
$$

where χ_{x+}, for example, is the eigenstate of the L_x operator with eigenvalue $+\hbar$. The problem is computing the eigenvectors. An alternative solution is to use rotation matrices. This takes about the same amount of calculation but is a bit trickier.

Simply solving the equation for the eigenvector $L_x \chi_{x-} = -\hbar \chi_{x-}$ gives

$$
|\chi_{x-}\rangle = \tfrac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}
$$

Now compute the L_u operator:

$$
L_u = \cos\theta L_x + \sin\theta L_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & e^{-i\theta} & 0 \\ e^{i\theta} & 0 & e^{-i\theta} \\ 0 & e^{i\theta} & 0 \end{pmatrix}
$$

Now again simply solve the three eigenvector problems like $L_u \chi_{u+} = \hbar \chi_{x+}$ yielding

$$
|\chi_{u+}\rangle = \tfrac{1}{2} \begin{pmatrix} e^{-i\theta} \\ \sqrt{2} \\ e^{i\theta} \end{pmatrix} \quad , \quad |\chi_{u0}\rangle = \tfrac{1}{2} \begin{pmatrix} \sqrt{2} e^{-i\theta} \\ 0 \\ \sqrt{2} e^{i\theta} \end{pmatrix} \quad , \quad |\chi_{u-}\rangle = \tfrac{1}{2} \begin{pmatrix} e^{-i\theta} \\ -\sqrt{2} \\ e^{i\theta} \end{pmatrix} \ .
$$

Plugging these into the formula above, we can compute the intensities:

$$
\begin{aligned}
I_+ &= \tfrac{1}{12} I_{\text{unpolarized}} \, (1 - \cos\theta)^2 \\
I_0 &= \tfrac{1}{6} I_{\text{unpolarized}} \, \sin^2\theta \\
I_- &= \tfrac{1}{12} I_{\text{unpolarized}} \, (1 + \cos\theta)^2
\end{aligned}
$$

Check that it adds up to $\frac{1}{3} I_{\text{unpolarized}}$ and that it agrees with the rotation matrix calulation.

#6 : UNDERGRADUATE QUANTUM MECHANICS

PROBLEM: The ground state of the hydrogen atom is split due to the electromagnetic interaction between the spin of the electon and the spin of the proton. The term in the Hamiltonian for this spin-spin interaction can be taken as

$$\mathcal{H}_{\text{hyperfine}} = \frac{4\pi}{3} \frac{e^2 g_{\text{p}}}{m_{\text{e}}\, m_{\text{p}}\, c^2}\, \delta(\boldsymbol{r})\, \boldsymbol{S} \cdot \boldsymbol{I} \ ,$$

where $\boldsymbol{S}$ is the spin operator for the electron, $\boldsymbol{I}$ is the spin operator for the proton, and $g_{\text{p}} = 5.58$ is the gyromagnetic ratio of the proton. The calculation is in CGS units for which $\alpha = e^2/\hbar c \simeq 1/137$. The unperturbed hydrogen ground state is

$$\psi_{100}(r, \theta, \phi) = (\pi a_{\text{B}}^3)^{-1/2}\, e^{-r/a_{\text{B}}} \ ,$$

where $a_{\text{B}} = \hbar^2/m_{\text{e}}\, e^2 = \hbar/\alpha\, m_{\text{e}}\, c$ is the Bohr radius. ($m_{\text{e}} = 0.511$ MeV$/c^2$ and $m_{\text{p}} = 938$ MeV$/c^2$).

(a) What quantum numbers describe the perturbed energy eigenstates? Calculate the energy shifts for each of these states, giving your answer in eV.

(b) Compute the energy shift in eV if a magnetic field of 100 Gauss is applied. Your answer need only be accurate to 10% so a weak field approximation is valid. (Recall that the Bohr magneton is $\mu_{\text{B}} = e\hbar/2m_{\text{e}}c = 0.579 \times 10^{-8}$ eV / Gauss and a nuclear magneton is $\mu_{\text{N}} = e\hbar/2m_{\text{p}}c$ is smaller by the ratio of the masses.)

#6 : UNDERGRADUATE QUANTUM MECHANICS

SOLUTION: We solve this problem using first order perturbation theory.

(a) The ground state of hydrogen has a spin-$\frac{1}{2}$ electron coupled to a spin-$\frac{1}{2}$ proton, yielding a total angular momentum of $j = 0$ or $j = 1$. The quantum numbers of the ground state are $n = 0$ (principal electronic quantum number), $s = \frac{1}{2}$, $l = \frac{1}{2}$, and $j = 0$ or $j = 1$. The energy shift, in first order perturbation theory, is

$$\begin{aligned}\Delta E &= \big\langle \Psi_0 \big| \mathcal{H}_{\text{hyperfine}} \big| \Psi_0 \big\rangle \\ &= \frac{4\pi}{3} \, \frac{e^2\, g_{\rm p}}{m_{\rm e}\, m_{\rm p}\, c^2} \, \frac{1}{\pi a_{\rm B}^3} \big[j(j+1) - \tfrac{3}{2} \big] \, \hbar^2 \\ &= \frac{2\, e^2\, g_{\rm p}}{3\, m_{\rm e}\, m_{\rm p}\, c^2} \, \frac{m_{\rm e}^3\, c^3\, \alpha^3}{\hbar} \big[j(j+1) - \tfrac{3}{2} \big] \\ &= \tfrac{2}{3}\, g_{\rm p}\, \alpha^4 \left(\frac{m_{\rm e}}{m_{\rm p}} \right) (m_{\rm e}\, c^2) \big[j(j+1) - \tfrac{3}{2} \big] \ .\end{aligned}$$

Thus,

$$\begin{aligned}\Delta E_{j=1} - \Delta E_{j=0} &= \tfrac{2}{3}\,(5.6) \cdot \left(\frac{1}{137} \right)^4 \cdot \left(\frac{0.51}{940} \right) \cdot (0.51\,\text{MeV}) \cdot 2 \\ &= 5.84 \times 10^{-6}\,\text{eV} \ .\end{aligned}$$

Note that

$$\begin{aligned}\boldsymbol{S} \cdot \boldsymbol{I} &= \tfrac{1}{2} \big[(\boldsymbol{S} + \boldsymbol{I})^2 - \boldsymbol{S}^2 - \boldsymbol{I}^2 \big] \\ &= \tfrac{1}{2} \, \big[j(j+1) - \tfrac{3}{2} \big] \, \hbar^2 \ .\end{aligned}$$

(b) We compute the first order splitting by the Zeeman Hamiltonian, $\mathcal{H}_{\text{Zeeman}} = \mu_{\rm B}\, B\, \sigma^z$. (Assume without loss of generality that the magnetic field is along the z-axis.) The problem here is that the eigenstates of $\boldsymbol{J}^2$ and J^z are linear combinations of products of spin wavefunctions for the electron and proton. Explicitly, we have

$$\begin{aligned}\big| j=1, m_j=+1 \big\rangle &= \big| m_s=+1, m_i=+1 \big\rangle \\ \big| j=1, m_j=0 \big\rangle &= \tfrac{1}{\sqrt{2}} \big| m_s=+1, m_i=-1 \big\rangle + \tfrac{1}{\sqrt{2}} \big| m_s=-1, m_i=+1 \big\rangle \\ \big| j=1, m_j=-1 \big\rangle &= \big| m_s=-1, m_i=-1 \big\rangle \\ \big| j=0, m_j=0 \big\rangle &= \tfrac{1}{\sqrt{2}} \big| m_s=+1, m_i=-1 \big\rangle - \tfrac{1}{\sqrt{2}} \big| m_s=-1, m_i=+1 \big\rangle \ .\end{aligned}$$

The first order (in B) energy shifts are then $\mu_{\rm B}\,B$, 0, $-\mu_{\rm B}\,B$, and 0, respectively, for these four states. In other words,

$$\Delta E_{j,m_j} = \mu_{\rm B} B\, m_j \ .$$

With $\mu_{\rm B} = 5.79 \times 10^{-9}$ eV/G, the shift in a 100 G field is about $0.579\,\mu$eV.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The surface tension σ of a liquid is the work required to increase the free surface area of the liquid by one unit of area. For pure water in contact with air at normal pressure, the surface tension has a constant value σ_0 at all temperatures for which the water is a liquid. If certain surfactant molecules are added to the water, they remain on the free surface and alter the surface tension. For water of area A containing N such molecules, one can measure this effect. Experiments show

$$\left(\frac{\partial\sigma}{\partial A}\right)_T = \frac{Nk_{\mathrm{B}}T}{(A-b)^2} - \frac{2a}{A}\left(\frac{N}{A}\right)^2$$

and

$$\left(\frac{\partial T}{\partial\sigma}\right)_A = -\frac{A-b}{Nk_{\mathrm{B}}} \ ,$$

where a and b are constants, and k_{B} is Boltzmann's constant. Find an expression for $\sigma(A,T)$. Your result should reduce to that for pure water (*i.e.* σ_0) in the limit $N \to 0$.

