PHYSICS DEPARTMENTAL EXAMINATION

SOLUTIONS

FALL, 1996

SCORE ________

Physics Departmental Examination - Fall 1996 - Part I

Student Identification # ________

PHYSICS DEPARTMENTAL EXAMINATION

SPECIAL INSTRUCTIONS

Please take a few minutes to read through all problems before starting to work. The proctor of the exam will attempt to clarify exam questions if you are uncertain about them. It is important to make an effort on every problem even if you do not know how to solve it completely. Partial credit will be given for partial solutions.

Problem 1.

In a region of space there is a uniform magnetic field $\mathbf{B} = \hat{y}B_0$. An uncharged copper sphere of radius R moves at constant velocity $\mathbf{v} = v_0\hat{x}$ through the magnetic field. Calculate the electric field in all of space, as seen in the laboratory frame. Is it necessary to exert a force on the sphere to maintain its motion? (Hint: Of course, v_0 is much smaller than c.)

Submitted by O'Neil

SOLUTION
Part I
Problem 1

3. In a region of space there is a uniform magnetic field $\mathbf{B}=\hat{y}\,B_0$. An uncharged copper sphere of radius R moves at constant velocity $\mathbf{v}=\mathrm{v}_0\hat{x}$ through the magnetic field. Calculate the electric field in all of space, as seen in the laboratory frame. Is it necessary to exert a force on the sphere to maintain its motion? (Hint: Of course, v_0 is much smaller than c.)

Let sphere be centered on origin of coordinates at the time of observation.

Inside sphere, the Lorentz force must be zero

$$0 = \underset{\sim}{E} + \frac{\underset{\sim}{V} \times \underset{\sim}{B}}{c} \qquad \therefore \quad \underset{\sim}{E} = -\frac{v_0 B_0}{c}\hat{z}$$

Inside and outside $\nabla^2 \varphi = 0$, but there is surface charge

let $\varphi = \varphi_{in}$ for $r < R$, $\varphi = \varphi_{out}$ for $r > R$

$$\varphi_{in} = v_0 B_0 \underbrace{r\cos\theta}_{z}$$

$$\varphi_{out} = \frac{A}{r^2}\cos\theta + \frac{B}{r}^{=0} \quad \text{since no net charge}$$

$$\varphi_{in}(R,\theta) = \varphi_{out}(R,\theta) \qquad \therefore \quad A = v_0 B_0 R^3$$

$$\underset{\sim}{E}_{out} = -\underset{\sim}{\nabla}\varphi_{out}$$

There is no net force on sphere, since B is uniform and $\int d^3r\, \underline{J}(\underline{r}) = \underset{\sim}{V}\cdot\underbrace{(\text{total charge})}_{=0}$.

SOLUTION
Part I
Problem 2

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Problem 2.

A radio pulsar emits a broad spectrum of radiation in the form of pulses of a few msec duration. Because of the dispersion of the interstellar plasma the frequency of the signal received at the earth drifts during the pulses. Assuming that the index of refraction of the plasma is given by $n = \sqrt{1-\omega_p{}^2/\omega^2}$:

a) Find an expression for the frequency drift $\frac{d\omega}{dt}$ during a pulse in terms of distance from the pulsar, x frequency of the pulsar, ω, and the plasma frequency, ω_p.
(Assume $\omega_- << \omega$)

b) The observed frequency shift in a pulse of duration 10 msec was 50 Khz and the mean frequency was 80 Mhz. $\omega_p = 10^4$ sec^{-1}. Find the distance to the pulsar.

Solution. For every frequency

$$x = v_g(\omega) \cdot t(\omega)$$

$\frac{dx}{dt} = 0$, Position of the observer is fixed.

$$0 = t \cdot \frac{dv_g}{d\omega} \cdot \frac{d\omega}{dt} + v_g$$

$$\underbrace{\frac{d\omega}{dt} = - \frac{v_g^2}{\frac{dv_g}{d\omega}} \quad \frac{1}{x}}$$

For the case under consideration $\omega_p \ll \omega$ we have

$$\omega \simeq ck + \frac{\omega_p^2}{2ck} \cong ck + \frac{\omega_p^2}{2\omega}$$

$$v_g = \frac{d\omega}{dk} \simeq c - \frac{\omega_p^2}{2c\ k^2} = c\left(1 - \frac{\omega_p^2}{2\omega^2}\right)$$

$$\frac{dv_g}{d\omega} \simeq \frac{c\ \omega_p^2}{\omega^3}$$

Finally:

$$\frac{d\omega}{dt} = - \frac{c}{X} \frac{\omega^3}{\omega_p^2}$$

b) If $N = 3 \cdot 10^{-2}\ cm^{-3}$

$$\omega_p^2 = 3 \cdot 10^9 \cdot 3 \cdot 10^{-2} = 10^8$$

$$\omega = 6.28 \cdot 8 \cdot 10^7$$

$$X = - \frac{c}{df/dt} f \cdot \frac{\omega^2}{\omega_p^2} =$$

$$= - \frac{3 \cdot 10^{10}}{5} \cdot 80 \cdot \frac{4\pi^2 \cdot 64 \cdot \cancel{10^{14}}\, 10^6}{\cancel{10^8}} =$$

$$= - 4,8 \cdot 10^{17} \cdot 4\pi^2 \cdot 64\ cm \simeq 10^{21}\ cm$$

$$\simeq 300\ ps$$

SOLUTION
Part I
Problem 3

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Physics Departmental Examination - Fall 1996 - Part I #3 Class. Mechanics

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Problem 3.

The captain of a small boat becalmed in the equatorial doldrums decides to resort to the expedience of raising the anchor ($m = 200$kg) to the top of the mast ($s = 20$ m). The rest of the boat has a mass of $M = 1000$ kg.

a) Why will the boat begin to move?

b) In which direction will it move?

c) How fast will it move?

