

Physics Departmental Examination - Fall 1997 - Part I

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PHYSICS DEPARTMENTAL EXAMINATION**SPECIAL INSTRUCTIONS**

Please take a few minutes to read through all problems before starting to work. The proctor of the exam will attempt to clarify exam questions if you are uncertain about them. It is important to make an effort on every problem even if you do not know how to solve it completely. Partial credit will be given for partial solutions.

Problem 1.

A globular cluster is a gravitationally bound association of stars.

- a) Derive a rough estimate of the relaxation time through two body interactions, that is, the time for the average star to experience one strong interaction which significantly changes its velocity. Assume that all stars have the same mass m , that the typical star velocity is v , and that the average number density of stars is n (stars/volume). Your answer should not depend on the size of the stars, which can be neglected.

Hint: A strong interactions will occur when potential energy is comparable to (or greater than) kinetic energy.

- b) Show that the escape velocity $V_e = 2v$. Use the virial theorem for a statistically steady, self-gravitating star cluster, which states that the total kinetic energy of N stars is minus 1/2 times the total gravitational potential energy.

1. Write down the gravitational potential energy for one pair of stars, with typical (not nearest neighbor) separation R , then
2. Write the virial equation in terms of m , N , v and R .
3. Write an expression for the velocity of escape from the cluster, where R can also be taken as the approximate size of the cluster.

[Note: The mean separation of a pair of stars - any pair, will be close to the size of the cluster used in (3) to get the potential energy. Hence only one size scale in the problem.]

4. Equate 2 and 3 to show $V_e = 2v$.
- c) How does the cluster evolve if the stars have a variety of masses?

Relaxation Time parts (a), (b) & (c)

Let each star have "sphere of influence" with cross-sectional area πr^2 , where r defined below

Strong interactions occur when a second star passes inside r .

define t_r = relaxation time = time between strong interactions.

cylindrical volume ~~swept~~ swept out by one star in time t_r is

$$\pi r^2 v t_r$$

where v = velocity.

By definition of t_r , $\pi r^2 v t_r = \frac{1}{n}$ stars/volume

$$\Rightarrow t_r = 1 / (\pi r^2 v n)$$

Choose r such that grav. potential energy of a pair of stars = typical random kinetic energy:

$$Gm^2/r = mv^2/2$$

$$\Rightarrow r = 2Gm/v^2$$

Then $\Rightarrow t_r = \frac{1}{\pi r^2 v n} = \frac{1}{\pi (2Gm/v^2)^2 v n} = \frac{v^3}{4\pi G^2 m^2 n}$

(1) Grav. Pot. Energy for one pair of stars = $-Gm^2/R$

(2) There are $N(N-1)/2$ pairs amongst N stars

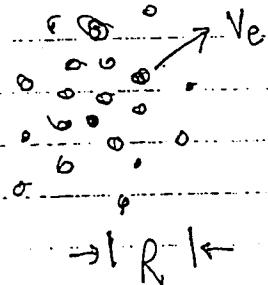
Total kinetic energy = $\frac{1}{2} \times$ total grav. pot. energy

$$Nm v^2/2 = \frac{1}{2} \frac{N(N-1)}{2} \left(\frac{-G m^2}{R} \right)$$

(3) ~~$m v_e^2/2 = G (N-1) m^2/R$~~

$$m v_e^2/2 = G [(N-1)m] m / R$$

mass of rest of cluster



(4) Re-write (3): $m v_e^2/2 = \frac{1}{4} G (N-1) m^2/R$

$$\text{Equate with (3)}: v_e^2 = 4 v^2$$

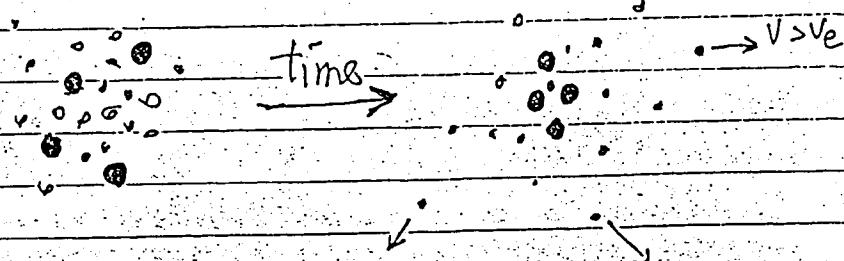
$$\Rightarrow v_e = 2v$$

In statistical equilibrium stars of all masses tend to have velocity distributions characterized by "equipartition of ~~energy~~ kinetic energy" - a general thermodynamic result.

If statistically $\frac{1}{2}mv^2$ is fixed, stars with low masses will tend to have larger v , they more often exceed the escape velocity and are lost.

preferentially. Heavier stars have lower v and sink to the core of the cluster.

The cluster core contracts, releasing binding energy, which requires the ~~loss of~~ heavy stars to move faster. They share kinetic energy with lighter stars, speeding up to "evaporation" of latter.



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Problem 2.

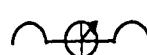
Imagine you are stranded in a desert island with your watch that measures time in seconds (very accurately), and a ruler that measures distance in centimeters. You also have two square metal plates, wire, a battery that supplies an unknown voltage, and a force meter that measures forces to very high accuracy in some unknown units. Suppose you have forgotten the values of all physical constants but not the rest of the physics you know. Explain how you would find out what the value of the speed of light is.



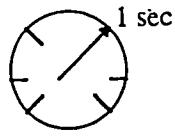
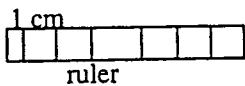
metal plates



battery
(unknown voltage)



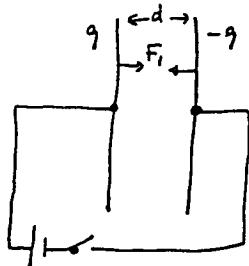
force meter
(unknown units)



watch

Solution

Put the two metal plates a distance d apart to form a parallel-plate capacitor:



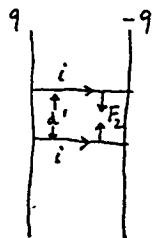
d should be much less than the side of the metal plate, $L \equiv \sqrt{A}$. Connect the battery to the metal plates for a while and then open the circuit. An unknown charge q will be on the metal plates. Measure the force between the plates, call it F_1 .

The energy of the capacitor is $U = \frac{q^2}{2C}$, capacitance $C = \frac{\epsilon_0 A}{d} \Rightarrow$

$$\Rightarrow U = \frac{q^2 d}{2\epsilon_0 A} \Rightarrow \text{force between plates, } F_1 = \frac{dU}{dd} = \frac{q^2}{2\epsilon_0 A}$$

So we measure $F_1 = \frac{q^2}{2\epsilon_0 A}$ as the force attracting one plate to the others.

Next, connect the two plates by two wires at a distance d' apart:



The capacitor will discharge through these wires. Measure the force F_2 (attractive) between these wires while the plates are discharging due to the magnetic field created by current flow.

It is given by (as function of time t):

$$F_2(t) = \frac{\mu_0 d}{2\pi d'} i^2(t)$$

with $i(t)$ the current flowing. The current is given by

$$i(t) = i_0 e^{-t/\tau} \quad \text{with } \tau \text{ a time constant } (\tau = RC \text{ but we don't need that}).$$

We can measure τ with what we have by measuring how long it takes for the force to disappear. More precisely, by measuring the ^{ratio of} forces at times t_1 and t_2 ,

$$\frac{F_2(t_1)}{F_2(t_2)} = e^{-(t_1 - t_2)/\tau}$$

$$\text{allows us to find } \tau = (t_2 - t_1) \ln \frac{F_2(t_1)}{F_2(t_2)}$$

We can find i_0 in terms of the initial charge q of the capacitor plates:

$$q = 2 \int_0^\infty dt i(t) = 2i_0 \int_0^\infty dt e^{-t/\tau} = 2i_0 \tau \Rightarrow i_0 = \frac{q}{2\tau}$$

so that the force between the wires initially is

$$F_2(t=0) = F_2 = \frac{\mu_0 d}{2\pi d'} i_0^2 = \frac{\mu_0 d q^2}{8\pi \tau^2 d'}$$

$$\text{so } F_2 = \boxed{\frac{\mu_0 d q^2}{8\pi \tau^2 d'}}$$

and we eliminate the unknown charge q by taking the ratio of electric and magnetic forces:

$$\frac{F_1}{F_2} = \frac{q^2 \cdot 8\pi \tau^2 d'}{2\epsilon_0 A \cdot \mu_0 d q^2} = \frac{4\pi d' \tau^2}{\epsilon_0 \mu_0 d A}; \text{ using } C = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \text{speed of light},$$

$$\boxed{C = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{F_1}{F_2}} \sqrt{\frac{d}{d'}} \frac{L}{\tau}}$$

, with L the side length of the metal plate.
 $\tau \approx$ time it takes capacitor to discharge
 d = distance between capacitor plates

d' = distance between current-carrying wires

F_1 = force between capacitor plates

F_2 = force between wires (initially).

Since only the ratio of forces enters we don't need to know in what units they are measured.

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Problem 3.

If $\psi(x)$ is a wave function which is such that the mean values of the position and momentum operators \hat{x} and \hat{p} are \bar{x} and \bar{p} , respectively,

1. Show that

$$\phi(x) = e^{-\frac{i\bar{p}x}{\hbar}} \Psi(x + \bar{x})$$

$$\text{has } \langle \hat{x} \rangle = 0$$

$$\langle \hat{p} \rangle = 0$$

where $\hat{x} = (\phi, \hat{x}\phi)/(\phi, \phi)$ and similarly for \hat{p}

2. For such a state $\phi(x)$, show that the wave function that minimizes the product $\overline{x^2 p^2}$ where

$$\overline{x^2} = (\phi, \hat{x}^2 \phi)/(\phi, \phi)$$

$$\overline{p^2} = (\phi, \hat{p}^2 \phi)/(\phi, \phi)$$

is the ground state of a harmonic oscillator with mass "m" = $1/\overline{x^2}$ and frequency "ω" = $\sqrt{\overline{x^2 p^2}}$ and that the minimum value of $\overline{x^2 p^2}$ is $\hbar^2/4$.

Jorge - This is pretty elementary: Murphy V6 question (1)
 If $\psi(x)$ is a wave function which is such
 that the mean values of \hat{x} and \hat{p} are \bar{x} and \bar{p} , respectively,

1. Show that

$$\phi(x)\psi(x) = e^{-i\bar{p}x/\hbar} \psi(x+\bar{x})$$

$$\text{has } \langle \hat{x} \rangle = 0$$

$$\langle \hat{p} \rangle = 0$$

2. For ~~this~~ such a state $\phi(x)$, show that the wave function that minimizes the product $\bar{x}^2 \bar{p}^2$ where

$$\bar{x}^2 = (\phi, \hat{x}^2 \phi) / (\phi, \phi)$$

$$\bar{p}^2 = (\phi, \hat{p}^2 \phi) / (\phi, \phi)$$

is the ground state of a harmonic oscillator with "mass" $m = 1/\bar{x}^2$ and frequency " $\omega = \sqrt{\bar{x}^2 \bar{p}^2}$ "

and that the minimum value of $\bar{x}^2 \bar{p}^2$ is $\hbar^2/4$.