#7 : UNDERGRADUATE STATISTICAL MECHANICS

`SOLUTION:` We write

$$d\sigma(A,T) = \left(\frac{\partial\sigma}{\partial T}\right)_A dT + \left(\frac{\partial\sigma}{\partial A}\right)_T dA$$

and integrate:

$$\begin{aligned}\sigma(A,T) &= \int \left(\frac{\partial\sigma}{\partial A}\right)_T dA + f(T) \\ &= \int dA \left\{ \frac{Nk_B T}{(A-b)^2} - \frac{2aN^2}{A^3} \right\} + f(T) \\ &= -\,\frac{Nk_B T}{A-b} + \frac{aN^2}{A^2} + f(T)\ .\end{aligned}$$

Holding A constant, we take the differential:

$$d\sigma\Big|_A = -\frac{Nk_B}{A-b}\,dT\ + f'(T)\,dT,$$

and comparing with $(\partial T/\partial\sigma)_A = -(A-b)/Nk_B$, we find $f'(T) = 0$, or $f(T) = f_0$ is a constant, which is determined from the condition setting $\sigma(N=0,T) = \sigma_0$, yielding $f_0 = \sigma_0$. Thus,

$$\sigma(A,T) = -\frac{Nk_B T}{A-b} + \frac{aN^2}{A^2} + \sigma_0\ .$$

We can derive the same result by first integrating with respect to temperature:

$$\begin{aligned}\sigma(A,T) &= \int \left(\frac{\partial\sigma}{\partial T}\right)_A dT + g(A) \\ &= -\int \frac{Nk_B}{A-b}\,dT + g(A) \\ &= -\,\frac{Nk_B T}{A-b} + g(A),\end{aligned}$$

where we have used $(\partial\sigma/\partial T)_A = \Big[(\partial T/\partial\sigma)_A\Big]^{-1}$. We next determine the unknown function $g(A)$:

$$\begin{aligned}\left(\frac{\partial\sigma}{\partial A}\right)_T &= \frac{Nk_{\rm B}T}{(A-b)^2} - g'(A) \\ &= \frac{Nk_{\rm B}T}{(A-b)^2} - \frac{2a}{A}\left(\frac{N}{A}\right)^2 .\end{aligned}$$

Thus,

$$g(A) = \frac{aN^2}{A^2} + g_0 ,$$

where the constant g_0 is determined from the limit

$$\lim_{N\to 0} \sigma = g_0 = \sigma_0 .$$

We thus recover the earlier result for $\sigma(A,T)$.

#8 : UNDERGRADUATE STATISTICAL MECHANICS

`PROBLEM:` The figure below illustrates a simple model of the surface of a two-dimensional 'solid' confined to a square lattice. The two ends of the surface are N lattice sites apart, with $N \gg 1$. The surface energy is proportional to the surface length, with energy $\varepsilon > 0$ per lattice length. The surface height can change by at most one lattice length at a time. (Overlaps are forbidden, so that outward-pointing surface normals never point downward.) Thus, the surface can be modeled by a Hamiltonian

$$H = \varepsilon \sum_{i=1}^{N} (1 + \sigma_i^2) \ ,$$

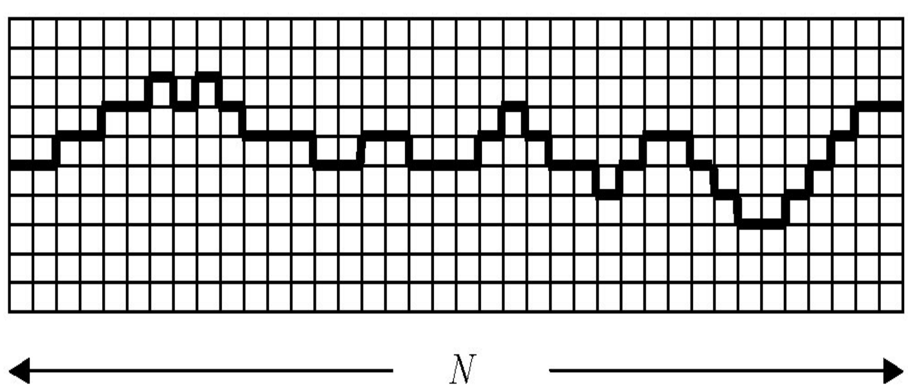
where $\sigma_i = +1$, 0, or -1 depending on whether the i^{th} 'column' contains a step up, no step, or a step down for the surface.

(a) Explain why this is Hamiltonian properly reflects the surface energy described above.

(b) Find the partition function $Z(T)$ for the surface.

(c) Find the free energy $F(T)$ for the surface, and sketch its temperature dependence. Physically interpret your result in the limits $k_{\text{B}}T \ll \varepsilon$ and $k_{\text{B}}T \gg \varepsilon$.

(d) Find the total length of the surface as a function of temperature, and sketch its temperature dependence.



#8 : UNDERGRADUATE STATISTICAL MECHANICS

`SOLUTION:` The surface consists of N horizontal steps, $N_\uparrow$ upward steps, and $N_\downarrow$ downward steps. The degrees of freedom the system possesses are whether after each horizontal step the surface goes upward, downward, or remains at the same level. Let us represent these three possibilities by a scalar variable $\sigma = +1$, 0, or -1, respectively. We further label each step by a subscript $i \in \{1, \ldots, N\}$.

(a) With $H = \varepsilon \sum_i (1 + \sigma_i^2)$, the energy is written as a sum over the N columns. The contribution from each column is ε if $\sigma = 0$, *i.e.* if there is no step, and 2ε if $\sigma = \pm 1$, *i.e.* if there is a step in either direction. Since each step adds an extra lattice length to the length of the surface, this Hamiltonian properly accounts for the surface energy of ε per lattice length.

(b) The partition function is a sum over all configurations. This may be represented as a product over the steps, *viz.*

$$\begin{aligned} Z = \mathrm{Tr}\, e^{-H/k_\mathrm{B}T} &= \prod_{i=1}^{N} \sum_{\sigma_i=-1}^{1} e^{-(1+\sigma_i^2)/k_\mathrm{B}T} \\ &= e^{-N\varepsilon/k_\mathrm{B}T} \left(1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}\right)^N \ . \end{aligned}$$

(c) The free energy is

$$\begin{aligned} F &= k_\mathrm{B}T \ln Z \\ &= N\varepsilon - N k_\mathrm{B}T \ln\left(1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}\right) \ . \end{aligned}$$

In the low temperature regime $k_\mathrm{B}T \ll \varepsilon$, we have $F \approx N\varepsilon$, which is the energy of a flat surface, whose length is the minimum value possible, N. In the high temperature regime $k_\mathrm{B}T \gg \varepsilon$, we have $-Nk_\mathrm{B}T \ln 3$, which reflects the fact that the surface is completely randomized, with 3^N equally probable configurations yielding an entropy $S = Nk_\mathrm{B} \ln 3$, as $T \to \infty$. The entropy term $-TS$ dominates the average energy E at these high temperatures.

(d) The total surface length is $L = N + N_\uparrow + N_\downarrow = N \cdot (1 + 2p)$, where p is the probability for an upward or downward step:

$$p = \frac{e^{-\varepsilon/k_\mathrm{B}T}}{1 + 2\, e^{-\varepsilon/k_\mathrm{B}T}} \ .$$

Thus, $\langle L \rangle_{T\to 0} = N$, while $\langle L \rangle_{T\to\infty} = \frac{5}{3}N$.

#9 : UNDERGRADUATE PHYSICAL ESTIMATES

PROBLEM: Suppose physicists had invented a rocket engine that was 100% efficient at converting a planet's rest mass energy into a change in its orbital energy around the Sun. What fraction of the Earth's rest mass would be required to raise its orbital energy to beyond the escape velocity of the solar system? You may need one or more of the following constants:

$$M_{\mathrm{Sun}} = 2.0 \times 10^{30}\,\mathrm{kg} \quad , \quad M_{\mathrm{Earth}} = 5.8 \times 10^{24}\,\mathrm{kg}$$
$$a_{\mathrm{Earth}} = 1.5 \times 10^{11}\,\mathrm{m} \quad , \quad G = 6.7 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$$

#9 : UNDERGRADUATE PHYSICAL ESTIMATES

`SOLUTION:` The potential energy of Earth in the Sun's gravitational field is

$$U = -\frac{GM_{\rm S}M_{\rm E}}{a_{\rm E}} \ ,$$

where $a_{\rm E}$ is the radius of the Earth's orbit, *i.e.* the Earth-Sun distance. The Earth's kinetic energy is $T = \frac{1}{2}M_{\rm E}V_{\rm E}^2$, where, balancing centrifugal and gravitational forces,

$$\frac{M_{\rm E}V_{\rm E}}{a_{\rm E}} = \frac{GM_{\rm S}M_{\rm E}}{a_{\rm E}^2} \quad \Rightarrow \quad V_{\rm E}^2 = \frac{GM_{\rm S}}{a_{\rm E}} \ .$$

This gives $T = -\frac{1}{2}U$, which follows from the virial theorem for homogeneous potentials: $2\bar{T} = k\bar{U}$, where k is the degree of homogeneity (*i.e.* $U(r) = Ar^k$; $k = -1$ for Kepler's problem) and the bars denote time averages. (For circular orbits, the time averages are the same as the instantaneous values.) The total energy of the Earth is then

$$E_{\rm E} = -\frac{GM_{\rm S}M_{\rm E}}{2a_{\rm E}} \ .$$

To liberate the Earth from the Sun's gravity and place it in an unbound orbit would require that $E(r=\infty) \geq 0$. Turning a fraction η of the Earth's rest mass into pure energy would work if

$$\eta M_{\rm E}c^2 = \frac{GM_{\rm S}M_{\rm E}}{2a_{\rm E}} \quad \Rightarrow \quad \eta = \frac{GM_{\rm S}}{2a_{\rm E}c^2} \simeq 4.9 \times 10^{-9}.$$

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

PROBLEM: The mass of the Sun is about $M_\odot \simeq 2.0 \times 10^{33}$ g, its luminosity is $L_\odot \simeq 3.8 \times 10^{33}$ erg/s, and its radius is $R_\odot \simeq 7.0 \times 10^{10}$ cm. The Sun is composed mostly of hydrogen. The central temperature of the Sun is approximately $k_B T_{\rm core} \simeq 1$ keV. You may also find it useful to recall that one mole of hydrogen has a mass of about 1.0 g.

