

SCORE \_\_\_\_\_

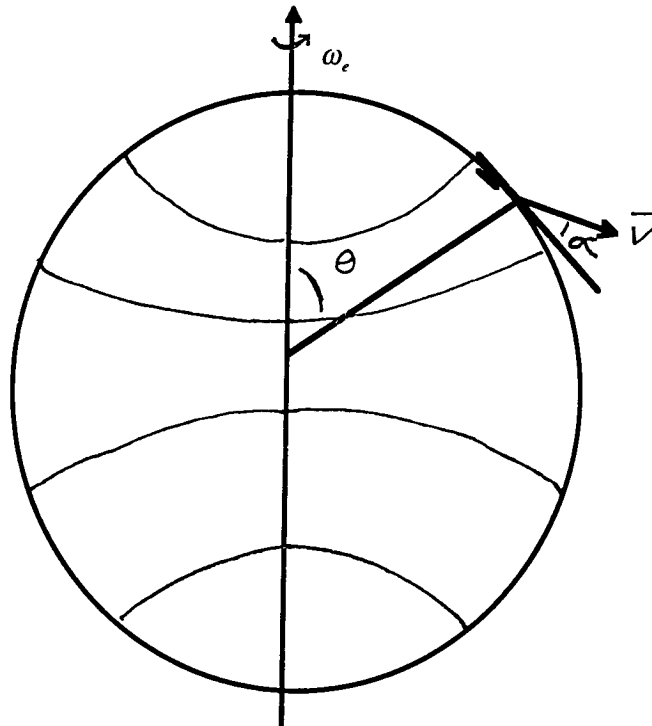
Physics Departmental Examination - Spring 1996 - Part I

Student Identification # \_\_\_\_\_

**Problem 1.**

A missile is fired from a point on the earth's surface with colatitude  $\theta$ . The missile's initial velocity  $V$  has a horizontal component which points due south, while its initial angle of inclination with respect to the horizontal is  $\alpha$  (see figure). Compute the magnitude and direction of the missile's Coriolis force deflection upon its landing.

Work to lowest nontrivial order in  $\omega_e$ , the earth's rotational frequency. You should assume that the acceleration due to gravity and colatitude do not change appreciably over the missile's trajectory. Your answer must *only* involve the parameters  $V$ ,  $\alpha$ ,  $\theta$ ,  $\omega_e$ , and  $g$ .



SOLUTION: PROB. 1

*Solution* – The equation of motion of a particle near the earth's surface is, to lowest order in  $\omega_e$  (i.e. neglecting the centrifugal term),

$$\ddot{\mathbf{r}} = -g \hat{\mathbf{z}} - 2\omega_e \times \dot{\mathbf{r}} ,$$

where  $\hat{\mathbf{z}}$  is the local surface normal (points up). We solve this equation perturbatively, writing  $\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\eta}$ , where  $\boldsymbol{\eta}$  is of order  $\omega_e$ . Thus,

$$\ddot{\mathbf{r}}_0 + \ddot{\boldsymbol{\eta}} = -g \hat{\mathbf{z}} - 2\omega_e \times (\dot{\mathbf{r}}_0 + \dot{\boldsymbol{\eta}})$$

which gives

$$\ddot{\mathbf{r}}_0 = -g \hat{\mathbf{z}}$$

$$\ddot{\boldsymbol{\eta}} = -2\omega_e \times \dot{\mathbf{r}}_0$$

with the solution

$$\mathbf{r}_0(t) = \mathbf{r}_0(0) + \mathbf{V}t - \frac{1}{2}g t^2 \hat{\mathbf{z}}$$

$$\boldsymbol{\eta}(t) = -\omega_e \times \mathbf{V} t^2 + \frac{1}{3}g t^3 \omega_e \times \hat{\mathbf{z}} .$$

Now writing  $\omega_e = \omega_e \cos \theta \hat{\mathbf{z}} - \omega_e \sin \theta \hat{\mathbf{x}}$ , we have

$$\begin{aligned} \omega_e \times \mathbf{V} &= (\omega_e \cos \theta \hat{\mathbf{z}} - \omega_e \sin \theta \hat{\mathbf{x}}) \times (V \cos \alpha \hat{\mathbf{x}} + V \sin \alpha \hat{\mathbf{z}}) \\ &= \omega_e V \cos(\theta - \alpha) \hat{\mathbf{y}} \end{aligned}$$

and substituting the time to fall,  $\Delta t = 2V \sin \alpha / g$ , into the solution, we obtain the Coriolis deflection

$$\boldsymbol{\eta}(\Delta t) = -\frac{4\omega_e}{g^2} V^3 \sin^2 \alpha \left( \frac{1}{3} \sin \alpha \sin \theta + \cos \alpha \cos \theta \right) \hat{\mathbf{y}} ,$$

which is in the east/west direction.

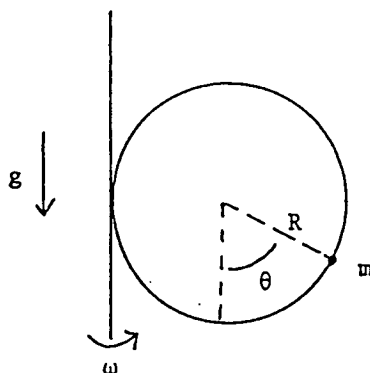
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Physics Departmental Examination - Spring 1996 - Part I

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**Problem 2.**

A bead of mass  $m$  slides without friction on a circular wire hoop of radius  $R$ , and the hoop is constrained to rotate about a vertical axle through a point on the hoop as shown in the figure. The rotation frequency ( $\omega$ ) is constant, and gravity is directed vertically down.



- Obtain a Lagrangian governing the motion of the bead.
- From this Lagrangian, obtain the Hamiltonian.
- Is the Hamiltonian equal to the energy of the mass?
- Is the Hamiltonian conserved? Explain your answer.
- Is the energy conserved? Explain your answer.

#16  
part 2

$$L = \frac{m}{2} [(R + r \cos \theta)^2 \dot{\phi}^2 + R^2 \dot{\theta}^2] - mgr \cos \theta$$

$$b. \quad p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = m(R + r \cos \theta)^2 \dot{\phi}$$

$$H = \sum_i \dot{q}_i p_i - L = \frac{p_{\phi}^2}{2m(R + r \cos \theta)^2} - \frac{m}{2} R^2 (\dot{\theta})^2 - mgr \cos \theta$$

c. no

should be -

$$E = \frac{m}{2} R^2 \dot{\theta}^2 + \frac{p_{\phi}^2}{2m(R + r \cos \theta)^2} + mgr \cos \theta$$

d. yes, since  $\frac{\partial H}{\partial t} = 0$ 

e. no, since forces of constraint do work on mass.

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**Physics Departmental Examination - Spring 1996 - Part I**

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**Problem 3.**

In studying the hydrogen atom one takes the proton to be a point charge of mass  $m_p$ .

Suppose the proton charge is distributed uniformly over a radius  $r_0 \approx 10^{-13}$  cm. Using perturbation theory, calculate approximately the shift in energy of the 1s level of hydrogen. Give an order of magnitude estimate of the ratio of the 2p and 1s level shifts.