Solutions:

$$1. \langle \hat{x} \rangle = \left\{ (e^{-i\bar{p}x/\hbar} \psi(x+\bar{x}), x e^{-i\bar{p}x/\hbar} \psi(x+\bar{x})) \right\} / (\psi, \psi)$$

$$= (\psi(x+\bar{x}), [x + \bar{x} - \bar{x}] \psi(x+\bar{x})) / (\psi, \psi)$$

$$= [(\psi(y), y \psi(y)) - \bar{x} (\psi(y), \psi(y))] / (\psi, \psi)$$

$$= \bar{x} - \bar{x} = 0$$

Change variable of integration to $y = x + \bar{x}$

Obviously $(\psi, \psi) = (\psi, \psi)$

$$\langle \hat{p} \rangle = (e^{-i\bar{p}x/\hbar} \psi(x+\bar{x}), \hat{p} e^{-i\bar{p}x/\hbar} \psi(x+\bar{x})) / (\psi, \psi)$$

$$= (\psi(x+\bar{x}), [\hat{p} - \bar{p}] \psi(x+\bar{x})) / (\psi, \psi) = 0.$$

(2)

$$\begin{aligned}
 \bar{x^2} \bar{p^2} &= \frac{(q, \dot{x}^2 \varphi) (q, \dot{p}^2 \varphi)}{(q, q) (q, q)} \\
 \delta(\bar{x^2} \bar{p^2}) &= \frac{[(\delta q, x^2 \varphi) + (q, x^2 \delta \varphi)] \bar{p^2}}{(q, q)} \\
 &\quad + \frac{\bar{x^2} [(\delta q, \dot{p}^2 \varphi) + (q, \dot{p}^2 \delta \varphi)]}{(q, q)} \\
 &- 2 \frac{\bar{x^2} \bar{p^2}}{(q, q)} [(q, \delta \varphi) + (\varphi, \delta q)] = 0
 \end{aligned}$$

Collecting coeff. of $(\delta \varphi, \dots) = 0$ we have

~~$\ddot{x^2} \ddot{p^2}$~~ $x^2 \varphi \bar{p^2} + \bar{x^2} \dot{p}^2 \varphi = 2 \bar{x^2} \bar{p^2} \varphi$

$$\text{or } \frac{\bar{x^2}}{2} \dot{p}^2 \varphi + \frac{\bar{p^2}}{2} \varphi = \bar{x^2} \bar{p^2} \varphi$$

$$\text{so with "m" = } \frac{1}{x^2} \quad E = \bar{x^2} \bar{p^2}$$

$$"m" \omega^2 = \bar{p^2}$$

$$\frac{\dot{p}^2}{2 "m"} \varphi + \frac{1}{2} "m" \omega^2 \varphi = E \varphi$$

where $E_{\min} = \frac{\hbar}{2} \omega$ = oscillator ground state

$$\omega^2 = \bar{x^2} \bar{p^2}, \text{ so } \bar{x^2} \bar{p^2} = \frac{\hbar}{2} \sqrt{\bar{x^2} \bar{p^2}}$$

$$\text{or } \sqrt{\bar{x^2} \bar{p^2}} = \frac{\hbar}{2}$$

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Problem 4.

A 500 km long hole is drilled between two points on the earth's surface along a straight line. Assume the density of the earth is uniform, and neglect friction and the earth's motion.

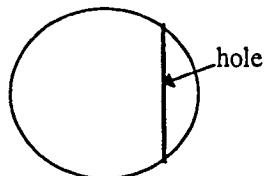
- (a) How long does it take for an object dropped into the hole at one end with zero initial velocity to reach the other end?
- (b) What is the maximum velocity that the object reaches?
- (c) Is there any other path for a hole between the given two points that will yield a shorter time for an object to go from one end to the other with zero initial velocity? Justify your answer.

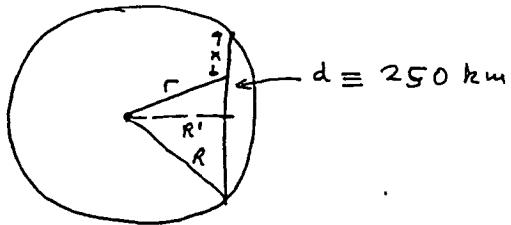
Data:

Radius of earth: $R=6400$ km.

$g=9.8$ m/sec² at earth's surface.

$$\int_0^R \frac{dx}{\sqrt{x(1-x)}} = \pi$$



Solution

$$\text{Conservation of energy: } \frac{1}{2}mv^2 + V(r) = \text{const.}$$

Potential energy at radius r :

$$\text{From b } F = -\frac{GmM(r)}{r^2} \quad (\text{Gauss law}), \quad M(r) = \frac{4}{3}\pi S r^3 = \frac{M r^3}{R^3}$$

$$\Rightarrow F = -\frac{GmM}{R^3} r = -mg \frac{r}{R}$$

$$\text{Potential energy: } V(r) = - \int_0^r dr F(r) = \frac{mg r^2}{2R}$$

$$\Rightarrow \text{conserv of energy gives } \frac{1}{2}mv^2 + \frac{mg r^2}{2R} = \frac{mg R}{2} \Rightarrow$$

$$v^2 = g \frac{(R^2 - r^2)}{R}$$

(b) Maximum velocity is at midpoint: $r^2 + d^2 = R^2 \Rightarrow R^2 - r^2 = d^2 \Rightarrow$

$$\Rightarrow v_{\max}^2 = \frac{gd^2}{R} \Rightarrow v_{\max} = \sqrt{\frac{gd^2}{R}} = \sqrt{\frac{9.8 \cdot (250 \cdot 10^3)^2}{6400 \cdot 10^3}} \frac{\text{m}}{\text{sec}}$$

$$\Rightarrow v_{\max} = 309.4 \text{ m/sec}$$

(a) Time taken:

$$dt = \frac{dx}{v} \Rightarrow t = \int_0^{2d} \sqrt{\frac{R}{g}} \frac{dx}{\sqrt{R^2 - r^2}}$$

$$r^2 = (d-x)^2 + R'^2; R'^2 = R^2 - d^2 \Rightarrow$$

$$r^2 = (d-x)^2 + R^2 - d^2 = d^2 - 2dx + x^2 + R^2 - d^2 = R^2 + x^2 - 2dx \Rightarrow$$

$$R^2 - r^2 = 2dx - x^2 = x(2d-x) \Rightarrow$$

$$t = \int_0^{2d} \sqrt{\frac{R}{g}} \frac{dx}{\sqrt{x(2d-x)}} = \sqrt{\frac{R}{g}} \int_0^1 \frac{dy}{\sqrt{y(1-y)}} = \pi \sqrt{\frac{R}{g}} = \pi \sqrt{\frac{6400.10^3}{9.8}} \text{ sec}$$

$$\Rightarrow t = 42.3 \text{ mins} \quad \text{independent of distance between points.}$$

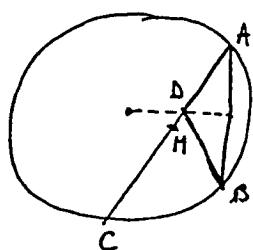
On, another: potential energy,

$$V(x) = \frac{mg}{2R} r^2 = \frac{mg}{2R} [R'^2 + (d-x)^2]$$

\rightarrow harmonic oscillation \Rightarrow frequency $\omega = \sqrt{\frac{g}{R}}$, period $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{R}{g}}$ \rightarrow time

to go from one end to other is $\frac{1}{2}$ period $\Rightarrow t = \pi \sqrt{\frac{R}{g}}$

(c) Yes, many. The time to go between ^{any} two points along a straight line is 42.3 mins, as calculated in (a). So,



To go from A to D takes $\frac{42.3}{2} = 21$ mins, if M is midpoint of AC. \therefore To go from A to D takes less than $\frac{42.3}{2}$ mins \Rightarrow the path ADB will take less time than the straight path AB.

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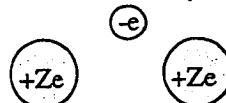
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Problem 5.

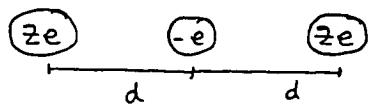
Consider a molecular ion formed by two nuclei of charge Ze each and one electron of charge ($-e$). The mass of each nucleus is ZM and that of the electron is m , with $M/m = 2000$. $e^2 = 14.4 \text{ eV} \text{ \AA}^2$, $\hbar^2/m = 7.62 \text{ eV} \text{ \AA}^2$

- a) Using classical electrostatics find the maximum value of Z ($=Z_m$) for which this molecular ion could be bound.
- b) Allowing for quantum mechanics, will the value of Z_m increase or decrease? Explain.
- c) Find an estimate for the size of this molecular ion (in \AA) as function of Z . Make a plot of size versus Z and find for which value(s) of Z (if any) it is an extremum. Is it a maximum or a minimum?



Solution

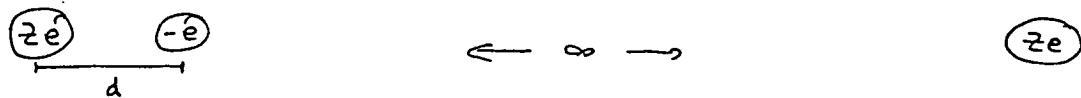
(a) Classically, lowest energy occurs when electron is in line joining both nuclei. Assume:



Energy of that configuration is:

$$E = \frac{z^2 e^2}{2d} - 2 \frac{ze^2}{d} = \frac{ze^2}{2d} (z-4)$$

If everything is unbound, $E=0 \Rightarrow$ this suggests that $\boxed{z < 4}$ gives binding. However, one should compare this with the energy of atom + a nucleus, i.e.



to see if other nucleus wants to bind. This gives

$$E' = -\frac{ze^2}{d} \Rightarrow E - E' = \frac{ze^2}{2d} (z-4+2) = \frac{ze^2}{2d} (z-2)$$

$$\Rightarrow \text{system is bound for } z < 2 \Rightarrow \boxed{z_m = 2}$$

(It can be shown that putting electron at unequal distance from both nuclei is less favourable).

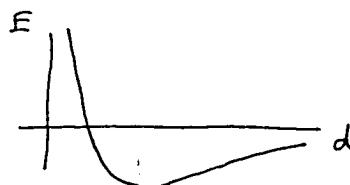
(b) It costs energy to localize electron \Rightarrow it will delocalize, attracting nuclei less
 \Rightarrow system will be less bound $\Rightarrow z_m$ will decrease.

(c) Estimate kinetic energy of electron. Assume 1d box of size $2d$

$$E_{kin} \sim \frac{\pi^2 \hbar^2}{2m(2d)^2} = \frac{\pi^2 \hbar^2}{8md^2}$$

So total energy is

$$E = \frac{\pi^2 \hbar^2}{8md^2} - \frac{ze^2(4-z)}{2d}$$



So we have:

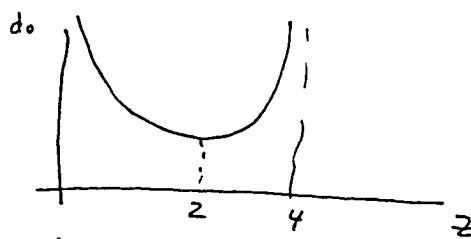
$$E(d) = \frac{C_1}{d^2} - \frac{C_2}{d}$$

Minimum energy at:

$$E'(d_0) = -\frac{2C_1}{d_0^3} + \frac{C_2}{d_0^2} = 0 \Rightarrow 2C_1 d_0^2 = C_2 d_0^3 \Rightarrow$$

$$d_0 = \frac{2C_1}{C_2} \Rightarrow d_0 = \frac{\cancel{2} \pi^2 \epsilon^2 \cdot \cancel{2}}{\cancel{8} m \cdot \cancel{2} e^2 (4-z)} = \frac{\epsilon^2}{m e^2} \frac{1}{2} \frac{\pi^2}{z(4-z)} = \frac{7.62 \text{ eV}\text{\AA}^3}{14.4 \text{ eV}\text{\AA}^3} \frac{\pi^2}{z(4-z)}$$

$$d_0 = \frac{2.61 \text{ \AA}}{z(4-z)}$$



Extremum:

$$d_0' = -\frac{1}{z^2(4-z)} + \frac{1}{z^2(4-z)^2} = 0 \Rightarrow 4-z = z \Rightarrow z = 2$$

So for $z=2$ (the largest z for which it binds) size is smallest. For smaller z ,

size of molecule increases due to increasing effect of zero-point motion. For $z=2$, $d_0 = 0.65 \text{ \AA}$

Note: motion of nuclei was neglected because $M/m > 1$.