SOLUTION
Part I
Problem 3

~~Data: velocity of earth: $\omega_0 = 0.0175$ sec^{-1}~~

(c) How fast will it move?

(*Chicago*)

Solution:

(a) The vertical motion of the anchor causes a Coriolis force $-2m\boldsymbol{\omega}\times\mathbf{v}$, where $\mathbf{v}$ is the velocity of the anchor and $\boldsymbol{\omega}$ the angular velocity of the earth, and so the boat moves.

(b) As $\boldsymbol{\omega}$ points to the north and $\mathbf{v}$ is vertically upward, the Coriolis force points toward the west. Hence the boat will move westward.

(c) As the total angular momentum of the boat and anchor with respect to the center of mass of the earth in an inertial frame is conserved, we have

$$(M+m)r^2\omega_0 = [Mr^2 + m(r+s)^2]\omega ,$$

where ω_0 and ω are the angular velocities of the earth and the boat respectively, r is the radius of the earth, giving

$$\frac{\omega}{\omega_0} \approx \frac{(M+m)r^2}{(M+m)r^2 + 2mrs} ,$$

or

$$\frac{\omega-\omega_0}{\omega_0} \approx \frac{-2ms}{(M+m)r+2ms} \approx \frac{-2ms}{(M+m)r} .$$

Hence the relative velocity of the boat with respect to the earth is

$$u = r(\omega-\omega_0) = \frac{-2ms\omega_0}{M+m} = -4.9\times10^{-4}\ \text{m/s} .$$

The negative sign indicates that the boat moves westward.

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A simple classical model of the CO_2 molecule would be a linear structure of three masses with the electrical forces between the ions represented by two identical springs of equilibrium length l and force constant k, as shown in Fig. 1.90. Assume that only motion along the original equilibrium line is possible, i.e. ignore rotations. Let m be the mass of O^- and M be the mass of C^{++}.

(a) How many vibrational degrees of freedom does this system have?

(b) Define suitable coordinates and determine the equation of motion of the masses.

(c) Seek a solution to the equations c
cillate with a common frequency (normal
frequencies.

(d) Calculate the relative amplitudes
cles for each of these modes and describ
mode. You may use a sketch as part of

(e) Which modes would you expect
what is the multipole order of each?

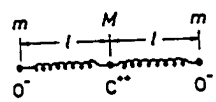

Fig. 1.90.

Solution:

(a) The system has two vibrational

(b) Let x_1, x_2 and x_3 be the dis
from their equilibrium positions respe
equations of motion are

$$m\ddot{x}_1 = k(x_2-x_1) ,$$
$$M\ddot{x}_2 = k(x_3-x_2) - k(x_2 -$$
$$m\ddot{x}_3 = -k(x_3-x_2) .$$

(c) Let $x_1 = A_1\cos\omega t$, $x_2 = A$
equations above. We have

$$(k - m\omega$$
$$-kA_1 + (2k - M\omega$$
$$-kA_1 + ($$

For A_1, A_2, A_3 not to be identically z

$$\begin{vmatrix} k-m\omega^2 & -k \\ -k & 2k-M \\ 0 & -k \end{vmatrix}$$

SOLUTION
Part I
Problem 4

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Problem 4.

An ideal gas consisting of particles of mass, m, is located in the spherically symmetric gravitational field of a massive star (mass M, radius r_0). The thermal conductivity coefficient of the gas is proportional to $T^{5/2}$. At the center of the star there is a heat source which maintains a constant temperature at the surface of the star so that $T(r_0) = T_0$. The density of the gas at the surface of the star is n_0. Assuming there are no heat sources in the gas, that T goes to zero at $r = \infty$, and that the gas is in hydrostatic equilibrium:

a) Construct a stationary solution for the density, n(r), and temperature, T(r) of the gas.

b) Consider this solution as a model for the solar corona and calculate the pressure of the corona at infinity. Compare this pressure to the cosmic ray pressure $10^{-12}\ dyne/cm^2$. On the basis of this comparison is this a reasonable model for the solar corona? Use the following numbers:

$$r_0 = 7.36\times10^{10}\ cm, \quad T_0 = 10^6\ K$$
$$n_0 = 10^5\ cm^{-3}, \quad M_\odot = 2\times10^{33}\ g$$
$$m = 10^{-24}\ g, \quad G = 6.67\times10^{-8}\ cm^3/g-\sec^2$$
(gravitational constant)

Solution

) For stationary gas $\quad div\,\vec{J}_T = 0$ (1)

$$J_{Tr} = -\varkappa \frac{dT}{dr} \text{ - heat flux}, \quad \varkappa = \varkappa_0 \left(\frac{T}{T_0}\right)^{5/2}$$

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \varkappa_0 \left(\frac{T}{T_0}\right)^{5/2} \frac{dT}{dr}\right] = 0$$

Solution satisfying conditions

$T = T_0$ at $r = r_0$, $T \to 0$ at $r \to \infty$

$$T = T_0 \left(r_0/r\right)^{2/7}$$

Balance of forces

$$-\frac{dP}{dr} - \frac{G M_\odot m n}{r^2} = 0$$

n - density $\qquad P = P_i + P_e = 2nT$

- pressure

Substituting $T(r)$ we have ~~the~~ following eqn for density profile: (2)

$$\frac{d}{dr}\left(\frac{n}{r^{2/7}}\right) = -\frac{G M \cdot m}{2 T_0 \cdot r_0^{2/7}} \frac{n}{r^2}$$

Solution

$$n(r) = n_0 \left(\frac{r}{r_0}\right)^{2/7} \exp\left\{\frac{7}{5} \frac{G M m}{2 T_0 \cdot r_0} \times \left[\left(r_0/r\right)^{5/7} - 1\right]\right\}$$

where $n_0 = n(r = r_0)$. For r close to r_0 we have usual barometric formulae

$$n(r) = n_0 \exp\left\{-\frac{G M m}{2 T_0 r_0} \frac{r - r_0}{r_0}\right\}$$

with the „atmospheric height"

