

SOLUTIONS

FALL, 1998-99

PART I

PHYSICS DEPARTMENTAL EXAMINATION SPECIAL INSTRUCTIONS

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. Please indicate with a "check" which of the (7) questions you wish to be graded below:

Section 1:	Problem 1 _____	Problem 2 _____
Section 2:	Problem 3 _____	Problem 4 _____
Section 3:	Problem 5 _____	Problem 6 _____
Section 4:	Problem 7 _____	Problem 8 _____
Section 5:	Problem 9 _____	Problem 10 _____

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Problem 1 (PART I, SECTION 1)

A satellite of mass 10^3 kg orbits the earth in a circular orbit of radius 20,000 km. The rockets on board the satellite are fired to give a net impulse of 10^6 N · m in a direction tangential to the orbit. (Neglect any change in the mass of the satellite). Determine the semimajor axis a and eccentricity ϵ of the resulting orbit. Also find the distances of the points of nearest and farthest approach.

For reference, the earth's mass is $M_{\oplus} = 5.97 \times 10^{24}$ kg, and the Cavendish constant is $G = 6.67 \times 10^{-11}$ m³/kg · s². Hint: The shape of a Keplerian orbit follows quickly from conservation of the Laplace-Runge-Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{l} - \mu \mathbf{k} \hat{\mathbf{r}}$, where μ is reduced mass and $V = -k/r$.

$$(6) \quad V = -\frac{k}{r} \quad k = G_N m_{\oplus} m = (6.67 \cdot 10^{-11})(6 \cdot 10^{24})(10^3) \\ = 4 \cdot 10^7 \text{ Nm}^2$$

$$\therefore V = -k/r = \frac{-4 \cdot 10^7}{2 \cdot 10^7} = -2 \cdot 10^{10} \text{ J}$$

$$T = \text{kinetic energy} = \frac{k}{2r} = 10^{10} \text{ J}$$

$$\therefore p = \text{momentum} = \sqrt{2mT} = 4.47 \cdot 10^6 \text{ kg m/s}$$

$$l = p \cdot r = 8.94 \cdot 10^{13} \text{ kg m}^2/\text{s}$$

$$\text{change in } p = \Delta p = 10^6 \text{ kg m/s}$$

$$\text{change in } L = \Delta l = \Delta p \cdot r = 2 \cdot 10^{13} \text{ kg m}^2/\text{s}$$

$$\therefore p' = 5.47 \cdot 10^6 \text{ kg m/s}$$

$$L' = 10.94 \cdot 10^{13} \text{ kg m}^2/\text{s}$$

$$\therefore \text{Final energy} = \frac{p'^2}{2m} + V = -5 \cdot 10^9 \text{ J}$$

$$\text{semimajor axis } a = -\frac{k}{2E} = 4 \cdot 10^7 \text{ m} = 40,000 \text{ km}$$

$$\text{semi eccentricity } e = \sqrt{1 + \frac{2EL^2}{mk^2}} = 0.5$$

$$\text{closest approach } a(1-e) = 20,000 \text{ km}$$

$$\text{farthest } a(1+e) = 60,000 \text{ km}$$

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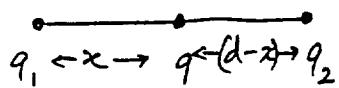
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Problem 2 (PART I, SECTION 1)

Two charges $q_1 > 0$ and $q_2 > 0$ are held fixed a distance d apart. A third charge q of mass m is constrained to move on the line joining q_1 and q_2 . Find its equilibrium position, and the frequency of small oscillations about the minimum.

(3)



$$\text{Potential} \quad V = \frac{q_1 q}{|x|} + \frac{q_2 q}{|d-x|}$$

$$\textcircled{a} \text{ Equilibrium: } \frac{\partial V}{\partial x} = 0 \Rightarrow \frac{q_1 q}{x^2} = \frac{q_2 q}{(d-x)^2} \quad (\text{or by equating forces})$$

$$\therefore \frac{x^2}{(d-x)^2} = \frac{q_1}{q_2} \Rightarrow \frac{x}{d-x} = \sqrt{\frac{q_1}{q_2}} \Rightarrow \frac{x}{d} = \frac{\sqrt{q_1}}{\sqrt{q_1} + \sqrt{q_2}}$$

let $x = x_0 + z$ x_0 = equilibrium position.

$$\text{Potential near minimum} = V_0 + \frac{1}{2} k z^2$$

$$k = \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} = \frac{2q_1 q}{x^3} + \frac{2q_2 q}{(d-x)^3}$$

$$= \frac{2q_1 q}{d^3} \left(\frac{\sqrt{q_1} + \sqrt{q_2}}{\sqrt{q_1}} \right)^3 + \frac{2q_2 q}{d^3} \left(\frac{\sqrt{q_1} + \sqrt{q_2}}{\sqrt{q_2}} \right)^3$$

$$= \frac{2q}{d^3} \left(\sqrt{q_1} + \sqrt{q_2} \right)^3 \left(\frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} \right)$$

$$= \frac{2q}{d^3} \cdot \frac{\left(\sqrt{q_1} + \sqrt{q_2} \right)^4}{\sqrt{q_1 q_2}}$$

$$\text{and } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2q}{md^3} \cdot \frac{\left(\sqrt{q_1} + \sqrt{q_2} \right)^2}{\left(q_1 q_2 \right)^{1/2}}}$$

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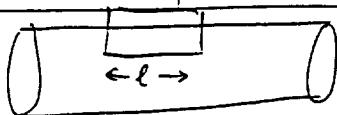
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Problem 3 (PART I, SECTION 2)

A loop of wire of radius R is placed inside a solenoid of length ℓ and radius a , with n turns per unit length. The plane of the loop is perpendicular to the axis of the solenoid. Find the mutual inductance.

3

3



magnetic field due to a current in the solenoid:

$$B \cdot l = \frac{4\pi}{c} \cdot n I l \quad (\text{Ampere's Law})$$

$$\Rightarrow B = \frac{4\pi n I}{c}$$

this is a uniform magnetic field.

$$\text{Flux through loop: } \Phi_B = (\pi R^2) \frac{4\pi n I}{c}$$

$$\text{Mutual inductance : } \quad (\pi R^2) \frac{4\pi n}{c^2}$$

$$\text{since} \quad \text{emf} = \frac{1}{c} \frac{d\Phi_B}{dt}$$

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Problem 4 (PART I, SECTION 2)

The dielectric tensor in a uniaxial crystal is given by the expression

$$\epsilon_{\alpha\beta} = \epsilon_{||} n_\alpha n_\beta + \epsilon_{\perp} (\delta_{\alpha\beta} - n_\alpha n_\beta) ,$$

where \hat{n} is the symmetry axis.

Compute the potential $\phi(\vec{r})$ from a point charge Q sitting at the origin of this uniaxial crystal. Hint: you may choose $\hat{z} = \hat{n}$.

Solution

We solve

$$\vec{\nabla} \cdot \vec{D} = 4\pi\varrho(\vec{r}) = 4\pi Q\delta(\vec{r})$$

with

$$\begin{aligned} D_\alpha &= \epsilon_{\alpha\beta} E_\beta \\ &= -\epsilon_{\alpha\beta} \frac{\partial\phi}{\partial x^\beta}. \end{aligned}$$

Thus, with $\hat{n} \equiv \hat{z}$,

$$-\epsilon_{||} \frac{\partial^2 \phi}{\partial z^2} - \epsilon_{\perp} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = -4\pi Q \delta(x) \delta(y) \delta(z).$$

The simplest solution involves a rescaling of z ,

$$z \equiv (\epsilon_{||}/\epsilon_{\perp})^{1/2} \bar{z},$$

so that Poisson's equation becomes

$$-\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial \bar{z}^2} \right) \phi = \frac{4\pi Q}{\sqrt{\epsilon_{||}\epsilon_{\perp}}} \delta(x) \delta(y) \delta(\bar{z}).$$

(Recall that $\delta(z/a) = a\delta(z)$.) Now the solution to this homogeneous equation is known to be

$$\phi(x, y, \bar{z}) = \frac{Q}{\sqrt{\epsilon_{||}\epsilon_{\perp}}} \frac{1}{\sqrt{x^2 + y^2 + \bar{z}^2}}.$$

Thus, the solution we seek is

$$\phi(x, y, z) = \frac{Q}{\sqrt{\epsilon_{||}\epsilon_{\perp}}} \frac{1}{\sqrt{x^2 + y^2 + (\epsilon_{\perp}/\epsilon_{||})z^2}}.$$

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Problem 5 (PART I, SECTION 3)

Two spin-one particles (which do not interact with each other) are in the ground state wavefunction $\psi_0(x)$ of some spherically symmetric potential $V(x)$. A small perturbation

$$H = JS_1 \cdot S_2$$

is turned on, where S_i are the spin operators for the two particles respectively. Find all possible spin-states, and determine their energy shift due to the perturbation. Which states are allowed if the two particles are identical?