Note to grader: I suggest giving a large fraction of the points of part (a) to those that find $z_m = 4$ instead of $z_m = 2$, as it is easy to make that mistake; e.g. 75% of points.

Suggested point distribution: (a) 4, (b) 1, (c) 5.

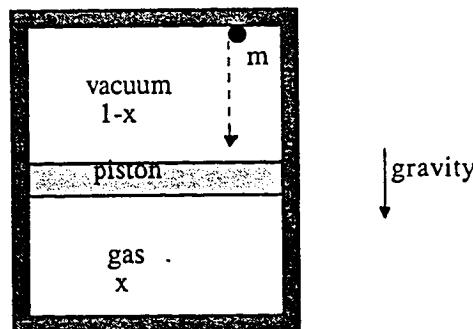
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Problem 6.

A monatomic ideal gas occupies a fraction x of the volume of a thermally insulated cylinder, the lower part, and is separated from the upper part (vacuum) by a movable piston of weight W . The system is initially in equilibrium.



A ball of mass m (and negligible dimension) sticks at the inside top of the cylinder. At some time it starts falling under the influence of gravity, bounces several times off the piston (which moves in the process) and stops after a while and the system reaches equilibrium again. Assume the heat capacities of the cylinder walls, piston and ball are negligible.

- For what value of x is the final x equal to the initial x ?
- Would the value of x found in (a) be larger, equal or smaller if the gas was diatomic instead? Justify.

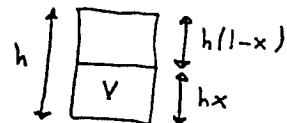
Solution

Let T be initial temperature of gas, T' final temperature. Energy of gas initially is $E = \frac{3}{2} n R T$ ($n = \# \text{ of moles}$)

When ball falls, it loses potential energy, gains kinetic energy which is transferred to the gas in the bouncing process, ~~and~~ heating it up. ~~Because~~ Energy conserving vs

$$\frac{3}{2} n R T + Mg h (1-x) = \frac{3}{2} n R T' \quad (h = \text{total height of cylinder})$$

$$\Rightarrow \boxed{R(T' - T) = \frac{2}{3} \frac{Mgh}{n} (1-x)} \quad (1)$$



Eq. of state initially $PV = nRT$; external pressure is applied by weight of piston.

Finally, assuming V is same: $(P + \frac{Mg}{A})V = nRT'$. ($A = \text{cross-sectional area of cylinder}$)

$$\Rightarrow Mg \frac{V}{A} = nR(T' - T) ; \text{ now } \frac{V}{A} = hx \Rightarrow$$

$$\boxed{R(T' - T) = \frac{Mghx}{n}} \quad (2)$$

$$\text{Equating (1) and (2)} \Rightarrow \frac{2}{3} \frac{Mgh}{n} (1-x) = \frac{Mghx}{n} \Rightarrow \frac{2}{3} (1-x) = x \Rightarrow \frac{2}{3} = \frac{5}{3}x \Rightarrow$$

$$\boxed{x = \frac{2}{5}}$$

(b) If gas is diatomic, spec. heat is larger. To heat it up to counteract extra weight of M , requires more energy $\Rightarrow M$ needs to fall more $\Rightarrow x$ is smaller.

$$\text{Finally, } \frac{3}{2} \rightarrow \frac{5}{2}, \Rightarrow \frac{2}{5}(1-x) = x \Rightarrow \frac{2}{5} = \frac{7}{5}x \Rightarrow \boxed{x = \frac{2}{7}}$$

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Problem 7.

A long solid cylindrical rod of radius R is uniformly magnetized. The magnetization M points along the axis of the rod. If the rod were cut in half along a plane that is perpendicular to the axis, what force would the two halves exert on each other?

ON

soln

$$\nabla \times H = \frac{4\pi}{c} J_F = 0$$

$$0 = \nabla \cdot B = \nabla \cdot H + 4\pi \nabla \cdot M$$

" 0 except
at ends

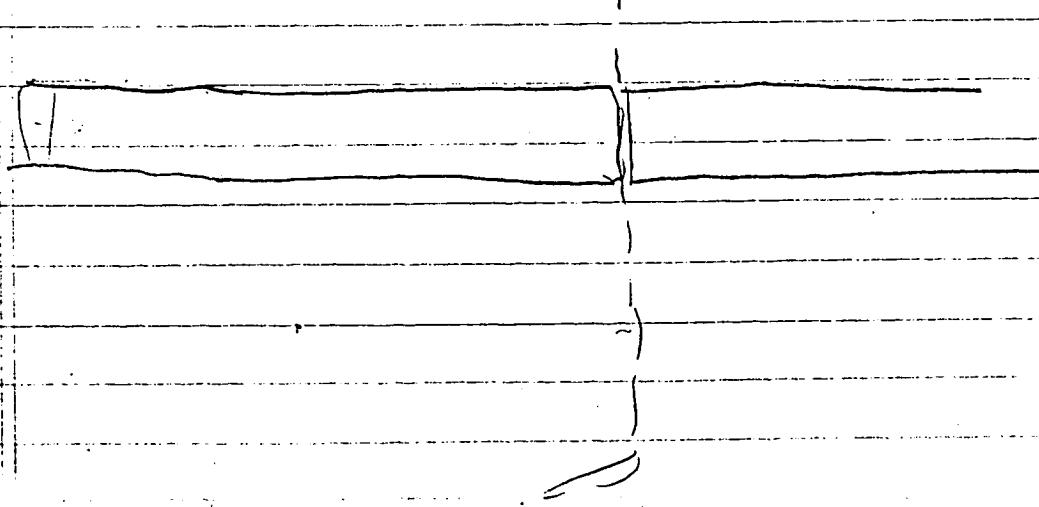
$H \approx 0$ except near ends

$$B = 4\pi M \quad \text{inside cylinder near
the cut}$$

B_n is continuous in the gap

at cut; so $B = 4\pi M$ in
the gap.

Construct surface S that goes
through cut and encloses (at ∞)
No left half of rod



Am I needed?

$$F = \int_{\text{gap}} T \cdot dS \underset{\substack{\text{q: cross} \\ \text{section}}}{\approx} \int_{\text{gap}} T \cdot dI = \frac{B^2}{8\pi} \pi R^2$$

negligible
contribution
from outside
of gap

$$F = \frac{(4\pi M)^2 \pi R^2}{8\pi} = 2\pi^2 M^2 R^2$$

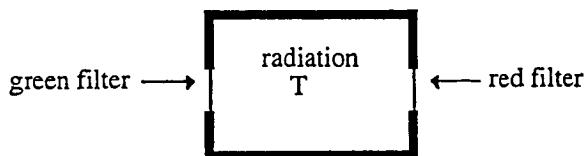
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Physics Departmental Examination - Fall 1997 - Part I

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Problem 8.

A box floating in outer space contains electromagnetic radiation in thermal equilibrium with its inner walls at temperature $T = 20,000\text{K}$ (never mind that no material known to date can withstand such temperature without vaporizing). The outer surfaces of the box's walls perfectly reflect all incident radiation. The box is initially at rest in a certain reference frame. At time $t = 0$, two windows, each of area A , are opened on walls at opposite sides of the box. Assume A is much smaller than the total inside surface area. The window on the left (right) has a green (red) filter that lets radiation through in a frequency range of $\Delta\omega = 10^{12}\text{ Hz}$ centered around $\lambda_{\text{green}} = 5200\text{\AA}$ ($\lambda_{\text{red}} = 6500\text{\AA}$) and reflects all others.



2 pts (a) In which direction does the box start to move in the given reference frame? Explain.

3 pts (b) Find an expression for the force acting on the box at time $t = 0^+$. Check and show explicitly that your expression is dimensionally correct.

3 pts (c) At some later time(s) the force acting on the box will vanish. Estimate the temperature inside the box at the earliest time that this happens.

2 pts (d) Make a qualitative plot of the force acting on the box for all times.

Data: $hc = 12400\text{eV\AA}$; $k_B = \frac{1}{11,600} \frac{\text{eV}}{\text{K}}$ (Boltzmann constant)

Hint: the equation $x = 3(1 - e^{-x})$ is easily solved by iteration.

Solution
④ see end note

- i) Momentum of a photon ω $p = \hbar k = \frac{\hbar \omega}{c}$, energy $\propto \hbar \omega$. Green photons carry more momentum than red ones. However we also need to know how many photons of each color go out to find out how box moves. The flux of photons through a window can be estimated by usual argument:

$\sim \frac{1}{6}$ of photons travel in each direction, then

$$\Phi = \frac{1}{6} n c \quad \text{is \# of photons crossing unit area in unit time;}$$

the correct answer doing ~~at~~ integration is $\frac{1}{4}$ instead of $\frac{1}{6}$, either is 0.4.

The flux of momentum out of a window of area A due to photons in frequency range $\Delta\omega$ around ω is then

$$P(\omega) = \frac{1}{4} n(\omega) \cdot c \cdot \frac{\hbar \omega}{c} \cdot \Delta\omega \cdot A$$

The energy density is $u(\omega) = \hbar \omega n(\omega) \Rightarrow P(\omega) = \frac{1}{4} u(\omega) \Delta\omega A$, or

$$P(\omega) = \frac{\hbar \omega^3}{4 \pi^2 c^3 (e^{\beta \hbar \omega} - 1)} \cdot \Delta\omega \cdot A$$

$$\begin{cases} \frac{1}{e^{\beta \hbar \omega} - 1} & \text{= occupation factor} \\ \hbar \omega & \text{= photon energy} \\ \omega^2 \Delta\omega \sim k^2 \Delta k & \text{= phase space factor} \end{cases}$$

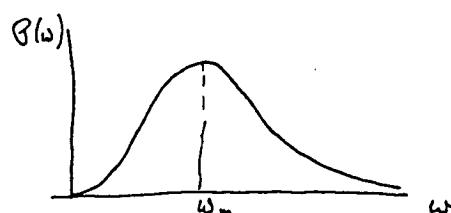
Now $\Delta\omega$ is the same for both windows, and much smaller than ω .