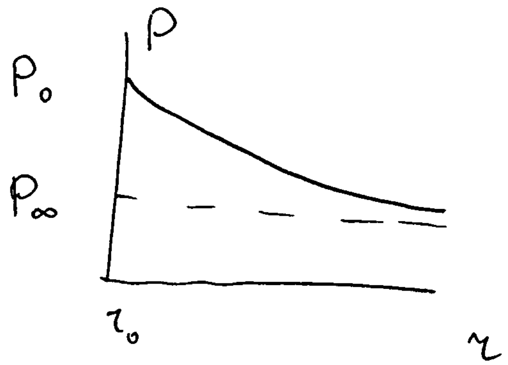
$h = \dfrac{2 T_0 r_0^2}{G M m}$. But for $r \gg r_0$

$n(r)$ increases as $r^{2/7}$ (due to

high temperature conductivity).

The pressure $P = 2nT$ is defined by the formulae

$$P(r) = P_0 \exp\left\{\frac{7}{5} \frac{G M_i}{2 T_0} \frac{m_i}{r_0} \left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]\right\}$$



For $r \to \infty$ the gas pressure in the proposed model approaches finite value

$$P_\infty = P_0 \exp\left\{-\frac{7}{5} \frac{G M_i m_i}{2 T_0 r_0}\right\}$$

b). If to consider obtained solution as a model for solar corona and to use values of parameters listed in formulation of the problem then:

$$P_0 = 2 n_0 T_0 = 2.76 \cdot 10^{-5} \ dyne/cm^2$$

$$P_\infty \simeq 3 \cdot 10^{-9} \ dyne/cm^2$$

The pressure at infinity exceeds the cosmic ray pressure completely isolating solar system from interstellar medium.

Hence the stationary atmosphere ~~is~~ cannot be considered as reasonable model of the solar corona.

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Physics Departmental Examination - Fall 1996 - Part I

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Problem 5.

The lowest (non-zero) rotational energy level of a H_2 molecule has energy

$$\varepsilon_{rot} / k = 350K$$
$$(k = \text{Boltzmann constant})$$

(a) Estimate the average rotational energy of H_2 molecules at temperature T=50K.

(b) Same as (a) for D_2 molecules, assuming the interatomic distance is the same as for H_2 molecules.

(c) Same as (a) for temperature T=3500K.

(d) Estimate the error of your result in (a).

Express all your answers in $^\circ K \times k$.

SOLUTION
Part I
Problem 5

Solution

Rotational energy levels are:

$$\varepsilon_{rot}(J) = \frac{\hbar^2}{2A} J(J+1) \quad ; \; J = 0, 1, 2, \ldots$$

A = moment of inertia, J = integer. Degeneracy of J-th level is $2J+1$.

For H_2:

$$\varepsilon_{rot}(1) = 350K \cdot k \equiv \varepsilon_{rot}$$

Rotational partition function:

$$Z_{rot} = \sum_{J=0}^{\infty} (2J+1)\, e^{-\beta \varepsilon_{rot}(J)}$$

At $T = 50K$; since $kT \ll \varepsilon_{rot}(1)$, consider first level only.

$$Z_{rot} \sim \sum_{J=0}^{1} (2J+1)\, e^{-\beta \varepsilon_{rot}(J)} = 1 + 3e^{-\beta \varepsilon_{rot}}$$

Average energy:

$$\bar{\varepsilon}_{rot} = -\frac{d}{d\beta} \ln Z_{rot} = \frac{3\varepsilon_{rot}\, e^{-\beta \varepsilon_{rot}}}{1 + 3e^{-\beta \varepsilon_{rot}}}$$

(a) $\Rightarrow \boxed{\bar{\varepsilon}_{rot} = 0.955\ K \times k}$ (0.957 if denominator is omitted, o.k. too).

(b) For D_2, mass is twice as large $\Rightarrow$ A is twice as large $\Rightarrow$

$$\varepsilon_{rot}(1) = 175\ K \times k = \varepsilon'_{rot}$$

$$\Rightarrow \boxed{\bar{\varepsilon}'_{rot} \sim 3\varepsilon'_{rot}\, e^{-\beta \varepsilon'_{rot}} = 15.85\ K \times k}$$

(c) At temperature $T = 3500K$, $T \gg \varepsilon_{rot}(1)/k \Rightarrow$ equipartition applies $\Rightarrow$

$$\bar{\varepsilon}_{rot} = 2 \times \frac{1}{2} kT = kT \Rightarrow \boxed{\bar{\varepsilon}_{rot} = 3500\,K \times k}$$

(d) To estimate error in (a), keep next level, $J = 2 \Rightarrow Z_{rot} \sim 1 + 3e^{-\beta \varepsilon_{rot}} + 5e^{-3\beta \varepsilon_{rot}}$

$$\Rightarrow \bar{\varepsilon}_{rot} = \frac{\varepsilon_{rot}\left(3e^{-\beta \varepsilon_{rot}} + 15e^{-3\beta \varepsilon_{rot}}\right)}{1 + \underbrace{\ldots}_{small}} \Rightarrow \text{error} \sim 15\varepsilon_{rot}\, e^{-3\beta \varepsilon_{rot}} \sim \boxed{4 \times 10^{-6}\ K \times k}$$

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Physics Departmental Examination - Fall 1996 - Part I

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Problem 6.

A thermally insulated cylinder contains in 1/2 of its volume one mole of water vapor, one mole of liquid water and one mole of ice in equilibrium. The other 1/2 of the volume is empty. The partition separating the two halves is removed, and the system reaches a new equilibrium state.