(a) How long does it take a photon to diffuse from the center of the Sun to its surface? *Hint: Assume photon-electron scattering dominates over all other photon scattering mechanisms. The Thomson cross section is* $\sigma_T \simeq 6.7 \times 10^{-25}$ cm^2. *Now, if only you knew the density of electrons...*

(b) Estimate the total internal energy of the Sun. Then estimate the time scale for energy loss in the sun if all nuclear reactions were somehow shut off. Is this result very different from that you found in part (a) ? Why or why not?

#10 : UNDERGRADUATE PHYSICAL ESTIMATES

SOLUTION:

(a) The photons diffuse, making a random walk of step length $\ell = 1/n\sigma_{\rm T}$, where n is the electron density. We call ℓ the photon 'mean free path'. A random walk of N steps of size ℓ extends an RMS distance of $d(N) = \sqrt{N}\,\ell$. We set $d = R_\odot$ and find that each photon undergoes an average of $N = (R_\odot/\ell)^2$ scatterings before it escapes the Sun. How long does this take? Each step takes a time ℓ/c, so

$$t \approx \frac{R_\odot^2}{\ell\, c} = \frac{n\,\sigma_{\rm T}\,R_\odot^2}{c} \ .$$

To estimate this, we need the electron density n. The number of electrons in the Sun should be the same as the number of protons, and if the Sun is mostly hydrogen,

$$n \simeq \frac{M_\odot}{\frac{4}{3}\pi R_\odot^3\, m_{\rm p}} = \frac{N_{\rm A}\, M_\odot}{\frac{4}{3}\pi R_\odot^3 \cdot 1\,{\rm g}} = 8.4 \times 10^{23}\,{\rm cm}^{-3} \ ,$$

where $m_{\rm p}$ is the mass of the proton, and $N_{\rm A} = 6.02 \times 10^{23}$ is Avogadro's number. Plugging this into the earlier expression, one finds

$$t \simeq 1.6 \times 10^{11}\,{\rm sec} = 5{,}300\,{\rm yrs} \ .$$

(b) The number density of protons and electrons is n, and the energy scale per particle is $k_{\rm B}T$. Thus, the total internal energy is approximated as

$$\mathcal{U}_{\rm int} \approx \tfrac{4}{3}\pi R_\odot^3 \cdot 2n \cdot k_{\rm B}T = 2 \times 10^{48}\,{\rm erg} \ .$$

Dividing this by the luminosity gives the time scale

$$t^* = \frac{U_{\rm int}}{L_\odot} \approx 5 \times 10^{14}\,{\rm s} = 15\,{\rm Myr} \ .$$

The results are very different because there is relatively very little energy in the radiation field in the Sun. Most of the Sun's energy is kinetic energy of its protons and electrons.

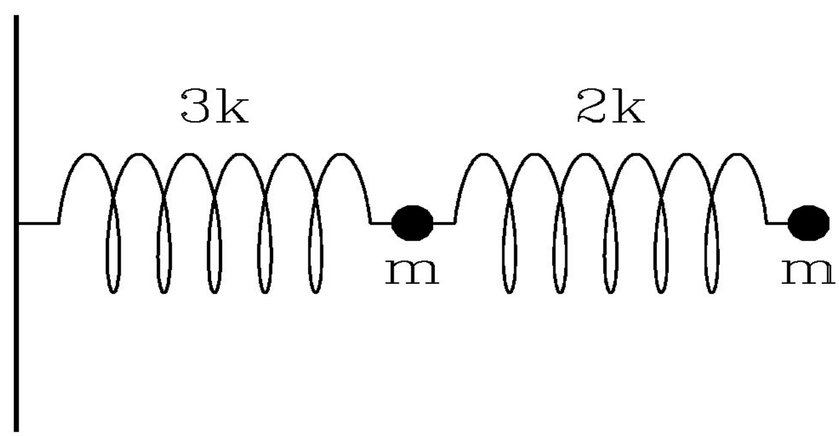
#11 : GRADUATE CLASSICAL MECHANICS

PROBLEM: Two identical balls of mass m are connected by springs as depicted in the diagram below. The spring constants are $3k$ and $2k$, respectively, and the spring at the left is attached to a fixed wall. All motion is in one dimension, along the axes of the springs.

(a) Choose as generalized coordinates the displacements of the springs from their equilibrium lengths. Write the Lagrangian for the system.

(b) Write the equations of motion.

(c) Find the frequencies of the normal modes of oscillation.



#11 : GRADUATE CLASSICAL MECHANICS

`SOLUTION:` Relative to the wall, the displacement of the left mass is $a_1 + x_1$, and that of the right mass is $a_1 + a_2 + x_1 + x_2$, where $a_{1,2}$ are the equilibrium lengths of the two springs. (The equilibrium lengths do not enter in to the problem, so it is fine to assume they are identical.)

(a) The Lagrangian is:

$$L = \tfrac{1}{2} m \dot{x}_1^2 + \tfrac{1}{2} m (\dot{x}_1 + \dot{x}_2)^2 - \tfrac{3}{2}\, k\, x_1^2 - k\, x_2^2 \ .$$

(b) The motion follows from the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \ .$$

This gives two coupled equations:

$$\begin{aligned} 2m\, \ddot{x}_1 + m\, \ddot{x}_2 &= -3k\, x_1 \\ m\, \ddot{x}_1 + m\, \ddot{x}_2 &= -2k\, x_2 \ . \end{aligned}$$

(c) There are several equivalent ways to obtain the normal modes. Here's a simple way to do so for the special case of $N = 2$ coupled oscillators. We add together the equations of motion, multiplying the second equation by an undetermined coefficient, α. This gives

$$m\big\{(2+\alpha)\, \ddot{x}_1 + (1+\alpha)\, \ddot{x}_2\big\} = -3k\, x_1 - 2\alpha k\, x_2 \ .$$

We now demand that the ratio of the coefficients of $\ddot{x}_2$ and $\ddot{x}_1$ on the LHS be the same as the ratio of the coefficients of x_2 and x_1 on the RHS. This means

$$\frac{1+\alpha}{2+\alpha} = \frac{2\alpha}{3} \ ,$$

which is a quadratic equation: $2\alpha^2 + \alpha - 3 = (\alpha - 1)(2\alpha + 3) = 0$. The roots are $\alpha = +1$ and $\alpha = -\frac{3}{2}$. The above equation of motion may now be cast as

$$\ddot{x}_1 + \tfrac{2}{3}\alpha\, \ddot{x}_2 = -\frac{3}{2+\alpha}\, \frac{k}{m}\big(\ddot{x}_1 + \tfrac{2}{3}\alpha\, \ddot{x}_2\big) \ ,$$

from which we glean that the two (unnormalized) normal modes are

$$\begin{aligned} y_1 &= x_1 + \tfrac{2}{3}\alpha\, x_2 && (\alpha = +1) \\ y_2 &= x_1 - x_2 && (\alpha = -\tfrac{3}{2})\ . \end{aligned}$$

The corresponding eigenfrequencies are given by

$$\omega^2 = \frac{3\,\omega_0^2}{2+\alpha} = \begin{cases} \omega_0^2 & \alpha = +1 \\ \\ 6\,\omega_0^2 & \alpha = -\frac{3}{2}\ , \end{cases}$$

where $\omega_0 \equiv \sqrt{k/m}$.

#12 : GRADUATE CLASSICAL MECHANICS

PROBLEM: Consider a particle of charge q moving in the (x, y) plane, and under the influence of a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and an external electric potential $\Phi(\boldsymbol{r})$. Choose a gauge in which $\boldsymbol{A} = Bx\,\hat{\boldsymbol{y}}$. In the large B limit, the inertial term in the equations of motion, $m\ddot{\boldsymbol{r}}$, can be neglected. This is equivalent to assuming the mass to be negligibly small ($m \to 0$), which you are to henceforth assume.

(a) Find the Lagrangian for the particle, in the $m \to 0$ limit.

(b) Identify a canonically conjugate coordinate-momentum pair and obtain the Hamiltonian function H. Verify that Hamilton's equations produce the expected result.

(c) Solve completely for the motion of a pair of equal particles (with $q = e$) moving in the presence of an external electric field $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, using a convenient choice of generalized coordinates. You should assume that the electric field from each charge is confined to the (x, y) plane and that the charges interact via a two-dimensional Coulomb potential $\phi(\boldsymbol{r}) = -2q \ln r$.

#12 : GRADUATE CLASSICAL MECHANICS

SOLUTION: We adopt the gauge $\boldsymbol{A} = Bx\,\hat{\boldsymbol{y}}$.