The net force on the box is the difference in momentum flowing out to right and left:



$$F = P_{\text{green}} - P_{\text{red}} = P(\omega_{\text{green}}) - P(\omega_{\text{red}})$$

The function $P(\omega)$ looks like

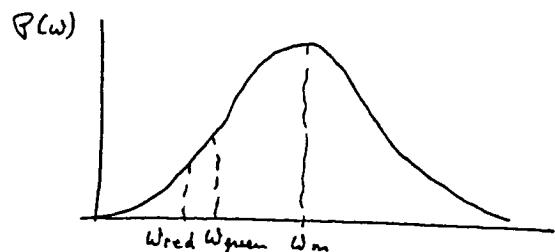


$$\text{with maximum at } \frac{\hbar \omega_m}{kT} \sim 2.82 \Rightarrow \frac{\hbar \omega_m}{k} = 56,400 \text{ K} \Rightarrow \boxed{\hbar \omega_m = 4.86 \text{ eV}}$$

The green and red wavelengths correspond to frequencies

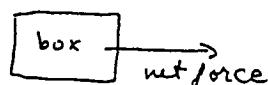
$$\hbar\omega_{\text{green}} = \frac{hc}{\lambda_{\text{green}}} = 2.38 \text{ eV} ; \quad \hbar\omega_{\text{red}} = 1.91 \text{ eV}$$

\Rightarrow both are smaller than $\hbar\omega_m$



Since more momentum is flowing out through the left window \Rightarrow

box starts to move to the right

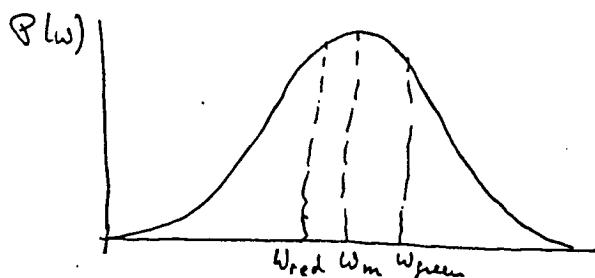


(b) Force acting on the box

$$F = P(\omega_{\text{green}}) - P(\omega_{\text{red}}) = \frac{\pi}{4\pi^2 c^3} \left(\frac{\omega_{\text{green}}^3}{e^{\beta\hbar\omega_{\text{green}}}-1} - \frac{\omega_{\text{red}}^3}{e^{\beta\hbar\omega_{\text{red}}}-1} \right) \Delta\omega A.$$

dimensions: $[F] = \frac{[t]}{[c^3]} [w^4] [A] = \frac{\text{energy} \cdot \text{time}^4}{\text{length}^3 \cdot \text{time}^4} \cdot \text{length}^2 = \frac{\text{energy}}{\text{length}} = \text{force}$

(c) As photons keep coming out the temperature inside the box starts to drop (because A is much smaller than total area we can assume that radiation inside remains in thermal equilibrium). At some point the distribution of momentum flux will look like



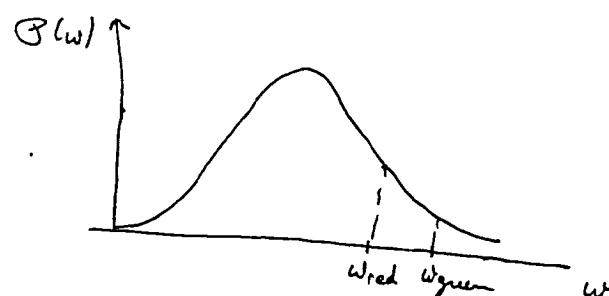
and the net force will be zero. That happens when

$$\omega_m \sim \frac{\omega_{green} + \omega_{red}}{2} \Rightarrow \hbar \omega_m = 2.15 \text{ eV} = 2.82 \text{ kT} \Rightarrow$$

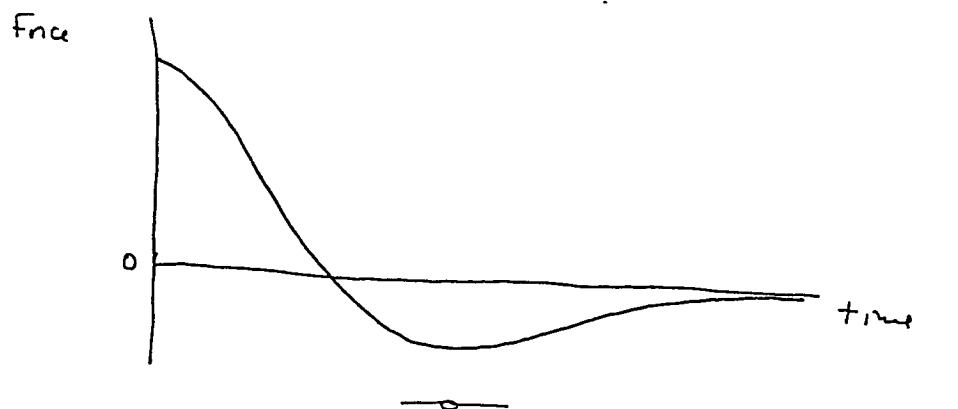
$$\Rightarrow T = \frac{2.15 \text{ eV}}{2.82 \text{ k}} = 8840 \text{ K}$$

(d)

After that time, it looks like



so more momentum flows out from the red window and the force acts to the left. Asymptotically for long times the photons inside the box will reach thermal equilibrium with the microwave background radiation and the force will vanish. So



④

Note: the fact that outer surfaces of box are perfectly reflecting \Rightarrow they don't emit any radiation (Kirchhoff's law) \Rightarrow radiation can only go out through windows.

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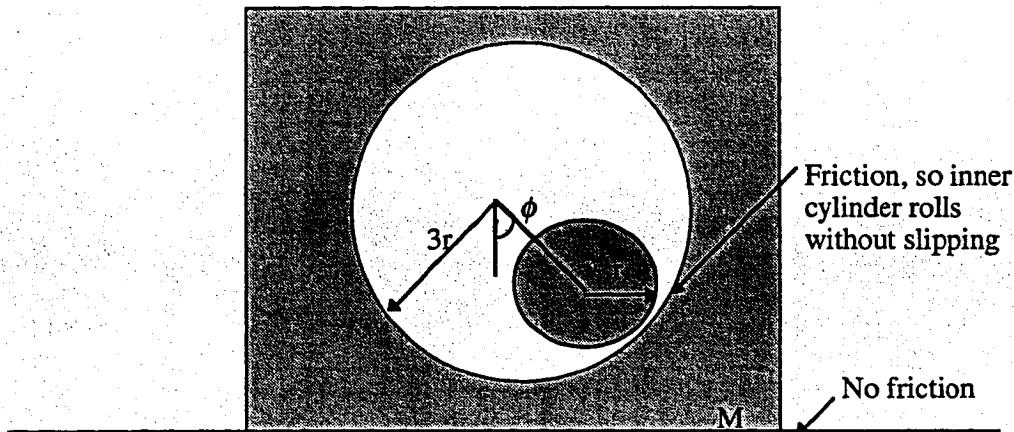
PHYSICS DEPARTMENTAL EXAMINATION

SPECIAL INSTRUCTIONS

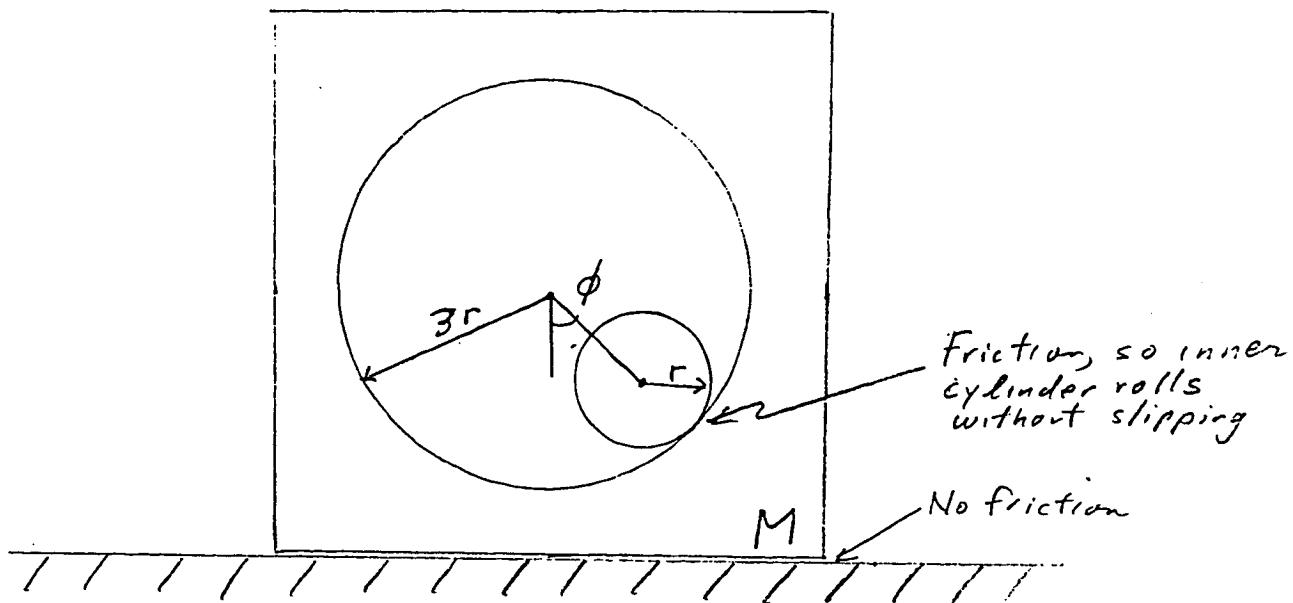
Please take a few minutes to read through all problems before starting to work. The proctor of the exam will attempt to clarify exam questions if you are uncertain about them. It is important to make an effort on every problem even if you do not know how to solve it completely. Partial credit will be given for partial solutions.

Problem 9.

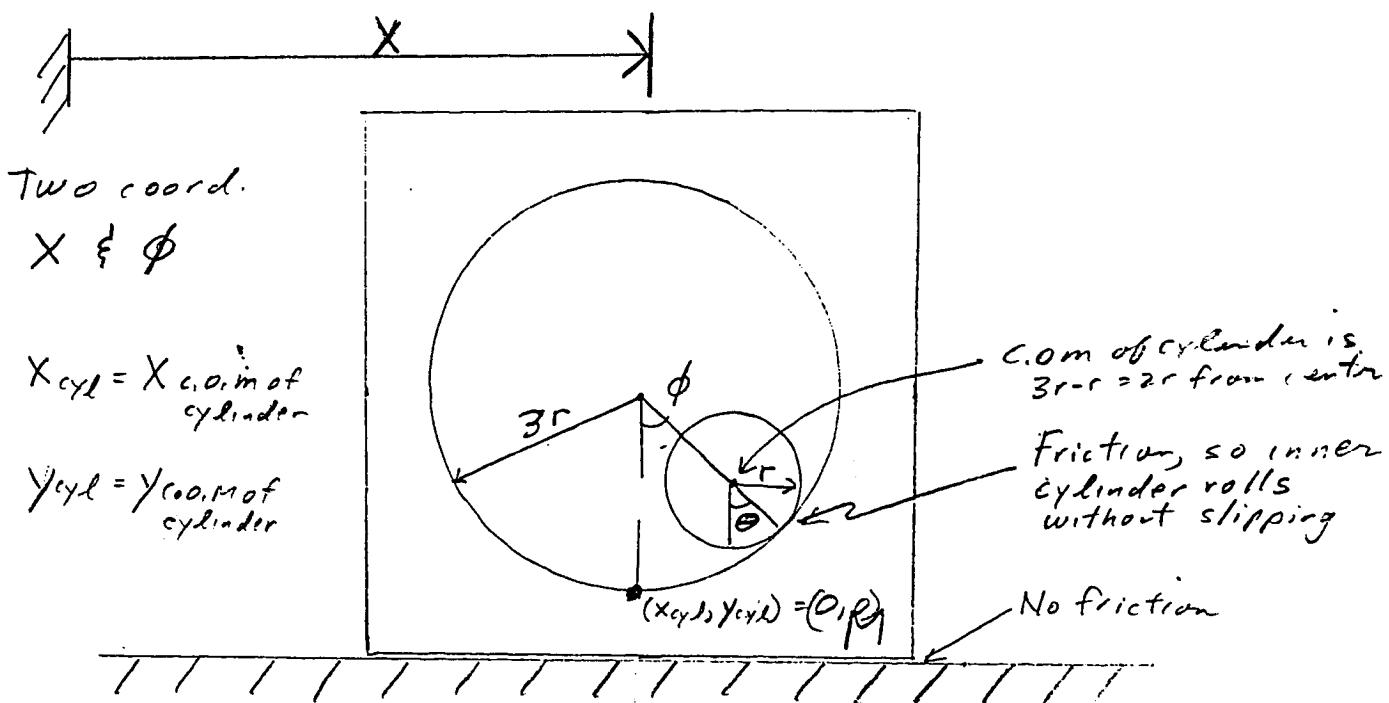
A cylinder of mass m and radius r rolls without slipping in a circular opening, of radius $3r$, within a block of mass M . The block slides without friction on a horizontal surface. Find the frequencies of the normal modes.