$$\rho(r) = \frac{q}{\frac{4\pi}{3}r_0^3} \quad \Theta(r_0 \leftarrow r_0)$$

$\Rightarrow$

$$\nabla^2 \phi = -4\pi\rho$$

$$\phi(\vec{x}) = \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} \rho$$

=



$$\int \vec{E} \cdot d\vec{S} = 4\pi Q = q \frac{4\pi}{3} r^3 / \frac{4\pi}{3} r_0^3 \quad 0 \leq r \leq r_0$$

$$= 4\pi q \quad r > r_0$$

$$4\pi r^2 E_r = 4\pi q \frac{r^3}{r_0^3} \quad 0 \leq r \leq r_0$$

$$= 4\pi q \quad r \geq r_0$$

$$0 \leq r \leq r_0 \quad E_r = \frac{q r}{r_0^3} \quad E_r = -\frac{\partial \phi}{\partial r}$$

$$= \frac{q}{r^2}$$

$$\text{so } \phi = -\frac{q r^2}{2 r_0^3} \quad 0 \leq r \leq r_0$$

$$\phi = -q/r \quad r > r_0$$

$$H_0 = \frac{p^2}{2m} - g^2/r$$

$$H_I = \left( \frac{g^2}{r} - \frac{g^2 r^2}{2r_0^3} \right) \quad 0 \leq r \leq r_0$$

$$= 0 \quad r > r_0$$

$$\psi_{1s} = N e^{-r/2a_0}$$

$$= \frac{1}{\sqrt{\pi}} \frac{e^{-r/2a_0}}{a_0^{3/2}}$$

$$\psi_{1p,m=0} = \frac{1}{(32\pi)^{1/2}} \frac{1}{a_0^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$$

$$\Delta E_{1s} = \int d^3r \psi_{1s}^* H_I \psi_{1s}$$

$$= 4\pi g^2 \int_0^{r_0} r^2 dr \left( \frac{1}{\pi a_0^3} \right) e^{-2r/a_0} \left( \frac{1}{r} - \frac{r^2}{2r_0^3} \right)$$

$$= \frac{4}{a_0^3} g^2 \int_0^{r_0} dr r^2 \left[ r - \frac{r^4}{2r_0^3} \right] e^{-2r/a_0}$$

$$r = r_0 x$$

$$\Delta E_{1s} = \frac{4}{a_0^3} g^2 \int_0^1 r_0 dx r_0 \left[ x - \frac{x^4}{2} \right] e^{-2x r_0/a_0}$$

$$\Delta E_{1s} = \frac{4q^2}{a_0^3} r_0^2 \int_0^1 dx \left( x - \frac{x^4}{2} \right) \quad \frac{r_0}{a_0} \ll 1$$

$$= \frac{4q^2}{a_0^3} \cdot \left( \frac{r_0}{a_0} \right)^2 \quad R_y = \frac{q^2}{2a_0} \quad \text{Order 1} = 1 - \frac{1}{10} = 0.9$$

$$= 8 \left( \frac{r_0}{a_0} \right)^2 R_y$$

$$a_0 = \frac{1}{2} \times 10^{-8} \text{ cm}$$

$$r_0 = 10^{-13} \text{ cm}$$

$$\frac{r_0}{a_0} = 2 \times 10^{-5}$$

$$\left( \frac{r_0}{a_0} \right)^2 = 4 \times 10^{-10}$$

$$\Delta E_{1s} = 32 \times 10^{-10} R_y$$

$$R_y = 13.59 \text{ eV}$$

$\Delta E_{2p}$  involves  $|\psi_{2p}|^2$  which has another factor of  $\left( \frac{r_0}{a_0} \right)^2$  more or less i.e. p-wave  $\psi$  is zero near origin!

$$\frac{\Delta E_{2p}}{\Delta E_{1s}} \approx \left( \frac{r_0}{a_0} \right)^2 \approx 10^{-10} !$$



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## Physics Departmental Examination - Spring 1996 - Part I

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**Problem 4.**

We consider an electron in the hydrogen atom with one unit of orbital angular momentum. We refer to the three eigenstates for  $L_z$  as  $Y_1^1$ ,  $Y_1^0$ ,  $Y_1^{-1}$ , with  $L_z Y_l^m = m\hbar Y_l^m$ . At  $t=0$ , a measurement is made with the result that the angular momentum in the x direction is exactly one unit of  $\hbar$ . What is an expression for the angular part of the wave function for the electron, just after the measurement, in terms of the above eigenfunctions?

Note that the functions

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta$$

Solution

$$\vec{L} = \vec{r} \times \vec{p} \quad ; \quad L_z = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

$$L_x = \frac{\hbar}{i} \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$L_y = \frac{\hbar}{i} \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

Eigenfunctions of  $L_z$ :

$$Y_{11} = c \frac{x+iy}{\sqrt{2}}$$

$$; \quad L_z Y_{11} = \hbar Y_{11}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta$$

$$Y_{10} = cz$$

$$L_z Y_{10} = 0$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1} = -c \frac{(x-iy)}{\sqrt{2}}$$

$$L_z Y_{1,-1} = -\hbar Y_{1,-1}$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta$$

$Y$ 's are orthonormal:  $\int d\Omega Y_{lm}^* Y_{l'm'} = \delta_{mm'}$  → determines  $c = \sqrt{\frac{3}{8\pi}}$ .  $Y_{l,-m} = (-1)^m Y_{lm}^*$  by convention

Solve for  $x, y, z$ :  $x = \frac{1}{c} \frac{Y_{11} - Y_{1,-1}}{\sqrt{2}} \quad ; \quad y = \frac{1}{c} \frac{Y_{1,1} + Y_{1,-1}}{\sqrt{2} i} \quad ; \quad z = \frac{1}{c} Y_{1,0}$

Construct matrix elements of  $L_x$  in  $Y_{lm}$  basis:

$$L_x Y_{11} = -\frac{\hbar c}{\sqrt{2}} z = -\frac{\hbar}{\sqrt{2}} Y_{10}$$

$$L_x Y_{10} = c \frac{\hbar}{i} y = -\frac{\hbar}{\sqrt{2}} \frac{Y_{11} + Y_{1,-1}}{\sqrt{2}}$$

$$L_x Y_{1,-1} = -\frac{\hbar c}{\sqrt{2}} z = -\frac{\hbar}{2} Y_{1,0}$$

$$L_x = \begin{matrix} & Y_{11} & Y_{10} & Y_{1,-1} \\ \begin{matrix} Y_{11} \\ Y_{10} \\ Y_{1,-1} \end{matrix} & \begin{pmatrix} 0 & -\frac{\hbar}{\sqrt{2}} & 0 \\ -\frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & -\frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \end{matrix}$$

So in this basis  $L_x$  is the matrix:  $L_x = -\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Eigenvectors of  $L_x$  with eigenvalues  $\pm \hbar$ :  $| \psi \rangle = \frac{1}{2} (1, -\sqrt{2}, 1)$  ;  $L_x | \psi \rangle = \hbar | \psi \rangle$

hence

$$| \psi \rangle = \frac{1}{2} ( Y_{1,1} + Y_{1,-1} - \sqrt{2} Y_{1,0} )$$

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**Physics Departmental Examination - Spring 1996 - Part I**

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**Problem 5.**

The temperature of the cosmic black body background radiation is currently 2.9 K. If the universe expanded quasi-statically from an initial state where the radiation temperature was  $T=3000$  K, what is the ratio of its current volume to its initial volume? Assume the universe is thermally insulated from its environment and has no matter.

Solution:

$$dE = -PdV \quad \text{adiabatic, quasi-static process.}$$

$$P = \frac{\mu(T)}{3} \quad \text{for photon gas, } \mu = \text{energy density}$$

$$E = \mu V$$

$$dE = d\mu \cdot V + dV \cdot \mu = -\frac{\mu}{3} \cdot dV \Rightarrow V d\mu = -\frac{4}{3} \mu dV \Rightarrow$$

$$\frac{d\mu}{\mu} = -\frac{4}{3} \frac{dV}{V} \Rightarrow \mu = \frac{C}{V^{4/3}} \Rightarrow \mu V^{4/3} = \text{constant}$$

$$\text{For photon gas, } \mu \propto T^4 \Rightarrow TV^{1/3} = \text{constant}$$

$$\Rightarrow \boxed{\frac{V_{\text{final}}}{V_{\text{initial}}} = \left( \frac{T_{\text{initial}}}{T_{\text{final}}} \right)^3 = 1.1 \times 10^9}$$

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**Physics Departmental Examination - Spring 1996 - Part 1**

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**Problem 6.**

- (a) Derive an expression for the vertical temperature gradient in an adiabatic atmosphere (processes in such an atmosphere are undergoing only adiabatic transformation). Use the properties of ideal gases and write the differential equation for the change of density with height. Your answer should be in terms of the specific heat at constant pressure and the acceleration of gravity.
  
- (b) Now assume that real temperature gradient differs from an adiabatic. What condition must this temperature gradient satisfy to assure the stability of atmosphere against vertical convection following the adiabatic law.

Solution.