(a) The Lagrangian is

$$
\begin{aligned}
L &= \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - q\,\Phi \\
&= \frac{qB}{c}\,x\dot{y} - q\,\Phi(x,y)\ .
\end{aligned}
$$

(b) Let us choose y as a coordinate. Its conjugate momentum is

$$
p = \frac{\partial L}{\partial \dot{y}} = \frac{qBx}{c}\ .
$$

Note that there is only one canonically conjugate pair: $\{y,p\}$. The x-coordinate is proportional to the y-momentum: $x = cp/qB$.

The Hamiltonian is

$$
H = p\,\dot{y} - L = q\,\Phi\Big(x = \frac{cp}{qB}, y\Big)\ ,
$$

and Hamilton's equations are

$$
\begin{aligned}
\dot{y} &= +\frac{\partial H}{\partial p} = \frac{c}{B}\,\frac{\partial \Phi}{\partial x} \\
\dot{p} &= -\frac{\partial H}{\partial y} = -q\,\frac{\partial \Phi}{\partial y}\ ,
\end{aligned}
$$

where $x = cp/qB$ throughout. Using x instead of p, the second of these gives $\dot{x} = -(c/B)(\partial\Phi/\partial y)$, which may be written together with the first equation in vector form as

$$
\dot{\boldsymbol{r}} = \frac{c}{B}\,\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}\Phi = \frac{c\,\boldsymbol{E}\times\boldsymbol{B}}{B^2}\ ,
$$

which is (or should be) the familiar formula for $\boldsymbol{E}\times\boldsymbol{B}$ drift.

(c) The electrical potential for a point charge of strength q in two-dimensions is $\phi(\boldsymbol{r}) = -2q\ln r$, as it satisfies $\boldsymbol{\nabla}^2\phi = -4\pi q\,\delta(\boldsymbol{r})$. Thus, the Lagrangian for the pair is

$$
L = \frac{eB}{c}\big(x_1\,\dot{y}_1 + x_2\,\dot{y}_2\big) + 2e^2\ln\big|\boldsymbol{r}_1 - \boldsymbol{r}_2\big| + eE\,\big(x_1 + x_2\big)\ .
$$

We define the relative and center-of-mass coordinates,

$$
\begin{aligned}
x &= x_1 - x_2 & X &= \tfrac{1}{2}(x_1 + x_2) \\
y &= y_1 - y_2 & Y &= \tfrac{1}{2}(y_1 + y_2) \ .
\end{aligned}
$$

In terms of these quantities,

$$
L = \frac{eB}{c}\left(\tfrac{1}{2}\,x\,\dot{y} + 2\,X\,\dot{Y}\right) + e^2 \ln(x^2 + y^2) + 2e\,E\,X \ .
$$

The Euler-Lagrange equations now give

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= 0 &&= \frac{eB\,\dot{y}}{2c} + \frac{2e^2\,x}{x^2+y^2} &&= \frac{\partial L}{\partial x} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) &= \frac{eB\,\dot{x}}{2c} &&= \frac{2e^2\,y}{x^2+y^2} &&= \frac{\partial L}{\partial y} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) &= 0 &&= \frac{2eB\,\dot{Y}}{c} + 2eE &&= \frac{\partial L}{\partial X} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) &= \frac{2eB\,\dot{X}}{c} &&= 0 &&= \frac{\partial L}{\partial Y}
\end{aligned}
$$

or, with $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$,

$$
\dot{r} = 0 \quad , \quad \dot{\theta} = \frac{4cE}{Br} \quad , \quad \dot{X} = 0 \quad , \quad \dot{Y} = -\frac{cE}{B} \ .
$$

This describes a pair of particles rotating about each other with angular velocity $\omega = 4cE/Ba$, where $r(0) = a$ is their constant separation. The center-of-mass of the pair moves in a straight line with velocity $\boldsymbol{V} = -(cE/B)\,\hat{\boldsymbol{y}}$.

#13 : GRADUATE ELECTROMAGNETISM

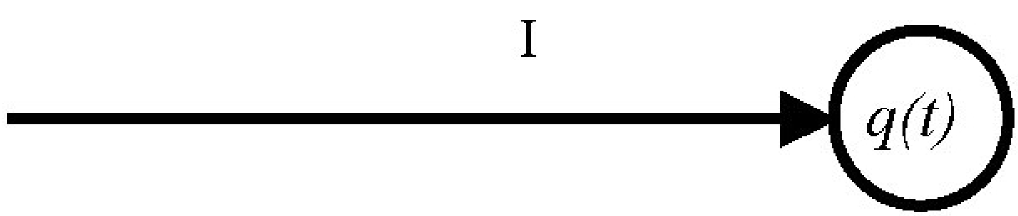
`PROBLEM:` This problem deals with displacement current.

(a) Consider the displacement current $\boldsymbol{j}_{\mathrm{d}} = \frac{1}{4\pi}\frac{\partial \boldsymbol{E}}{\partial t}$ of an electromagnetic field, and using Maxwell's equations show that the sum $\boldsymbol{J} = \boldsymbol{j} + \boldsymbol{j}_{\mathrm{d}}$ is divergenceless: $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$. (Here, $\boldsymbol{j}$ is the current of charges.)

(b) A conducting sphere of radius a is being charged through a straight wire, carrying current I, so that the charge on the sphere q obeys $\dot{q} = I$. Assuming a symmetric distribution of charge over the sphere's surface, find the electric field outside the sphere. Determine also the displacement current, and verify the conservation law $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$.

(c) Using the Ampére-Maxwell law in the form $\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}$, and taking advantage of the cylindrical symmetry in this problem, find the magnetic field everywhere in space.

(d) By appropriately limiting your result from (c), verify that close to the wire, the answer has the familiar form for an infinite straight wire.



#13 : GRADUATE ELECTROMAGNETISM

SOLUTION:

(a) From Maxwell, we have

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{4\pi}{c}\,\boldsymbol{j} + \frac{1}{c}\,\frac{\partial\boldsymbol{E}}{\partial t} \\ &= \frac{4\pi}{c}\,(\boldsymbol{j}+\boldsymbol{j}_{\rm d})\ ,\end{aligned}$$

and since $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})=0$, it follows that $\boldsymbol{\nabla}\cdot\boldsymbol{J}=0$.

(b) From $\boldsymbol{\nabla}\cdot\boldsymbol{E}=4\pi\rho$, we have $\boldsymbol{E}=q\,\hat{\boldsymbol{r}}/r^2$ for a spherical distribution of charges. Thus,

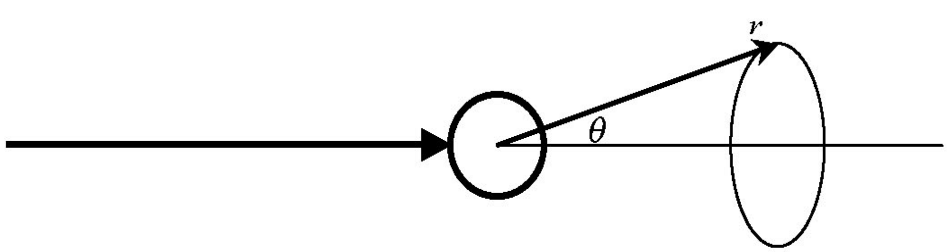
$$\boldsymbol{j}_{\rm d} = \frac{I}{4\pi r^2}\,\hat{\boldsymbol{r}}\ .$$

Note that $\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\rm d} = I\,\delta(\boldsymbol{r})$, which vanishes outside the sphere. Since $\boldsymbol{\nabla}\cdot\boldsymbol{j}=0$ outside the sphere as well, we have that $\boldsymbol{\nabla}\cdot\boldsymbol{J}=0$.

(c) From axial symmetry, we expect circular magnetic field lines. So use the integral form of Ampère's law,

$$\oint\limits_{\partial\Sigma}\boldsymbol{B}\cdot d\boldsymbol{\ell} = \frac{4\pi}{c}\int\limits_{\Sigma}dA\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\ ,$$

where Σ is any two-dimensional surface, and $\hat{\boldsymbol{n}}$ is the local surface normal. Consider the $\boldsymbol{B}$ field along a circular loop a distance r from the center of the sphere, at an angle θ with respect to the wire's axis: Since there is no physical charge flowing through the loop, the total



current is just the displacement current from part (b). Let Σ be the cap of a sphere of radius r, subtending a solid angle Ω. We therefore have

$$2\pi B r \sin\theta = \frac{4\pi}{c} \cdot \frac{\Omega}{4\pi} \cdot I = \frac{\Omega\, I}{c} \ ,$$

where $r\sin\theta$ is the radius of the loop, and $\Omega = 2\pi(1-\cos\theta)$ is the solid angle subtended by the loop. We therefore have

$$B(r,\theta) = \frac{(1-\cos\theta)\, I}{cr\,\sin\theta} = \frac{I}{cr}\,\tan(\tfrac{1}{2}\theta)\ .$$

Note that there are two choices we could make for our cap. The complementary region Σ' would subtend solid angle $4\pi-\Omega$, and is pierced by the wire. In this case, both $\boldsymbol{j}$ and $\boldsymbol{j}_{\mathrm{d}}$ contribute to $\boldsymbol{J}$, and after considering the opposite orientation of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{r}}$ on Σ', we obtain

$$2\pi B r \sin\theta = \frac{4\pi}{c}\bigg\{ -\frac{4\pi-\Omega}{4\pi}\cdot I + I \bigg\} = \frac{\Omega\, I}{c} \ ,$$

as before.