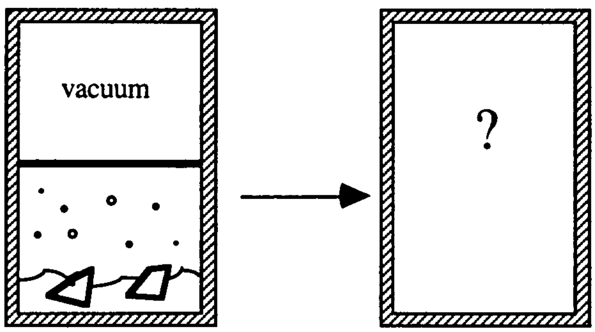
(a) How much (in moles) of each phase is there in the final state?

(b) What is the pressure (in mm Hg) in the final state?

(c) What is the temperature (in °C) in the final state?

Your answers don't need to be accurate to more than 1%. Assume water vapor is an ideal gas. If you make any approximation, justify it.

Hint: Assume that the change in temperature is small for parts (a) and (b).



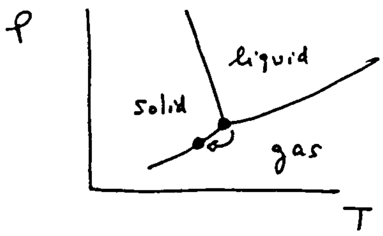
initial final

Some of the following information should be useful:

- Temperature and pressure of water at triple point: $T_{tr} = 0.01\ ^\circ C = 273.16\ ^\circ K$, $P_{tr} = 4.58$ mm Hg
- Latent heat of fusion of ice: 80 cal/mole
- Latent heat of vaporization of water: 540 cal/mole
- Specific heat of water: 18 cal / °C mole
- Phase-equilibrium lines for water near the triple point:

$$\frac{dP}{dT} = \begin{cases} 27 \text{ mm Hg / °C} & \text{liquid-vapor} \\ 101{,}840 \text{ mm Hg / °C} & \text{solid-liquid} \\ 32 \text{ mm Hg / °C} & \text{solid-vapor} \end{cases}$$

Solution



SOLUTION
Part I
Problem 6

Pressure becomes lower ⇒ moves away from triple point, on solid-gas line.

Neglect change in T for parts (a) and (b).

Assume: x moles of liquid → ice ; l fusion = 80 cal/mole

1−x " " " → vapor ; l vapn = 540 cal/mole

Energy conservation ⇒ $80x = 540(1-x) \Rightarrow x = 0.87$

⇒ (a) $\boxed{\text{1.87 moles of ice, 1.13 moles of vapor.}}$

(b) $\frac{P_i V_i}{n_i T_i} = \frac{P_f V_f}{n_f T_f}$; assume $T_i = T_f$; $V_f = 2V_i$, $n_f = 1.13\, n_i$ ⇒

$P_f = \frac{1.13}{2} P_{in} \Rightarrow \boxed{P_f = 2.59 \text{ mm Hg}}$

(c) Find T on the gas-solid coexistence for which P = 2.59 mm Hg.

$\frac{dP}{dT} = 32 \frac{\text{mm Hg}}{°C} \Rightarrow \Delta T = \frac{\Delta P}{32} = \frac{2.59 - 4.58}{32} = -0.062\ °C$

⇒ $\boxed{T_f = -0.052\ °C}$

The temperature changed by only $\frac{0.062}{273} = 0.02\ \%$. In part (a), the energy conservation eq. should have included the change in temperature × heat capacity, but to 1% accuracy can be ignored. In part (b) similarly, the difference between T_i and T_f can be ignored to 1% accuracy.

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Physics Departmental Examination - Fall 1996 - Part I

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Problem 7.

Let $|\ell m>$ be an eigenstate of angular momentum such that

$$L_z|\ell m\rangle = \hbar m|\ell m\rangle$$

with

$$L^2|\ell m\rangle = \hbar^2\ell(\ell+1)|\ell m\rangle$$

and

$$L^2 = L_x^{\,2} + L_y^{\,2} + L_z^{\,2}$$

a) Find $L_\pm|\ell m\rangle$ where $L_\pm = L_x \pm iL_y$

b) Find $\langle \ell m|L_x^{\,2}|\ell m\rangle$

Justify all steps.

Solution

First prove identity

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y]$$

but $[L_x, L_y] = i\hbar L_z$ so $L_+L_- = L_x^2 + L_y^2 + \hbar L_z$

$L_x^2 + L_y^2 = L^2 - L_z^2 \Rightarrow$

$\boxed{L_+L_- = L^2 - L_z^2 + \hbar L_z}$ Likewise $\boxed{L_-L_+ = L^2 - L_z^2 - \hbar L_z}$

(b) Note $[L_z, L_\pm] = [L_z, L_x \pm iL_y] = [L_z, L_x] \pm i[L_z, L_y]$
$= i\hbar L_y \pm i(-i)\hbar L_x = \pm\hbar L_\pm$

So $L_z(L_+|lm\rangle) = L_+L_z|lm\rangle + \hbar L_+|lm\rangle$
$= \hbar m L_+|lm\rangle + \hbar L_+|lm\rangle$
$= \hbar(m+1)(L_+|lm\rangle)$

So $L_+|lm\rangle$ is an eigenvector of $\underline{|l, m+1\rangle}$.

$\boxed{L_+|lm\rangle = c|l, m+1\rangle}$ Find constant c.