1. For the adiabatic process  $p \sim \rho^\gamma$  where  $\gamma = \frac{C_p}{C_v}$  specific heat ratio.

2. Ideal gas law  $p = \rho RT$ ,  $R = C_p - C_v$ .

Hence for the adiabatic process  $T \sim \rho^{\gamma-1}$

3. From hydrostatic pressure balance  $dp = -\rho g dz$ . Since

$$p = p_0 \left( \frac{\rho}{\rho_0} \right)^\gamma \quad (p_0, \rho_0 \text{ - pressure and density at } z=0 \text{ level}), \text{ we}$$

can write  $\gamma \frac{p_0}{\rho_0^\gamma} \rho^{\gamma-2} d\rho = -g dz$  and after integration to obtain:

$$\frac{\gamma}{\gamma-1} RT_0 \left[ \left( \frac{\rho}{\rho_0} \right)^{\gamma-1} - 1 \right] = -gz.$$

$$\text{Since } \frac{\gamma}{\gamma-1} R = \frac{C_p - C_v}{C_p - C_v} C_p = C_p, \quad \left( \frac{\rho}{\rho_0} \right)^{\gamma-1} - 1 = \frac{T}{T_0} - 1,$$

we can finally write adiabatic law for the temperature:

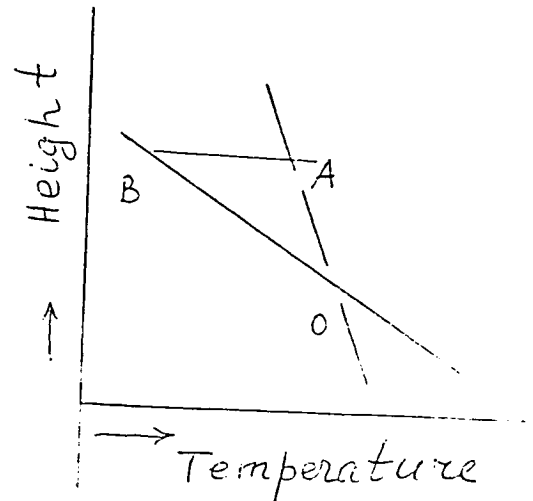
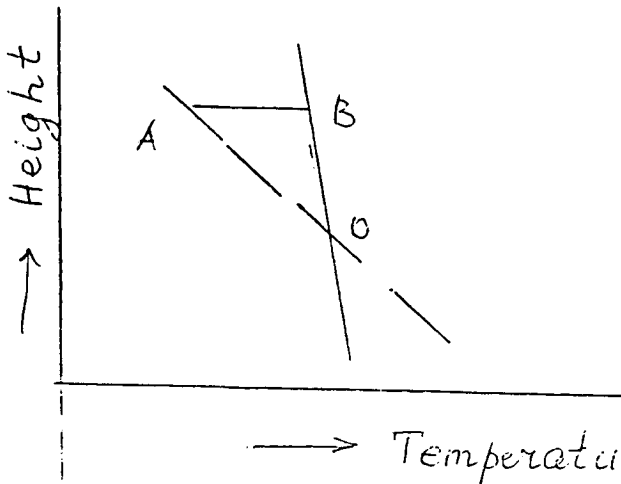
$$T - T_0 = -\frac{g}{C_p} z$$

Temperature gradient:

$$\frac{dT}{dz} = -\frac{g}{C_p} = -\Gamma_{ad}, \quad \Gamma_{ad} \text{ is called adiabatic lapse rate.}$$

Consider a layer of atmosphere in which the actual lapse rate  $\Gamma$  is less than  $\Gamma_{ad}$  (see Fig. A). If a test parcel of air originally located at a level 0 is raised slightly to the level defined by points A and B, its temperature will fall to  $T_A$ , which is lower than environmental temperature  $T_B$  at this level. Since the parcel immediately adjusts to the pressure of its environment, it follows from the ideal gas law that the colder parcel must be denser than the warmer environmental air. Therefore, the parcel tends to return to its original level. It is a stable atmosphere. Repeating the same arguments for the case  $\Gamma > \Gamma_{ad}$ , it is easy to see that the test parcel will be less dense than its environment and continue to raise - unstable atmosphere.

(A) Stable atmosphere (B) Unstable atmosphere.



Solid - actual temperature profile,  
 Dashed - adiabatic temperature profile.

The larger is positive difference

$$\Gamma_{ad} - \Gamma$$

the greater is the restoring force for a given displacement and the greater is atmosphere stability.

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**Physics Departmental Examination - Spring 1996 - Part I**

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**Problem 7.**

An electric dipole of strength  $p$  is oriented perpendicular to and at a distance  $d$  from an infinite conducting plane. Calculate the force exerted on the plane by the dipole.



REPLACE PLANE  
BY IMAGE DIPOLE

FORCE ON  $\vec{p}$

$$\vec{F} = (\vec{p} \cdot \nabla) \vec{E}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \left[ \frac{3(\vec{p}_1 \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{p}_1}{r^3} \right]$$

$$4\pi\epsilon_0 \vec{F} = \frac{3(\vec{p}_1 \cdot \vec{p})\vec{r}}{r^5} + \frac{3(\vec{p}_1 \cdot \vec{r})\vec{p}}{r^5} - \frac{15(\vec{p}_1 \cdot \vec{r})(\vec{p} \cdot \vec{r})\vec{r}}{r^7} + \frac{3\vec{p}_1(\vec{p} \cdot \vec{r})}{r^5}$$

$$\vec{p}_1 = \vec{p} \quad \frac{\vec{p}}{|\vec{p}|} = \frac{\vec{r}}{|\vec{r}|} = \hat{p} \quad |\vec{p}| = p \quad |\vec{r}| = r = 2d$$

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \left[ \frac{3p^2\vec{r}}{r^5} + \frac{3p^2\vec{r}}{r^5} - \frac{15p^2\vec{r}}{r^5} + \frac{3p^2\vec{r}}{r^5} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \left[ \frac{-6p^2}{r^4} \right] \hat{p}$$

$$= -\frac{3p^2}{32\pi\epsilon_0 d^4} \hat{p}$$

FORCE ON PLANE IS NEGATIVE OF THIS

$$\vec{F}_{\text{PLANE}} = -\vec{F} = \frac{3p^2}{32\pi\epsilon_0 d^4} \hat{p} \quad (\text{ATTRACTIVE})$$

POINT DISTRIBUTION

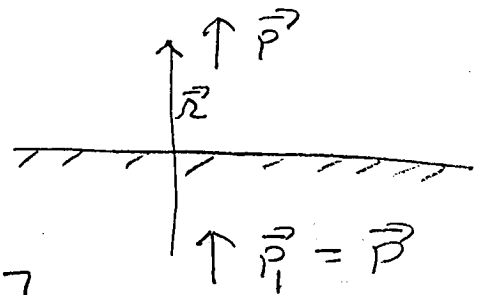
CORRECT IMAGE +2

CORRECT FORCE +2

DEPENDENCE ON  $p, d$  +2

SIGN +2

NUMERICAL FACTOR +2



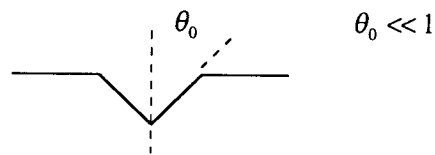
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Physics Departmental Examination - Spring 1996 - Part I

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**Problem 8.**

Above the flat metallic surface an electric field is homogeneous and normal to the surface. At a point on the surface is a conical depression of angle  $\theta_0$ . Find the electric field as function of position inside of the depression



Hint: Due to the boundary conditions the electric field is a rapid function of  $r$  inside of the depression. The Laplacian in spherical coordinates is

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{d\phi}{d\theta} \right)$$

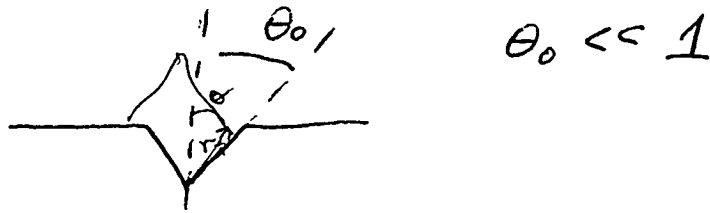
(A)

E. M. part 1

SOLUTION: PROB. 8

problem in electrostatic.

Determine the field near the end of a sharp conical depression on the surface of conductor (see fig.)



Hint: Use spherical coordinates with the origin at the vertex of the cone and the polar axis along its axis. Expression for the Laplacian in spherical coordinates could be useful:

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right)$$

Solution.