① $\text{spin } 1 \otimes \text{spin } 1 \rightarrow \text{spin } 2, 1, 0$ are possible total spins

$$S_1 \cdot S_2 = \frac{1}{2} [(S_1 + S_2)^2 - S_1^2 - S_2^2]$$

$$S^2 = s(s+1)k^2 = \begin{cases} 0 & \text{spin 0} \\ 2 & \text{spin 1} \\ 6 & \text{spin 2} \end{cases}$$

$$\therefore \Delta E = \langle H \rangle = \langle J \cdot S_1 \cdot S_2 \rangle$$

$$= \frac{J}{2} (S^2 - S_1^2 - S_2^2)$$

$$= \frac{J}{2} \begin{cases} (0-2-2) & \text{spin 0} \\ (2-2-2) & \text{spin 1} \\ (6-2-2) & \text{spin 2} \end{cases}$$

$$= \begin{cases} -2J & \text{spin 0} \\ -J & \text{spin 1} \\ J & \text{spin 2} \end{cases}$$

Spin 1 particles are bosons. Since they are in the same spatial wavefunction $\psi_0(x)$, the spin wavefunction must be totally symmetric \Rightarrow only spin 2, 0 are allowed

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Problem 6 (PART I, SECTION 3)

A three-dimensional harmonic oscillator with Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 \mathbf{r}^2}{2}$$

is subject to a quadrupolar perturbation

$$\mathcal{H}' = \lambda(x^2 + y^2 - 2z^2) .$$

Compute the energy shift of the ground state to second order in perturbation theory.

Compare with the result you get by solving the problem exactly.

(5)

Use creation and annihilation operators for x, y, z directions.

$$H = (\hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y + \hat{a}_z^\dagger \hat{a}_z + 3/2) \hbar\omega$$

$$\delta H = \lambda \frac{\hbar}{2m\omega} \left[(\hat{a}_z + \hat{a}_z^\dagger)^2 + (\hat{a}_y + \hat{a}_y^\dagger)^2 - 2(\hat{a}_x + \hat{a}_x^\dagger)^2 \right]$$

Shift in the ground state energy:

$$\text{at first order } E^{(1)} = \langle 000 | \delta H | 000 \rangle$$

$$= \frac{\lambda \hbar}{2m\omega} \cdot [1 + 1 - 2] = 0$$

$$\text{at second order } \delta E^{(2)} = \sum_{\substack{(n_x n_y n_z) \\ (0,0,0)}} \frac{|\langle n_x n_y n_z | \delta H | 000 \rangle|^2}{E_{(000)} - E_{(n_x n_y n_z)}}$$

The only states that have non-zero matrix element are $|200\rangle$, $|020\rangle$ and $|002\rangle$

For each of these, the energy denominator is $-2\hbar\omega$

The matrix elements of δH are:

$$\begin{aligned} \langle 200 | \delta H | 000 \rangle &= \frac{\lambda \hbar}{2m\omega} \cdot \langle 200 | (\hat{a}_x^\dagger)^2 | 000 \rangle \\ &= \frac{\sqrt{2} \lambda \hbar}{2m\omega} \end{aligned}$$

$$\text{Similarly } \langle 020 | \delta H | 000 \rangle = \frac{\sqrt{2} \lambda \hbar}{2m\omega} \quad \langle 002 | \delta H | 000 \rangle = (-2) \frac{\sqrt{2} \lambda \hbar}{2m\omega}$$

$$\therefore \delta E^{(2)} = -\frac{1}{2\hbar\omega} \cdot \left[\frac{2\lambda^2 \hbar^2}{4m^2\omega^2} + \frac{2\lambda^2 \hbar^2}{4m^2\omega^2} + \frac{8\lambda^2 \hbar^2}{4m^2\omega^2} \right]$$

$$\Rightarrow E^{(2)} = -\frac{3}{2} \frac{\lambda^2 \hbar}{m^2 \omega^3}$$

Exact solution:

$$H + \delta H = \frac{p_x^2}{2m} + z^2 \left(\frac{m\omega^2}{2} + \lambda \right) + \frac{p_y^2}{2m} + y^2 \left(\frac{m\omega^2}{2} + \lambda \right)$$

$$+ \frac{p_z^2}{2m} + z^2 \left(\frac{m\omega^2}{2} - 2\lambda \right)$$

The ground state energy = $\frac{1}{2} \hbar (\omega_x + \omega_y + \omega_z)$ where $\omega_x, \omega_y, \omega_z$ are the frequencies of the x, y, z oscillators.

$$\therefore E = \frac{1}{2} \hbar \cdot \left\{ \sqrt{\omega^2 + \frac{2\lambda}{m}} + \sqrt{\omega^2 + \frac{2\lambda}{m}} + \sqrt{\omega^2 - \frac{4\lambda}{m}} \right\}$$

$$= \frac{1}{2} \hbar \omega \left\{ 2 \left(1 + \frac{2\lambda}{m\omega^2} \right)^{1/2} + \left(1 - \frac{4\lambda}{m\omega^2} \right)^{1/2} \right\}$$

$$= \frac{1}{2} \hbar \omega \left\{ 2 \left(1 + \frac{\lambda}{m\omega^2} - \frac{1}{8} \left(\frac{2\lambda}{m\omega^2} \right)^2 + \dots \right) \right.$$

$$\left. + \left(1 - \frac{2\lambda}{m\omega^2} - \frac{1}{8} \left(\frac{4\lambda}{m\omega^2} \right)^2 + \dots \right) \right\}$$

$$= \frac{1}{2} \hbar \omega \left\{ 3 - \frac{3 \lambda^2}{m^2 \omega^4} \right\} = \frac{3}{2} \hbar \omega - \frac{3}{2} \frac{\lambda^2 \hbar}{m^2 \omega^2}$$

National Brand
MOSHEETS FILLED 1.5 SQUARE
SQUARES 1.5 FISH 3.5 SQUARE
ROSELINE LIGHT 3.5 SQUARE
RECTIFIED WHITE 3.5 SQUARE
20 RECYCLED WHITE 3.5 SQUARE
Made in U.S.A.

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Problem 7 (PART I, SECTION 4)

UCSD scientists discover a new material and name it Bobdynesium in honor of our Chancellor. The internal energy of Bobdynesium satisfies the following relation:

$$E(S, V, N) = \frac{a S^5}{V^2 N^2} ,$$

where S is entropy, V volume, and N particle number. The numerical value of the constant a is 2.00×10^{48} in MKS units.

- (a) What are the MKS units of the constant a ?
- (b) Derive the analog of the ideal gas law for this system – an equation of state relating p , T , N , and V .
- (c) How much work is required to isothermally expand 3.00 moles of Bobdynesium from $V_i = 2.00 \text{ m}^3$ to $V_f = 3.00 \text{ m}^3$ at a temperature of $T = 300 \text{ K}$? Recall $N_0 = 6.02 \times 10^{23}$.
- (d) At a pressure of $p = 1.00 \times 10^5 \text{ Pa}$, a quantity of Bobdynesium is placed in a sealed chamber. The chamber is thermally insulated so that no heat is exchanged between sample and environment. Additional pressure is applied and the change in volume is recorded. What is the measured compressibility $\kappa = -V^{-1} \partial V / \partial p$?

Solution(a) Since $[S] = \text{J/K}$,

$$[a] = \frac{\text{K}^5 \text{m}^6}{\text{J}^4} = \frac{\text{K}^5 \text{s}^8}{\text{kg}^4 \text{m}^2} .$$

(b) From $E = aS^5/V^2N^2$ and

$$dE = \bar{S} dT - p dV + \mu dN$$

we derive

$$\begin{aligned} T &= \frac{\partial E}{\partial S} \Big|_{V,N} = \frac{5aS^4}{V^2N^2} \\ p &= -\frac{\partial E}{\partial V} \Big|_{V,N} = \frac{2aS^5}{V^3N^2} . \end{aligned}$$

The first of these gives

$$S = \left(\frac{TV^2N^2}{5a} \right)^{1/4}$$

hence

$$p = \frac{2}{5^{5/4}\sqrt[4]{a}} T^{5/4} V^{-1/2} N^{1/2} .$$

(c) Now we compute

$$\begin{aligned} W_{i-f} &= \int_{V_i}^{V_f} p dV \\ &= \frac{2}{5^{5/4}\sqrt[4]{a}} T^{5/4} N^{1/2} \int_{V_i}^{V_f} dV V^{-1/2} \\ &= \frac{4}{5^{5/4}\sqrt[4]{a}} T^{5/4} \sqrt{N} \left(\sqrt{V_f} - \sqrt{V_i} \right) . \end{aligned}$$

Plugging in $N = 3.00N_0$, $V_i = 2.00 \text{ m}^3$, $V_f = 3.00 \text{ m}^3$, and $T = 300 \text{ K}$, we obtain $W_{i-f} = 240 \text{ J}$.(d) Sealed means no particle transfer, i.e. $dN = 0$, and no heat exchanged means adiabatic, i.e. $dS = 0$. From the relation for $p(S, V, N) = -(\partial E / \partial V)_{S,N}$, we find $p \propto V^{-3}$, so the (adiabatic) compressibility is

$$\kappa_s = -\frac{1}{V} \frac{\partial V}{\partial p} \Big|_{S,N} = \frac{1}{3p} .$$

Thus, the measured compressibility is $3.33 \times 10^{-6} (\text{Pa})^{-1}$.