$(\langle lm|L_+)^\dagger(L_+|lm\rangle) = |c|^2\langle l, m+1|l, m+1\rangle = |c|^2$

$|c|^2 = \langle lm|L_-L_+|lm\rangle$
$= \langle lm|(L^2 - L_z^2 - \hbar L_z)|lm\rangle = \langle L^2\rangle - \langle L_z^2\rangle - \hbar\langle L_z\rangle$

$|c|^2 = \hbar^2 l(l+1) - \hbar^2 m^2 - \hbar^2 m$

So

$$c = \hbar\sqrt{l(l+1) - m^2 - m} = \hbar\sqrt{(l-m)(l+m+1)}$$

So

$$\boxed{L_+|lm\rangle = \hbar\sqrt{(l-m)(l+m+1)}\,|l, m+1\rangle}$$

Likewise using $L_-|lm\rangle = c'|l, m-1\rangle$ and other identity gives

$$L_-|lm\rangle = \hbar\sqrt{(l+m)(l-m+1)}\,|l, m-1\rangle$$

Solution

SOLUTION
Part I
Problem 7

(b) $\langle lm | L_x^2 | lm \rangle$.

$$L_+ + L_- = 2L_x \Rightarrow L_x^2 = \frac{1}{4}\left(L_+^2 + L_-^2 + L_+L_- + L_-L_+\right)$$

$$\langle L_x^2 \rangle = \frac{1}{4}\left[\langle L_+^2 \rangle + \langle L_-^2 \rangle + \langle L_+L_- \rangle + \langle L_-L_+ \rangle\right]$$

$$\langle L_+^2 \rangle = \langle lm | L_+ L_+ | lm \rangle = c_m \langle lm | L_+ | l, m+1 \rangle = c_m c_{m+1} \langle lm | l, m+2 \rangle$$
$$= 0$$

Likewise $\langle L_-^2 \rangle = 0$

$$\langle L_x^2 \rangle = \frac{1}{4}\langle L_+L_- + L_-L_+ \rangle = \frac{1}{4}\langle L^2 - L_z^2 + \hbar L_z + L^2 - L_z^2 - \hbar L_z \rangle$$

$$= \frac{1}{2}\langle L^2 - L_z^2 \rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

SOLUTION
Part I
SCORE ___ Problem 8

Physics Departmental Examination - Fall 1996 - Part I

Student Identification # ___________

Problem 8.

A particle moves in a one-dimensional potential defined by

$$V(x)=\begin{cases} cx^2 & \text{for } x>0 \\ +\infty & \text{for } x<0 \end{cases}$$

With c > 0
Find the energy eigenvalues.

UG QM Part I #8

Quantum

A particle of mass M, moves in a one-dimensional potential defined by

SOLUTION
Part I
Problem 8

$$V(x) = \begin{cases} C x^2 & \text{for } x > 0 \\ +\infty & \text{for } x < 0 \end{cases}$$

Find the energy eigenvalues.

<u>Solution</u>:

Eigenvalues of harmonic oscillator are $\left(V(x) = \frac{1}{2} M\omega^2 x^2\right)$

$$E = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2$$

$$\omega = \sqrt{\frac{2C}{M}}$$

even (odd) n correspond to even and odd wave functions.

Since $V = \infty$ for $x < 0$, wave functions satisfy the boundary condition $V(x=0) = 0$ and are the same as for h.o. for $x > 0 \Rightarrow$ only odd solutions of harmonic oscillator $\Rightarrow$ odd $n \Rightarrow$

$$n \to 2n + 1 \Rightarrow n + \frac{1}{2} \to 2n + \frac{3}{2} = \frac{4n+3}{2} \Rightarrow$$

$$\boxed{E = \frac{\hbar\omega}{2}(4n+3) \; ; \; n = 0, 1, 2, \ldots}$$

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Physics Departmental Examination - Fall 1996 - Part II

Student Identification # ________

PHYSICS DEPARTMENTAL EXAMINATION
SPECIAL INSTRUCTIONS
Please take a few minutes to read through all problems before starting to work. The proctor of the exam will attempt to clarify exam questions if you are uncertain about them. It is important to make an effort on every problem even if you do not know how to solve it completely. Partial credit will be given for partial solutions.

Problem 9.

A long cylindrical copper rod is of radius R. For time, $t \leq 0$, the rod is immersed in a uniform external magnetic field $\mathbf{B} = \hat{z}B_0$, where (z, r, θ) is a cylindrical coordinate system with the z-axis coincident with the axis of the rod. At $t = 0$, the external field is switched off (or the rod is suddenly jerked out of the field). You may assume that $\sigma R^2/c^2$ is large compared to the time for the external field to be switched off, where σ is the conductivity of the copper rod.

a. Obtain an equation that governs the evolution of $B_z(r,\theta,t)$ in a cross section of the rod that is far from either end. What boundary condition does $B_z(r,\theta,t)$ satisfy at $r = R$?

b. Find $B_z(r,\theta,t)$ for $t > 0$ and $r < R$.

Submitted by O'Neil

7. A long cylindrical copper rod is of radius R. For $t \le 0$, the rod is immersed in a uniform external magnetic field $\mathbf{B} = \hat{z} B_0$, where (z, r, θ) is a cylindrical coordinate system with the z-axis coincident with the axis of the rod. At $t = 0$, the external field is switched off (or the rod is suddenly jerked out of the field). You may assume that $T = \sigma R^2/c^2$ is large compared to the time for the external field to be switched off, where σ is the conductivity of the copper rod.

a. Obtain an equation that governs the evolution of $B_z(r,\theta,t)$ in a cross section of the rod that is far from either end. What boundary condition does $B_z(r,\theta,t)$ satisfy at $r = R$?

$$\nabla \times B = \frac{4\pi}{c} J + \underbrace{\cancel{\frac{1}{c}\frac{\partial E}{\partial t}}}_{\text{neglect}} \simeq \frac{4\pi}{c}\sigma E$$

$$\underbrace{\nabla \times \nabla \times B}_{\nabla \underbrace{\nabla \cdot B}_{0} - \nabla^2 B} = \frac{4\pi}{c}\sigma \nabla \times E = -\frac{4\pi\sigma}{c^2}\frac{\partial B}{\partial t}$$

$$\nabla^2 B_z = \frac{4\pi\sigma}{c^2}\frac{\partial B_z}{\partial t}, \quad B_z(R,t) = 0 \text{ for } t > 0$$

b. Find $B_z(r,\theta,t)$ for $t > 0$ and $r < R$.