Separating variables in the Laplace equation  $\nabla^2 \phi = 0$

$$\phi = R(r) Y(\theta),$$

we have two equations

$$\frac{d}{dr} (r^2 \frac{dR}{dr}) = C R(r)$$

$$\frac{1}{\sin\theta} \frac{d}{d\theta} (\sin\theta \frac{dY}{d\theta}) = -C Y$$

C - const. Solution of the first equation

$$R(r) = r^\alpha$$

where  $\alpha$  is a positive root of a quadratic equation

$$\alpha(\alpha+1) = C$$

Second equation is to be solved for  $0 < \theta < \theta_0 \ll 1$ , i.e. outside conductor. Hence  $\sin\theta \rightarrow \theta$  and  $Y(\theta)$  is fastly changing function of  $\theta$ , i.e.  $C \gg 1$ . In that case

and  $\alpha \approx \sqrt{C}$  equation for  $Y$  acquires a form of standard Bessel equation.

$$\frac{d^2 Y}{d\theta^2} + \frac{1}{\theta} \frac{dY}{d\theta} + \alpha^2 Y = 0$$

(regular at  $\theta=0$ )

Solution Boundary

$$Y = Y_0(\alpha\theta)$$

condition  $\phi(\theta=\theta_0)=0$

i.e.

$$Y_0(\alpha\theta_0) = 0$$

Only smallest root of this equation has a physical sense. (Inside the hole  $E_\theta$  can't change the sign or  $\phi$  must be monotonic function of  $\theta$ ).

Hence

$$\alpha = \frac{2.405}{\theta_0}$$

$$\phi = r^\alpha \cdot Y_0(\alpha \theta)$$

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**Physics Departmental Examination - Spring 1996 - Part I**

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**Problem 9.**

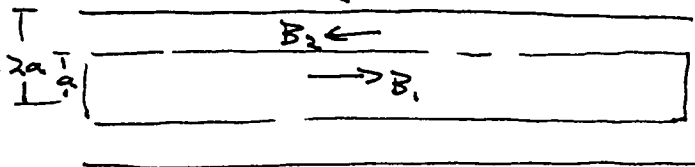
A long solenoid having radius  $a$ , length  $\ell$  and  $N$  turns is placed inside a perfectly conducting cylindrical shell of radius  $2a$ , and length  $\ell$ , and is coaxial with it.\* When a current  $I$  is passed through the solenoid, what force per unit area is exerted on the outer shell?

\*Initially there is no flux through the cylinder.

A long solenoid having radius  $a$ , length  $l$  and  $N$  turns is placed inside a perfectly conducting cylindrical shell of radius  $2a$ , and length  $l$ , and is coaxial with it.

When a current  $I$  is passed through the solenoid, what total force <sup>per unit area</sup> is exerted on the outer shell?

ans: Total flux through the outer shell must remain zero



Thus,

$$\pi a^2 B_1 = -B_2 \cdot \pi((2a)^2 - a^2) = -3\pi a^2 B_2$$

or  $B_1 = -3B_2$

By Ampere's law,

$$B_1 - B_2 = \frac{\mu_0 N I}{l} = -4B_2$$

Force <sup>per unit area</sup> on the outer shell is magnetic pressure ~~times area~~:

$$\begin{aligned} F &= \frac{B_2^2}{2\mu_0} \cdot 4\pi a l = \frac{1}{2\mu_0} \cdot \left(\frac{\mu_0 N I}{4l}\right)^2 \cdot 4\pi a l \\ &= \frac{\mu_0 (N I)^2 \cdot \pi a}{32 l^2} \end{aligned}$$

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I = \frac{4\pi}{c} NI$$

where  $c = 9 \times 10^9$

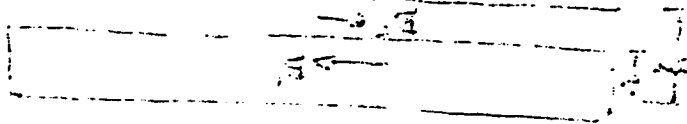
$$\therefore \mu_0 \rightarrow \frac{4\pi}{c}$$

$$\text{or } \frac{\mu_0}{4\pi} \rightarrow \frac{1}{c}$$

Then  $\frac{\mu_0}{4\pi} \left( \frac{NI}{c} \right)^2 \rightarrow \frac{4\pi}{8c} \left( \dots \right)$

$$B = \mu_0 \frac{NI}{c} = \frac{4\pi}{c} \frac{NI}{c}$$

$$\frac{\mu_0 (NI)^2}{32 c}$$



$$\int \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$\frac{4\pi}{c} \frac{NI}{c} \cdot l = \mu_0 I$$

$$\frac{4\pi}{c} \frac{NI}{c} \cdot l = \mu_0 I$$

$$\frac{4\pi}{c} \frac{(NI)^2}{c} \cdot l = \mu_0 I^2$$



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**Physics Departmental Examination - Spring 1996 - Part II**

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**Problem 10.**

Consider the following differential equation for  $y(x)$  :

$$\frac{d^2y}{dx^2} - x^2y = 0.$$

- (a) What is the form of the two possible asymptotic behaviors of  $y(x)$  as  $x \rightarrow +\infty$ ?
- (b) How does the general solution for  $y(x)$  behave near  $x=0$ ?
- (c) Give the formal solution for  $y(x)$  [which derives from your answer to part (b)] as an infinite series expansion about the point  $x=0$ . [This will contain two arbitrary constants].

We are considering the equation

$$\frac{d^2 y}{dx^2} - x^2 y = 0$$

a) Near  $\infty$  look for a WKB solution  $y \sim e^{\int k dx}$ .

We find  $k^2 + k' = x^2$ . For large  $x$ ,  $k = \pm x$  ( $k^2 \gg k'$  for  $|x| > 1$ ).

Hence  $y \sim e^{\pm \frac{x^2}{2}}$ . (Next order WKB (not required) would

give  $y \sim \frac{1}{\sqrt{|x|}} e^{\pm \frac{x^2}{2}}$ .)

b) Near  $x = 0$  the solution is clearly  $a + bx$ .

c) In a power series, successive terms will differ by  $x^4$  as can be seen from the D.E.

$$\text{Try } y_1 = \sum_{n=0}^{\infty} a_n x^{4n} \quad \text{and } y_2 = \sum_{n=0}^{\infty} b_n x^{4n+1} \text{ which}$$

have the proper behavior near  $x=0$ . On substitution

$$(4n)(4n-1) a_n = a_{n-1}$$

$$a_n \approx \prod_{m=1}^n \frac{1}{4m \times 4m-1} \approx \frac{1}{(16)^n n! n^{-1/4}!}$$

$$\text{Hence } y_1 = \alpha \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{4n}}{n! n^{-1/4}!} \quad \text{and } y_2 = \beta \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{4n+1}}{n! n+1/4!}$$

d) We want to investigate the behavior ~~near~~ for large  $x$ . The ~~series obtained~~ terms in the series obtained in c will have a maximum for large  $n \sim O(x^2)$  and many terms contribute. Hence it is approximately correct to replace the sum by an integral

$$y_0 = \alpha \int_0^{\infty} dn \frac{\left(\frac{x}{2}\right)^{4n}}{n! n^{-1/4}!} + \beta \int_0^{\infty} dn \frac{\left(\frac{x}{2}\right)^{4n+1}}{n! n^{1/4}!}$$

Both integrals become large for large  $x$ . Hence if we want  $y$  to be small at  $\infty$  we must choose the appropriate  $\beta/\alpha$  so the two ~~integrals~~ expressions cancel. Changing the variable in the second integral  $n = n' - 1/4$  we see readily that the cancellation requires  $\alpha = -\beta$ . Hence near

$$x=0 \quad y = \alpha \left[ \frac{1}{(-1/4)!} + \frac{x}{2(1/4)!} \right]$$

and:

$$\frac{y'}{y} = - \frac{(-1/4)!}{2(1/4)!}$$

## Physics Departmental Examination - Spring 1996 - Part II

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**Problem 11.**

If the solar system were immersed in a uniformly dense spherical cloud of weakly-interacting massive particles (WIMPs), then objects in the solar system would experience gravitational forces from both the sun and the cloud of WIMPs such that

$$F_r = -\frac{k}{r^2} - br.$$

Assume that the extra force due to the WIMPs is very small (i.e.,  $b \ll k/r^3$ ).

- a) Find the frequency of radial oscillations for a nearly circular orbit.
- b) Find the average angular velocity.
- c) Find the rate of precession of the perihelion to lowest order in (b).