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Problem 8 (PART I, SECTION 4)

A magnetic system consists of N independent spins σ_i , each of which can take one of three values: $\sigma = -1, 0, +1$. The system is in a magnetic field H , and the Hamiltonian is

$$\mathcal{H} = -\mu_0 H \sum_i \sigma_i ,$$

where $\mu_0/k_B = 5.00 \times 10^{-3}$ K/T. (Recall 1 T = 10^4 G).

- (a) Initially the system is in equilibrium at $T = 1.00$ mK (10^{-3} K) in a field of $H = 1.00$ kG = 0.100 T. What is the average $\langle \sigma \rangle$?
- (b) The system is then adiabatically demagnetized by reducing the external field to $H = 10$ G. What is the final temperature?
- (c) Suppose that instead of the adiabatic demagnetization, the system were isothermally demagnetized at $T = 1.00$ mK from $H_i = 0.100$ T to $H_f = 0$ T. Compute the change in the free energy in Joules per particle. Recall $k_B = 1.38 \times 10^{-32}$ J/K.

Solution

(a) The thermal average $\langle \sigma \rangle$ is given by

$$\begin{aligned}\langle \sigma \rangle &= \frac{\exp(\mu_0 H/k_B T) - \exp(-\mu_0 H/k_B T)}{\exp(\mu_0 H/k_B T) + 1 + \exp(-\mu_0 H/k_B T)} \\ &= \frac{2 \sinh(\mu_0 H/k_B T)}{1 + 2 \cosh(\mu_0 H/k_B T)}.\end{aligned}$$

Substituting

$$\begin{aligned}\mu_0 H/k_B T &= \frac{(5.00 \times 10^{-3} \text{ K/T}) \cdot (0.100 \text{ T})}{10^{-3} \text{ K}} \\ &= \frac{1}{2},\end{aligned}$$

we have

$$\begin{aligned}\langle \sigma \rangle &= \frac{2 \sinh(\frac{1}{2})}{1 + 2 \cosh(\frac{1}{2})} \\ &= 0.320.\end{aligned}$$

(b) The specific entropy S/N is a function of the dimensionless ratio $\mu_0 H/k_B T$. Thus, $\mu_0 H/k_B T$ is a constant for adiabatic processes. A hundredfold adiabatic decrease from $H_i = 10^3 \text{ G}$ to $H_f = 10 \text{ G}$ results in a concomitant temperature decrease by the same factor. Thus, if $T_i = 1 \text{ mK}$, then $T_f = 10 \mu\text{K}$.

(c) Now we need the entropy. The partition function is

$$\begin{aligned}Z &= \text{Tr } e^{-H/k_B T} \\ &= \left(e^{\mu_0 H/k_B T} + 1 + e^{-\mu_0 H/k_B T} \right)^N\end{aligned}$$

hence

$$\begin{aligned}F &= -k_B T \ln Z \\ &= -N k_B T \ln(1 + 2 \cosh x).\end{aligned}$$

where $x \equiv \mu_0 H/k_B T$. Thus, $x_i = \frac{1}{2}$ and $x_f = 0$, and the free energy change per particle is

$$\begin{aligned}\frac{\Delta F}{N} &= k_B T \ln \left(\frac{1 + 2 \cosh x_i}{1 + 2 \cosh x_f} \right) \\ &= k_B T \ln \left(\frac{1}{3} + \frac{2}{3} \cosh \frac{1}{2} \right) \\ &= 0.08166 k_B T = 1.127 \times 10^{-27} \text{ J/particle}.\end{aligned}$$

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Problem 9 (PART I, SECTION 5)

- (a) The period T of the Earth's rotation can be measured with radio interferometry (Very Long Baseline Interferometry). The position of a radio quasar is measured twice with a baseline of 10,000 km at a frequency of 8.4 GHz. An atomic clock is used to ensure that the second measurement is made exactly 24 hours after the first. What is the relative precision $\Delta T/T$ for this measurement? What if the second measurement is one year later?
- (b) Show that the moment of inertia of a homogeneous sphere is $\frac{2}{5}MR^2$ and estimate the Earth's moment of inertia. (*Hint:* $\sin^3 x = \sin x - \sin x \cos^2 x$)
- (c) At latitude α a mass m is lifted from the surface of the Earth to a height h . How much does the Earth's rotation rate change? (Assume $h \ll R_e$.)
- (d) Which of the following events can be detected by the radio interferometer?
- (i) a major volcano eruption ($V = 10 \text{ km}^3$, $h = 2 \text{ km}$)
 - (ii) construction of the Great Wall (5000 km long \times 6 m wide \times 10 m high)
 - (iii) cooling of the atmosphere by 30°C
 - (iv) the ice age ($5 \times 10^7 \text{ km}^2$ of ice, 3 km thick)
 - (v) plant growth in the Northern summer ($30 \times 10^6 \text{ km}^2$ of forest, 10^4 trees/km^2 , 1 m^3 of wood per tree)

Solution

9.1

$$a) \Delta\varphi = \frac{\lambda}{D} = \frac{c}{vD} = 3.6 \text{ nrad} = 740 \mu\text{s}$$

$$\text{Globus } 15 \text{ mas} \hat{=} 1 \text{ ms} \Rightarrow \Delta\varphi \hat{=} 50 \mu\text{s}$$

$$\frac{\Delta T}{T} = \frac{50 \mu\text{s}}{86400 \text{s}} = 5 \cdot 10^{-10} \text{ for 1 day}, \quad \frac{\Delta T}{T} = \frac{50 \mu\text{s}}{3 \cdot 10^7 \text{s}} = 1.6 \cdot 10^{-12} \text{ for 1 year}$$

$$\begin{aligned} b) J &= \int_0^{2\pi} \int_0^R \int_0^r \rho r^2 \sin^2 \vartheta^2 dV \\ &= \int_0^{2\pi} \int_0^R \int_0^r \rho r^4 \sin^3 \vartheta d\vartheta dr d\vartheta \\ &= 2\pi \rho \int_0^R r^4 dr \cdot \int_0^{2\pi} \sin^3 \vartheta d\vartheta \\ &= 2\pi \rho \left(\int_0^{2\pi} \sin^2 \vartheta d\vartheta - \int_0^{2\pi} \cos^2 \vartheta d\vartheta \right) \int_0^R r^4 dr \\ &= 2\pi \rho \left(\left[-\cos \vartheta \right]_0^{2\pi} - \left[-\frac{1}{3} \cos^3 \vartheta \right]_0^{2\pi} \right) \left[\frac{1}{5} r^5 \right]_0^R \\ &= 2\pi \rho \left(2 - \frac{2}{3} \right) \cdot \frac{1}{5} R^5 \\ &= \frac{8}{15} \pi \rho R^5 \\ &= \frac{2}{5} M R^2 \end{aligned}$$

$$\begin{aligned} J_{\oplus} &= \frac{2}{5} \cdot 6 \cdot 10^{24} \text{ kg} \cdot (6.4 \cdot 10^6 \text{ m})^2 \\ &= 10^{38} \text{ kg m}^2 \end{aligned}$$

alternatively:

$$\begin{aligned} J_{\oplus} &= \frac{8}{15} \pi \cdot 5.5 \text{ g cm}^{-3} \cdot (6.4 \cdot 10^6 \text{ m})^5 \\ &= 10^{38} \text{ kg m}^2 \end{aligned}$$

$$c) \Delta J = (m(R+h)^2 - mR^2) \cos\varphi \approx 2mrRh \cos\varphi \quad q_2$$

conservation of angular momentum:

$$J\omega = \text{const.} \Rightarrow \frac{J}{T} = \text{const.} \Rightarrow \frac{\Delta J}{J} = -\frac{\Delta T}{T} \Rightarrow$$

$$\frac{\Delta T}{T} = -\frac{2mrRh \cos\varphi}{J}$$

d) (i) $3 \cdot 10^{13} \text{ kg}$ $\frac{\Delta T}{T} \approx -5 \cdot 10^{-15}$ not detectable

(ii) 10^{12} kg $\frac{\Delta T}{T} \approx -5 \cdot 10^{-13}$ not detectable

(iii) the scale height of the atmosphere drops from 8 to 7.2 km. Over 1 cm there is one kg of air.

