DEPARTMENTAL WRITTEN EXAM SOLUTIONS

SPRING 2003

ID Number ________ PART I Score ________

PHYSICS DEPARTMENTAL EXAM – SPRING 2003

DEPARTMENT OF PHYSICS
DEPARTMENTAL EXAMINATION – SPRING 2003

PART I

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. **Please indicate with a "check" which of the (7) questions you wish to be graded below:**

Section 1:	**Problem 1** ____________	**Problem 2** ____________
Section 2:	**Problem 3** ____________	**Problem 4** ____________
Section 3:	**Problem 5** ____________	**Problem 6** ____________
Section 4:	**Problem 7** ____________	**Problem 8** ____________
Section 5:	**Problem 9** ____________	**Problem 10** ____________

Section 1, problem 1

A mass m is attached to the end of a string and rotating in a circle on a frictionless table, with initial kinetic energy E_0. The string passes through a hole in the center of the table, and someone below is keeping the string taut and slowly pulling down, until the radius is halved. How much work was done?

$$W = -\int_R^{R/2} F\,dr \quad , \qquad F = \frac{mv^2}{r} = \frac{\ell^2}{mr^3}$$

$$\hookrightarrow \quad W = \frac{\ell^2}{2mr^2}\Bigg|_R^{R/2} = \frac{3\ell^2}{2mR^2} = \frac{3}{2} m V_0^2$$

$$= 3E_0$$

Section 1, problem 2

A rope of mass M and length L is suspended vertically, with its lower end touching a scale. The rope is released and falls onto the scale. What is the reading of the scale when a length x of the rope has fallen?

$$\mu \equiv \frac{M}{L} \qquad F = \mu g x + \text{impulse}$$

$$\text{impulse} = \frac{\Delta p}{\Delta t} = \mu \frac{\Delta \ell \, V}{\Delta t} = \mu V^2 = 2\mu g x$$

$\Delta \ell$ = little segment of rope, after one end hits scale, other end hits & stops at time Δt later, $\Delta \ell = V \Delta t$

velocity of piece of rope.

So $F = 3\mu g x$

Section 2, problem 3

The region $0 \le x \le L$ is filled with a material with a material with x dependent conductivity, $\sigma = a/x$, where a is a constant. The plane $x = 0$ is held at zero potential (grounded), while the plane $x = L$ is held at constant potential $V_0 > 0$. Consider the steady state situation, where all quantities are time independent. Find the current density $\vec{J}$, the electric field $\vec{E}$, and the charge density ρ in the entire region $0 \le x \le L$.

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \{ \quad \frac{\partial \rho}{\partial t} = 0$$

$$\hookrightarrow \vec{\nabla} \cdot \vec{J} = 0 \qquad \hookrightarrow \vec{J} = J \hat{x}$$

$\uparrow$ constant!

$$\vec{J} = \sigma \vec{E} \qquad \hookrightarrow \vec{E} = \frac{Jx}{a} \hat{x}$$

$$V_0 = -\int_0^L \vec{E} \cdot dx = -\frac{JL^2}{2a} \longrightarrow J = -\frac{2aV_0}{L^2}$$

so $\boxed{\vec{J} = -\frac{2aV_0}{L^2} \hat{x}}$ $\boxed{\vec{E} = -\frac{2x}{L^2} V_0 \hat{x}}$

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \Rightarrow \boxed{\rho = \frac{-2\varepsilon_0 V_0}{L^2}}$$

Section 2, problem 4

A fat wire, of radius a, carries a constant current I, uniformly distributed over its cross sectional area. A narrow gap in the wire, of width $w \ll a$, forms a parallel plate capacitor. The gap is filled with an insulating material with permittivity ϵ and permeability μ. Find the magnetic field $\vec{B}(\vec{r}, t)$ in the gap, with distance r from the central axis of the wire small compared with a, so that edge effects can be ignored.

In gap: $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f = 0$

$\hookrightarrow \vec{\nabla} \times \vec{H} = \frac{d}{dt}\left(\frac{Q_f}{A}\right) = \frac{I}{\pi a^2}$

area

$\vec{H} = H(r)\,\hat{\phi}$ $\qquad \oint \vec{H} \cdot d\vec{l} = H(r)\,2\pi r = \frac{I r^2}{a^2}$

cylindrical coords

$\hookrightarrow \vec{H} = \frac{I r}{2\pi a^2}\,\hat{\phi}$

$\hookrightarrow \vec{B} = \frac{\mu I r}{2\pi a^2}\,\hat{\phi}$

Section 3, problem 5

For this problem you are to regard the $2s$ and $2p$ states of a hydrogen atom as being exactly degenerate in energy, with energy ΔE above the ground state (1s). An unpolarized beam of photons, having exactly the energy ΔE is incident on a hydrogen atom in its ground state. Determine the relative probabilities of excitation to the 2s state and to each of the 2p states. Be sure to specify your choice of basis for the 2p states. Assume that you need to only consider electric dipole transitions.