The kinetic energy, which separates into a term due to the bead's motion along the wire and a term due to the rotation of the bead with the wire, is

$$T = \frac{1}{2}m\dot{\alpha}^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2. \quad (10.80)$$

The Lagrangian is  $L = T - V$ . Using Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \quad (10.81)$$

we find that

$$a\ddot{\theta} + g\sin\theta - a\omega^2\cos\theta\sin\theta = 0. \quad (10.82)$$

At an equilibrium point  $\ddot{\theta} = 0$ , so  $g = a\omega^2\cos\theta$ , or  $\omega^2 = g/a\cos\theta$ . This equation has a solution for  $\omega$  only if  $\omega^2 \geq g/a$ , so the critical angular velocity is

$$\omega_c = \sqrt{\frac{g}{a}}, \quad (10.83)$$

and the equilibrium angle is

$$\theta_0 = \cos^{-1}\left(\frac{g}{a\omega^2}\right). \quad (10.84)$$

b) If the mass makes small oscillations around the equilibrium point  $\theta_0$ , then we can describe the motion in terms of a small parameter  $\phi = \theta - \theta_0$ . The equation of motion (10.82) becomes

$$a\ddot{\phi} + g\sin(\theta_0 + \phi) - a\omega^2\cos(\theta_0 + \phi)\sin(\theta_0 + \phi) = 0. \quad (10.85)$$

Using standard trigonometric identities, the small angle approximations  $\sin\phi \approx \phi$  and  $\cos\phi \approx 1$ , and our solution for  $\theta_0$  (10.84), it is easy to show that

$$\ddot{\phi} + \omega^2\left(1 - \frac{g^2}{a^2\omega^4}\right)\phi = 0. \quad (10.86)$$

This has the general solution

$$\phi = A\cos\Omega t + B\sin\Omega t, \quad (10.87)$$

### 1.10. SOLAR SYSTEM WIMPS

where

$$\Omega = \omega\sqrt{1 - \frac{g^2}{a^2\omega^4}}, \quad (10.88)$$

and  $A$  and  $B$  are arbitrary constants. The period of oscillation is  $2\pi/\Omega$ .

SOLUTION: Prob. 11

**Solution 1.10.** In plane-polar coordinates, the Lagrangian for a particle moving in a central potential  $V(r)$  is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \quad (10.89)$$

where  $m$  is the mass of the particle. The potential is given in the question as

$$V(r) = -\frac{k}{r} + \frac{1}{2}br^2. \quad (10.90)$$

The  $\theta$ -component of Lagrange's equation is

$$\frac{\partial L}{\partial \theta} = mr^2\dot{\theta} = \text{constant} \equiv l. \quad (10.91)$$

The hamiltonian of our system is then

$$H = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} + V(r) = \frac{p_r^2}{2m} + V_{\text{eff}}(r), \quad (10.92)$$

with  $p_r = m\dot{r}$  and

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r). \quad (10.93)$$

The term  $l^2/2mr^2$  is referred to as an "angular momentum barrier." Solving the equations of motion for this hamiltonian is equivalent to solving Lagrange's equations for the Lagrangian:

$$L = \frac{1}{2}m\dot{r}^2 - V_{\text{eff}}(r). \quad (10.94)$$

This is a completely general result for the motion of a particle in a central potential and could easily have been our starting point in this problem (e.g., Goldstein, Chapter 3).

It may seem unnecessarily long-winded to go through this procedure, but note that the sign of the angular momentum barrier in (10.94) is *opposite* to what we would have gotten if we had naively replaced  $\theta$  with  $l/mr^2$  in the Lagrangian (10.89). This is due to the fact that the Lagrangian is a function of the time derivative of the position, and not of the canonical momentum.

The equation of motion from (10.94) is

$$m\ddot{r} = -\frac{d}{dr} V_{\text{eff}}(r). \quad (10.95)$$

If the particle is in a circular orbit at  $r = r_0$  we require that the force on it at that radius should vanish,

$$\left. \frac{dV_{\text{eff}}}{dr} \right|_{r=r_0} = 0. \quad (10.96)$$

Using our expression for  $V_{\text{eff}}$  (10.93), we derive an expression relating the angular momentum  $l$  to the radius of the orbit  $r_0$ :

$$\frac{l^2}{mr_0^3} - \frac{k}{r_0^2} - br_0 = 0. \quad (10.97)$$

We are interested in perturbations about this circular orbit. Provided the perturbation remains small, we can expand  $V_{\text{eff}}(\tau)$  about  $r_0$ ,

$$V_{\text{eff}}(\tau) = V_{\text{eff}}(r_0) + (\tau - r_0)V'_{\text{eff}}(r_0) + \frac{1}{2}(\tau - r_0)^2 V''_{\text{eff}}(r_0) + \dots \quad (10.98)$$

If we use this expansion in the Lagrangian (10.94) together with the condition (10.96), we find

$$L = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}(\tau - r_0)^2 V''_{\text{eff}}(r_0), \quad (10.99)$$

where we have dropped a constant term. This is just the Lagrangian for a simple harmonic oscillator, describing a particle undergoing radial oscillations with frequency

$$\omega^2 = \frac{1}{m} V''_{\text{eff}}(r_0). \quad (10.100)$$

### 1.10. SOLAR SYSTEM WIMPS

Differentiating  $V_{\text{eff}}(\tau)$  twice gives us

$$\frac{3l^2}{mr_0^4} - \frac{2k}{r_0^3} + b = m\omega^2. \quad (10.101)$$

We can eliminate  $l$  between equations (10.101) and (10.97) to give the frequency of radial oscillations:

$$\omega = \left( \frac{k}{mr_0^3} + \frac{4b}{m} \right)^{1/2}. \quad (10.102)$$

To find the rate of precession of the perihelion, we need to know the period of the orbit. From the definition of angular momentum equation (10.91), we have an equation for the orbital angular velocity  $\omega_1$ ,

$$\omega_1 \equiv \frac{d\theta}{dt} = \frac{l}{mr^2}. \quad (10.103)$$

Let us write  $\tau(t) = r_0 + \epsilon(t)$ , where  $\epsilon(t)$  is sinusoidal with frequency  $\omega$  and average value zero. We substitute  $\tau(t)$  into equation (10.103) and expand in  $\epsilon(t)$ :

$$\frac{d\theta}{dt} = \frac{l}{mr_0^2} \left( 1 - \frac{2\epsilon}{r_0} + \mathcal{O}(\epsilon^2) \right). \quad (10.104)$$

To zeroth order in the small quantities  $br_0^3/k$  and  $\epsilon/r_0$ , the period of the orbit  $T_1$  is the same as the period of oscillations  $T_2 = 2\pi/\omega$ . Therefore we can average  $\epsilon$  over  $T_1$  rather than  $T_2$  and still get zero, to within terms of second order, which we are neglecting. The average angular velocity is therefore

$$\bar{\omega}_1 = \frac{2\pi}{T_1} \approx \frac{l}{mr_0^2} = \sqrt{\frac{k}{mr_0^3} + \frac{b}{m}}, \quad (10.105)$$

where we have made use of (10.97).

Now consider one complete period of the radial oscillation. This takes place in time  $T_2 = 2\pi/\omega$ . In this time the particle travels along its orbit through an angle of

$$\theta = 2\pi \frac{\bar{\omega}_1}{\omega} = 2\pi \frac{\sqrt{k/mr_0^3 + b/m}}{\sqrt{k/mr_0^3 + 4b/m}}$$

$$\approx 2\pi \left(1 - \frac{3br_0^3}{2k}\right). \quad (10.106)$$

In other words, the particle does not quite orbit through  $2\pi$  before the radial oscillation is completed. Each time around the perihelion precesses backwards through an angle

$$\delta\theta = 3\pi \frac{br_0^3}{k}, \quad (10.107)$$

and it gets around in time  $T_2$ , so the precession rate is

$$\begin{aligned} \alpha &= \frac{\delta\theta}{T_2} = \frac{3\pi br_0^3}{k} \frac{\sqrt{k/mr_0^3 + 4b/m}}{2\pi} \\ &\approx \frac{3b}{2} \sqrt{\frac{r_0^3}{mk}}. \end{aligned} \quad (10.108)$$

b) When  $r$  is large enough that  $F_r \approx -br$ , we see that the force is like that of a linear spring. In this case the planar motion of the orbit can be resolved into simple harmonic motion in each of its three cartesian components. Thus the orbits will in general be ellipses; however, in each case the sun will be at the *center* of the ellipse rather than at one of the foci (as is the case for Newtonian gravity).