$5 \cdot 10^{18} \text{ kg}$ $\frac{\Delta T}{T} \approx +2 \cdot 10^{-10}$ detectable within 1 week

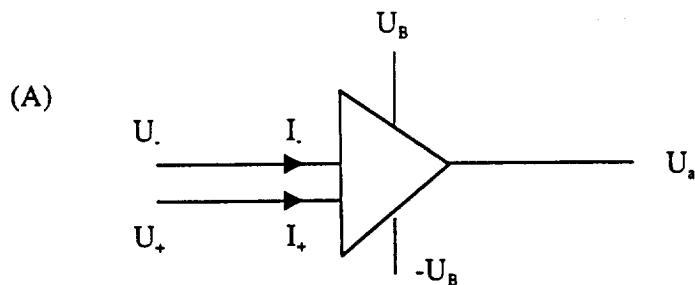
(iv) $3 \cdot 10^{15} \text{ kg}$ $\frac{\Delta T}{T} \approx -2 \cdot 10^{-9}$ detectable within 1 day

(v) $3 \cdot 10^{14} \text{ kg}$ $\frac{\Delta T}{T} \approx -5 \cdot 10^{-16}$ not detectable

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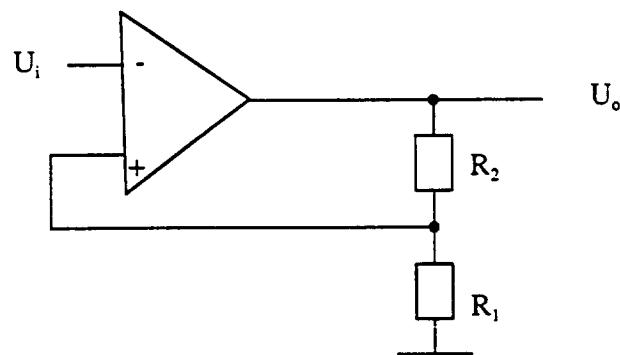
Problem 10 (PART I, SECTION 5)

An Operational Amplifier (OpAmp, see figure (A)) acts as a difference amplifier, i.e.,

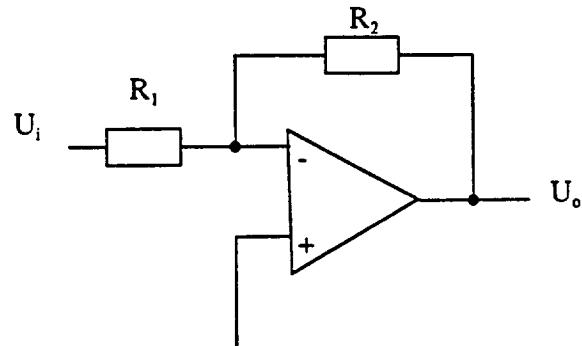
$$U_o = A_o(U_+ - U_-)$$

An ideal OpAmp has "infinite" input impedance and "infinite" amplification A_o . We consider the circuits (B) through (G) on the attached sheet.

(B)



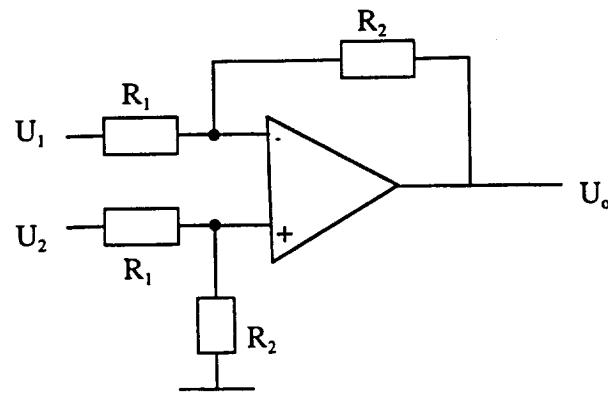
(C)



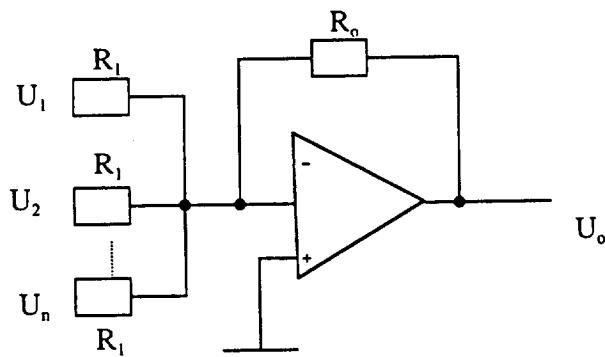
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Problem 10 (part I, Section 5) CONTINUED

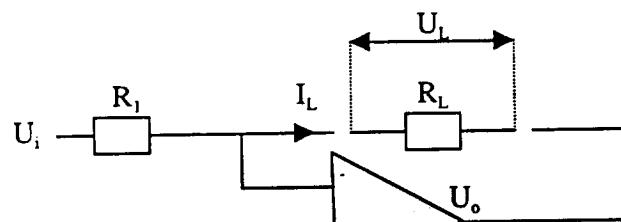
(D)



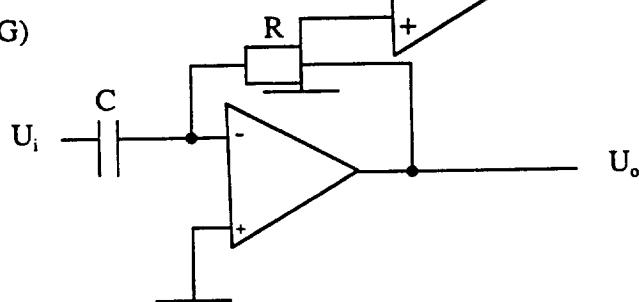
(E)



(F)



(G)



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Problem 10 (part I, Section 5) CONTINUED

Answer the following questions:

- (a) Which conditions on U_+ , U_- , I_+ , I_- hold in an "ideal" OpAmp?
- (b) How does U_o depend on U_i in (B)?
- (c) How does U_o depend on U_i in (C)?
- (d) How does U_o depend on U_1 and U_2 in (D)?
- (e) How does U_o depend on U_1, U_2, \dots, U_n in (E)?
- (f) How does I_L depend on R_L in (F)?
- (g) How does U_o depend on U_i in (G)?
- (h) Assign the labels (1) "adder", (2) "inverting amplifier", (3) "constant-current supply", (4) "subtracting amplifier", (5) "differentiating circuit", (6) "non-inverting amplifier" to the circuits (B) through (G).

Hint: The answer to question (a) is the key to all other questions.

(1V)

a) $U_+ = U_- , \quad I_+ = I_- = 0$

b) $\left. \begin{array}{l} U_+ - U_- = U_i \\ U_- = \frac{R_1}{R_1 + R_2} U_o \end{array} \right\} \Rightarrow U_o = \left(1 + \frac{R_2}{R_1}\right) U_i$

c) $\left. \begin{array}{l} U_+ - U_- = 0 \\ I_- = 0 \end{array} \right\} \Rightarrow \frac{U_i}{R_1} = - \frac{U_o}{R_2} \Rightarrow U_o = - \frac{R_2}{R_1} U_i$

d) $\left. \begin{array}{l} \frac{U_+ - U_-}{R_1} = \frac{U_- - U_o}{R_2} \\ U_- = U_+ = \frac{R_2}{R_1 + R_2} U_2 \end{array} \right\} \Rightarrow U_o = \frac{R_2}{R_1} U_1 - \left(1 + \frac{R_2}{R_1}\right) U_- = \frac{R_2}{R_1} (U_1 - U_2)$

e) $\left. \begin{array}{l} U_+ = U_- = 0 \\ I_- = 0 \end{array} \right\} \Rightarrow \frac{U_1}{R_1} + \frac{U_2}{R_1} + \dots + \frac{U_n}{R_1} = - \frac{U_o}{R_o} \Rightarrow U_o = - \frac{R_o}{R_1} (U_1 + \dots + U_n)$

f) $U_+ = U_- = 0 \Rightarrow \frac{U_1}{R_1} = - \frac{U_o}{R_L} = I_L$

I_L depends only on U_1, R_1 , not on R_L .

g) $U_+ - U_- = 0 \Rightarrow Q = C U_i \Rightarrow I_c = \dot{Q} = C \dot{U}_i \quad \left. \begin{array}{l} \\ \\ I_- = 0 \Rightarrow I_c = I_R \end{array} \right\} \Rightarrow$

$C \dot{U}_i = - \frac{U_o}{R} \Rightarrow U_o = - R C \dot{U}_i$

h) 1E, 2C, 3F, 4D, 5G, 6B

PART II

PHYSICS DEPARTMENTAL EXAMINATION SPECIAL INSTRUCTIONS

Please take a few minutes to read through all problems before starting the exam. The proctor of the exam will attempt to clarify example questions if you are uncertain about them. Please attempt seven (7) of the (10) questions. The questions are grouped in five Sections. You must attempt at least one question from each of the five (5) Sections. E.g. Section 1: problem 1 or problem 2. Partial credit will be given for partial solutions for seven (7) questions only. Please indicate with a "check" which of the (7) questions you wish to be graded below:

Section 1:	Problem 11 _____	Problem 12 _____
Section 2:	Problem 13 _____	Problem 14 _____
Section 3:	Problem 15 _____	Problem 16 _____
Section 4:	Problem 17 _____	Problem 18 _____
Section 5:	Problem 19 _____	Problem 20 _____

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PART II

Score: _____

Problem 11 (PART II, SECTION 1)

A unsuspecting solid spherical planet of mass M_0 rotates with angular velocity ω_0 . Suddenly, a giant asteroid of mass αM_0 smashes (radially) into and sticks to the planet at a location which is at colatitude θ (*i.e.* the latitude is $90^\circ - \theta$). Millions perish. The new mass distribution is no longer spherically symmetric, so the rotational axis should precess. Recall Euler's equation

$$\frac{d\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N}^{(\text{ext})}$$

for rotations in a body-fixed frame.