Solution Assume that we need only consider electric dipole transitions. Then the 2s state is never excited. If the 2p states are characterized by their component of angular momentum in the beam direction, the $m = \pm 1$ states are equally likely to be excited and the $m = 0$ state is never excited.

Section 3, problem 6

A two level system has as its Hamiltonian

$$\mathcal{H} = \begin{pmatrix} 0 & ig \\ -ig & \Delta \end{pmatrix}$$

in some basis. At time zero, the quantity D, described by

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

is measured and found to have the value zero.

1. What is the probability that a measurement of D at a later time t will yield the value one?

2. If a measuring apparatus monitors the value of D continuously, what is the probability that its value will be one at the later time t?

Solution #6

1. The state vector is

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$$

and the Schrödinger equation is

$$\begin{aligned} i\hbar\dot{\psi}_0 &= ig\psi_1 \\ i\hbar\dot{\psi}_1 &= -ig\psi_0 + \Delta\,\psi_1 \end{aligned}$$

from which it follows that

$$-\hbar^2\ddot{\psi}_1 = g^2\psi_1 + i\hbar\Delta\,\psi_1$$

and therefore, since $\psi_0(0) = 1$ and $\psi_1(0) = 0$,

$$\psi_1(t) = A\left(e^{-i\omega_1 t} - e^{-i\omega_2 t}\right)$$

for some constant A and

$$\omega_{1,2} = \frac{\Delta}{2\hbar} \pm \sqrt{\frac{\Delta^2}{4\hbar^2} + \frac{g^2}{\hbar^2}}.$$

From the Schrödinger equation for ψ_0 we can now deduce that

$$\psi_0(t) = \frac{iAg}{\hbar}\left(\frac{1}{\omega_1}e^{-i\omega_1 t} - \frac{1}{\omega_2}e^{-i\omega_2 t}\right)$$

and therefore

$$\frac{iAg}{\hbar}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right) = 1$$

Problem 6 Continued:

or

$$A = \frac{-i/2}{\sqrt{1+\frac{\Delta^2}{4g^2}}}.$$

The desired probability is

$$|\psi_1(t)|^2 = |A|^2\left[2 - 2\cos(\omega_1 - \omega_2)t\right] = \frac{\sin^2 \frac{gt}{2\hbar}\sqrt{1+\frac{\Delta^2}{4g^2}}}{1+\frac{\Delta^2}{4g^2}}.$$

2. Continuous measurement of D forces the system to stay in one eigenstate of D. From the initial condition, this eigenstate is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

If you have the solution to the first part in hand, then you can consider a sequence of measurements separated in time by ϵ; we eventually take the limit $\epsilon \to 0$. For sufficiently small ϵ, the probability that the first measurement yields one is

$$p_1 = \frac{g^2\epsilon^2}{4\hbar^2},$$

so the probability of remaining in the initial state is

$$p_0 = 1 - p_1 = 1 - \frac{g^2\epsilon^2}{4\hbar^2}.$$

All we really need is that $p_1 \propto \epsilon^2$, which is a consequence of the Schrödinger equation; the detailed solution is not necessary. After n measurements, the probability of still being in the initial state is at least p_0^n. For fixed time $t > 0$, choose $\epsilon = t/n$. Then the probability of being in initial state at time t is

$$P_0(t) \geq \left(1 - \frac{g^2t^2}{4\hbar^2n^2}\right)^n$$

and, for continuous measurement,

$$\lim_{n\to\infty} \log P_0(t) \geq n \log\left(1 - \frac{g^2t^2}{4\hbar^2n^2}\right) = 0.$$

We have $\lim_{n\to\infty} P_0(t) = 1$, and the system never leaves the initial state.

Section 4, problem 7

A photon of frequency ω_0 would have energy $1ev$. An ideal blackbody emits power per unit area in the frequency range between ω and $\omega + d\omega$ equal to $s(\omega, T)d\omega$. Find

$$\frac{s(\omega_0, 600K)}{s(\omega_0, 300K)}.$$

Recall that $k_B(300K) \approx \frac{1}{40}eV$.