Solution 1.9. a) The potential energy of the bead is

$$V = -mga \cos \theta. \quad (10.79)$$



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Physics Departmental Examination - Spring 1996 - Part II

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**Problem 12.**

A system of  $N$  bosons in two dimensions has an energy-momentum relation

$$\epsilon_p = cp^{3/2}$$

and density  $n = N/A$  ( $A = \text{area}$ ).

- (a) Show that at low temperatures the system will Bose-condense, and that the Bose condensation temperature  $T_c \sim n^\alpha$ . Find  $\alpha$ .
- (b) Show that the entropy below  $T_c$  goes as  $S \sim T^\beta$ , and the “pressure” below  $T_c$  (i.e. its equivalent in two dimensions) goes as  $P \sim T^\gamma$ . Find  $\beta$  and  $\gamma$ .

Hint: The differential of the grand potential is  $d\theta = -sd - pdA + nd\mu$  where  $\mu$  is the chemical potential.

Solution

The number equation is

$$N = \sum_p \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} = \frac{A}{(2\pi)^2} \int d^2p \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1}$$

Bose condensation occurs if the integral is finite for  $\mu \rightarrow 0$ . At small  $p$ ,

$$\int \frac{d^2p}{e^{\beta\epsilon_p} - 1} = \int \frac{p dp}{\beta \epsilon_p} = \int \frac{p dp}{c p^{3/2}} = \int \frac{dp}{p^{1/2}} \text{ is finite} \Rightarrow \text{it Bose condenses.}$$

 $T_c$  is given by:

$$N = \frac{A}{(2\pi)^2} \int d^2p \frac{p}{e^{\beta_c c p^{3/2}} - 1} \quad ; \text{ change variables: } \beta_c c p^{3/2} = x \Rightarrow p = \left(\frac{A T_c}{c}\right)^{2/3} x^{2/3} = 1$$

$$N = \frac{A}{(2\pi)^2} \cdot \left(\frac{A T_c}{c}\right)^{4/3} \cdot \frac{2}{3} \int dx \frac{x^{1/3}}{e^x - 1} \Rightarrow \boxed{T_c \sim N^{3/4}} \Rightarrow \boxed{\alpha = 3/4}$$

(b) Calculate grand-partition function

$$\mathcal{Z} = \sum e^{-\beta(\epsilon_p - \mu) N}$$

$$Q = -kT \ln \mathcal{Z} \quad \text{grand potential}$$

$$dQ = -S dT - P dA + N d\mu \quad (\text{here, } V \rightarrow A)$$

$$\Rightarrow \left. \left( \frac{\partial Q}{\partial V} \right)_{T, \mu} \right|_{S, \mu} = S = - \left. \frac{\partial Q}{\partial T} \right|_{A, \mu} \quad ; \text{ below } T_c, \mu = \text{const}, S = - \left. \frac{\partial Q}{\partial T} \right|_A$$

$$P = - \left. \frac{\partial Q}{\partial A} \right|_{T, \mu} \quad ; \text{ below } T_c, P = - \left. \frac{\partial Q}{\partial A} \right|_T$$

So,

$$\mathcal{Z} = \prod_p \sum_{n_p} e^{-\beta(\epsilon_p - \mu) n_p} = \prod_p \frac{1}{1 - e^{-\beta(\epsilon_p - \mu)}}$$

$$Q = +kT \sum_p \ln(1 - e^{-\beta(\epsilon_p - \mu)}) = \frac{kTA}{(2\pi)^2} \int d^2p \ln(1 - e^{-\beta(\epsilon_p - \mu)})$$

Below  $T_c$ ,  $\mu = 0$ . Changing variables,  $p = \left(\frac{kT}{c}\right)^{2/3} \chi^{2/3} \Rightarrow$

$$Q = kT \cdot T^{4/3} \cdot A \cdot \#s \sim T^{7/3} A$$

$$\Rightarrow S = -\frac{\partial Q}{\partial T} \sim T^{4/3} \Rightarrow \boxed{\beta = 4/3}$$

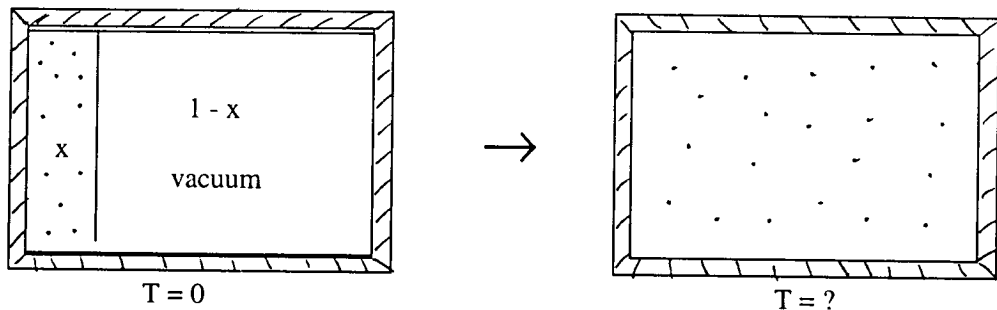
$$P = -\frac{\partial Q}{\partial A} \sim T^{7/3} \Rightarrow \boxed{\gamma = 7/3}$$

## Physics Departmental Examination - Spring 1996 - Part II

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**Problem 13.**

A gas of fermions occupies initially a fraction  $x$  of the volume of a container. It is separated by a partition from the rest of the container where there is vacuum. The system is thermally insulated and at  $T=0$ . Assume the fermions are non-interacting point particles, spin  $1/2$ , and have no internal degrees of freedom. The Fermi temperature of the fermion system is  $T_f = 75,000$  K. The partition is then removed, and the gas undergoes free expansion to occupy the entire volume.



Calculate the final temperature for (i)  $x=0.01$  and (ii)  $x=0.99$ . Justify any approximations you make.

Hints: In the initial state, it would take an amount of energy  $Q=2733$  ergs to raise the temperature of the system by 1 K at constant volume, and it would take  $Q=4.98$  joules to raise the Fermi temperature of the system by 1 K at constant temperature.

$$\begin{aligned} \text{Boltzmann constant:} \quad k &= 1.38 \times 10^{-16} \text{ ergs/K} \\ &= 1.38 \times 10^{-23} \text{ joules/K} \end{aligned}$$

$$\text{Avogadro's number:} \quad N_A = 6.02 \times 10^{23}$$

Solution

Initially, total energy is ground state energy of fermion system:

$$E_0 = c N k T_F$$

Can find  $c$  from hint: to raise  $T_F$  by 1K,  $Q = 4.98$  joules  $\Rightarrow$

$$4.98 \text{ joules} = c \cdot N_A \cdot k \Delta T_F \Rightarrow c = \frac{4.98}{6.02 \times 10^{23} \cdot 1.38 \times 10^{-23}} = 0.6 = \frac{3}{5}$$

$$\text{So, } E_0 = \frac{3}{5} N k T_F = \frac{3}{5} N E_F$$

When system expands,  $E_F$  decreases:

$$E_F \propto n^{2/3}$$

Since energy is conserved

(no work is performed),  $T$  increases.

(i). System expands by a factor of 100  $\Rightarrow$

$$E_F' = \frac{E_F}{100^{2/3}} = \frac{E_F}{21.5} = 3481 \text{ K}$$

So system will be classical in this regime. Mean energy is given by equipartition

$$E = \frac{3}{2} NkT = E_0 = \frac{3}{5} NkT_F \Rightarrow T = \frac{2}{5} T_F = \boxed{T = 30,000 \text{ K}}$$

{Note:  $30,000 \gg 3481 \Rightarrow$  classical approx. is valid}

(ii) System expands by 1% only  $\Rightarrow$  it is still degenerate. Specific heat is

$$C_V = c Nk \frac{T}{T_F}$$

Calculate  $c$  from hint:

$$Q = \int_0^1 dT C_V = \frac{c Nk T^2}{2 T_F} = 2733 \Rightarrow T = \frac{2 \times 2733}{6.02 \times 10^{23} \times 1.38 \times 10^{-23}} \quad c = \frac{2 \times 2733 \times 75,000}{6.02 \times 10^{23} \cdot 1.38 \times 10^{-23}}$$

$$\Rightarrow \boxed{c = 4.93} \quad (= \pi^2/2)$$

So after expansion: ground state energy is  $E_0'$ , thermal energy is  $\frac{c Nk T^2}{2 T_F}$

$$E(T) = E_0' + \frac{c Nk T^2}{2 T_F} = E_0$$

Since  $n' = 0.99 n \Rightarrow T_F' = 0.99^{2/3} T_F = 74,500 \text{ K} \Rightarrow$

$$500 \text{ K} \times \frac{3}{5} = \frac{c \cdot T^2}{2 T_F} \Rightarrow \boxed{T = 3021 \text{ K}}$$

Since  $T \ll T_F$ , using low-temperature form of  $C_V$  is justified.