- (a) What is the new inertia tensor I_{ij} along principle center-of-mass frame axes? Don't forget that the CM is no longer at the center of the sphere! Recall $I = \frac{2}{5}MR^2$ for a solid sphere.
- (b) What is the period of precession of the rotational axis in terms of the original length of the day $2\pi/\omega_0$? *Hint:* One of the components of the angular velocity $\boldsymbol{\omega}$ is conserved.

Solution

Let's choose body-fixed axes with \hat{z} pointing from the center of the planet to the smoldering asteroid.

The CM lies a distance

$$d = \frac{\alpha M_0 \cdot R + M_0 \cdot 0}{(1 + \alpha)M_0} = \frac{\alpha}{1 + \alpha} R$$

from the center of the sphere. Thus, relative to the center of the sphere, we have

$$I = \frac{2}{5}M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where we have used the definition

$$I_{ij} = \int d^3r \rho(\vec{r}) (\vec{r}^2 \delta_{ij} - r_i r_j) .$$

Now we shift to a frame with the CM at the origin, using the parallel axis theorem,

$$I_{ij}(\vec{d}) = I_{ij}^{\text{CM}} + M(\vec{d}^2 \delta_{ij} - d_i d_j) ,$$

which follows immediately from the definition of I_{ij} . Thus, with $\vec{d} = d\hat{z}$,

$$\begin{aligned} I_{ij}^{\text{CM}} I &= \frac{2}{5}M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - (1 + \alpha)M_0d^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= M_0R^2 \begin{pmatrix} \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 & 0 \\ 0 & \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} . \end{aligned}$$

In the absence of external torques, Euler's equations along principal axes read

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\omega_2\omega_3 \\ I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\omega_3\omega_1 \\ I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

Since $I_1 = I_2$, $\omega_3(t) = \omega_3(0) = \omega_0 \cos \theta$ is a constant. We then obtain $\dot{\omega}_1 = \Omega\omega_2$, $\dot{\omega}_2 = -\Omega\omega_1$, with

$$\Omega = \frac{I_2 - I_3}{I_1} \omega_3 = \frac{5\alpha}{7\alpha + 2} \omega_3 .$$

The period of precession τ in units of the pre-cataclysmic day is

$$\frac{\tau}{T} = \frac{\omega}{\Omega} = \frac{7\alpha + 2}{5\alpha \cos \theta} .$$

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Problem 12 (PART II, SECTION 1)

What is the general equation for a geodesic on the surface

$$z(x, y) = \frac{2x^{3/2}}{3\sqrt{a}},$$

where a is a dimensionful constant and $x > 0$?

Solution

The distance of a path from (x_0, y_0) to (x_1, y_1) is a functional of $y(x)$:

$$\begin{aligned} D[y(x)] &= \int_{x_0}^{x_1} dx \sqrt{(dx)^2 + (dy)^2 + \left(\frac{dz}{dx}\right)^2 (dx)^2} \\ &= \int_{x_0}^{x_1} dx \sqrt{1 + \frac{x}{a} + [y'(x)]^2} \end{aligned}$$

where $y'(x) = dy/dx$. Varying with respect to $y(x)$ yields the Euler-Lagrange equations,

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right) = 0$$

where the 'Lagrangian' in this case is the integrand,

$$L(y, y', x) = \sqrt{1 + \frac{x}{a} + (y')^2} .$$

Since L is only a function of y' and x , we conclude that

$$\frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + \frac{x}{a} + (y')^2}} = C_1$$

where C_1 is a constant. Solving for y' , we obtain the first order equation

$$\frac{dy}{dx} = \frac{C_1}{\sqrt{1 - C_1^2}} \sqrt{1 + \frac{x}{a}} ,$$

which may be integrated easily to give the general result

$$y(x) = \frac{C_1}{\sqrt{1 - C_1^2}} \frac{2}{3\sqrt{a}} (a + x)^{3/2} + C_2 .$$

Redefining constants,

$$y(x) = A \left(1 + \frac{x}{a}\right)^{3/2} + B .$$

The constants A and B are used to set $y(x_0) = y_0$ and $y(x_1) = y_1$.

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PART II

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Problem 13 (PART II, SECTION 2)

A metallic sphere of radius a encased in a shell of outer radius R and dielectric constant ϵ is placed in an external electric field $\mathbf{E} = E\hat{x}$. (The region outside the dielectric is vacuum.) What is the maximum field amplitude on the surface of the conductor?

Recall that the general solution to Laplace's equation is

$$\phi(r, \theta) = \sum_m \left[A_m r^m + B_m r^{-(m+1)} \right] P_m(\cos \theta),$$

and that $P_1(x) = x$. Another potentially useful factoid:

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r \sin \theta} \frac{\partial}{\partial \varphi}.$$

Solution

The general azimuthally symmetric solution to Laplace's equation is

$$\phi(r, \theta) = \sum_m \left[A_m r^m + B_m r^{-(m+1)} \right] P_m(\cos \theta),$$

where $P_m(x)$ is a Legendre polynomial. Divide space into regions I ($a < r < R$) and II ($r > R$). As $r \rightarrow \infty$, the potential is

$$\phi_{\text{II}}(r \rightarrow \infty, \theta) = -Ez = -Er \cos \theta,$$

hence we only need consider the $m = 1$ term in the general series. We can now write

$$\begin{aligned}\phi_I(r, \theta) &= Ar \cos \theta + \frac{B}{r^2} \cos \theta \\ \phi_{\text{II}}(r, \theta) &= Cr \cos \theta + \frac{D}{r^2} \cos \theta.\end{aligned}$$

The corresponding electric fields are then

$$\begin{aligned}\vec{E}_I &= \left(-A + \frac{2B}{r^3} \right) \cos \theta \hat{r} + \left(A + \frac{B}{r^3} \right) \sin \theta \hat{\theta} \\ \vec{E}_{\text{II}} &= \left(-C + \frac{2D}{r^3} \right) \cos \theta \hat{r} + \left(C + \frac{D}{r^3} \right) \sin \theta \hat{\theta}.\end{aligned}$$

Examining the $r \rightarrow \infty$ limit gives $C = -E$. We need to solve for A , B , and D .

Now we apply boundary conditions at $r = R$. We know that $\epsilon \vec{E} \cdot \hat{n}$ and $\vec{E} \times \hat{n}$ are continuous, where $\hat{n} = \hat{r}$ is the surface normal. This gives us two equations,

$$\begin{aligned}\epsilon \left(-A + \frac{2B}{R^3} \right) &= E + \frac{2D}{R^3} \\ A + \frac{B}{R^3} &= -E + \frac{D}{R^3}.\end{aligned}$$

At the surface of the conductor ($r = a$), $\vec{E} \times \hat{r}$ must vanish, which says $B = -a^3 A$. Solving for A , we find

$$A = \frac{-3E}{(\epsilon + 2) + \frac{2a^3}{R^3}(\epsilon - 1)}.$$

On the surface of the conductor, $\vec{E} = -3A \cos \theta \hat{r}$, so

$$E_{\text{max}} = 3A = \frac{9E}{(\epsilon + 2) + \frac{2a^3}{R^3}(\epsilon - 1)}.$$

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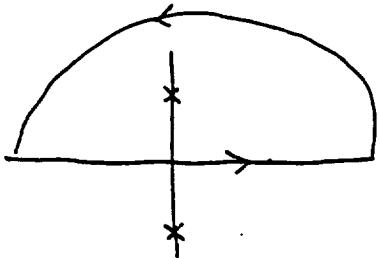
Problem 14 (PART II, SECTION 2)

Evaluate the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{(k^2 + 4)^2}$$

(?) $I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k^2+4)^2}$ has double poles at $k=\pm 2i$

if $x > 0$ can complete contour in the upper half plane



- if $f(z)$ has a double pole at z_0 , the residue is
 $\left[\frac{d}{dz} (z-z_0)^2 f(z) \right]_{z=z_0}$

so we need

$$\begin{aligned} & \frac{d}{dk} \left. \frac{(k-2i)^2 e^{ikx}}{(k^2+4)^2} \right|_{k=2i} \\ &= \left. \frac{d}{dk} \frac{e^{ikx}}{(k+2i)^2} \right|_{k=2i} = \frac{ix e^{-2x}}{(4i)^2} - \frac{2e^{-2x}}{(4i)^3} \end{aligned}$$

$$I = \frac{1}{2\pi i} \times \text{residue} = \frac{1}{2\pi i} e^{-2x} \left[-\frac{ix}{16} - \frac{2i}{64} \right]$$

$$= -\frac{e^{-2x}}{\pi} \left[\frac{x}{32} + \frac{1}{64} \right]$$

if $x < 0$ can compute using contour in the lower half-plane

$$\begin{aligned} I &= \left(-\frac{1}{2\pi i} \right) \left. \frac{d}{dk} \frac{e^{ikx}}{(k-2i)^2} \right|_{k=-2i} = -\frac{1}{2\pi i} \left[\frac{ix e^{2x}}{(-4i)^2} - \frac{2e^{2x}}{(-4i)^3} \right] \\ &= \frac{e^{2x}}{\pi} \left[-\frac{x}{32} + \frac{1}{64} \right] \end{aligned}$$

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PART II

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Problem 15 (PART II, SECTION 3)

Tritium β -decay $T \rightarrow He_3^+ + e^- + \bar{\nu}_e$ is used to place an upper limit of 3 eV on the mass of the electron neutrino. Assume that the β -decay of tritium happens instantaneously on the time scale of atomic physics, and that the electron produced in the β -decay is ejected from the system. The reaction can then be thought of as a tritium atom converting to singly ionized He_3^+ . Assume that the tritium atom is originally in its ground state. What is the probability that the He_3^+ ion is also in its ground state? (The tritium nucleus has charge e , with a mass $3m_p$, where m_p is the proton mass.)