A cylinder of mass m and radius r rolls without slipping in a circular opening of radius $3r$, within a block of mass M . The block slides without friction on a horizontal surface. Find the frequencies of the normal modes.



A cylinder of mass m and radius r rolls without slipping in a circular opening of radius $2r$, within a block of mass M . The block slides without friction on a horizontal surface. Find the frequencies of the normal modes.



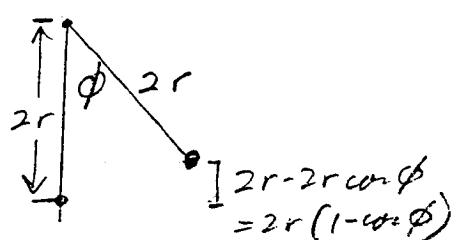
$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m \dot{X}_{\text{cyl}}^2 + \frac{1}{2} m \dot{X}_{\text{cyl}}^2$$

$$I = \int_0^r dm \rho^2 = \frac{M}{\pi r^2} \int \rho d\rho d\theta \rho^2 = \frac{M}{\pi r^2} \cdot \frac{\rho^4}{4} \cdot 2\pi = \frac{Mr^2}{2}$$

$$\dot{\theta} = \frac{V_{\text{com}}}{r} = \frac{(3r - r)\dot{\phi}}{r} = 2\dot{\phi}$$

$$X_{\text{cyl}} = X + 2r \sin \phi$$

$$Y_{\text{cyl}} = 2r(1 - \cos \phi)$$



(2)

$$\dot{x}_{cyl} = \dot{x} + (2r\cos\phi) \dot{\phi}$$

$$\dot{y}_{cyl} = (2r\sin\phi) \dot{\phi}$$

$$\therefore T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} \left(\frac{1}{2} m r^2 \right) (2\dot{\phi})^2 + \frac{1}{2} m (\dot{x} + 2r\dot{\phi}\cos\phi)^2 \\ + \frac{1}{2} m (2r\dot{\phi}\sin\phi)^2$$

$$= \frac{1}{2} M \dot{x}^2 + m r^2 \dot{\phi}^2 + \frac{1}{2} m \dot{x}^2 + 2m r^2 \dot{\phi}^2 \cos^2\phi \\ + 2m r \dot{x} \dot{\phi} \cos\phi + 2m r^2 \dot{\phi}^2 \sin^2\phi$$

$$= \frac{1}{2} (M+m) \dot{x}^2 + 3m r^2 \dot{\phi}^2 + 2m r \dot{x} \dot{\phi} \cos\phi$$

$$V = mg y = mg 2r(1 - \cos\phi)$$

$$H = -2rmg \cos\phi + \text{Const}$$

$$L = T - V$$

$$= \frac{1}{2} (M+m) \dot{x}^2 + 2m r \dot{x} \dot{\phi} \cos\phi + 3m r^2 \dot{\phi}^2 - 2rmg \cos\phi + C$$

① Consider linear motion of system first.

$$\frac{dL}{dx} = (M+m) \ddot{x} + 2m r \dot{\phi} \cos\phi$$

$$\frac{dL}{dx} = 0$$

$$\therefore \frac{d}{dt} \frac{dL}{dx} = (M+m) \ddot{x} + 2m r \dot{\phi} \cos\phi - 2m r \dot{\phi}^2 \sin\phi = 0$$

(3)

Normal mode approx: $\cos\phi \approx 1$; $\dot{\phi} \approx 0$

$$(M+m)\ddot{x} + 2mr\ddot{\phi} \approx 0$$

② consider angular motion of cylinder

$$\frac{2L}{2\dot{\phi}} = 2mr\dot{x}\cos\phi + 6mr^2\dot{\phi}$$

$$\frac{2L}{2\phi} = -2mr\dot{x}\dot{\phi}\sin\phi - 2rmg\sin\phi$$

$$\begin{aligned} \therefore \frac{d}{dt}\left(\frac{2L}{2\dot{\phi}}\right) &= 2mr\ddot{x}\cos\phi - 2mr\dot{x}\dot{\phi}\sin\phi + 6mr^2\ddot{\phi} \\ &= -2mr\dot{x}\dot{\phi} - 2rmg\sin\phi \end{aligned}$$

Again, normal mode approx: $\sin\phi \approx \phi$, $\dot{\phi} \approx 0$
 $\cos\phi \approx 1$; $\dot{x} \approx 0$

$$2mr\ddot{x} + 6mr^2\ddot{\phi} + 2rmg\phi \approx 0$$

③ Assume normal mode solutions

$$x = X_0 e^{i\omega t} \quad \phi = \Phi_0 e^{i\omega t}$$

$$\begin{pmatrix} -(M+m)\omega^2 & -2mr\omega^2 \\ -2mr\omega^2 & -(6mr^2\omega^2 - 2rmg) \end{pmatrix} \begin{pmatrix} X_0 \\ \Phi_0 \end{pmatrix} = 0$$

$$\begin{vmatrix} (M+m)\omega^2 & 2mr\omega^2 \\ 2mr\omega^2 & 6mr^2\omega^2 - 2rmg \end{vmatrix} = 0$$

(7)

$$(M+m) \cdot 6mr^2 \omega^4 - 2r mg(M+m) \omega^2 - 4m^2 r^2 \omega^4 = 0$$

$$\omega^2 \left[\frac{6m^2 r^2 \omega^2 + 6mMr^2 \omega^2 - 4m^2 r^2 \omega^2}{2} - 2r mg(m+M) \right] = 0$$

$$\omega^2 \left[\cancel{2} (m+3M) \cancel{r} \cancel{\omega^2} - \cancel{2} \cancel{r} \cancel{mg} (m+M) \right] = 0$$

$$\omega^2 \left[\omega^2 = \frac{g(m+M)}{r(m+3M)} \right] = 0$$

$$\therefore \omega_1 = 0$$

Uniform translation.

$$\omega_2 = \sqrt{\frac{g}{r} \frac{(m+M)}{(m+3M)}}$$

Check, for $M \rightarrow \infty$
should get pendulum

$$\begin{aligned} E &= \frac{1}{2} m(2r)^2 \dot{\phi}^2 + \frac{1}{2} \left(\frac{mr^2}{2}\right) \dot{\theta}^2 + mg r \dot{\phi} \\ &= 2mr^2 \dot{\phi}^2 + mr^2 \dot{\phi}^2 + mg r \dot{\phi}^2 \\ &= \frac{1}{2} (6m) (r \dot{\phi}^2) + \left(\frac{2mg}{r}\right) (r \dot{\phi})^2 \\ \omega &= \sqrt{\frac{2mg}{r6m}} = \sqrt{\frac{g}{3r}} \quad \checkmark \end{aligned}$$

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Problem 10.

An electron is incident with impact parameter p and speed v_0 on a proton at rest. Calculate the energy radiated during the collision assuming the ordering $e^2/p \ll m v_0^2 \ll mc^2$.
(Hint: for this ordering there is a simple approximation for the orbit.)

FINAL EXAM
June 10, 1994Each problem is worth 10 points.

1. An electron is incident with impact parameter ρ and speed v_0 on a proton at rest. Calculate the energy radiated during the collision assuming the ordering $e^2/\rho \ll mv_0^2 \ll mc^2$. (Hint: for this ordering there is a simple approximation for the orbit.)

Can use dipole radiation, since non-relativistic

$$\therefore W = \int_{-\infty}^{+\infty} dt \frac{2}{3} \frac{(\vec{P})^2}{c^3} \quad \text{What about?}$$

Let $r = r_p - r_e$ be relative coordinate

$$\ddot{r} = -\frac{e^2 r}{r^2}, \quad \ddot{r} = -\frac{e^3}{4r^2}$$

$$W = \int_{-\infty}^{+\infty} dt \frac{2}{3} \frac{e^6}{4r^2 c^3} \frac{1}{r^4(t)}$$

~~approximate~~ nearly straight line orbit since
 $r^2/s \ll mv_0^2 \Rightarrow r^2 = s^2 + v_0^2 t^2$

$$W = \frac{2}{3} \frac{e^6}{4r^2 c^3} \int_{-\infty}^{+\infty} dt \frac{1}{r^4(s^2 + v_0^2 t^2)} = \frac{2}{3} \frac{e^6}{4r^2 c^3 s^3} \int_{-\infty}^{+\infty} \frac{dt}{1+t^2}$$

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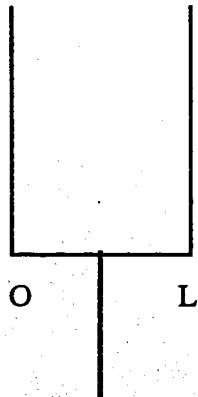
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Problem 11.

A particle of mass m is in a one-dimensional infinite square well of width L that has an attractive δ -function potential at the center, given by,

$$V(x) = -\alpha\delta(x - \frac{L}{2})$$



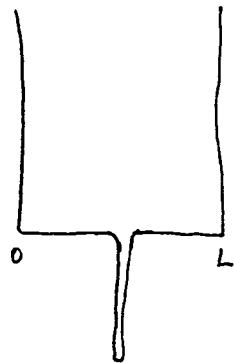
Find upper and lower bounds to the ground state energy of this particle that differ by less than $2\frac{\alpha}{L}$, for all α . You may use the variational principle without proving it.

Hint: Prove and use the theorem that if $H = H_1 + H_2$, then $(\epsilon_1 + \epsilon_2)$ is a lower bound to the ground state energy of H , with ϵ_1 and ϵ_2 the ground state energies of H_1 and H_2 respectively.

Hint: The ground state energy of a particle of mass m in the δ -function potential given above is $\epsilon = -m\alpha^2/2\hbar^2$.

A particle of mass m is in a one-dimensional infinite square well that has an attractive δ -function potential at the center, given by

$$V(x) = -\alpha \delta\left(x - \frac{L}{2}\right)$$



Find upper and lower bounds to the ground state energy of this particle that differ by less than ~~$10^{-2} \frac{\alpha}{L}$~~ , for all α .

~~Hint:~~ You may use the variational principle without proving it.

Hint: ~~prove and~~ use the theorem that if $H = H_1 + H_2$, then $(E_1 + E_2)$ is a lower bound to the ground state energy of H , with E_1 and E_2 the ground state energies of H_1 and H_2 respectively.)