(d) Near the wire, we have $\theta \to \pi$, and $\cos\theta \to 1$, and we recover the familiar expression

$$B(r,\theta) \approx \frac{2I}{cr\sin\theta} = \frac{2I}{cR} \ ,$$

where $R = r\sin\theta$ is the perpendicular distance from the wire.

#14 : GRADUATE ELECTROMAGNETISM

`PROBLEM:` A wave guide has rectangular cross section with sides of length $2a$ and a.

(a) What is the dispersion relation for the TM modes?

(b) What is the dispersion relation for the TE modes?

(c) For what frequency range does a single mode alone propagate?

You may assume $\mu = \epsilon = 1$ inside the wave guide.

#14 : GRADUATE ELECTROMAGNETISM

`SOLUTION:` For a derivation of waveguide physics (not required) write

$$\begin{aligned} \boldsymbol{E}(x,y,z) &= \big[\boldsymbol{E}_\perp(x,y) + \hat{\boldsymbol{z}}\,E_z(x,y)\big]\,e^{i(kz-\omega t)} \\ \boldsymbol{B}(x,y,z) &= \big[\boldsymbol{B}_\perp(x,y) + \hat{\boldsymbol{z}}\,B_z(x,y)\big]\,e^{i(kz-\omega t)}\ , \end{aligned}$$

where $\hat{\boldsymbol{z}}$ is the waveguide axis. Separate the gradient operator into transverse and longitudinal components,

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp + \hat{\boldsymbol{z}}\,\frac{\partial}{\partial z}\ ,$$

where $\hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}_\perp = 0$, and plug into Maxwell's equations. The divergence equations are

$$\begin{aligned} ik\,E_z + \boldsymbol{\nabla}_\perp\cdot\boldsymbol{E}_\perp &= 0 \\ ik\,B_z + \boldsymbol{\nabla}_\perp\cdot\boldsymbol{B}_\perp &= 0\ , \end{aligned}$$

while the curl equations are each split into longitudinal and transverse components, yielding

$$\begin{aligned} \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}_\perp\times\boldsymbol{E}_\perp &= \frac{i\omega}{c}\,B_z \\ ik\hat{\boldsymbol{z}}\times\boldsymbol{E}_\perp - \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp E_z &= \frac{i\omega}{c}\,\boldsymbol{B}_\perp \\ \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}_\perp\times\boldsymbol{B}_\perp &= -\frac{i\omega}{c}\,E_z \\ ik\hat{\boldsymbol{z}}\times\boldsymbol{B}_\perp - \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp B_z &= -\frac{i\omega}{c}\,\boldsymbol{E}_\perp\ . \end{aligned}$$

These last four equations may be used to write $\boldsymbol{E}_\perp$ and $\boldsymbol{B}_\perp$ in terms of E_z and B_z:

$$\begin{aligned} \boldsymbol{E}_\perp &= \frac{ik\boldsymbol{\nabla}_\perp E_z - ik_0\,\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp B_z}{k_0^2-k^2} \\ \boldsymbol{B}_\perp &= \frac{ik\boldsymbol{\nabla}_\perp B_z + ik_0\,\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp E_z}{k_0^2-k^2}\ , \end{aligned}$$

where $k_0 = \omega/c$. Substituting these into the two scalar equations, one finds that E_z and B_z obey wave equations of the form

$$\begin{aligned} \Big(\boldsymbol{\nabla}_\perp^2 - k^2 + \frac{\omega^2}{c^2}\Big)\,E_z &= 0 \\ \Big(\boldsymbol{\nabla}_\perp^2 - k^2 + \frac{\omega^2}{c^2}\Big)\,B_z &= 0\ . \end{aligned}$$

Boundary conditions:

$$\text{TM modes:} \quad B_z = 0, \quad E_z\big|_{\text{surface}} = 0$$

$$\text{TM modes:} \quad E_z = 0, \quad \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\perp} B_z\big|_{\text{surface}} = 0 \ ,$$

where $\hat{\boldsymbol{n}}$ is the surface normal. Only these last two sets of equations are needed to solve the problem. We assume the long $(2a)$ direction is the x-direction, and the short (a) direction the y-direction.

(a) For TM modes, we write

$$E_z(x,y) = E_0 \sin\left(\frac{m\pi x}{2a}\right) \sin\left(\frac{n\pi y}{a}\right) \ ,$$

where m and n are positive integers. This solution properly vanishes on the boundaries at $x = 0$, $x = 2a$, $y = 0$, and $y = a$. The wave equation then yields the dispersion branches

$$\omega_{mn}(k) = \sqrt{c^2 k^2 + \left(\frac{m\pi c}{2a}\right)^2 + \left(\frac{n\pi c}{a}\right)^2} \ .$$

Any mode with $k > 0$ will propagate.

(b) For TE modes, we need $\hat{n} \cdot \boldsymbol{\nabla}_{\perp} B_z = 0$ on the boundaries. Thus,

$$B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{2a}\right) \cos\left(\frac{n\pi y}{a}\right) \ .$$

Both m and n are nonnegative integers, but for a nontrivial solution at least one of them must be positive. The dispersion branches $\omega_{mn}(k)$ are the same as in (a).

(c) The lowest frequency branch is $\text{TE}_{1,0}$ (*i.e.* $m = 1$ and $n = 0$), with $\omega_{1,0}(k = 0) = \pi c/2a$. The next lowest branches are $\text{TE}_{2,0}$ and $\text{TE}_{1,1}$, which are degenerate, and for which $\omega_{0,1}(k = 0) = \pi c/a$. Thus, a single mode propagates at all frequencies in the range

$$\frac{\pi c}{2a} \leq \omega < \frac{\pi c}{a} \ .$$

$\#$ 15 : GRADUATE QUANTUM MECHANICS

`PROBLEM:` A hydrogen atom in its ground state is placed in an electric field $\boldsymbol{E}(t) = \boldsymbol{E}_0 \cos(\omega t)$, where $\omega > me^4/2\hbar^3$. Find the probability per unit time that the atom will be ionized. You should assume that the wavefunctions of the electron in the ionized states are plane waves. The ground state wavefunction of hydrogen is $\psi_0(\boldsymbol{r}) = (\pi a_{\rm B}^3)^{-1/2} \exp(-r/a_{\rm B})$, where $a_{\rm B} = \hbar^2/me^2$ is the Bohr radius.

15 : GRADUATE QUANTUM MECHANICS

SOLUTION: We use Fermi's Golden Rule for the transition rate,

$$\Gamma_{\mathrm{i}\to\mathrm{f}} = \frac{2\pi}{\hbar}\, \big|\langle \mathrm{f}\big|\hat{V}(\omega)\big|\mathrm{i}\rangle\big|^2\, \delta(E_\mathrm{f} - E_\mathrm{i} - \hbar\omega) \ ,$$

valid for harmonic perturbations of the form $V(t) = \hat{V}(\omega)\, e^{-i\omega t}$. (For a real harmonic potential, sum over positive and negative frequency components.) Our potential is

$$V(t) = -\, e\boldsymbol{E}_0 \cdot \boldsymbol{r}\, \cos(\omega t) \ ,$$

so $\hat{V}(\omega) = \hat{V}(-\omega) = -e\boldsymbol{E}_0 \cdot \boldsymbol{r}$. The matrix element we seek is then

$$\begin{aligned}\mathcal{M} &= -e\big\langle \psi_{\boldsymbol{k}}\big|\boldsymbol{E}_0 \cdot \boldsymbol{r}\big|\psi_0\big\rangle \\ &= -eE_0\, (\pi a_\mathrm{B}^3)^{-1/2}\, V^{-1/2} \!\int\! d^3\!r\, e^{-r/a_\mathrm{B}}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \ ,\end{aligned}$$

where we take $\boldsymbol{E}_0 = E_0\, \hat{\boldsymbol{e}}$. We may, without loss of generality, take $\boldsymbol{k}$ to lie along $\hat{\boldsymbol{z}}$. Writing $\boldsymbol{r}$ and $\hat{\boldsymbol{e}}$ in polar coordinates, we then have

$$\hat{\boldsymbol{e}} \cdot \boldsymbol{r} = r\cos\theta\cos\vartheta + r\sin\theta\sin\vartheta\cos(\phi - \boldsymbol{\varphi}) \ ,$$

where (θ, ϕ) and $(\vartheta, \boldsymbol{\varphi})$ are the polar and azimuthal angles for $\boldsymbol{r}$ and $\hat{\boldsymbol{e}}$, respectively. The last term integrates to zero. The matrix element is then

$$\mathcal{M} = -2\pi e E_0\, (\pi a_\mathrm{B}^3 V)^{-1/2}\, \cos\vartheta \int\limits_0^\infty \! dr\, r^2 \int\limits_0^\pi \! d\theta\, \sin\theta\, r\cos\theta\, e^{-r/a_\mathrm{B}}\, e^{-ikr\cos\theta} \ .$$

The double integral is straightforward:

$$\begin{aligned}\int\limits_{-1}^{1}\! d\mu\, \mu \int\limits_0^\infty \! dr\, r^3\, e^{-(a_\mathrm{B}^{-1} + ik\mu)r} &= 6\int\limits_{-1}^{1}\! d\mu\, \frac{\mu}{(a_\mathrm{B}^{-1} + ik\mu)^4} \\ &= -\frac{6}{k^2} \int\limits_{a_\mathrm{B}^{-1} - ik}^{a_\mathrm{B}^{-1} + ik} \frac{s - a_\mathrm{B}^{-1}}{s^4} \\ &= -\frac{16i\, k\, a_\mathrm{B}^{-1}}{(a_\mathrm{B}^{-2} + k^2)^3} \ .\end{aligned}$$