(2) Initial wavefunction $\psi = 1s$ state of Hydrogen

$$= 2 \left(\frac{z}{a_0} \right)^{3/2} e^{-zr/a_0} Y_{00} \text{ with } z=1$$

Final wavefunction = $1s$ state of He_3^+

= same with $z \rightarrow 2$.

~~REDO REDO~~

\therefore amplitude that state remains in the ground state

$$\langle 1s He_3^+ | 1s H \rangle = \int_0^\infty r^2 dr \int d\Omega \cdot \left[2 \left(\frac{1}{a_0} \right)^{3/2} e^{-r/a_0} Y_{00} \right]$$

$$\times \left[2 \left(\frac{2}{a_0} \right)^{3/2} e^{-2r/a_0} Y_{00} \right]^*$$

$$= \int_0^\infty r^2 dr \cdot 4 e^{-3r/a_0} \underbrace{\frac{2^{3/2}}{a_0^3} \int d\Omega |Y_{00}|^2}_1$$

$$= 4 \cdot \frac{2^{3/2}}{a_0^3} \cdot \frac{2}{(3/a_0)^3} = \frac{16\sqrt{2}}{27}$$

$$\therefore \text{Probability} = \left| \frac{16\sqrt{2}}{27} \right|^2 = 0.7$$

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PART II

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Problem 16 (PART II, SECTION 3)

A three-dimensional harmonic oscillator with Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2 \mathbf{r}^2}{2}$$

is in an excited state $|n, 0, 0\rangle$. (The states are labelled $|n_x, n_y, n_z\rangle$.) In the dipole approximation, what are the allowed final states for single-photon emission? Compute the total rate for spontaneous emission to each of the allowed final states.

④

$$H = \frac{1}{2m} (\vec{p} - \frac{e\vec{A}}{c})^2 = \frac{\vec{p}^2}{2m} - \frac{e}{2mc} (\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) + \frac{e^2 \vec{A}^2}{2mc^2}$$

$$\vec{A} = \int \sqrt{\frac{2\pi\hbar}{\omega_k}} c [\vec{e}_k a_k e^{-ik \cdot x} + \vec{e}_k^\dagger a_k^\dagger e^{ik \cdot x}] \frac{d^3 k}{(2\pi)^3}$$

where ω_k = frequency of the photon

single photon emission: take the \vec{a} term linear
in \vec{A} , and for dipole approximation $e^{ik \cdot x} \approx 1$.

$$\therefore H_{\text{int}} = -\frac{e}{2mc} \cancel{\int d^3 k} \int \frac{d^3 k}{(2\pi)^3} \sqrt{\frac{2\pi\hbar}{\omega_k}} c (\vec{p} \cdot \vec{e}_k) a_k$$

is the relevant term.

By Fermi's Golden rule, the decay rate is

$$\Gamma = \frac{2\pi}{\hbar} \delta(\hbar\omega + E_f - E_i) |\langle f | H_{\text{int}} | i \rangle|^2 \times \text{density of states}$$

The atomic matrix element is

$$\langle n'_x n'_y n'_z | \vec{p} | \cancel{n_0 0} n_0 0 \rangle$$

ω_0 = oscillator frequency

A, A^\dagger = oscillator operators

$$= i \sqrt{\frac{m\omega_0}{2}} \langle n'_x n'_y n'_z | (\vec{A}^\dagger - \vec{A}) | n_0 0 \rangle$$

The final state must have energy less than the initial state, so only the A term is relevant

$$n'_x = n-1$$

$$n'_y = 0$$

$$n'_z = 0$$

$$\text{and } \langle n-1 0 0 | \vec{p} | n 0 0 \rangle = \hat{x} i \sqrt{\frac{m\omega_0}{2}} \sqrt{n}$$

$$\Gamma = \sum_{\text{pol}} \int \frac{2\pi}{k} \delta(\omega_k - \omega_0) \frac{e^2}{m^2} \frac{2\pi\hbar}{\omega_k} \frac{m\omega_0\hbar}{2} n |\hat{x} \cdot \epsilon_k|^2 \frac{d^3k}{(2\pi)^3}$$

$$\text{Now } d^3k = k^2 dk d\Omega \quad k = \omega/c$$

$$\Gamma = \sum_{\text{pol}} \int \frac{2\pi}{k^2} \delta(\omega_k - \omega_0) \frac{e^2}{m^2} \frac{2\pi\hbar}{\omega_k} \frac{m\omega_0\hbar}{2} n \frac{\omega^2 d\omega d\Omega}{(2\pi)^3} |\hat{x} \cdot \epsilon_k|^2$$

The δ function sets the photon frequency $\omega_k = \omega_0$ on doing the ω integral

$$\Gamma = \sum_{\text{pol}} \int d\Omega \frac{1}{4\pi} \frac{e^2 \omega_0^2 n}{m} |\hat{x} \cdot \epsilon_k|^2$$

$$\text{The integral } \int d\Omega |\hat{x} \cdot \epsilon_k|^2 = \frac{1}{3} \cdot 4\pi$$

and $\sum_{\text{pol}} = \text{factor of 2}$

$$\therefore \Gamma = \frac{2}{3} \frac{e^2 \omega_0^2 n}{m}$$

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Problem 17 (PART II, SECTION 4)

The free energy of mixing of a polymer solution is, in the mean field approximation, given by the formula

$$\frac{a^3 F_{\text{mix}}}{V k_B T} = \frac{1}{N} \phi \ln \phi + (1 - \phi) \ln(1 - \phi) + \chi \phi(1 - \phi).$$

In the above formula, V is the volume of the system, a^3 is the volume per *monomer*, N is the length of the *polymer* (the number of constituent monomers), and ϕ is the dimensionless *monomer* concentration (the number of monomers per volume a^3). χ is the polymer-solute interaction parameter (the solute concentration is $(1 - \phi)$).

- (a) What is the number of polymers N_p in terms of V , a , and N .
- (b) The osmotic pressure Π is defined by

$$\Pi = -\left. \frac{\partial F_{\text{mix}}}{\partial V} \right|_{N_p},$$

i.e. the variation of the free energy of mixing with volume *holding the number of polymers constant*. Derive the equation of state $\Pi = \Pi(\phi, T, \chi)$ (your expression may involve the volume unit a^3).

- (c) Examine and comment on the limit $\phi \rightarrow 0$.
- (d) For $N^{-1} \ll \phi \ll 1$, derive an expression for Π .

Solution

(a) The number of polymers is the number of monomers divided by N :

$$N_p = \frac{V\phi}{a^3 N} .$$

(b) We must compute

$$\Pi = -\left. \frac{\partial F_{\text{mix}}}{\partial V} \right|_{N_p} .$$

We use

$$\begin{aligned} \left. \frac{\partial}{\partial V} \right|_{N_p} &= \left. \frac{\partial \phi}{\partial V} \frac{\partial}{\partial \phi} \right|_{N_p} \\ &= -\left. \frac{\phi}{V} \frac{\partial}{\partial \phi} \right|_{N_p} = -\left. \frac{\phi^2}{N N_p a^3} \frac{\partial}{\partial \phi} \right|_{N_p} . \end{aligned}$$

Now

$$F_{\text{mix}} = \frac{N N_p k_B T}{\phi} \left\{ \frac{1}{N} \phi \ln \phi + (1 - \phi) \ln(1 - \phi) + \chi \phi(1 - \phi) \right\}$$

so

$$\begin{aligned} \Pi &= -\frac{k_B T \phi^2}{a^3} \frac{\partial}{\partial \phi} \left[\frac{1}{N} \ln \phi + (\phi^{-1} - 1) \ln(1 - \phi) + \chi(1 - \phi) \right] \\ &= \frac{k_B T}{a^3} [(N^{-1} - 1) \phi - \ln(1 - \phi) - \chi \phi^2] . \end{aligned}$$

(c) Taking $\phi \rightarrow 0$ we obtain

$$\Pi = \frac{\phi}{a^3 N} k_B T .$$

This is simply the ideal gas law for polymers – the density of polymers is $n_p = N_p/V = \phi/a^3 N$.