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**Physics Departmental Examination - Spring 1996 - Part II**

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**Problem 14.**

A 3-dimensional harmonic oscillator with potential energy  $V(\vec{r}) = \frac{1}{2}m\omega^2 r^2$  is in its ground state at  $t = +\infty$ . The quantum mechanical particle subject to the oscillator force has a charge  $e$  and mass  $m$ . A time dependent electric field constant in space is turned on along the  $z$ -direction. The magnitude of the electric field is given by

$$E(t) = \varepsilon \cdot e^{-\frac{t}{\tau}}$$

where  $\varepsilon$  and  $\tau$  are constant parameters. What is the probability that the oscillator is in an excited state at  $t = +\infty$ ? Assume that  $\varepsilon$  is small and first order perturbation theory is applicable.

solution

The time dependent perturbation is

$$H_{int}(t) = -e \cdot \mathcal{E} \cdot Z \cdot e^{-\frac{t^2}{\tau^2}}$$

The oscillator Hamiltonian is

$$H_0 = \left( a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + \frac{1}{2} \right) \hbar \omega$$

$$Z = \left( \frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} (a_3 + a_3^\dagger)$$

There will be excitation in first order into the first excited level of the third mode. The probability amplitude of the excitation is

$$d = \frac{-i}{\hbar} \int_{-\infty}^{+\infty} (-e\mathcal{E}) \langle n_1=0, n_2=0, 1 | Z | n_1=0, n_2=0, n_3=0 \rangle \cdot e^{-\frac{t^2}{\tau^2}} e^{i\omega t} dt$$

$$d = \frac{i e \mathcal{E}}{\hbar} \left( \frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{\tau^2}} e^{i\omega t} dt$$

$$= \frac{i e \mathcal{E}}{\hbar} \left( \frac{\hbar}{2m\omega} \right)^{\frac{1}{2}} (\pi \tau^2)^{\frac{1}{2}} \cdot e^{-\frac{\omega^2 \tau^2}{4}}$$



$$P_{0 \rightarrow 1} = |d|^2 = \frac{e^2 \mathcal{E}^2 \pi \tau^2}{2m \omega \hbar} e^{-\frac{\omega^2 \tau^2}{2}}$$

**Physics Departmental Examination - Spring 1996 - Part II**

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**Problem 15.**

A box containing a particle is divided into a right and left compartment by a this partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket  $|R\rangle$  ( $|L\rangle$ ), where we have neglected spatial variations within each half of the box. The most general state vector can be written as

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle.$$

The particle can tunnel through the partition. This tunneling effect is characterized by the perturbation Hamiltonian

$$H = \Delta \left[ |L\rangle\langle R| + |R\rangle\langle L| \right]$$

where  $\Delta$  is a real number with the dimension of energy.

- (a) Find the normalized energy eigenkets and corresponding eigenvalues.
- (b) Suppose at  $t=0$  the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?
- (c) Suppose the printer made an error and wrote  $H$  as

$$H = \Delta |L\rangle\langle R|.$$

Show that probability conservation is violated in the time-evolution problem in this case.

$$\text{Det} \begin{bmatrix} 0-\lambda & \Delta \\ \Delta & 0-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - \Delta^2 = 0 \quad \lambda = \pm \Delta$$

$$\begin{bmatrix} 0 & \Delta \\ \Delta & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \Delta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \Delta \\ \Delta & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\Delta \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a.  $\pm \Delta$  eigenvalues  $\frac{1}{\sqrt{2}} (|R\rangle \pm |L\rangle)$  ← eigenvectors  $|\pm \Delta\rangle$

b.  $t=0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow |R\rangle$

$$|R\rangle = \frac{1}{\sqrt{2}} [|\Delta\rangle + |-\Delta\rangle] \quad |L\rangle = \frac{1}{\sqrt{2}} [|\Delta\rangle - |-\Delta\rangle]$$

$$|t\rangle = \frac{1}{\sqrt{2}} \left( e^{-\frac{i}{\hbar} \Delta \cdot t} |\Delta\rangle + e^{\frac{i}{\hbar} \Delta \cdot t} |-\Delta\rangle \right)$$

$$\langle L|t\rangle = \frac{1}{2} \left( e^{-\frac{i}{\hbar} \Delta \cdot t} + e^{\frac{i}{\hbar} \Delta \cdot t} \right) = \frac{1}{2} \cancel{\cos \frac{\Delta \cdot t}{\hbar}} - i \sin \frac{\Delta \cdot t}{\hbar}$$


$$\text{Probability} = |\langle L|t\rangle|^2 = \sin^2 \frac{\Delta \cdot t}{\hbar}$$

$$c. \quad |\psi(0)\rangle = c_1 |R\rangle + c_2 |L\rangle \quad |c_1|^2 + |c_2|^2 = 1$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \Delta t H} |\psi(0)\rangle \Rightarrow |\psi(t)\rangle = |\psi(0)\rangle - \frac{i}{\hbar} \Delta t \cdot \Delta \cdot c_1 |L\rangle = c_1 |R\rangle + (c_2 - \frac{i}{\hbar} \Delta t \cdot \Delta \cdot c_1) |L\rangle$$

$$\langle \psi(t) | \psi(t) \rangle = |c_1|^2 + |c_2 - \frac{i}{\hbar} \Delta t \cdot \Delta|^2 =$$

$$= 1 + \frac{i}{\hbar} \Delta t (c_2 - c_2^*) + o(\Delta t^2)$$

  
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**Problem 16.**

Consider a gas of dust particles having a form of spheres of radius  $a$  and dielectric constant  $\epsilon$ , and  $\mu = 1$ . This gas is occupying a half space  $z < 0$  and has density  $n_0$ . An electromagnetic linearly polarized wave is incident on this half-space in  $z$ -direction with a wavenumber  $k$  satisfying the condition  $ka \ll 1$ . Assume the electric field of the wave is oriented in  $x$ -direction and has a magnitude of  $E_0$ .

- (a) Calculate the potential from a single dust particle in an electric field  $E_0$ .
- (b) Using the result obtained in (a) calculate the power radiated into solid angle  $d\Omega$ .
- (c) Calculate the decay length for the intensity for propagation of the electromagnetic wave through the gas.

Part II Problem E.M.

a) If  $ka \ll 1$  ( $\frac{\omega\sqrt{\epsilon}}{c} \ll a^{-1}$ ) Helmholtz equation for wave potential  $\nabla^2\phi + \frac{\omega^2}{c^2}\epsilon\phi = 0$  is

reduced to static problem  $\nabla^2\phi = 0$ . Solution of this equation outside sphere  $r > a$

$$\phi_1 = -E_0 r \cos\theta + \sum_{\ell=1}^{\infty} A_{\ell} r^{-\ell-1} P_{\ell}(\cos\theta)$$

$E_0$  - electric field of the incident wave. We introduced spherical coordinates with axes along  $x$ -direction of  $E$  field polarization.  $P_{\ell}(\cos\theta)$  - Legendre polynomial. Inside the sphere  $r < a$

everywhere  $e \rightarrow \ell(\ell+1)$

$$\phi_2 = \sum_{\ell=1}^{\infty} B_{\ell} r^{\ell} P_{\ell}(\cos\theta)$$

Boundary conditions

$\phi_1 = \phi_2$  | - tangential  $E$  is continuous on a sphere

$\frac{\partial\phi_1}{\partial r} = \epsilon \frac{\partial\phi_2}{\partial r}$  | - normal component of electric induction is also continuous. Since

driving term ( $\sim E_0$ ) contains  $\cos\theta$  only terms with  $\ell = 1$  are important in sums. Then

$$\phi = -E_0 r \cos\theta + \frac{A}{r^2} \cos\theta \quad r > a$$

$$\phi = -Br \cos\theta \quad r < a$$

$$-E_0 a + \frac{A}{a^2} = B a \quad A = \frac{E_0 a^3 (\epsilon - 1)}{\epsilon + 2}$$

$$-E_0 - \frac{2A}{a^3} = \epsilon B \quad B = \frac{3E_0 a^3}{\epsilon + 2}$$

$\epsilon - 1$   
not in power  
of exp.