Use $\vec{S} \propto$ energy density for light

so $$\frac{s(\omega_0, T_1)}{s(\omega_0, T_2)} = \frac{\bar{E}(\omega_0, T_1)}{\bar{E}(\omega_0, T_2)}$$

average energy $$\bar{E}(\omega, T) = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

so $$\frac{s(\omega_0, 600K)}{s(\omega_0, 300K)} = \frac{e^{1eV/\frac{1}{40}eV} - 1}{e^{1eV/\frac{1}{20}eV} - 1} \approx e^{20}$$

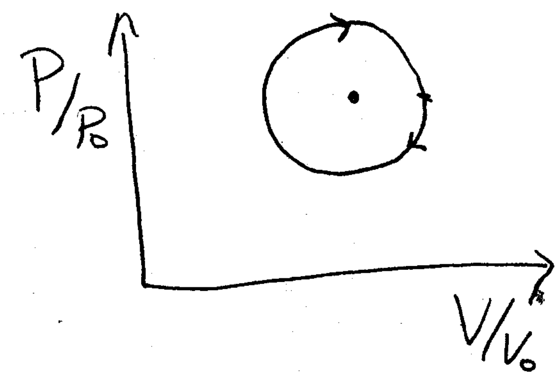
Section 4, problem 8

A mole of ideal diatomic gas, of molar specific heat $\frac{5}{2}R$, is in a piston of variable volume $V(t) = V_0(1 + \alpha\cos\omega t)$. The pressure of the gas is $P(t) = P_0(1 - \alpha\sin\omega t)$. The frequency ω is sufficiently small so that the process is always in equilibrium and reversible, and α is a small, dimensionless, quantity.

1. Find the net work done by the gas in the time from $t = 0$ to $t = \tau \equiv 2\pi/\omega$. The sign should be positive if the gas does net work or negative if net work is done on the gas.

2. The gas has internal energy $E(t)$ at time t. Find $E(\frac{1}{2}\tau) - E(0)$.

3. How much heat is absorbed by the gas (with appropriate sign) between time $t = \frac{1}{2}\tau$ and $t = \tau$?



$W = \int p\,dV$

1) Work done by gas is positive & equal to area enclosed $= \pi P_0 V_0 \alpha^2$

2) $E = \frac{5}{2}RT$

$E = \frac{5}{2}PV$ so $E(\frac{\tau}{2}) - E(0) = \frac{5}{2}P_0V_0\left((1-\alpha)-(1+\alpha)\right)$

$$\boxed{E(\tfrac{\tau}{2}) - E(0) = -5\alpha P_0 V_0}$$

3) $\Delta Q = \Delta E + \Delta W$

so $\Delta Q = 5\alpha P_0V_0 + \frac{\pi}{2}P_0V_0\alpha^2 + P_0V_0(2\alpha) = \boxed{P_0V_0\left(7\alpha + \frac{\pi\alpha^2}{2}\right)}$

Section 5, Problem 9. Know The Error of Thy Way ! : Systematic error somestimes arise when the experimenter unwittingly measures the wrong quantity. "Hawkeye" jones sets out to measure g, the acceleration due to gravity. The experimental setup consists of a pendulum made of a steel ball of radius 1.0 cm suspended by a light (relatively massless) string of length ℓ. Jones performs five measurements and gathers the following data:

String Length ℓ (in CM)	Period of Oscillation (in Sec)
51.2	1.448
59.7	1.566
68.2	1.669
79.7	1.804
88.3	1.896

(a) For each pair of measurements, calculate the value of **g**.

(b) Calculate the mean value of **g** and the standard deviation (σ_g). Assuming that all measurement errors performed by Jones are of random nature, express the measured value in the form $\mathbf{g_{obs}} = \mathbf{g_{mean}} \pm \sigma_{\mathbf{g}}$.

(c) Compare $\mathbf{g}_{obs}$ with the well known value of $\mathbf{g} = 979.6 \pm 0.0000001\ cm/s^2$. Is the agreement statistically acceptable ? Explain your argument clearly. If you think there is a discrepency, its time to interrogate Jones. At your request Jones repeats his measurements and manages to reproduce the table above with an accuracy of 1/1000. Perhaps the error is systematic in nature ! Identify the possible error, correct for it and recalculate $\mathbf{g}_{obs}$. Is the agreement any better?

Section 5, problem 9

Experimental problem

The correct distance to use is that from the pivot to the center of the ball, not the string length. Using string length gives result for g which is 10σ away from true value. If they correct for this mistake, they get another chance to use correct length & recalculate g.

Section 5, problem 10

The following questions do not require any knowledge of hydrodynamics or hydrodynamic numerical constants. They do require some sense and the back of an envelope on which to make calculations. Dimensionless constants of order of magnitude unity can and should be omitted.

1. Give the dispersion relation, ω vs k, for water waves in deep water.
2. How deep must the water be in part 1?
3. A storm in the middle of the ocean creates waves of all wave numbers, up to some cutoff K. A surfer in La Jolla, several thousand miles away from the storm, sees essentially monochromatic waves. Why?
4. On consecutive days, the surfer of part 3 sees the wavelength change. Does it increase or decrease? Explain.

Solution #10

1. The only possibly relevant physical quantities are the water density ρ and the acceleration of gravity g, since water is essentially incompressible and the depth cannot enter because it is "infinite". By dimensional analysis, omitting a possible constant factor,

$$\omega = \sqrt{gk}.$$

2. The only length in the problem, other than the wavelength, is the depth of the water. So "deep" means much deeper than the wavelength.