Makes sense!!

(d) For $N^{-1} \ll \phi \ll 1$, we expand the logarithm and obtain

$$\begin{aligned} \frac{a^3 \Pi}{k_B T} &= \frac{1}{N} \phi + \frac{1}{2}(1 - 2\chi)\phi^2 + \mathcal{O}(\phi^3) \\ &\approx \frac{1}{2}(1 - 2\chi)\phi^2 . \end{aligned}$$

Note that $\Pi > 0$ only if $\chi < \frac{1}{2}$, which is the condition for a “good solvent”.

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Problem 18 (PART II, SECTION 4)

Consider an interface between a liquid of density ρ_L and a gas of density ρ_G . The height $z(x, y)$ of the surface is a function of transverse coordinates (x, y) . The energy is

$$E = \int dx \int dy \left[\frac{1}{2}(\rho_L - \rho_G) g z^2 + \frac{1}{2}\sigma (\vec{\nabla} z)^2 \right] ,$$

where g is the acceleration due to gravity and σ is the surface tension.

- (a) Explain why the gravitational potential energy density is $\frac{1}{2}(\rho_L - \rho_G) g z^2$.
- (b) Compute the mean square height ξ , defined by

$$\xi^2 = \frac{1}{A} \int dx \int dy \langle z^2(x, y) \rangle$$

where $\langle z^2 \rangle$ represents a (classical) thermal average and A is the area of the system.

Hint: Assume periodic boundary conditions on a $L_x \times L_y$ rectangle,

$$z(x + L_x, y) = z(x, y + L_y) = z(x, y) ,$$

in which case one can write the Fourier transform

$$z(\mathbf{r}) = \frac{1}{\sqrt{L_x L_y}} \sum_{\mathbf{q}} \hat{z}_{\mathbf{q}} \exp(i\mathbf{q} \cdot \mathbf{r}) ,$$

where

$$\mathbf{q} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y} \right) .$$

with integer n_x and n_y . Further assume $|\mathbf{q}| < q_0 \equiv 2\pi/a_0$, where a_0 is a molecular diameter.

Solution

(a) The potential energy density is

$$\begin{aligned} \mathcal{U}(z, y) &= \int dz' \Delta\rho(z') g z' \\ &= \int_0^z dz' (\rho_L - \rho_G) g z' \\ &= \frac{1}{2}(\rho_L - \rho_G) g z^2. \end{aligned}$$

Here, $\Delta\rho(z)$ is the deviation of the local density from the equilibrium value. In equilibrium, we assume $\rho = \rho_L$ for $z < 0$ and $\rho = \rho_G$ for $z > 0$.

(b) In terms of $\dot{z}_{\vec{q}}$,

$$\begin{aligned} E &= \int dx \int dy \left[\frac{1}{2}(\rho_L - \rho_G) g z^2 + \frac{1}{2}\sigma(\vec{\nabla}z)^2 \right] \\ &= \frac{1}{2} \sum_{\vec{q}} \left[(\rho_L - \rho_G) g + \sigma \vec{q}^2 \right] |\dot{z}_{\vec{q}}|^2. \end{aligned}$$

The thermal expectation of $|\dot{z}_{\vec{q}}|^2$ is

$$\langle |\dot{z}_{\vec{q}}|^2 \rangle = \frac{k_B T}{\Omega_{\vec{q}}}$$

where

$$\Omega_{\vec{q}} \equiv (\rho_L - \rho_G) g + \sigma \vec{q}^2.$$

Thus,

$$\begin{aligned} \xi^2 &= \frac{1}{L_x L_y} \sum_{\vec{q}} \langle |\dot{z}_{\vec{q}}|^2 \rangle \\ &= \int \frac{d^2 q}{(2\pi)^2} \frac{k_B T}{\Omega_{\vec{q}}} \\ &= \frac{k_B T}{4\pi\sigma} \ln \left(1 + \frac{\sigma q_0^2}{(\rho_L - \rho_G) g} \right). \end{aligned}$$

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PART II

Score: _____

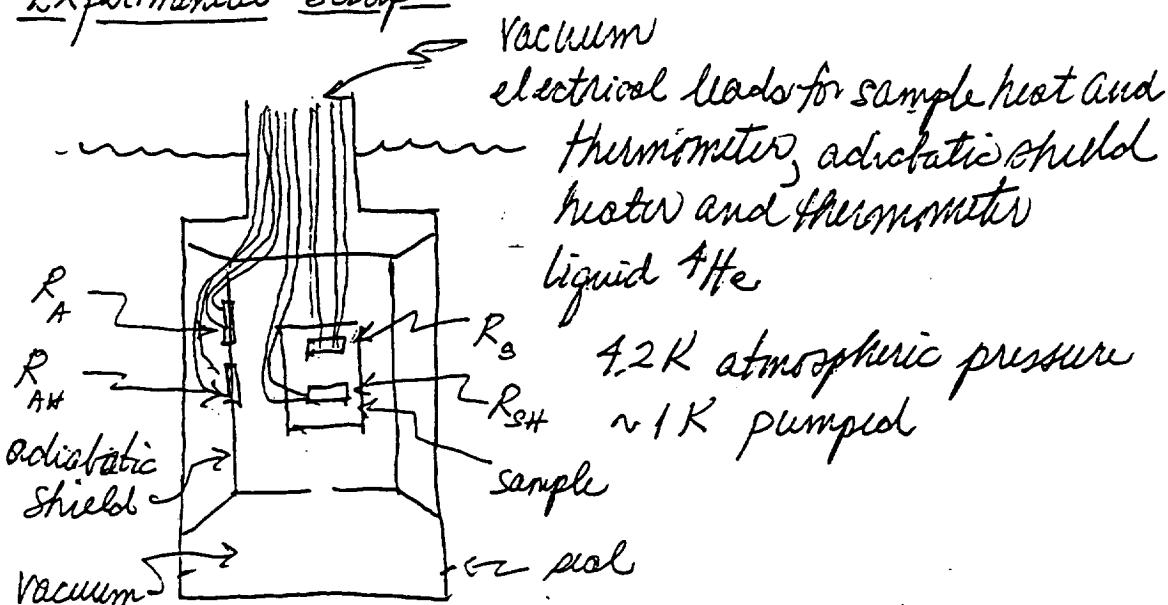
Problem 19 (PART II, SECTION 5)

Describe how you would measure the specific heat of a material at low temperatures (*i.e.*, in the range 1 K to 10 K). Make a schematic drawing of your experimental set-up and explain how you would measure the relevant quantities needed to give the specific heat as a function of temperature.

It is found that C *versus* T for a normal (non-superconducting) metal behaves as $C(T) = \gamma T + AT^3$ for low temperatures. Suppose you measure $C(T)$ for several temperatures. How would you plot your data so that you could graphically determine the coefficients γ and A ? Which information about the electronic excitations and the lattice vibrations (phonons) can be gleaned from γ and A ?

Experimental setup

National® Brand
13-401 GASKIN ELITE POLY. SQUARIC
42-101 GASKIN ELITE POLY. SQUARIC
42-102 GASKIN ELITE POLY. SQUARIC
42-103 GASKIN ELITE POLY. SQUARIC
42-104 UNPOLISHED WHITE SQUARIC
42-105 PRECUTTED WHITE SQUARIC
42-106 SQUARIC



Set temperature T_A of adiabatic shield close to temperature T_s of sample to minimize heat leak to surroundings by radiation.

Heat leak by conduction is minimized by evacuating chamber and supporting sample and shield with thin threads (cotton, nylon, etc.)

Sample

R_s - resistance thermometer
 R_{sh} - resistance heater

Adiabatic Shield

R_A - resistance thermometer
 R_{AH} - resistance heater

Sample is cooled

to low temperature

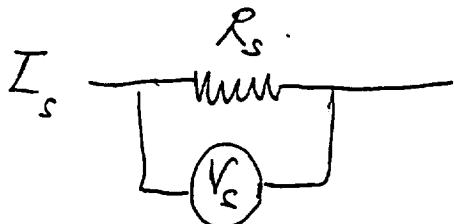
through small heat leak for long period of time or by including heat switch (mechanical, superconducting, etc.)