The matrix element must be squared, then summed over all final $\boldsymbol{k}$ states. Recalling the relation

$$\sum_{\boldsymbol{k}} A(\boldsymbol{k}) = V \int \frac{d^3k}{(2\pi)^3}\, A(\boldsymbol{k}) \ ,$$

we obtain for the total transmission rate

$$\begin{aligned}\Gamma &= \frac{2\pi}{\hbar} \int_0^\infty \frac{dk\, k^2}{4\pi^2} \int_0^\pi d\vartheta\, \sin\vartheta \cdot \frac{4\pi^2\, e^2 E_0^2 \cos^2\vartheta}{\pi a_{\mathrm{B}}^3} \cdot \frac{256\, k^2\, a_{\mathrm{B}}^{-2}}{(a_{\mathrm{B}}^{-2} + k^2)^6} \cdot \delta\!\left(\frac{\hbar k^2}{2m} - \hbar\omega + \frac{\hbar^2}{2ma_{\mathrm{B}}^2}\right) \\ &= \frac{256}{3\hbar}\, e^2 E_0^2\, a_{\mathrm{B}}^3 \left(\frac{\omega_0}{\omega}\right)^3 \left(\frac{\omega}{\omega_0} - 1\right)^{3/2} \ ,\end{aligned}$$

where $\omega_0 \equiv \hbar/2ma_{\mathrm{B}}^2 = me^4/2\hbar^3$ is the lowest ionization frequency. Note that $\Gamma \to 0$ at the ionization edge, $\omega = \omega_0$. The approximation of ionized states by plane waves is accurate only for $\omega \gg \omega_0$.

#16 : GRADUATE QUANTUM MECHANICS

PROBLEM: Consider a two-state system in which the 'interaction' eigenstates $|i\rangle$ and $|j\rangle$ are related to the energy eigenstates $|E_1\rangle$ and $|E_2\rangle$, with $E_1 < E_2$, according to the unitary transformation,

$$|i\rangle = \cos\theta|E_1\rangle + \sin\theta|E_2\rangle$$

$$|j\rangle = -\sin\theta|E_1\rangle + \cos\theta|E_2\rangle ,$$

where θ is a real number. The energy eigenvalues $E_{1,2}$ satisfy $\mathcal{H}_{\text{vac}}|E_a\rangle = E_a|E_a\rangle$, where $a = 1, 2$, and where $\mathcal{H}_{\text{vac}}$ is the vacuum Hamiltonian of the system.

(a) The system is prepared to be in the state $|i\rangle$ at time $t = 0$. What is the minimum lapse of time until the system is again in the state $|i\rangle$?

(b) Imagine that our system is now subjected to an additional interaction, $\mathcal{H}_{\text{new}}(t) = A(t)\,|i\rangle\langle i|$, where $A(t)$ has dimensions of energy. Write down the Hamiltonian as a 2×2 matrix in the 'interaction' basis.

(c) Write down the Hamiltonian from part (b) as a 2×2 matrix in the original energy eigenbasis.

#16 : GRADUATE QUANTUM MECHANICS

SOLUTION:

(a) We have

$$\begin{aligned}\big|\psi(t)\big\rangle &= \cos\theta\,\big|E_1\big\rangle\,e^{-iE_1 t/\hbar} + \sin\theta\,\big|E_2\big\rangle\,e^{-iE_2 t/\hbar} \\ &= e^{-iE_1 t/\hbar}\Big(\cos\theta\,\big|E_1\big\rangle + \sin\theta\,\big|E_2\big\rangle\,e^{-i\Delta E\,t/\hbar}\Big)\ ,\end{aligned}$$

where $\Delta E = E_2 - E_1$. The probability that the system is in state $\big|i\big\rangle$ is

$$P_i = \big|\big\langle i\big|\psi(t)\big\rangle\big|^2\ ,$$

and so we first compute the probability amplitude

$$\begin{aligned}\big\langle i\big|\psi(t)\big\rangle &= \big\langle \psi(0)\big|\psi(t)\big\rangle \\ &= \big(\cos^2\!\theta + \sin^2\!\theta\,e^{-i\Delta E\,t/\hbar}\big)\,e^{-iE_1 t/\hbar}\ .\end{aligned}$$

We now have

$$\begin{aligned}P_i &= \big(\cos^2\!\theta + \sin^2\!\theta\,e^{-i\Delta E\,t/\hbar}\big)\big(\cos^2\!\theta + \sin^2\!\theta\,e^{+i\Delta E\,t/\hbar}\big) \\ &= \cos^4\!\theta + \sin^4\!\theta + 2\sin^2\!\theta\cos^2\!\theta\cos(\Delta E\,t) \\ &= (\cos^2\!\theta + \sin^2\!\theta)^2 - 2\sin^2\!\theta\cos^2\!\theta\big(1-\cos(\Delta E\,t)\big) \\ &= 1 - \sin^2(2\theta)\,\sin^2\big(\tfrac{1}{2}\Delta E\,t\big)\ .\end{aligned}$$

Thus, the minimum time interval is $\Delta t = 2\pi\hbar/(E_2 - E_1)$.

(b) We need to express the vacuum Hamiltonian $\mathcal{H}_{\rm vac}$ in terms of the interaction eigenstates $\big|i\big\rangle$ and $\big|j\big\rangle$. We know

$$\mathcal{H}_{\rm vac} = E_1\,\big|E_1\big\rangle\big\langle E_1\big| + E_2\,\big|E_2\big\rangle\big\langle E_2\big|\ .$$

We invert the relation between the interaction eigenstates and the energy eigenstates to obtain

$$\begin{aligned}\big|E_1\big\rangle &= \cos\theta\,\big|i\big\rangle - \sin\theta\,\big|j\big\rangle \\ \big|E_2\big\rangle &= \sin\theta\,\big|i\big\rangle + \cos\theta\,\big|j\big\rangle\ .\end{aligned}$$

We may now write

$$\mathcal{H}_{\text{vac}} = (E_1 \cos^2\theta + E_2 \sin^2\theta)\,\big|i\big\rangle\big\langle i\big| + (E_2 - E_1)\sin\theta\cos\theta\,\big|i\big\rangle\big\langle j\big|$$

$$+ (E_2 - E_1)\sin\theta\cos\theta\,\big|j\big\rangle\big\langle i\big| + (E_1 \cos^2\theta + E_2 \sin^2\theta)\,\big|j\big\rangle\big\langle j\big| \ .$$

Thus, we may write

$$\big[\mathcal{H}\big]_{\text{interaction}} = \begin{pmatrix} E_1 \cos^2\theta + E_2 \sin^2\theta + A(t) & (E_2 - E_1)\,\sin\theta\cos\theta \\ (E_2 - E_1)\,\sin\theta\cos\theta & E_1 \sin^2\theta + E_2 \cos^2\theta \end{pmatrix} \ .$$

(c) The Hamiltonian is

$$\mathcal{H} = E_1\,\big|E_1\big\rangle\big\langle E_1\big| + E_2\,\big|E_2\big\rangle\big\langle E_2\big| + A(t)\,\big|i\big\rangle\big\langle i\big|$$

$$= E_1\,\big|E_1\big\rangle\big\langle E_1\big| + E_2\,\big|E_2\big\rangle\big\langle E_2\big|$$

$$+ A(t)\,\big(\cos\theta\big|E_1\big\rangle + \sin\theta\big|E_2\big\rangle\big)\big(\cos\theta\big\langle E_1\big| + \sin\theta\big\langle E_2\big|\big)$$

$$\big[\mathcal{H}\big]_{\text{energy}} = \begin{pmatrix} E_1 + A(t)\,\cos^2\theta & A(t)\sin\theta\,\cos\theta \\ A(t)\,\sin\theta\,\cos\theta & E_2 + A(t)\,\sin^2\theta \end{pmatrix} \ .$$

#17 : GRADUATE STATISTICAL MECHANICS

PROBLEM: A three-dimensional gas of noninteracting bosonic particles obeys the dispersion relation $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{1/2}$.

(a) Obtain an expression for the density $n(T,z)$ where $z = \exp(\mu/k_B T)$ is the fugacity. Simplify your expression as best you can, adimensionalizing any integral or infinite sum which may appear. You may find it convenient to define

$$g_\nu(z) \equiv \frac{1}{\Gamma(\nu)} \int_0^\infty \! dx \, \frac{x^{\nu-1}}{z^{-1}\, e^x - 1} = \sum_{k=1}^\infty \frac{z^k}{k^\nu} \ .$$

Note $g_\nu(1) = \zeta(\nu)$, the Riemann zeta function. *On the exam, the factor $1/\Gamma(\nu)$ was inadvertently left out of the above identity. No points were deducted from anyone as a result.*

(b) Find the critical temperature for Bose condensation, $T_c(n)$. Your expression should only include the density n, the constant A, physical constants, and numerical factors (which may be expressed in terms of integrals or infinite sums).

(c) What is the condensate density n_0 when $T = \frac{1}{2}\,T_c$?

(d) Do you expect the second virial coefficient to be positive or negative? Explain your reasoning. (You don't have to do any calculation.)