Soln by separation of variables

$$B_z = \sum_n A_n J_0\left(\frac{r}{R}\chi_{0n}\right) e^{-\lambda_n t} \qquad J_0(\chi_{0n}) = 0$$

where $\left(\frac{\chi_{0n}}{R}\right)^2 = \frac{4\pi\sigma\lambda_n}{c^2}$

A'ns determined by initial condition

$$A_n = \frac{\int_0^R 2\pi r dr\, B_0 J\left(\frac{r}{R}\chi_{0n}\right)}{\int_0^R 2\pi r dr\, J_0^2\left(\frac{r}{R}\chi_{0n}\right)}$$

SOLUTION
Part II
Problem 10

SCORE

Physics Departmental Examination - Fall 1996 - Part II

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Problem 10.

A plane electromagnetic wave in vacuum with

$$\underline{E}(r,t) = \mathrm{Re}\,\underline{E}e^{i(kz - \omega t)}$$

is incident on and scatters off of a free electron. The electron motion is non-relativistic. Does the electron experience a time averaged acceleration in the z direction? If so, what is the magnitude of the acceleration?

Hint: Consider conservation of momentum.

not valid

think ahead

Sc T

SOLUTION Part II Problem 10

by

4. A plane electromagnetic wave,

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\,\mathbf{E}\,e^{ikz - i\omega t}$$

$$\mathbf{B}(\mathbf{r},t) = \cancel{\left(\frac{c\mathbf{k}}{\omega}\right)\times\mathbf{E}(\mathbf{r},t)}, \quad \frac{ck}{\omega}\hat{z}\times \underset{\sim}{E}(r,t)$$

is incident on and scatters off of a free electron. The electron motion is non-relativistic. Does the electron experience a time-average acceleration in the ~~k~~ $\hat{z}$-direction? If so, what is the magnitude of the acceleration? (Hint: consider conservation of momentum.)

the scattered radiation is isotropic so momentum is conserved from incident wave and transfered to electron.

Time-average Thomson scattering

$$\langle \text{Power} \rangle = \frac{2}{3}\frac{e^2}{c^3}\langle \dot{v}_i^2 \rangle = \frac{2}{3}\frac{e^2}{c^3}\left(\mathrm{Re}\,\frac{e}{m}\underset{\sim}{E}\,e^{-i\omega t}\right)^2 = \frac{e^4|E|^2}{2m^2c^3}$$

$$\langle \text{momentum}/\text{time} \rangle = \langle \text{Power} \rangle / c$$

$$m\dot{V}_z = \text{momentum}/\text{time}$$

$$\dot{V}_z = \frac{1}{2}\frac{e^4}{m^2c^4}\frac{|E|^2}{m}$$

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Problem 11.

Evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4 + a^4}.$$

The solution should contain neither integrals nor infinite series and should be manifestly real when a is real.

D Friedkin

SOLUTION
Part II
Problem 11

Mathematical Physics Compute the sum

$$\sum_{n=\infty}^{\infty} \frac{1}{n^4 + a^4}.$$

The solution should contain neither integrals nor infinite series and should be manifestly real when a is real.
Solution Use the residue theorem to write

$$S = \sum_{n=\infty}^{\infty} \frac{1}{n^4 + a^4} = \oint_C \frac{dz}{2\pi i} \frac{\pi \cot \pi z}{z^4 + a^4},$$

where the path C encloses the real axis in the counterclockwise sense. In addition to the real poles from the cot, there are four poles

$$\begin{aligned} r_n &= a e^{i(2n+1)\pi/4}, \qquad n = 0 \ldots 3 \\ &= r_0,\ ir_0,\ -r_0,\ -ir_0 \end{aligned}$$

and $r_0 = a e^{i\pi/4}$. Close C "the other way" to obtain

$$\begin{aligned} S &= -\sum_{n=0}^{3} \frac{\pi \cot \pi r_n}{4 r_n^3} = -\frac{\pi}{2} \sum_{n=0}^{1} \frac{\cot \pi r_n}{r_n^3} \\ &= -\frac{\pi}{2 r_0^3} [\cot \pi r_0 + i \cot i\pi r_0]. \end{aligned}$$

Straightforward algebra leads to

$$S = \frac{\pi}{\sqrt{2} a^3} \frac{\sinh \sqrt{2}\pi a + \sin \sqrt{2}\pi a}{\cosh \sqrt{2}\pi a - \cos \sqrt{2}\pi a}.$$

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Problem 12.

You know that at most one fermion may occupy any given single particle state. A *parafermion* is a particle for which the maximum occupancy of any given single-particle state is k, where k is an integer greater than zero. (For $k = 1$, parafermions are regular everyday fermions; for $k = \infty$ parafermions are regular everyday bosons.) Consider a system with one single-particle level whose energy is ε, i.e. the Hamiltonian is simply $H = \varepsilon n$, where n is the particle number.

a) Compute the partition function $\Xi(\mu, T)$ in the grand canonical ensemble for parafermions.

b) Compute the occupation function $n(\mu, T)$. What is n when $\mu = -\infty$? When $\mu = \varepsilon$? When $\mu = +\infty$? Show that $n(\mu, T)$ reduces to the Fermi and Bose distributions in the appropriate limits.

c) Sketch $n(\mu, T)$ as a function of μ for both $T = 0$ and $T > 0$.

d) Can a gas of ideal parafermions condense in the sense of Bose condensation?

Solution

We have that $N = n$ (since there is only one level, there is no state index associated with n), hence

$$\begin{aligned}\Xi &= \operatorname{Tr}\exp\left(\frac{\mathcal{H} - \mu N}{k_B T}\right)\\ &= 1 + y + y^2 + \ldots + y^k = \frac{1 - y^{k+1}}{1 - y}\end{aligned}$$

where $y = e^{(\mu-\varepsilon)/k_B T}$. Note that there is no singularity as a function of y (even at $y = 1$).