3. If the beach is at a distance D from the storm, which ended a time T in the past, D and T must be related by the group velocity, $D = v_g T$. The group velocity depends on the wave number,

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}},$$

so we see waves for which

$$\begin{aligned} \frac{D}{T} &= \frac{1}{2}\sqrt{\frac{g}{k}}, \\ k &= \frac{gT^2}{2D^2}. \end{aligned} \tag{1}$$

4. On consecutive days T increases so, according to (1), k increases and the wavelength decreases.

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PART II

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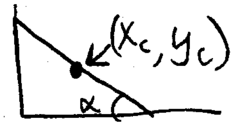
Section 6:	**Problem 11**________	**Problem 12** ________
Section 7:	**Problem 13**________	**Problem 14** ________
Section 8:	**Problem 15**________	**Problem 16**________
Section 9:	**Problem 17** ________	**Problem 18** ________
Section 10:	**Problem 19** ________	**Problem 20** ________

Section 6, problem 11

A stick of length L and mass M (uniformly distributed) is placed with one end on a frictionless wall and the other end on a frictionless floor. The initial angle is α. Because of gravity, the stick slides. Write down the Lagrangian for the angle α and use it to get the equations of motion for $\alpha(t)$. Write down all conserved quanitities.



$$T = \frac{M}{2}\left(\dot{x}_c^2 + \dot{y}_c^2\right) + \frac{1}{2} I_c \dot{\alpha}^2$$

$$I_c = \frac{M}{12} L^2 \qquad \dot{x}_c^2 + \dot{y}_c^2 = \frac{L^2}{4} \dot{\alpha}^2$$

so $$T = ML^2\dot{\alpha}^2\left(\frac{1}{8} + \frac{1}{24}\right) = \frac{ML^2\dot{\alpha}^2}{6}$$

$$\mathcal{L} = \frac{M}{6} L^2 \dot{\alpha}^2 - \frac{1}{2} MgL\sin\alpha$$

E.O.M. $$\frac{L}{3}\ddot{\alpha} = -\frac{1}{2} g\cos\alpha$$

Energy is conserved $$E = \frac{ML^2}{6}\dot{\alpha}^2 + \frac{1}{2} MgL\sin\alpha$$

evd. at $t=0$

$\hookrightarrow E = \frac{1}{2} mgL\sin\alpha_0$ //

Section 6, problem 12

A system has as phase space the two dimensional surface $x^2 + y^2 + z^2 = 1$ (as seen from three dimensional space in which the phase space is embedded). The coordinates x, y, and z are phase space functions, and they have Poisson brackets

$$\{x, y\} = z, \qquad \{y, z\} = x, \qquad \{z, x\} = y.$$

1. If the Hamiltonian is z, find the orbits in phase space.

2. Find canonical coordinates p and q on at least part of the phase space, which you may specify. Express q and p in terms of x, y, and z, or you may give the inverse of this relationship.

3. As a surface embedded in three dimensional space, regions of phase space are endowed with *area*. What, if anything, does Liouville's theorem say about the change of this area with time?

Solution #12

1. The equations of motion are

$$\dot{x} = \{x, z\} = -y$$
$$\dot{y} = \{y, z\} = x,$$

with general solution

$$x(t) = x_0 \cos t - y_0 \sin t$$
$$y(t) = x_0 \sin t + y_0 \cos t.$$

The orbits are circles on the sphere lying in planes perpendicular to the z-axis.

2. There are many possibilities. For example, one can try to find a function $f(z)$ such that $q = f(z)x$, $p = -f(z)y$, and it turns out that $f(z) = z/\sqrt{1 - z^2}$ does the job on the domain $0 < z < 1$. Another possibility is $x = \sqrt{1 - q^2}\cos p$, $y = -\sqrt{1 - q^2}\sin p$, which works on the same domain. For this choice, $z = q$.

3. The area measured using canonical coordinates is invariant. Use the second choice in part 2. Then

$$dx\, dy = q\, dq\, dp,$$

so $dx\, dy/z$ is invariant. This is exactly the area element on the sphere, as inherited from three dimensional space.

Section 7, problem 13

In a region of space there is a uniform magnetic field $\vec{B} = \hat{x}B_0$. An uncharged copper sphere of radius R moves with a velocity $\vec{v} = v_0\hat{y}$ through the magnetic field. Compute the electric field everywhere (inside and outside the sphere), as seen in the lab frame. Also, find the surface charge density on the surface of the sphere. Take $v_0 \ll c$.

SOLUTION # 13

submitted by O'Neil

SOLUTION
Part I
Problem 1

3. In a region of space there is a uniform magnetic field $\mathbf{B}=\hat{y}\,B_0$. An uncharged copper sphere of radius R moves at constant velocity $\mathbf{v}=v_0\hat{x}$ through the magnetic field. Calculate the electric field in all of space, as seen in the laboratory frame. Is it necessary to exert a force on the sphere to maintain its motion? (Hint: Of course, v_0 is much smaller than c.)