Heat pulse ΔQ applied by passing current I through sample resistance heater R_{SH} in time interval Δt

$$\Delta Q = P_{SH} \Delta t = I^2 R_{SH} \Delta t$$

Temperature change ΔT inferred from R_s - measured with voltmeter using 4 lead technique (2 current leads, 2 voltage leads)

i.e.,

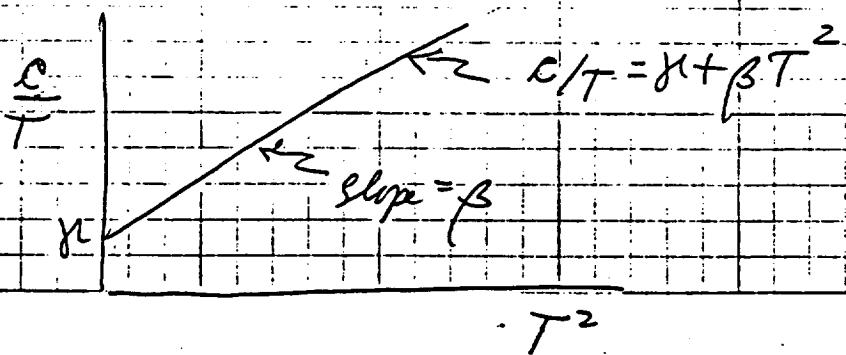


$$\text{Heat capacity : } C = \frac{\Delta Q}{\Delta T}$$

$$\text{Specific heat : } c = \frac{C}{m}; m-\text{mass}$$

Normal (nonsuperconducting) metal

$$C = C_{\text{electronic}} + C_{\text{lattice}} = C_{el} + C_{el} = \gamma T + \beta T^3$$



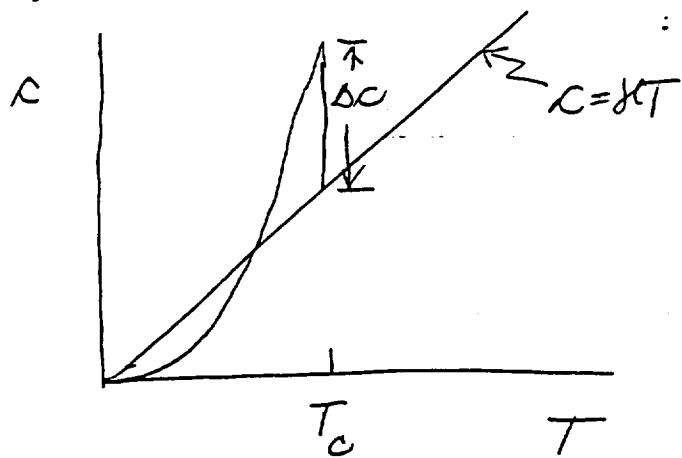
δ - electronic specific heat coefficient

$\delta \propto N(E_F)$ - density of states at Fermi level E_F

$\beta \propto \Theta_D^{-3}$; Θ_D - Debye temperature - characteristic temperature of phonons (Quantized lattice vibrations).

Superconducting metal

electronic
specific
heat \Rightarrow



$\Delta C \sim \delta T_c$; T_c - superconducting critical temperature

$$C_s \propto e^{-\frac{\Delta T}{T}} \text{ for } T < T_c$$

Δ - superconducting energy gap

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PART II

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Problem 20 (PART II, SECTION 5)

Give a rough (order-of-magnitude) estimate of the following. Show/discuss how you reach your conclusions. (Do any 5.)

- (a) The energy generated by the Sun during one second.
- (b) The number of H atoms per second that need to be converted to He (net fusion reaction $4 \text{H}^+ \longrightarrow +2\bar{\nu}_e$) to generate 1 W of fusion power.
- (c) The longest wavelength of light that can be detected with a detector made of (undoped) silicon.
- (d) The size of an organ pipe for the note a' (440 Hz).
- (e) The minimum energy of an electron that emits Čerenkov radiation in water.
- (f) The maximum field (in square arc minutes) that can be observed with a 10 m telescope using a modern CCD detector (2048×2048 pixels, pixel size is $15 \mu\text{m}$).
- (g) The optical depth of hydrogen gas heated to 2000 K in a cavity of 1 m length at 1 bar pressure, to H α -line (6563 Å) absorption.
- (h) The frequency of the lowest rotational transition of the CO molecule.

$$\begin{aligned}
 a) \quad P &= \sigma A T^4 \\
 &= \sigma \cdot 4\pi R_0^2 T_0^4 \\
 &= 5.7 \cdot 10^{-8} \text{ W m}^{-2} \text{ K}^{-4} \cdot 4\pi \cdot (700,000 \text{ km})^2 \cdot (5780 \text{ K})^4 \\
 &= 4 \cdot 10^{26} \text{ W}
 \end{aligned}$$

remarks: you can get R_0 from $\vartheta_0 = 30^\circ$ and $d_0 = 150 \cdot 10^6 \text{ km}$

you can get T_0 from $\lambda_{\max} \approx 5000 \text{ Å}$ and Wien's Law

Alternatively:

The solar constant $S = 1300 \text{ W m}^{-2}$

$$\begin{aligned}
 P &= 4\pi d_0^2 \cdot S \\
 &= 4\pi \cdot (150 \cdot 10^6 \text{ km})^2 \cdot S \\
 &= 4 \cdot 10^{26} \text{ W}
 \end{aligned}$$

- b) The reaction $4\text{H} \rightarrow \text{He} + 2e^+ + 2\nu$ generates $27 \text{ MeV} = 4.3 \cdot 10^{-12} \text{ J}$.
 Need $4 \cdot \frac{1}{4.3 \cdot 10^{-12}} = 10^{12}$ H atoms per second to generate 1W.

Alternatively:

The efficiency of H fusion is 0.7%. The H mass needed is given by

$$E = \gamma \cdot mc^2 \quad \text{We therefore get:}$$

$$E = \gamma \cdot mc^2 = \gamma \cdot n_p \cdot m_p c^2 = \gamma \cdot n_p \cdot 1837 \cdot m_e c^2 \Rightarrow$$

~~$n_p = 10^{12} \text{ cm}^{-3}$~~

$$n_p = 10 \cdot \frac{1}{0.007} \cdot \frac{1}{1837} \cdot \frac{1}{511 \text{ keV}} \cdot \frac{1}{1.6 \cdot 10^{-19} \text{ eV/J}}$$

$= 10^{12}$ H atoms per second

band gap = 1.2 eV

$$\lambda = \frac{c}{\gamma} = \frac{hc}{E} = 1 \mu m$$

Alternatively: $912 \text{ \AA} \stackrel{\circ}{=} 13.6 \text{ eV}$

$$\lambda = 912 \text{ \AA} \cdot \frac{13.6 \text{ eV}}{1.2 \text{ eV}} = 1 \mu m$$

d) open pipe: $l = \frac{\lambda}{2} = \frac{c}{2v}$

$$= \frac{230 \text{ m s}^{-1}}{2 \cdot 440 \text{ Hz}} = 37.5 \text{ cm}$$

stopped pipe: $l = \frac{\lambda}{4} = \frac{c}{4v} = 18.8 \text{ cm}$

e) $v \geq \frac{c}{n}$ or $\beta \geq \frac{1}{n}$

$$E = m_0 c^2 \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right) \geq m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{1}{n^2}}} - 1 \right)$$

$$= 511 \text{ keV} \cdot \left(\frac{1}{\sqrt{1-(\frac{3}{4})^2}} - 1 \right)$$

$$= 262 \text{ keV}$$

f) detector diagonal: $\sqrt{2} \cdot 2048 \cdot \frac{15}{28} \mu m = \frac{43}{70} \text{ mm}$

A Ω through system is constant, detector accepts $\leq 2\pi$ sterad

$$(\frac{43}{70} \text{ mm})^2 \cdot 2\pi = (10 \text{ m})^2 \cdot \Omega \Rightarrow$$

$$\Omega = \frac{1.2}{10^4} \text{ sterad} = \frac{1400}{3500} \text{ arcmin}^2$$

g) H α absorption is the transition ~~when~~ $n=2 \rightarrow n=3$. Only atoms in $n=2$ can absorb H α . This level is $\frac{3}{4} \cdot 13.6 \text{ eV} = 10.2 \text{ eV}$

above the ground state.

$$n^* = n_0 e^{-\frac{E}{kT}}$$
$$= n_0 \cdot e^{-\frac{10.2 \text{ eV}}{1.38 \cdot 10^{-23} \text{ J K}^{-1} \cdot 2000 \text{ K}}}$$
$$= n_0 \cdot 2 \cdot 10^{-26}$$

There are no atoms in the $n=2$ level. \Rightarrow The optical depth is 0.
(We don't have to know anything about the ${}^2\text{H} \leftrightarrow {}^3\text{H}_2$ equilibrium)

h) $L = I \omega = \sqrt{J(J+1)} \hbar$

$$E = \frac{1}{2} I \omega^2 = \frac{1}{2} \frac{\hbar^2}{I} J(J+1)$$

$$J=0 \Rightarrow J=1 \Rightarrow \Delta E = \frac{\hbar^2}{I}, \quad \Delta v = \frac{1}{2\pi} \frac{\hbar}{I}$$

$$I = \mu r^2 = \frac{m_1 m_2}{m_1 + m_2} r^2$$

$$\text{CO: } \mu = \frac{12 \cdot 16}{12+16} m_p, \quad r = 1.13 \text{ \AA} \Rightarrow I = 1.5 \cdot 10^{-46} \text{ kg m}^2 \Rightarrow$$

$$\Delta v = 115 \text{ GHz}$$