The occupation $n(\mu, T)$ is given by

$$n = \frac{k_B T}{\Xi}\frac{\partial \Xi}{\partial \mu} = y\frac{\partial \ln \Xi}{\partial y} = \frac{1}{y^{-1} - 1} - \frac{(k+1)}{y^{-(k+1)} - 1} \,.$$

Note that for $k = 1$ we have

$$n_{k=1} = \frac{1}{y^{-1} - 1} - \frac{2}{y^{-2} - 1} = \frac{1}{y^{-1} + 1} \,,$$

while for $k = \infty$, where μ must be less than ε in order for the partition sum to converge, we have $y < 1$ and $y^{-(k+1)} \to \infty$ (much faster than $k \to \infty$) and hence

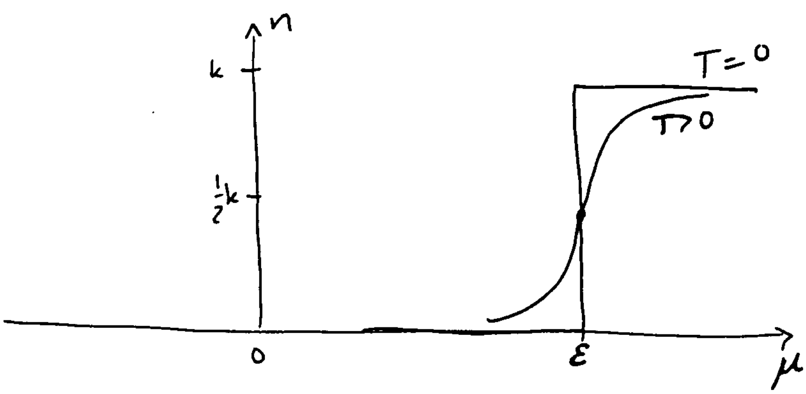
$$n_{k=\infty} = \frac{1}{y^{-1} - 1} \,.$$

Thus, the Fermi and Bose functions are recovered.

Checking some values of μ, when $\mu = -\infty$ we have $y = 0$ and the formula gives $n = 0$. This makes sense because the chemical potential is so negative that there are no particles in the system. When $\mu = +\infty$, we have $y = \infty$ and $n = k$, which says that we are at maximum occupancy, and this also makes sense. When $\mu = \varepsilon$ ($y = 1$), the chemical potential is set exactly to the energy of the single particle level, which means the probability of occupancy is 50%, and we find $n = \frac{1}{2}k$. This can be extracted from the general formula by setting $y = 1 + \epsilon$ with $\epsilon \to 0$, or by noting that

$$n = \frac{0 \cdot y^0 + 1 \cdot y^1 + 2 \cdot y^2 + \ldots + k \cdot y^k}{y^0 + y^1 + y^2 + \ldots + y^k} = \frac{0 + 1 + 2 + \ldots + k}{1 + 1 + 1 + \ldots + 1} = \frac{\frac{1}{2}k(k+1)}{k+1} = \tfrac{1}{2}k \,.$$

Since the maximum occupancy is k, there is no phase transition in the sense of Bose condensation.



($\varepsilon > 0$ sketched)

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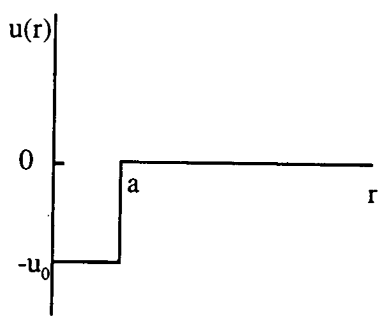
Problem 13.

Consider a system of 2 atoms of mass m in a one-dimensional box of length L. The interaction energy between the 2 atoms is

$$u(|r_1 - r_2|) = \begin{cases} -u_0 & \text{for } |r_1 - r_2| < a \\ 0 & \text{for } |r_1 - r_2| > a \end{cases}$$

Assume $a \ll L$, and $u_0 > 0$. The system is in a heat reservoir at temperature T. Treat the system classically.



(a) Find an expression for the "pressure" of this system

$$P = kT\frac{\partial \ln Z}{\partial L}$$

at temperature T (Z is the partition function). Your answer should reduce to the ideal gas answer P=NkT/L for $a \to 0$.

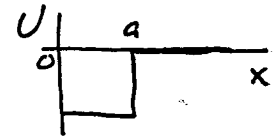
(b) Give limiting values of the result in (a) for (i) $T \ll u_0$ and (ii) $T \gg u_0$. If any of these reduces to the ideal gas answer give the next order correction. Explain the physical meaning of the results.

II - 13 Solution

The Hamiltonian is

$$\mathcal{H} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + U(|x_1 - x_2|)$$

$$U(x) = -u_0\, \Theta(a - |x|)$$

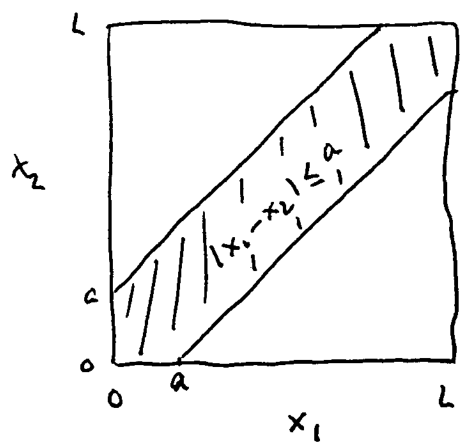


Thus, the partition f^n is

$$Z_1 = \frac{1}{2h^2}\int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_0^L dx_1 \int_0^L dx_2\, e^{-\beta\mathcal{H}(p_1, p_2, x_1, x_2)}$$

$$= \frac{1}{2}\left[\frac{1}{h}\int_{-\infty}^{\infty} dp\, e^{-\beta p^2/2m}\right]^2 \int_0^L dx_1 \int_0^L dx_2\, e^{-\beta U(x_1 - x_2)} \quad (3)$$

If U were zero, the spatial integrals would give L^2.
Instead, $U = -u_0$ whenever $|x_1 - x_2| < a$:



$(L-a)^2$ = area over which $|x_1 - x_2| > a$

$L^2 - (L-a)^2 = 2aL - a^2$
= area over which $|x_1 - x_2| < a$

Thus,

$$\int_0^L dx_1 \int_0^L dx_2\, e^{-\beta U(x_1 - x_2)} = (L-a)^2 + e^{\beta u_0}(2aL - a^2) \quad (1)$$

To order a this is just $\quad L^2 + 2aL\,(e^{\beta u_0} - 1) + \mathcal{O}(a^2)$.