#17 : GRADUATE STATISTICAL MECHANICS

`SOLUTION:` We use Bose quantum statistical mechanics.

(a) The density for Bose particles are given by

$$
\begin{aligned}
n(T,z) &= \int\!\frac{d^3\!k}{(2\pi)^3}\,\frac{1}{z^{-1}\,\exp(Ak^{1/2}/k_{\rm B}T)-1}\\
&= \frac{1}{\pi^2}\bigg(\frac{k_{\rm B}T}{A}\bigg)^6\int\limits_0^\infty\!\!ds\,\frac{s^5}{z^{-1}\,e^s-1}\\
&= \frac{120}{\pi^2}\bigg(\frac{k_{\rm B}T}{A}\bigg)^6 g_6(z)\ ,
\end{aligned}
$$

where we have changed integration variables from k to $s = Ak^{1/2}/k_{\rm B}T$, and we have defined the functions

$$
g_\nu(z) \equiv \frac{1}{\Gamma(\nu)}\int\limits_0^\infty\!\!ds\,\frac{s^{\nu-1}}{z^{-1}\,e^s-1} = \sum_{k=1}^\infty \frac{z^k}{k^\nu}\ .
$$

Note $g_\nu(1)=\zeta(\nu)$, the Riemann zeta function.

(b) Bose condensation sets in for $z=1$, *i.e.* $\mu=0$. Thus, the critical temperature $T_{\rm c}$ and the density n are related by

$$
n = \frac{120\,\zeta(6)}{\pi^2}\bigg(\frac{k_{\rm B}T_{\rm c}}{A}\bigg)^6,
$$

or

$$
T_{\rm c}(n) = \frac{A}{k_{\rm B}}\bigg(\frac{\pi^2\,n}{120\,\zeta(6)}\bigg)^{1/6}\ .
$$

(c) For $T<T_{\rm c}$, we have

$$
\begin{aligned}
n &= n_0 + \frac{120\,\zeta(6)}{\pi^2}\bigg(\frac{k_{\rm B}T}{A}\bigg)^6\\
&= n_0 + (T/T_{\rm c})^6\,n\ ,
\end{aligned}
$$

where n_0 is the condensate density. Thus, at $T=\frac{1}{2}\,T_{\rm c}$,

$$
n_0\big(T=\tfrac{1}{2}T_{\rm c}\big) = \tfrac{63}{64}\,n.
$$

(d) The virial expansion of the equation of state is

$$p = nk_\mathrm{B}T\big\{1 + B_2(T)\,n + B_3(T)\,n^2 + \ldots\big\} \ .$$

We expect $B_2(T) < 0$ for noninteracting bosons, reflecting the tendency of the bosons to condense. (Correspondingly, for noninteracting fermions we expect $B_2(T) > 0$.)

For the curious, we compute $B_2(T)$ by eliminating the fugacity z from the equations for $n(T,z)$ and $p(T,z)$. First, we find $p(T,z)$:

$$\begin{aligned} p(T,z) &= -k_\mathrm{B}T \int\!\frac{d^3\!k}{(2\pi)^3} \,\ln\Big(1 - z\,\exp(-Ak^{1/2}/k_\mathrm{B}T)\Big) \\ &= -\frac{1}{\pi^2}\bigg(\frac{k_\mathrm{B}T}{A}\bigg)^7 \int\limits_0^\infty\!\! ds\, s^5\,\ln\big(1 - z\,e^{-s}\big) \\ &= \frac{120}{\pi^2}\bigg(\frac{k_\mathrm{B}T}{A}\bigg)^7 g_7(z). \end{aligned}$$

Expanding in powers of the fugacity, we have

$$\begin{aligned} n &= \frac{120}{\pi^2}\bigg(\frac{k_\mathrm{B}T}{A}\bigg)^6\bigg\{z + \frac{z^2}{2^6} + \frac{z^3}{3^6} + \ldots\bigg\} \\ p &= \frac{120}{\pi^2}\bigg(\frac{k_\mathrm{B}T}{A}\bigg)^6\bigg\{z + \frac{z^2}{2^7} + \frac{z^3}{3^7} + \ldots\bigg\} \ . \end{aligned}$$

Solving for $z(n)$ using the first equation, we obtain, to order n^2,

$$z = \bigg(\frac{\pi^2 A^6\, n}{120\, k_\mathrm{B}^6\, T^6}\bigg) - \frac{1}{2^6}\bigg(\frac{\pi^2 A^6\, n}{120\, k_\mathrm{B}^6\, T^6}\bigg)^2 + \mathcal{O}(n^3) \ .$$

Plugging this into the equation for $p(T,z)$, we obtain the first nontrivial term in the virial expansion, with

$$B_2(T) = -\frac{\pi^2}{15360}\,\frac{A^6}{k_\mathrm{B}^6\, T^6} \ ,$$

which is negative, as expected. Note also that the ideal gas law is recovered for $T \to \infty$, for fixed n.

#18 : GRADUATE STATISTICAL MECHANICS

`PROBLEM:` A collection of spin-$\frac{1}{2}$ particles is adsorbed on a surface. There are N adsorption sites. For each site, let $\sigma = 0$ if there is no adsorbed particle, $\sigma = +1$ if there is particle present with spin up, and $\sigma = -1$ if there is a particle present with spin down. The particles are non-interacting, and the energy for each adsorption site is given by $\varepsilon = -W\sigma^2$, where $-W < 0$ is the binding energy.

(a) Let $Q = N_\uparrow + N_\downarrow$ be the number of adsorbed particles, and N_0 be the number of vacant adsorption sites. The surface magnetization is $M = N_\uparrow - N_\downarrow$. Compute, in the microcanonical ensemble, the statistical entropy $S(Q, M)$.

(b) Let $q = Q/N$ and $m = M/N$ be the dimensionless adsorbate density and magnetization density, respectively. Assuming that we are in the thermodynamic limit, where N, Q, and M all tend to infinity, but with q and m finite, Find the temperature $T(q, m)$. Recall Stirling's formula

$$\ln(N!) = N \ln N - N + \mathcal{O}(\ln N) \ .$$

(c) Show explicitly that T can be negative for this system. What does negative T mean? What physical degrees of freedom have been left out that would avoid this strange property?

#18 : GRADUATE STATISTICAL MECHANICS

`SOLUTION:` There is a constraint on $N_\uparrow$, N_0, and $N_\downarrow$:

$$N_\uparrow + N_0 + N_\downarrow = Q + N_0 = N \ .$$

The total energy of the system is $E = -WQ$.

(a) The number of states available to the system is

$$\Omega = \frac{N!}{N_\uparrow!\, N_0!\, N_\downarrow!} \ .$$

Fixing Q and M, along with the above constraint, is enough to completely determine $\{N_\uparrow, N_0, N_\downarrow\}$:

$$N_\uparrow = \tfrac{1}{2}\,(Q+M) \qquad , \qquad N_0 = N - Q \qquad , \qquad N_\downarrow = \tfrac{1}{2}\,(Q-M) \ ,$$

whence

$$\Omega(Q,M) = \frac{N!}{\big[\frac{1}{2}(Q+M)\big]!\,\big[\frac{1}{2}(Q-M)\big]!\,(N-Q)!} \ .$$

The statistical entropy is $S = k_{\rm B} \ln \Omega$:

$$S(Q,M) = k_{\rm B} \ln(N!) - k_{\rm B} \ln\big[\tfrac{1}{2}(Q+M)!\big] - k_{\rm B} \ln\big[\tfrac{1}{2}(Q-M)!\big] - k_{\rm B} \ln\big[(N-Q)!\big] \ .$$

(b) Now we invoke Stirling's rule,

$$\ln(N!) = N \ln N - N + \mathcal{O}(\ln N) \ ,$$

to obtain

$$\begin{aligned}
\ln\Omega(Q,M) &= N\ln N - N - \tfrac{1}{2}(Q+M)\ln\big[\tfrac{1}{2}(Q+M)\big] + \tfrac{1}{2}(Q+M) \\
&\quad - \tfrac{1}{2}(Q-M)\ln\big[\tfrac{1}{2}(Q-M)\big] + \tfrac{1}{2}(Q-M) - (N-Q)\ln(N-Q) + (N-Q) \\
&= N\ln N - \tfrac{1}{2}Q\ln\Big[\tfrac{1}{4}(Q^2-M^2)\Big] - \tfrac{1}{2}M\ln\bigg(\frac{Q+M}{Q-M}\bigg) \\
&= -Nq\ln\Big[\tfrac{1}{2}\sqrt{q^2-m^2}\Big] - \tfrac{1}{2}Nm\ln\bigg(\frac{q+m}{q-m}\bigg) - N(1-q)\ln(1-q) \ ,
\end{aligned}$$

where $Q = Nq$ and $M = Nm$. Note that the entropy $S = k_\mathrm{B} \ln \Omega$ is extensive. The statistical entropy per site is thus

$$s(q,m) = -k_\mathrm{B}\, q \ln\left[\tfrac{1}{2}\sqrt{q^2 - m^2}\right] - \tfrac{1}{2} k_\mathrm{B}\, m \ln\left(\frac{q+m}{q-m}\right) - k_\mathrm{B}\,(1-q)\ln(1-q)\ .$$

The temperature is obtained from the relation

$$\begin{aligned} \frac{1}{T} &= \frac{\partial S}{\partial E}\bigg|_M = \frac{1}{W}\frac{\partial s}{\partial q}\bigg|_m \\ &= \frac{1}{W}\ \ln(1-q) - \frac{1}{W}\ \ln\left[\tfrac{1}{2}\sqrt{q^2-m^2}\right]\ . \end{aligned}$$

Thus,

$$T = \frac{W/k_\mathrm{B}}{\ln\big[2(1-q)/\sqrt{q^2-m^2}\big]}\ .$$

(c) We have $0 \le q \le 1$ and $-q \le m \le q$, so T is real (thank heavens!). But it is easy to choose $\{q, m\}$ such that $T < 0$. For example, when $m = 0$ we have $T = W/k_\mathrm{B} \ln(2q^{-1} - 2)$ and $T < 0$ for all $q \in (\frac{2}{3}, 1]$. The reason for this strange state of affairs is that the entropy S is bounded, and is not an monotonically increasing function of the energy E (or the dimensionless quantity Q). The entropy is maximized for $N \uparrow = N_0 = N_\downarrow = \frac{1}{3}$, which says $m = 0$ and $q = \frac{2}{3}$. Increasing q beyond this point (with $m = 0$ fixed) starts to reduce the entropy, and hence $(\partial S/\partial E) < 0$ in this range, which immediately gives $T < 0$. What we've left out are kinetic degrees of freedom, such as vibrations and rotations, whose energies are unbounded, and which result in an increasing $S(E)$ function.