Finally  $\phi = -E_0 r \sin\theta + \frac{E_0 a^3}{r^2} \cdot \frac{\epsilon-1}{\epsilon+2} \cdot \cos\theta$  for  $r > a$ . Last term is electric potential from

dy-pole  $d = E_0 a^3 \cdot \frac{(\epsilon-1)}{\epsilon+2}$  - electric dy-pole induced by an incident wave in the dielectric

sphere. Power radiated in a solid angle  $d\Omega$  (electric dipole radiation) is

$$\frac{dP}{d\Omega} = \frac{c}{8\pi} \cdot k^4 |d|^2 \sin^2\theta =$$

$$\frac{\omega^4}{8\pi c^3} |E_0|^2 a^6 \sin^2\theta \cdot \frac{(\epsilon-1)^2}{(\epsilon+2)^2}$$

Total radiated power can be obtained by integrating over solid angle  $\Omega$ .

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{1}{3} \frac{\omega^4}{c^3} |E_0|^2 a^6 \frac{(\epsilon-1)^2}{(\epsilon+2)^2}$$

b) E.M. power in an incident wave is

energy flux averaged over time

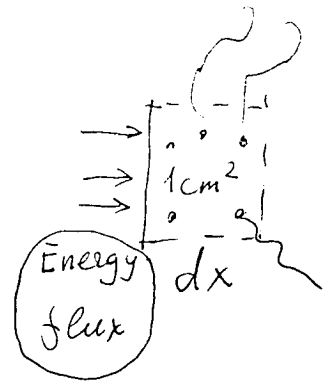
$$I_i = \frac{1}{2} \frac{c |E_0|^2}{4\pi}$$

~~( $\frac{1}{2}$  from averaging  $\cos^2 \omega t$ )~~

At a distance  $dx$  wave meets  $N dx$  scattering centers ( $N$  - density) and part of the power

$$|dI| = P \cdot N \cdot dx$$

is lost by radiation



$$\frac{dI}{dx} = -\frac{8\pi}{3} \frac{\omega^4}{c^4} a^6 \frac{(\epsilon-1)^2}{(\epsilon+2)^2} I = -\alpha I$$

$$I = I_0 e^{-\alpha x}, \quad \text{where}$$

$$\alpha = \frac{8\pi}{3} \frac{\omega^4 a^6}{c^4} \frac{(\epsilon-1)^2}{(\epsilon+2)^2} \quad \text{Extension length } \frac{1}{\alpha}$$

## Physics Departmental Examination - Spring 1996 - Part II

Student Identification # \_\_\_\_\_

**Problem 17.**

- (a) A medium consists of particles with charge,  $e$ , and density,  $n_0$  (*particles/cm<sup>3</sup>*). Show that it can be described as a dielectric medium with dielectric constant,  $\epsilon(\omega) = 1 - (\omega_{p0}/\omega)^2$  and find the expression for the plasma frequency,  $\omega_{p0}$ . Assume that the density is sufficiently low to treat the particles as classical and ignore thermal motion. Show that a wave will propagate in this medium only if its frequency,  $\omega$ , is greater than  $\omega_{p0}$ .
- (b) Assume that the density suddenly increases at  $t=0$  from  $n_0$  to  $n_1$ . Construct a solution to Maxwell's equations describing the evolution in time of a wave of frequency  $\omega$ , for  $t < 0$  such that  $\omega_{p0} < \omega < \omega_{p1}$  where  $\omega_{p1}$  is the plasma frequency corresponding to  $n_1$ . Note that the solution requires two oppositely propagating waves with the same wave vector as the incident wave.



Solution - Part II Problem E.M.

From Maxwell equations for a linearly polarized electromagnetic wave

$$\nabla^2 E_x - \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = + \frac{4\pi}{c^2} \frac{\partial J_x}{\partial t} =$$

$$= - \frac{4\pi e n_0(t) \partial U_x}{c^2 \partial t} - \frac{4\pi e}{c^2} U_x \frac{dn_0}{dt}$$

$j_x = -en_0 U_x$   $\text{div } \vec{E} = 0, \frac{\partial U_x}{\partial t} = \frac{e}{m} E_x$  - equation for electron motion

- current density

$$\omega_p^2 = \frac{4\pi e^2 n_0}{m}$$

Substituting  $\frac{\partial U_x}{\partial t}$  from equation of motion, we have

$$\nabla^2 E_x - \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} - \frac{\omega_p^2(t) E_x}{c^2} =$$

$$= - \frac{4\pi e}{c^2} U_x \frac{dn_0}{dt}$$

$$\nabla^2 E_x - \frac{1}{c^2} \epsilon \frac{\partial^2 E}{\partial t^2} = 0$$

$$\nabla^2 E_x + \frac{\omega^2}{c^2} \epsilon E$$

For  $t < t_0$  solution in a form of a plane monochromatic wave

$$E_x = A e^{i(kz - \omega t)}$$

$$\frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2} = \frac{\omega^2}{c^2} \epsilon$$

$$\omega_1^2 = c^2 k^2 + \omega_{p1}^2; \omega_{p1}^2 = \frac{4\pi e^2 n_{01}}{m}$$

Initial value

Boundary conditions at  $t = t_0$ . From eqn. for  $E_x$ :

$$(1) \quad \left. \frac{\partial E_x}{\partial t} \right|_{t_0+0} - \left. \frac{\partial E_x}{\partial t} \right|_{t_0-0} = 4\pi e U_x(t_0)(n_{02} - n_{01})$$

$$(2) \quad E_x \Big|_{t_0+0} - E_x \Big|_{t_0-0} = 0,$$

from eqn. for  $U_x$  :

$$\epsilon = \left( - \frac{\omega_p^2}{\omega^2} \right)$$

$$(3) \quad U_x(t = t_0 - 0) = U_x(t = t_0 + 0)$$

In order to satisfy these conditions it is necessary to take into account the following:

a) Conditions are to be satisfied at every  $z$ . So the solution for  $t > t_0$  also is proportional to  $e^{ikz}$ .

b) Both transmitted and reflected from the density jump waves are to be included into the solution:

$$E_x = B_1 \exp(ikz - i\omega_2 t) + B_2 \exp(ikz + i\omega_2 t),$$

$$\omega_2^2 = c^2 k^2 + \omega_{p2}^2, \quad \omega_{p2}^2 = \frac{4\pi e^2 n_{02}}{m}.$$

From condition (1) we can write relationship between amplitudes:

$$\omega_2 \left[ B_1 e^{-i\omega_2 t_0} - B_2 e^{i\omega_2 t_0} \right] = \omega_1 A e^{-i\omega_1 t_0} + \frac{4\pi e^2}{m\omega_1} (n_{02} - n_{01}) A e^{-i\omega_1 t_0}. \quad (4)$$

From condition (2) we have:

$$B_1 e^{-i\omega_2 t_0} + B_2 e^{i\omega_2 t_0} = A e^{-i\omega_1 t_0}. \quad (5)$$

Since  $\omega_{p2}^2 - \omega_{p1}^2 = \omega_2^2 - \omega_1^2$ , (4) can be also rewritten as

$$\frac{1}{\omega_2} B_1 e^{-i\omega_2 t_0} - \frac{1}{\omega_2} B_2 e^{i\omega_2 t_0} = \frac{1}{\omega_1} A e^{-i\omega_1 t_0}$$

equivalent to the condition (3)

.From equations (4) and (5) we can write:

$$B_1 = \frac{1}{2} A \exp[-i(\omega_1 - \omega_2)t_0] \left(1 + \frac{\omega_2}{\omega_1}\right),$$
$$B_2 = \frac{1}{2} A \exp[-i(\omega_1 + \omega_2)t_0] \left(1 - \frac{\omega_2}{\omega_1}\right).$$

Possible application in electronics: generation of a high frequency signal with an amplification in amplitude in the case  $\omega_2 \gg \omega_1$ .