Let sphere be centered on origin of coordinates at the time of observation.

Inside sphere, the Lorentz force must be zero

$$0 = \underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \qquad \therefore \quad \underset{\sim}{E} = -\frac{v_0 B_0}{c}\hat{z}$$

Inside and outside $\nabla^2\varphi = 0$, but there is surface charge

let $\varphi = \varphi_{in}$ for $r < R$, $\varphi = \varphi_{out}$ for $r > R$

$$\varphi_{in} = v_0 B_0 \underbrace{r\cos\theta}_{z}$$

$$\varphi_{out} = \frac{A}{r^2}\cos\theta + \frac{\cancelto{0}{B}}{r} \quad \text{since no net charge}$$

$$\varphi_{in}(R,\theta) = \varphi_{out}(R,\theta) \qquad \therefore \quad A = v_0 B_0 R^3$$

$$\underset{\sim}{E}_{out} = -\underset{\sim}{\nabla}\varphi_{out}$$

There is no net force on sphere, since B is uniform and $\int d^3r\, \underset{\sim}{J}(\underset{\sim}{r}) = \underset{\sim}{V}\cdot \underbrace{(\text{total charge})}_{\shortparallel\ 0}$.

Section 7, problem 14

Concentric circular loops of wire of radii R_1 and R_2, with $R_1 < R_2$, carry currents $I_1(t)$ and $I_2(t)$ given by

$$I_1(t) = \nu R_2^2 \cos \omega t,$$
$$I_2(t) = -\nu R_1^2 \cos \omega t.$$

Suppose the frequency ω is low, $\omega R_2/c \ll 1$. Compute the total rate of radiation (in Watts) of this system, in terms of R_1, R_2, ν, ω and c, the velocity of light in vacuum.

#14

Grad E&M Concentric circular loops of wire of radii R_1 and R_2, with $R_1 < R_2$, carry currents $I_1(t)$ and $I_2(t)$ given by

$$I_1(t) = \nu R_2^2 \cos \omega t,$$
$$I_2(t) = -\nu R_1^2 \cos \omega t.$$

Suppose the frequency ω is low: $\omega R_2/c \ll 1$. Compute the total rate of radiation (in Watts) from this system. Give the solution in terms of R_1, R_2, ν, ω and the velocity of light in vacuum, c.

[Remark: I made up this problem for an exam *many* years ago and it is probably safe to recycle it now.]

Solution Recall that, in SI units, when the current density is of the form $\Re[\vec{J}(\vec{r}) \exp -i\omega t]$, the vector potential has the similar form $\Re[\vec{A}(\vec{r}) \exp -i\omega t]$ with

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 c^2} \int d^3r' \, \vec{J}(\vec{r}\,')e^{ik|\vec{r}-\vec{r}\,'|},$$

and the radiation field is

$$\vec{A}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 c^2 r} e^{ikr} \vec{\mathcal{J}}(k\hat{r})$$

where $\vec{\mathcal{J}}$ is the Fourier transform of the current density,

$$\vec{\mathcal{J}}(\vec{k}) = \int d^3r' \, \vec{J}(\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}\,'}.$$

Problem 14 Continued

Looking ahead, we have, in the radiation zone,

$$
\begin{aligned}
\vec{B}(\vec{r}) &= \frac{ik}{4\pi\varepsilon_0 c^2 r}\hat{r} \times \vec{\mathcal{J}}(k\hat{r}) \\
\vec{E}(\vec{r}) &= c\vec{B}(\vec{r}) \times \hat{r} \\
\vec{S}(\vec{r}) &= \frac{\varepsilon_0 c^2}{2}\Re\vec{E}(\vec{r}) \times \vec{B}^*(\vec{r}) \\
&= \frac{k^2}{32\pi^2\varepsilon_0 c r^2}|\hat{r} \times \vec{\mathcal{J}}(k\hat{r})|^2\,\hat{r}.
\end{aligned}
$$

Now we must compute $\vec{\mathcal{J}}(\vec{k})$ in the limit of small k. Choose coordinates so that the current loops are in the x-y plane, centered at the origin, and $\hat{r} = \cos\psi\hat{z} + \sin\psi\hat{x}$. We find

$$
\begin{aligned}
\vec{\mathcal{J}}(k\hat{r}) &= \int_0^{2\pi} d\theta\,(\sin\theta\hat{x} - \cos\theta\hat{y})\nu\left[R_2^2R_1e^{-ikR_1\sin\psi\cos\theta} - R_1^2R_2e^{-ikR_2\sin\psi\cos\theta}\right] \\
&= \int_0^{2\pi} d\theta\,(\sin\theta\hat{x} - \cos\theta\hat{y})\nu\left(R_2^2R_1^4 - R_1^2R_2^4\right)\frac{(-ik\sin\psi)^3}{3!}\cos^3\theta
\end{aligned}
$$