#19 : GRADUATE MATHEMATICAL PHYSICS

`PROBLEM:` The *Fermi integral* of order k is given by the expression

$$F_k(\eta) = \int_0^\infty dx \, \frac{x^k}{\exp(x-\eta)+1} \ .$$

(a) Express $F_k(\eta)$ as an infinite sum.

(b) Find $dF_k(\eta)/d\eta$ in terms of Fermi integrals.

(c) Use an integral form of the relation from (a) to express $F_2(\eta) - F_2(-\eta)$ as a polynomial in η. You may need to use $F_1(0) = \frac{1}{12}\pi^2$. *Hint: You will need to compute* $F_0(\eta)$.

#19 : GRADUATE MATHEMATICAL PHYSICS

SOLUTION: Slightly tedious, but straightforward.

(a) We write

$$\frac{1}{\exp(x-\eta)+1} = \sum_{\ell=1}^{\infty} (-1)^{\ell}\, e^{-\ell(x-\eta)} \ ,$$

whence

$$F_k(\eta) = \Gamma(k+1) \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\, e^{\ell\eta}}{\ell^{k+1}} \ ,$$

where $\Gamma(k+1) = k!$ for integer k.

(b) Using the result from part (a),

$$\frac{dF_k(\eta)}{d\eta} = \Gamma(k+1) \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\, e^{\ell\eta}}{\ell^{k}} = k\, F_{k-1}(\eta) \ .$$

Note we have used $k\Gamma(k) = \Gamma(k+1)$. One may also derive this directly from the definition of $F_k(\eta)$ via integration by parts:

$$\begin{aligned}
\frac{dF_k(\eta)}{d\eta} &= \int_0^{\infty} \! dx\, x^k\, \frac{d}{d\eta} \left[\frac{1}{\exp(x-\eta)+1}\right] \\
&= -\int_0^{\infty} \! dx\, x^k\, \frac{d}{dx} \left[\frac{1}{\exp(x-\eta)+1}\right] \\
&= -\frac{x^k}{\exp(x-\eta)+1}\Bigg|_0^{\infty} + k \int_0^{\infty} \! dx\, \frac{x^k}{\exp(x-\eta)+1} \\
&= k\, F_{k-1}(\eta) \ .
\end{aligned}$$

(c) The integral form of the differential relation we derived is

$$F_k(\eta) - F_k(0) = k \int_0^{\infty} \! d\eta'\, F_{k-1}(\eta') \ .$$

Note that

$$F_0(\eta) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}\, e^{\ell\eta}}{\ell} = \ln(1+e^{\eta}) \ .$$

(c) Now we wish to evaluate

$$
\begin{aligned}
F_2(\eta) - F_2(-\eta) &= 2\int_0^{\eta} d\eta'\, F_1(\eta') - \ 2\int_0^{-\eta} d\eta'\, F_1(\eta') \\
&= 2\int_0^{\eta} d\eta' \left\{ F_1(\eta') + F_1(-\eta') \right\} .
\end{aligned}
$$

We therefore must calculate

$$
\begin{aligned}
F_1(\eta) + F_1(-\eta) &= 2F_1(0) + \int_0^{\eta} d\eta'\, F_0(\eta') + \int_0^{-\eta} d\eta'\, F_0(\eta') \\
&= 2F_1(0) + \int_0^{\eta} d\eta' \left\{ F_0(\eta') - F_0(-\eta') \right\} \\
&= 2F_1(0) + \tfrac{1}{2}\eta^2 \ ,
\end{aligned}
$$

where we have used $F_0(\eta) - F_0(-\eta) = \eta$, which is derived from our earlier result for $F_0(\eta)$. Thus,

$$
\begin{aligned}
F_2(\eta) - F_2(-\eta) &= 4F_1(0)\,\eta + \tfrac{1}{3}\,\eta^3 \\
&= \tfrac{1}{3}\pi^2 + \tfrac{1}{3}\,\eta^3 \ .
\end{aligned}
$$

#20 : GRADUATE MATHEMATICAL PHYSICS

`PROBLEM:` We know that the all observables are unchanged if we make a global change of the phase of the electron wavefunction, $\psi \to e^{i\Lambda}\,\psi$. We could call this *global phase symmetry.* All relative phases (say for amplitudes to go through different slits in a diffraction experiment) remain the same and no physical observable changes. This is a symmetry in the theory which we already know about.

Let's postulate that there is a bigger symmetry and see what the consequences are. This is a *local phase symmetry* embodied by the transformation

$$\psi(\boldsymbol{r},t) \to e^{i\Lambda(\boldsymbol{r},t)}\,\psi(\boldsymbol{r},t)\ .$$

That is, we can change the phase by a *different amount* at each point in spacetime and the physics will remain unchanged. This local phase symmetry is much "bigger" than the global one.

It is clear that this transformation leaves the absolute square of the wavefunction the same, but what about the Schrödinger equation? It must also be unchanged:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2\psi - e\phi\,\psi\ .$$

(a) Find the corresponding tranformation of $\boldsymbol{A}$ and ϕ needed to leave the Schrödinger equation invariant under local phase transformations of the wave function.

(b) How does this transformation change Maxwell's equations?

#20 : GRADUATE MATHEMATICAL PHYSICS

SOLUTION: The local gauge transformation is given by

$$\psi(\boldsymbol{r},,t) \longrightarrow e^{i\Lambda(\boldsymbol{r},t)}\,\psi(\boldsymbol{r},t)\ .$$

(a) Under the local gauge transformation, space and time derivatives transform as

$$\begin{aligned}\boldsymbol{\nabla} &\longrightarrow e^{-i\Lambda}\,\boldsymbol{\nabla}\,e^{i\Lambda} = \frac{\partial}{\partial\boldsymbol{r}} + i\boldsymbol{\nabla}\Lambda \\ \frac{\partial}{\partial t} &\longrightarrow e^{-i\Lambda}\,\frac{\partial}{\partial t}\,e^{i\Lambda} = \frac{\partial}{\partial t} + i\frac{\partial\Lambda}{\partial t}\ .\end{aligned}$$

Therefore, with $\tilde{\psi} \equiv e^{i\Lambda}\,\psi$, the Schrödinger equation becomes

$$i\hbar\frac{\partial\tilde{\psi}}{\partial t} - \hbar\frac{\partial\Lambda}{\partial t}\,\tilde{\psi} = \frac{1}{2m}\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A} + \hbar\boldsymbol{\nabla}\Lambda\Big)^2\,\tilde{\psi} - e\phi\,\tilde{\psi}\ .$$

The additional terms may be cancelled by the following *covariant* changes in the 4-potential $(\phi, \boldsymbol{A})$:

$$\begin{aligned}\phi &\to \phi + \frac{\hbar}{e}\frac{\partial\Lambda}{\partial t} \\ \boldsymbol{A} &\to \boldsymbol{A} - \frac{\hbar c}{e}\,\boldsymbol{\nabla}\Lambda\ .\end{aligned}$$

(b) Maxwell's equations don't change because although the 4-potential changes *covariantly*, *viz.*

$$\begin{aligned}\phi &\to \phi + \frac{1}{c}\frac{\hbar c}{e}\frac{\partial\Lambda}{\partial t} \\ \boldsymbol{A} &\to \boldsymbol{A} - \frac{\hbar c}{e}\boldsymbol{\nabla}\Lambda\ ,\end{aligned}$$

the *fields* themselves are *invariant* and do not change:

$$\begin{aligned}\boldsymbol{E} &= -\vec{\boldsymbol{\nabla}}\phi - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} \\ &\to -\boldsymbol{\nabla}\phi - \frac{\hbar}{e}\boldsymbol{\nabla}\frac{\partial\Lambda}{\partial t} - \frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} + \frac{\hbar}{e}\frac{\partial}{\partial t}\boldsymbol{\nabla}\Lambda = \boldsymbol{E}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A} \\ &\to \boldsymbol{\nabla}\times\boldsymbol{A} - \frac{\hbar c}{e}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Lambda = \boldsymbol{B}\ .\end{aligned}$$