Hint': the ^{ground state} energy of a particle of mass m in the δ -function potential given above is $\epsilon = -m\alpha^2/2\pi^2$

Lower bound: take $H = H_1 + H_2$

$$H_1 = \frac{p^2}{2m_1} + V_{sq.w.}, \quad H_2 = \frac{p^2}{2m_2} - \alpha \delta(x); \quad ; \quad \underbrace{\frac{1}{m_1} + \frac{1}{m_2}}_{\frac{1}{m}}$$

Exact p. st. energies of H_1 and H_2 are:

$$\varepsilon_1 = \frac{\pi^2 \hbar^2}{2m_1 L^2}, \quad \varepsilon_2 = -\frac{m_2 \alpha^2}{2\hbar^2}, \quad \text{hence, using } \frac{1}{m_1} = \frac{1}{m} - \frac{1}{m_2},$$

$$E_{l.b.} = \varepsilon_1 + \varepsilon_2 = \frac{\pi^2 \hbar^2}{2m L^2} - \frac{\pi^2 \hbar^2}{2m_2 L^2} - \frac{m_2 \alpha^2}{2\hbar^2} \quad \left(\begin{array}{l} \text{Clearly, choosing either } m_2 = m, m_1 \rightarrow \infty \\ \text{or } m_1 = m, m_2 \rightarrow \infty, \text{ won't work for all } \alpha. \end{array} \right)$$

$$\text{Choose: } m_2 = \frac{C \hbar^2}{L \alpha} \Rightarrow \frac{\pi^2 \hbar^2}{2m_2 L^2} = \frac{\pi^2 \hbar^2}{2L^2 C \alpha^2} \chi_\alpha = \frac{\pi^2 \alpha}{2C} \frac{\chi_\alpha}{L}; \quad \frac{m_2 \alpha^2}{2\hbar^2} = \frac{C \chi_\alpha \alpha^2}{L \hbar^2 2\hbar^2} = \frac{C}{2} \frac{\alpha^2}{L}$$

$$E_{l.b.} = \frac{\pi^2 \hbar^2}{2m L^2} - \frac{\alpha}{L} \left(\frac{\pi^2}{2C} + \frac{C}{2} \right)$$

For example, for $C = 3$, $\frac{\pi^2}{2C} + \frac{C}{2} = 3.14 \Rightarrow E_{l.b.}$ will differ from $E_{u.b.}$ by less than required.

Optimal C is obtained by maximizing $E_{l.b.}$.

$$\frac{\partial E_{l.b.}}{\partial C} = 0 = \frac{\pi^2}{2C^2} - \frac{1}{2} \Rightarrow C = \pi \Rightarrow$$

$$E_{l.b.} = \frac{\pi^2 \hbar^2}{2m L^2} - \frac{\pi \alpha}{L}$$

prob 11
grader notes

upper bound 4 points

lower bound 6 points

2 pts calculating E_1 and E_2

2 pts optimising

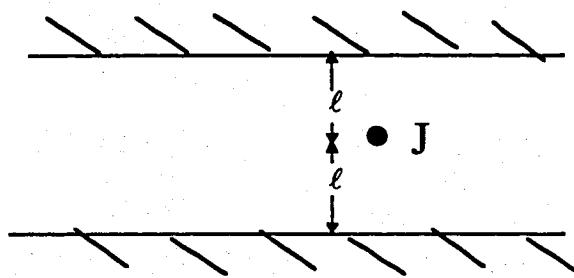
2 pts proof

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Problem 12.

A linear current J is located at the center of two infinite massive iron plates, as at distance ℓ of each, as shown in the figure:



The current extends from $-\infty$ to ∞ in direction perpendicular to the paper.

Assuming that the magnetic permeability of iron $\mu \rightarrow \infty$ calculate the magnetic field between the two plates. Draw qualitatively the magnetic field lines between the plates.

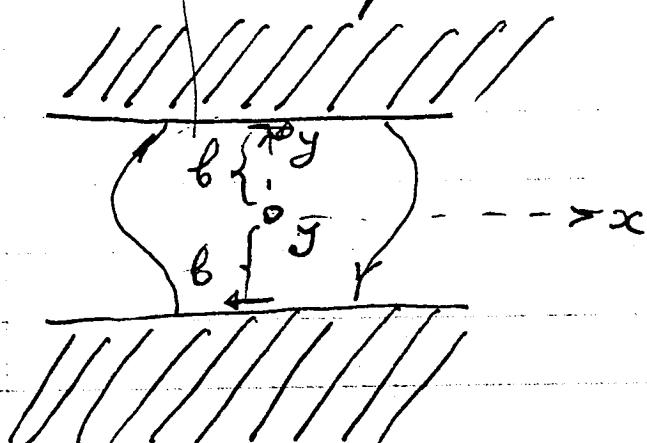
Hint: Use Fourier transformation to find a solution satisfying appropriate boundary conditions.

(is it ferromagnetic)

Prob 12

Problem EM, p. Q?

A ^{linear} current I is located between two iron massive plates, as it is shown in a figure



Assuming that magnetic permeability of iron $\mu \rightarrow \infty$ calculate magnetic field components between two plates.

Hint: Use Fourier transformation to find solution satisfying appropriate boundary conditions.

Solution:

Magnetic field of a linear conductor current in infinite medium

$$B_y = \frac{2I}{\pi R}, \quad B_{x0} = -\frac{2I}{\pi R} \sin \varphi = -\frac{2Iy}{\pi R^2 + y^2}$$

$$B_y = B_0 \cos \varphi = \frac{2J}{c} \frac{x}{x^2 + y^2} \quad (*)$$

Solution is a superposition of Green function $(*)$ and solution of homogeneous problem chosen in a way to satisfy boundary conditions.

B.C.:

$H_t = H_x$ - continuous at $y = \pm b$

$$\frac{B_x^{int}}{M} = B_x^{ext}, \quad u \rightarrow \infty, \quad \underline{\underline{B_x = 0 \text{ for } y = \pm b}}$$

For current free problem

$$B = -\nabla \psi$$

, $\nabla^2 \psi = 0$. Using Fourier transformation we have

$$\psi(x, y) = \int_0^\infty dk \psi_k(y) \sin kx \quad (**) \quad \text{using}$$

$$\frac{d^2 \psi_k}{dy^2} - k^2 \psi_k = 0$$

From B.C.

$$\mp \frac{2y}{c} \frac{b}{x^2 + y^2} - \int_k \psi_k(\pm b) \cos kx dk = 0$$

Both terms are even in x , that is why in $**$ we used expansion in \sin only. $\psi_k(y)$ - odd function of y , i.e.

$$\psi_k(g) = c_k \operatorname{sh} k g$$

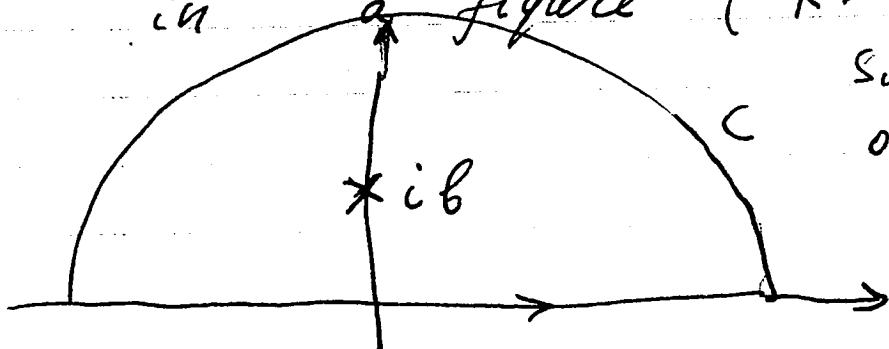
$$\int_0^\infty dk \cdot k \cdot c_k \operatorname{sh} kb \cos kx = -\frac{2J}{c} \frac{b}{x^2 + b^2}$$

Using relationship for inverse Fourier transform, we have

$$k c_k \operatorname{sh} k b =$$

$$= -\frac{y}{\pi c} b \int_{-\infty}^{\infty} dx \frac{\cos kx}{x^2 + b^2}$$

To calculate $I = \operatorname{Re} \int_{-\infty}^{\infty} dx \frac{e^{ikx}}{x^2 + b^2}$
 close contour of integration in a complex plane the way it is shown
 in a figure ($k > 0$!)



Since integral over $C \rightarrow 0$ when ϵ

$$I = 2\pi i \operatorname{Res}(ib) = \frac{\pi}{b} e^{-kb}$$

Hence

$$K C_K = - \frac{J}{\pi C b} \pi b \frac{e^{-kb}}{\sinh kb}$$

Final answer:

$$\left\{ \begin{array}{l} B_x = - \frac{2J}{c} \frac{y}{x^2+y^2} - \int_0^\infty dk K C_k \cdot \sinh ky \\ \qquad \qquad \qquad \cos kx = \\ = - \frac{2J}{c} \frac{y}{x^2+y^2} + \frac{2J}{c} \int_0^\infty dk \frac{\sinh ky}{\sinh kb} \cdot e^{-kb} \cdot \cos kx \end{array} \right.$$

$$\left\{ \begin{array}{l} B_y = \frac{2J}{c} \frac{x}{x^2+y^2} - \int_0^\infty dk K C_k \\ \qquad \qquad \qquad \sin kx \cdot \cosh ky = \\ = \frac{2J}{c} \frac{x}{x^2+y^2} + \frac{2J}{c} \int_0^\infty dk \frac{\cosh ky}{\sinh kb} \cdot e^{-kb} \sin kx \end{array} \right.$$

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Problem 13.

Consider a gas of non-interacting spin 1/2 fermions at zero temperature, each with magnetic moment μ and no charge. The magnetization of the system is measured in the presence of a magnetic field H of 1 Tesla and again in the presence of a magnetic field of 2 Tesla. If it is found that

- a) The magnetization for $H = 2T$ is precisely twice as large as the one for 1T;
- b) The magnetization for $H = 2T$ is larger than twice the one for 1T;
- c) The magnetization for $H = 2T$ is smaller than twice the one for 1T;

what can you conclude about the dimensionality of the system in each case? Justify your answers.

(Part 13)

Consider a system of non-interacting spin $1/2$ fermions at zero temperature, each with magnetic moment μ and no charge. The magnetization of the system is measured in the presence of a magnetic field H of 1 Tesla and again in the presence of a magnetic field H of 2 Tesla. If it is found that

- (a) The magnetization at $H=2T$ is precisely twice as large as for $1T$;
- (b) " " " " " " ^{larger} ~~smaller~~ than twice as large as for $1T$;
- (c) " " " " " " smaller " " " " " "

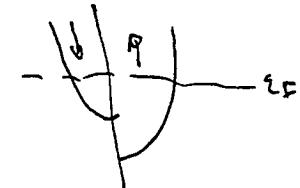
What can you conclude about the dimensionality of the system in each case? Justify your answers.

Solution

Magnetic density is given by

$$m = \mu(n_+ - n_-)$$

with n_σ the density of fermions with mag. moment in direction σ . In the presence of magnetic field H , energy of fermion shifts by $-\mu H$ ($+\mu H$) for spin σ parallel (antiparallel) to H . So,



$$\epsilon_{\sigma}(k) = \epsilon(k) - \sigma \mu H \quad ; \quad \epsilon(k) = \frac{\hbar^2 k^2}{2m}$$

At zero temperature, states up to energy E_F are occupied, so the Fermi wavevector for \uparrow and \downarrow fermions is obtained from:

$$\begin{aligned} \frac{\hbar^2 k_{F\uparrow}^2}{2m} - \mu H &= E_F & k_{F\uparrow} &= \hbar \left(1 + \frac{\mu H}{E_F} \right)^{1/2} & \text{with } \hbar_F = \sqrt{\frac{2m_F}{\hbar^2}} \\ \frac{\hbar^2 k_{F\downarrow}^2}{2m} + \mu H &= E_F & k_{F\downarrow} &= \hbar \left(1 - \frac{\mu H}{E_F} \right)^{1/2} \end{aligned}$$

The relation between density and Fermi wavevector is, in d dimensions:

$$n_\sigma = \frac{1}{(2\pi)^d} \sqrt{d} k_F \quad \text{with } \sqrt{d} k_F = \text{volume of } d\text{-dimensional sphere of radius } k_F.$$

We have $\sqrt{d} k_F = C_d R_F^d$, with $C_d = \text{constant}$ ($C_1 = 2, C_2 = \pi, C_3 = \frac{4}{3}\pi, \dots$)

$$\Rightarrow m = \mu(n_\uparrow - n_\downarrow) = \frac{\mu}{(2\pi)^d} C_d R_F^d \left[\left(1 + \frac{\mu H}{E_F} \right)^{d/2} - \left(1 - \frac{\mu H}{E_F} \right)^{d/2} \right]$$

$$\text{Let } x = \frac{\mu H}{E_F}, \text{ so}$$

$$m(x) = C \left[(1+x)^{d/2} - (1-x)^{d/2} \right]$$

In particular, for $d=2 \Rightarrow m(x) = 2Cx \Rightarrow m$ is linear in $H \Rightarrow$ case (a).