(where we kept the lowest power of k whose coefficient does not vanish)

$$
= \hat{y}\frac{\pi i k^3\sin^3\psi}{8}\nu R_1^2R_2^2(R_2^2 - R_1^2).
$$

Now assemble the pieces and integrate over solid angles to get the total radiated power:

$$
P = \int_0^{\pi} 2\pi\sin\psi\,d\psi\,\frac{k^2}{32\pi^2\varepsilon_0 c}\frac{\pi^2k^6\sin^6\psi}{64}\nu^2R_1^4R_2^4(R_2^2 - R_1^2)^2
$$

(using $|\hat{r} \times \hat{y}|^2 = 1$)

$$
\begin{aligned}
&= \frac{\pi k^8}{1024\varepsilon_0 c}\nu^2R_1^4R_2^4(R_2^2 - R_1^2)^2\int_0^{\pi}\sin^7\psi\,d\psi \\
&= \frac{\pi k^8}{1120\varepsilon_0 c}\nu^2R_1^4R_2^4(R_2^2 - R_1^2)^2.
\end{aligned}
$$

[The numerical factor is not guaranteed.)

Section 8, problem 15

Consider a particle of mass m moving on one dimension in a potential well $V(x)$. The Hamiltonian has a complete discrete spectrum, with eigenvalues E_n and wavefunctions ψ_n. Show the validity of the sum rule

$$\sum_n \frac{2m}{\hbar^2}|X_{ni}|^2(E_n - E_i) = 1,$$

where i labels any of the bound states and

$$X_{ni} \equiv \langle\psi_n|\widehat{x}|\psi_i\rangle.$$

SOLUTION $\boxed{15}$

Consider a particle of mass m moving in one dimension in a potential well $V(x)$. The Hamilton operator has a complete discrete spectrum with eigenvalues E_n and wavefunctions ψ_n.

Show the validity of the sum rule:

$$\sum_n \frac{2m}{\hbar} |X_{ni}|^2 (E_n - E_i) = 1$$

where i labels any of the bound states and

$$X_{ni} = \langle \psi_n | \hat{x} | \psi_i \rangle$$

Solution:

$$\dot{x} = \frac{p}{m}$$

$$[x, p] = i\hbar$$

$$\dot{x} = \frac{i}{\hbar}[H, x]$$

$$[x, [H, x]] = \frac{\hbar^2}{m}$$

$$\langle \psi_i | 2xHx - x^2 H - Hx^2 | \psi_i \rangle = \frac{\hbar^2}{m}$$

$$\langle \psi_i | xHx | \psi_i \rangle = \sum_n \langle \psi_i | x | \psi_n \rangle \langle \psi_n | x | \psi_i \rangle E_n = \sum_n |X_{ni}|^2 E_n$$

$$\langle \psi_i | x^2 H | \psi_i \rangle = \langle \psi_i | H x^2 | \psi_i \rangle = \sum_n \langle \psi_i | x | \psi_n \rangle \langle \psi_n | x | \psi_i \rangle E_i$$

$$= \sum_n |X_{ni}|^2 E_i$$

$$[x, [H, x]] = \frac{\hbar^2}{m} \implies \frac{2m}{\hbar^2} \sum_n |X_{ni}|^2 (E_n - E_i) = 1$$

Section 8, problem 16

Calculate the energy shift in the ground state of the hydrogen atom due to the finite extension of the proton. Treat the proton as a sphere of radius R with homogeneous charge distribution. Hint: use first order perturbation theory.

SOLUTION 16

Calculate the energy shift in the ground state of the hydrogen atom from the finite extension of the proton. Treat the proton as a sphere of radius R with homogeneous charge distribution.

Hint: use first order perturbation theory

Solution: perturbing potential

$$H' = \frac{e^2}{R}\left(\frac{R}{r} - \frac{3}{2} + \frac{1}{2}\frac{r^2}{R^2}\right) \quad \text{if } r \le R$$

$$V(r) = \begin{cases} -\dfrac{e^2}{R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right) & r \le R \\ -\dfrac{e^2}{r} & r \ge R \end{cases}$$

$$\Delta E = \int \psi_{100}^* \, H' \, \psi_{100} \, d^3 r = \frac{e^2}{a_H}\left[1 - 3\,\frac{(4+4x+x^2)e^{-x} - (4-x^2)}{x^3}\right]$$

$$x = \frac{2R}{a_H} \qquad a_H = \frac{\hbar^2}{me^2}$$

for small x $\quad \Delta E = \dfrac{e^2}{a_H}\dfrac{x^2}{10}$

Section 9, problem 17

A particle of mass m, called A, undergoes the reaction

$$A + A \leftrightarrow A_2$$

where the bound state A_2 has mass $2m$ and binding energy W (i.e. the energy of a stationary A_2 composite is $-W < 0$). The system is at equilibrium at temperature T, in the thermodynamic limit, with N total particles A in volume V, and $n = N/V$ finite. The N total particles are distributed as N_1 unbound As and N_2 composites. The particles are all nonrelativistic and you can ignore any internal rotational or vibrational degrees of freedom in A_2. Each A is indistinguishable from the others, and each A_2 is indistinguishable from the others.

Find the partition function of the system and the free energy per particle. Use this to find $x \equiv N_1/N$, the fraction of the particles which are unpaired.

#17

SOLUTION:

We have $N_1 + 2N_2 = N = nV$, and $N_1 = xN$. Therefore $N_2 = \frac{1}{2}(1-x)N$. We now write down the partition function

$$Z = \frac{Z_1^{N_1}}{N_1!} \cdot \frac{Z_2^{N_2}}{N_2!} \ ,$$

where $Z_1 = (V/\lambda^3)^{N_1}$ where $\lambda = (2\pi\hbar^2/mk_{\rm B}T)^{1/2}$ is the thermal wavelength. For the composites, we clearly have

$$Z_2 = \left(2^{3/2}\, e^{W/k_{\rm B}T}\, \frac{V}{\lambda^3}\right)^{N_2} \ .$$

Note that the thermal wavelength of the composites is $2^{3/2}$ times smaller than that of the monomers due to the mass difference ($2m$ *versus* m). The free energy per particle is then $f = -N^{-1} k_{\rm B}T \ln Z$:

$$\frac{f}{k_{\rm B}T} = -\tfrac{1}{2}(1+x) + x\ln x + \frac{1-x}{2}\ln\left(\frac{1-x}{2}\right) + \tfrac{1}{2}(1+x)\ln(n\lambda^3) - \tfrac{1}{2}(1-x)\left(\tfrac{3}{2}\ln 2 + \frac{W}{k_{\rm B}T}\right) \ .$$

We now minimize with respect to x, setting $\partial f/\partial x = 0$, which yields

$$2\ln x - \ln(1-x) + \tfrac{5}{2}\ln 2 + \frac{W}{k_{\rm B}T} + \ln(n\lambda^3) = 0 \ ,$$

the solution of which is

$$x = \frac{2}{1 + \left[1 + 8\sqrt{2}\, n\lambda^3\, \exp(W/k_{\rm B}T)\right]^{1/2}} \ .$$

Interpretation: At low values of the density n we have $x = 1$ and none of the monomers bind, despite the fact that $W > 0$. The reason is that entropy favors them to remain unbound. For fixed n, we correctly obtain $x(T \to 0) = 0$.

Section 9, problem 18

A gas of classical identical particles is interacting with some two body potential $U(|\vec{r}|)$. Write a formal expression for the grand partition function Ξ, as a function of the temperature T and volume V and potential $U(r)$. Determine the pressure $p(z,T)$ and density $n(z,T)$, both to second order in the fugacity $z \equiv e^{\mu/kT}$.

Solution

To second order in the density, one has

$$p/k_BT = n - B_2\,n^2 + \cdots \,,$$

where

$$B_2 = -\tfrac{1}{2}\int d^3\!r\,\left[e^{-U(r)/k_BT} - 1\right] \,.$$

Interlude – This result is easily derived by taking the logarithm of the grand partition function,

$$\begin{aligned}\Xi &= \sum_{N=0}^{\infty} \frac{1}{N!}\,e^{N\mu/k_BT}\,Z_N(T,V)\\ &= 1 + zV\lambda_T^{-3} - \tfrac{1}{2}z^2\,V\lambda_T^{-6}\int d^3r\,e^{-U(r)/k_BT} + \cdots \,,\end{aligned}$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$ is the thermal wavelength, which appears upon performing the momentum integrals, *viz.*

$$\int \frac{d^3\!p}{(2\pi\hbar)^3}\,\exp(-p^2/2mk_BT) = \lambda_T^{-3}$$

Recalling that $\ln\Xi = \Omega/k_BT = -pV/k_BT$ and carrying out the Taylor series expansion to second order in the fugacity, one immediately obtains the expansion for $p(z,T)$:

$$p/k_BT = z\,\lambda_T^{-3} - \tfrac{1}{2}z^2\,\lambda_T^{-6}\int d^3\!r\,f(r) + \cdots \,,$$

where $f(r) \equiv e^{-U(r)/k_BT} - 1$ is the Mayer function. Next, one invokes $n = -V^{-1}(\partial\Omega/\partial\mu)_{T,V} = \partial(p/k_BT)/\partial\ln z$ to obtain a series expansion for $n(z,T)$:

$$n = z\,\lambda_T^{-3} + z^2\,\lambda_T^{-6}\int d^3\!r\,f(r) + \cdots \,.$$

Finally, one inverts $n(z,T)$ to find $z(n,T)$ and substitutes into the pressure equation, yielding

$$p/k_BT = n + B_2\,n^2 + \cdots \,,$$

with $B_2 = -\frac{1}{2}\int d^3\!r\,f(r)$.