We can verify numerically that $m(2x) > 2m(x)$ for $d=1$ (case (b)) and $m(2x) < 2m(x)$ for $d=3$ (case (c)).

Or, from a Taylor expansion:

$$(1+x)^{d/2} = 1 + \frac{d}{2}x + \frac{1}{2} \cdot \frac{d}{2} \left(\frac{d}{2}-1 \right) x^2 + \frac{1}{6} \frac{d}{2} \left(\frac{d}{2}-1 \right) \left(\frac{d}{2}-2 \right) x^3$$

$$\Rightarrow (1+x)^{d/2} - (1-x)^{d/2} = d \cdot x + \frac{4}{3} \frac{d(d-2)(d-4)}{24} x^3 =$$

$$m(x) = C \cdot d \cdot \left(x + \frac{2}{3} (d-2)(d-4) x^3 + \dots \right)$$

For $d=1$, $(d-2)(d-4) > 0 \Rightarrow$ correction to linear term is positive $\Rightarrow m(2x) > 2m(x)$

" " " " negative $\Rightarrow m(2x) < 2m(x)$

For $d=3$, $(d-2)(d-4) < 0 \Rightarrow$

General answer is:

Case (a) : $d=2$ or $d=4$.

Case (b) : $d < 2$ or $d > 4$

Case (c) : $2 < d < 4$

to be
as shown follows from Taylor expansion argument above and can be shown more generally valid

for the relevant range of x .

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Physics Departmental Examination - Fall 1997 - Part II

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Problem 14.

Consider a weakly ionized plasma consisting of three components - electrons, ions and neutral atoms. Evolution of the electron density is described by the following diffusion equation:

$$\frac{\partial n}{\partial t} = D \nabla^2 n + v n$$

Here, D is the diffusion coefficient and v governs the rate at which electrons are created by collisions.

- a. Use separation of variables to construct a solution of the diffusion equation for a plasma inside an infinitely long cylinder having radius a . Assume an arbitrary radial distribution of the initial density $n(t=0, z) = f(z)$ and boundary condition corresponding to perfectly absorbing walls.

$$n(t, a) = 0 \quad \text{for all } t \geq 0$$

- b. Use the solution of part (a) to find the critical radius a_c such that the electron density will grow exponentially with time if $a > a_c$.

Hint: The Laplacian in cylindrical coordinates is equal to:

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Consider a weakly ionized plasma consisting of three components - electrons, ions and neutrals. Evolution of electron density is described by the following diffusion equation

$$\frac{\partial n}{\partial t} = D \nabla^2 n + V n$$

Here D is the diffusion coefficient and V governs the rate at which electrons are created by collisions.

- a) Use separation of variables to construct a solution of the diffusion equation for a plasma inside an ~~perfectly~~ infinitely long cylinder with ~~radius~~ ^{having} radius a . ~~and~~
- ~~per~~ Assume an arbitrary radial distribution of the initial density $n(0, r) = f(r)$ and boundary condition corresponding to the perfectly absorbing walls.
- $n(t, \alpha) = 0$ for all $t \geq 0$

b) Use solution of $\nabla^2 \psi = 0$ to find the critical radius a_{cr} such that electron density will grow exponentially with time if $a > a_{cr}$.

Hint: Laplacian in cylindrical coordinates is equal to:

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

Solution.

Separation of variables:

$$n = \psi(t) \psi(r)$$

$$\frac{1}{\psi} \frac{d\psi}{dt} = \frac{D}{\psi r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) + V$$

Hence $\psi \frac{D}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) = -\lambda^2 = \text{Const}$

$$\psi = J_0(\lambda r) ; \quad x = \frac{\lambda}{\sqrt{D}}$$

From B.C. $x = \frac{\beta_n}{a} , J_0(\beta_n) = 0$

$$\frac{d\psi}{dt} = V\psi - \lambda^2 \psi$$

$$\psi = e^{(V - \frac{D}{a^2} \beta_n^2)t}$$

By superposition
solution

$$n(t, x) = \sum_n f_n \times J_0(\beta_n x/a)$$

From initial

$$n(0, x) = f(x) = \sum f$$

From S. L. theories

$J_0(\beta_n x/a)$ are the form the complete a of functions and on the interval $0, a$ satisfying condition written in

$$f_n = \frac{\int_0^a f(x) J_0(\beta_n x/a) dx}{\int_0^a J_0^2(\beta_n x/a) dx}$$

b) Instability s.
harmonic. If
amplitude grows.

$$\alpha_{nh} = \sqrt{D_1/B_1} \cdot B_2 / B_1$$

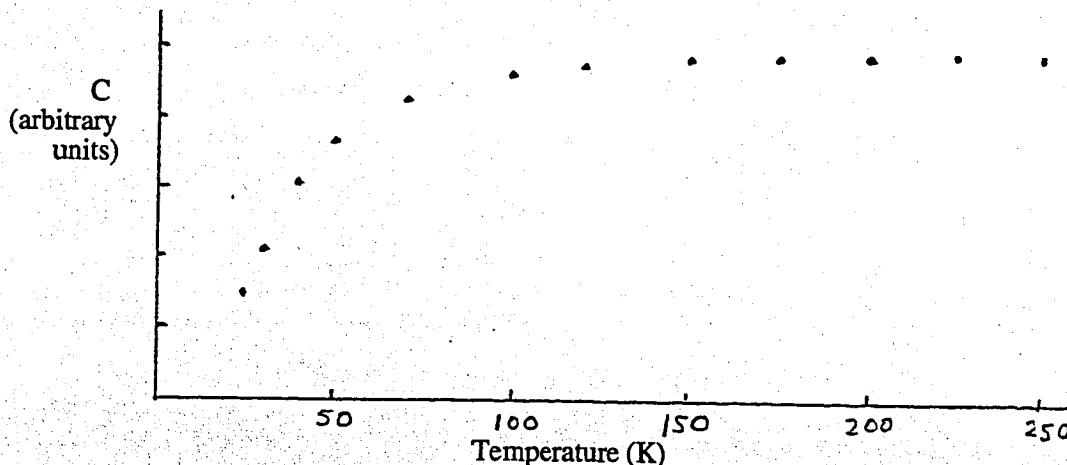
SCORE _____

Physics Departmental Examination - Fall 1997 - Part II

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Problem 15.

The Lindemann melting criterion states that a solid will melt when the average displacement of an atom due to its vibrations becomes a certain fraction of the interatomic distance. Assume that fraction is $1/10$, and that you have an elemental solid for which X-ray diffraction reveals that the interatomic spacing is 1 \AA . You measure the heat capacity versus temperature and obtain the graph given below:



Assume that the heat capacity is only due to atomic vibrations, and that they can be described by an Einstein model.

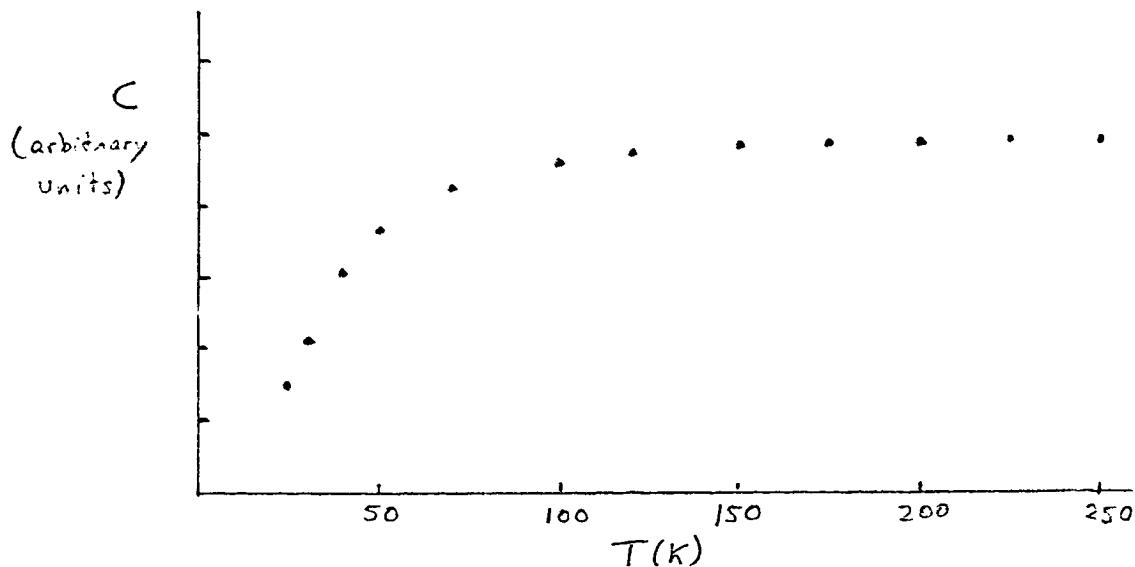
- Can you estimate the value of the melting temperature of this solid from the information given above?
- What can you conclude about the atomic weight of the atoms forming this solid from the fact that it is not a liquid at zero temperature?
- From what you know about this solid, what is the maximum melting temperature it could have? Justify your answer.

Data: $M_p c^2 = 2 \times 10^9 \text{ eV}$, M_p = mass of proton or neutron

$$\hbar c = 1973 \text{ eV \AA}$$

$$k_B = \frac{1}{11,600} \frac{\text{eV}}{\text{^\circ K}}$$
 Boltzmann constant

The Lindemann melting criterion states that a solid will melt when the average displacement of an atom due to its vibrations becomes a certain fraction of the interatomic distance. Assume that fraction is $1/10$, and that you have an elemental solid for which X-ray diffraction reveals that the interatomic spacing is 1\AA . You measure the heat capacity versus temperature and obtain the graph given below:



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- What can you conclude about the atomic weight of the atoms forming this solid from the fact that it is not a liquid at zero temperature?
- From what you know about this solid, what is the maximum melting temperature it could have? Justify your answer.