(a) The Mayer function for our problem is

$$f(r) = \left[\left(\frac{r}{a}\right)^{U_0/k_BT} - 1\right]\,\Theta(a-r) \,.$$

This is continuous and strictly nonpositive for $r \in [0,\infty)$. Integrating, we find

$$B_2(T) = \tfrac{2}{3}\,\pi\,a^3\,\frac{U_0}{U_0 + 3k_BT} \,.$$

Section 10, problem 19

Energy in a star is produced my nuclear reactions. The number of collisions with center of momentum kinetic energy in the interval from E to $E + dE$ is

$$Ne^{-E/kT} E dE$$

per unit time, with N a constant. The probability that a collision with CM energy E will result in a nuclear reaction is

$$Me^{-\alpha/\sqrt{E}},$$

where M and α are constants. Find an *approximate* expression for the total number of nuclear reactions, per unit time, assuming that

$$\left(\frac{kT}{\alpha^2}\right)^{1/3} \ll 1.$$

$$I = \int_0^\infty NM e^{-E/kT} e^{-\alpha/\sqrt{E}} E\, dE$$

$$\downarrow \quad E^2 = (kT\alpha)^{4/3} z$$

$$I = \frac{NM}{2}(kT\alpha)^{4/3} \int_0^\infty e^{-f(z)} dz$$

with $\quad f(z) \equiv \frac{1}{\epsilon}\left(z^{1/2} + z^{-1/4}\right) \qquad \epsilon \equiv \left(\frac{kT}{\alpha^2}\right)^{1/3} \ll 1$

expand $\quad f(z) \simeq f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2$

$f'(z_0) = 0 \quad \longrightarrow \quad z_0 = 2^{-4/3} \quad$ so $\quad f(z_0) = \frac{1}{\varepsilon}\frac{3}{2^{2/3}}$

$f''(z_0) = 9/\varepsilon$

$$I \approx \frac{NM}{4}(kT\alpha)^{4/3} e^{-3/\epsilon 2^{2/3}} \sqrt{\frac{2\pi\epsilon}{9}}$$

Section 10, problem 20

Evaluate

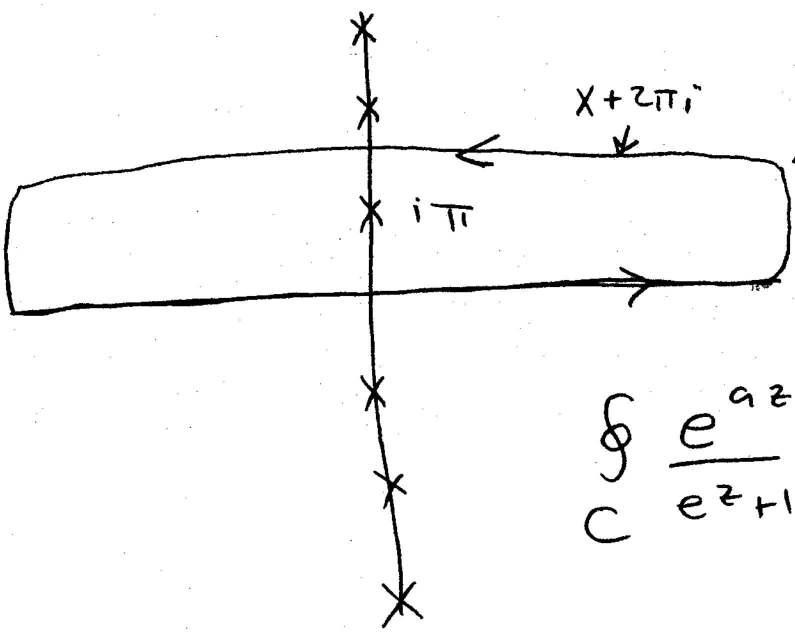
$$I(a) = \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x + 1} \qquad (0 < a < 1).$$

Section 10, problem 20

Evaluate

$$I(a) = \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1} dx \quad (0 < a < 1).$$

$$\int \frac{e^{az}}{e^z+1} dz \qquad \text{poles @} \quad z = i\pi + 2\pi n$$



$$\oint_C \frac{e^{az}}{e^z+1} dz = (1-e^{2\pi i a}) I$$

$$= 2\pi i \, (\text{residue @ } i\pi)$$

$$z = i\pi + \varepsilon$$

$$\frac{e^{az}}{e^z+1} \approx \frac{e^{i\pi a}}{-\varepsilon} \quad \text{so res} = -e^{i\pi a}$$

$$(1-e^{2\pi i a}) I = -2\pi i \, e^{i\pi a}$$

$$\hookrightarrow \boxed{I = \pi / \sin \pi a}$$