Data: $M_p c^2 = 2 \times 10^9 \text{ eV}$, M_p = mass of proton in electron units
 $\hbar c = 1973 \text{ eV \AA}$

Solution

- (a) Can't estimate value of melting temperature because we don't know atomic weight
 (b) Zero point vibrations don't destroy crystalline state \Rightarrow

$$\langle x \rangle_{\text{rms}} = \sqrt{\langle x^2 \rangle} < 0.1 \text{ \AA} \quad \text{according to Lennard-Jones}$$

Assume simple harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{1}{2} Kx^2, \quad \omega = \sqrt{\frac{K}{M}}$$

$$\text{Vibrational} \Rightarrow \langle H \rangle = 2 \times \frac{1}{2} K \langle x^2 \rangle \Rightarrow \langle x^2 \rangle = \frac{E}{K}$$

$$\text{At } T=0, \quad E = \frac{\hbar\omega}{2} \Rightarrow \langle x^2 \rangle = \frac{\hbar\omega}{2K} = \frac{\hbar}{2M\omega}$$

Mass of atom $M = A \cdot M_p$, $A = \text{atomic weight}$, $M_p = \text{mass of nucleus}$

$$\text{So, } \langle x^2 \rangle = \frac{\hbar}{2A M_p \omega} = \frac{(\hbar c)^2}{2A M_p c^2 \hbar \omega} = (0.1 \text{ \AA})^2 \Rightarrow \boxed{A = \frac{(\hbar c)^2}{2 M_p c^2 \hbar \omega (0.1 \text{ \AA})^2}}$$

$$\text{or } A = \frac{1128}{\frac{\hbar\omega}{k_B}}$$

To find ω , we assume vibrations are described by Einstein model; every 1)

$$E = 3N\hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right)$$

$$C = \frac{\partial E}{\partial T} = 3N \left(\frac{\hbar\omega}{kT} \right)^2 k \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} = 3Nk \frac{x^2 e^x}{(e^x - 1)^2} \quad x = \frac{\hbar\omega}{kT}$$

$$\text{So, } \frac{C}{3Nk} = \frac{x^2 e^x}{(e^x - 1)^2}; \quad \text{for } x \rightarrow 0 \quad (T \rightarrow \infty), \quad \frac{C}{3Nk} \rightarrow 1$$

$$\text{For } x = 3, \quad \frac{C}{3Nk} \approx 0.50 \Rightarrow \frac{C(x=3)}{C(x \rightarrow 0)} = 0.50; \quad \text{from graph, we see that } C$$

$$\text{is } \sim \frac{1}{2} \text{ of its asymptotic value for } T \sim 33 \text{ K} \Rightarrow \frac{\hbar\omega}{k} = 3 \Rightarrow \boxed{\frac{\hbar\omega}{k} \sim 100 \text{ K}}$$

$$\text{Hence we get } A = \frac{1128}{100} \approx 11$$

So the fact that it is not a liquid at $T=0 \Rightarrow$ atomic weight is larger than 11.

(c) At temperatures much larger than the Einstein temperature $\frac{\hbar\omega}{k} = T_E = 100\text{ K}$, equipartition says

$$kT = \hbar^2 M \langle x^2 \rangle \Rightarrow \text{using } \langle x^2 \rangle = (0.1\text{ \AA})^2 \text{ as criterion for melting}$$

$$kT = \frac{(\hbar\omega)^2 A \pi c^2}{(hc)^2} \times (0.1\text{ \AA})^2 \Rightarrow$$

$$\Rightarrow T = \left(\frac{\hbar\omega}{k}\right)^2 \frac{\pi c^2 \cdot A}{(hc)^2} (0.1\text{ \AA})^2 = \frac{100^2 \text{ K}^2 \cdot 2 \cdot 10^9 \text{ eV \cdot \AA}}{1973^2 \text{ eV}^2 \text{ \AA}^2} \cdot \frac{1}{11,600} \frac{\text{eV}}{\text{K}} \cdot (0.1)^2 \text{ \AA}^2$$

$$\Rightarrow \boxed{T = 4.43 \text{ A} (\text{K})}$$

The maximum atomic weight in nature is ≈ 250 , so the maximum melting temperature for this solid is $\boxed{T_{\max} \approx 1100\text{ K}}$

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Problem 16.

A charged particle with charge q and mass m is moving in a homogeneous magnetic field B pointing in the z direction. Construct the Schrodinger equation for this particle, choosing a gauge so that the vector potential has components both along the x and y directions. Separate variables in cylindrical coordinates. In the equation for the radial part of the wavefunction $R(\rho)$ (ρ = radial coordinate) use the substitution $u(\rho) = R(\rho)\sqrt{\rho}$. Then, use the WKB approximation for bound states to calculate the particle energy spectrum and describe the physical meaning of the different terms in the energy.

Hints:

- 1) See hint of problem 14.
- 2) If you don't remember the WKB approximation use instead Bohr's quantization condition for the radial momentum and coordinate.

Hint: choose a gauge so that the vector potential has components both along the x and y directions.

Hint:

Part 16

Stephan

Problem 1.

1. Charged particle with charge e and mass μ is gyrating in a homogeneous magnetic field B_0 pointing in the z direction. Construct Schrödinger equation for this particle.

Separating variables in polar coordinates, with an axes along

method

use WKB approximation

For the calculation of the radial part of the wave function,

use condition $u(s) = R(s)^{1/2}$. Then

In this approximation calculate particle energy spectrum and describe the physical meaning sense of the different terms.

(1)

Quantum, part B

problem 1.

1. In a WKB approximation calculate the energy levels of charged particle gyrating in homogeneous magnetic field

Solution.

With a choice of the components of vector potential

$$A_x = -\frac{B_0 y}{2}, \quad A_y = +\frac{B_0 x}{2}$$

$$(B_{0z} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

we can write Hamiltonian of the particle with mass μ , charge e in a homogeneous magnetic field $B_0 \hat{H}0z$ in a following form

$$\mathcal{H} = \frac{1}{2\mu} \left(\hat{p}_x + \frac{e}{c} A_x \right)^2 + \frac{1}{2\mu} \left(\hat{p}_y + \frac{e}{c} A_y \right)^2 + \frac{1}{2\mu} \hat{p}_z^2$$

(2)

where $\hat{p} = \frac{\hbar}{i} \nabla$ - operator of the quantum momentum. Hence:

$$\hat{H} = -\frac{\hbar^2}{2\mu} \nabla^2 + \frac{\hbar}{i} \frac{eB_0}{2\mu c} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) + \frac{e^2 B_0^2}{8\mu c^2} (x^2 + y^2)$$

Introducing polar coordinates

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan \frac{y}{x}$$

$$\left(\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial r} + \frac{\sin\varphi}{r} \frac{\partial}{\partial \varphi} \right)$$

$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\varphi}{r} \frac{\partial}{\partial \varphi}$$

we can write Schrödinger equation in a following form:

$$\left[-\frac{\hbar^2}{2\mu} \nabla_{\perp}^2 \psi - \frac{\hbar^2}{2\mu} \frac{\partial^2 \psi}{\partial z^2} + i\hbar eB_0 \frac{\partial \psi}{\partial \varphi} + \frac{e^2 B_0^2}{8\mu c^2} r^2 \psi \right] = E\psi$$

$$\text{where } \nabla_{\perp}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

is the transverse Laplacian.

Separating variables we can write ψ = function in a form

(3)

$$\psi(p, \varphi, z) = R(r) e^{im\varphi + ik_z z}$$

The equation for the radial part of the wave function has the form

$$\frac{d^2 R}{dp^2} + \frac{1}{p} \frac{dR}{dp} + \left[\frac{2\mu}{\hbar^2} E - k_z^2 - \frac{m^2}{p^2} - \frac{e^2 B_0^2}{4\hbar^2 c^2 p^2} \right] R = 0$$

Substituting $R = u/\sqrt{p}$, we obtain an equation that can be easily solved in WKB approximation

$$u'' + \frac{2\mu}{\hbar^2} [E^* - V_{eff}(r)] u = 0$$

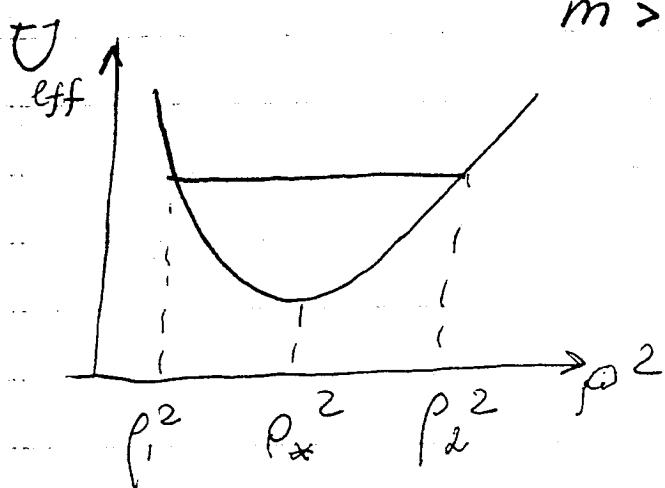
where $E^* = E - \frac{\hbar^2 k_z^2}{2\mu}$ and

$$V_{eff} = \frac{\hbar^2}{2\mu p^2} \left[m + \frac{e B_0}{2\hbar c} p^2 \right]^2$$

- effective potential for the radial motion. In WKB approximation

(4)

tion $(m \gg 1)$ we changed $m^2 - \frac{1}{4}$ into m .



$$m > 0$$

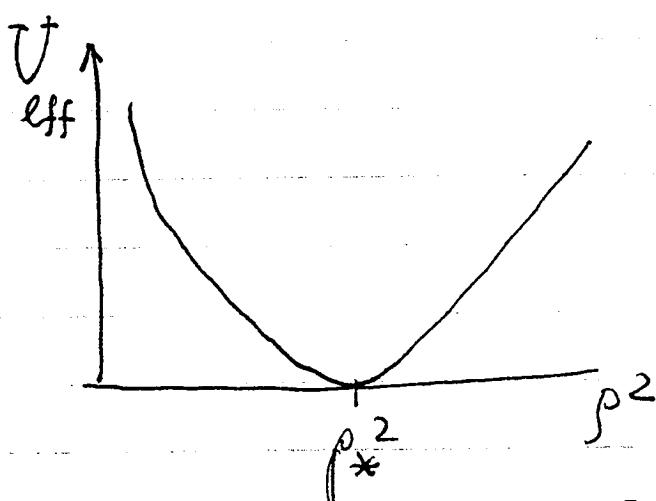
$$p^{*2} = \frac{2\hbar c}{eB_0} m$$

$$V_{\text{eff}}^{\min} = \hbar \omega_c \cdot m$$

$$\text{where } \omega_c = \frac{eB_0}{\mu c}$$

- cyclotron frequency

$$m < 0$$



$$p^{*2} = -\frac{2\hbar c}{eB_0} m$$

$$V_{\text{eff}}^{\min} = 0$$

$$u(p) \sim \frac{1}{\sqrt{K_p}} e^{i \int_{p_1}^p K_p d\beta}$$

$$K_p^2 = \frac{2M}{t^2} (E' - V_{\text{eff}}(p))$$

(5)

From condition of quantization

$$\oint k_p dp = \pi \left(n + \frac{1}{2}\right)$$

$$\text{or } \int_{P_1}^{P_2} dp \sqrt{\frac{2M}{\hbar^2} E' - \frac{1}{p^2} \left(m + \frac{eB_0}{2\hbar c} p^2\right)^2} = \pi \left(n + \frac{1}{2}\right)$$

we have an energy spectrum

$$E = \frac{\hbar^2 k_z^2}{2\mu} + V_{\text{eff}}^{\min} + \hbar\omega_c \left(n + \frac{1}{2}\right)$$

The first term - continuous part of the energy spectrum corresponding to uniform motion along B_0 .
Second part

$\hbar\omega_c \left(\frac{m}{2} + \frac{1}{2}\right)$ - energy spectrum of gyration

and $\hbar\omega_c \left(n + \frac{1}{2}\right)$ - energy spectrum corresponding to radial oscillations.

Classical circular orbits correspond to the case $n \ll m$ ($m > 0$)