Now $\frac{1}{h}\int_{-\infty}^{\infty} dp\, e^{-\beta p^2/2m} = \frac{1}{h}\sqrt{\frac{2\pi m}{\beta}} = (2\pi\hbar^2/mk_BT)^{-1/2} = \lambda_T^{-1}$.

Thus,

$$Z_1 = \tfrac{1}{2}\lambda_T^{-2}\left\{(L-a)^2 + (2aL-a^2)\,e^{u_0/k_BT}\right\}$$

The pressure is given by

(a)
$$p = -\frac{\partial F}{\partial L} = k_BT\,\frac{\partial \ln Z_1}{\partial L}$$

$$= k_BT\,\frac{\partial}{\partial L}\ln\left\{(L-a)^2 + (2aL-a^2)e^{u_0/k_BT}\right\}$$

$$= k_BT\cdot\frac{2(L-a) + 2a\,e^{\beta u_0}}{(L-a)^2 + e^{\beta u_0}(2aL-a^2)} \qquad (2)$$

As $a\to 0$, we have $p \to 2k_BT/L \Rightarrow$ ideal gas law$_{(N=2)}$. (1)

(b) $$\lim_{\beta u_0\to\infty} p = k_BT\cdot\frac{2a}{2aL-a^2} = \frac{2k_BT}{2L-a} = \frac{k_BT}{L-\frac{1}{2}a}$$

Since $a \ll L$, we obtain $p(u_0 \gg k_BT) \simeq k_BT/L$, i.e. it is as if only one particle (a bound pair) is rattling around in the "box". Size of pair is $\simeq a/2$. (3)

$$\lim_{\beta u_0\to 0} p = k_BT\cdot\frac{2L}{L^2} = \frac{2k_BT}{L} = \text{ideal gas law}$$

So the ideal gas law is recovered in the limit of zero interaction. For finite but weak interactions, $e^{\beta u_0}-1 \simeq \beta u_0$, and

$$p = k_BT\cdot\frac{2L + 2a\beta u_0}{L^2 + 2aL\,\beta u_0 + \mathcal{O}(a^2)} = \frac{2k_BT}{L}\left\{1 - \frac{a}{L}\frac{u_0}{k_BT} + \ldots\right\}$$

$$p = \frac{2k_BT}{L} - \frac{2au_0}{L^2} + \ldots$$

pressure reduced due to attractive interactions (cf. Van der Waals)

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Problem 14.

A particle of mass m moves under gravity on a smooth surface, the equation of which is $z = x^2 + y^2 - xy$, the z-axis being vertical, pointing upwards.

a) Find the equations of motion of the particle.

b) Find the frequencies of the normal-modes for small oscillations about the position of stable equilibrium.

c) If the particle is displaced from equilibrium slightly and then released, what must be the ratio of the x and y displacements to guarantee that only the higher frequency normal-mode is excited?

SOLUTION
Part II
Problem 14

$$V = \frac{1}{2}M\dot{\xi}^2 - \frac{aK}{2e}\xi^2 .$$

$$\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} = 0$$

$$\ddot{\xi} + \frac{aK}{e}\xi = 0 .$$

frequency of small oscillations about the

$$\omega = \sqrt{\frac{aK}{Me}} ,$$

$$\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{Me}{aK}} .$$

, then aK is positive and the above results

e then

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_{x_0} < 0 ,$$

l at equilibrium is a maximum and the no oscillation occurs. This can also be n, which would give an imaginary ω.

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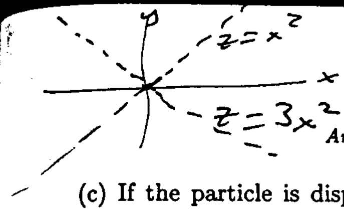
under gravity on a smooth surface the $- xy$, the z-axis being vertical, pointing

tion of the particle.

normal-modes for small oscillations about



(c) If the particle is displaced from equilibrium slightly and then released, what must be the ratio of the x and y displacements to guarantee that only the higher frequency normal-mode is excited?

$(x+y)(x -$

(Wisconsin)

Cylindrical coords: z, r, θ

$x = r\cos\theta$; $x^2 = r^2\cos^2\theta$

$y = r\sin\theta$; $y^2 = r^2\sin^2\theta$

$x^2 + y^2 - xy = r^2 - r^2\cos\theta\sin$

$z = z$

Solution:

(a) As

$$z = x^2 + y^2 - xy ,$$
$$\dot{z} = 2x\dot{x} + 2y\dot{y} - \dot{x}y - x\dot{y} = \dot{x}(2x - y) + \dot{y}(2y - x) .$$

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{x}^2(2x - y)^2 + \dot{y}^2(2y - x)^2 + 2\dot{x}\dot{y}(2x - y)(2y - x)] \\ &\quad - mg(x^2 + y^2 - xy) . \end{aligned}$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

give

$$\begin{aligned} &\frac{d}{dt}[\dot{x} + \dot{x}(2x - y)^2 + \dot{y}(2x - y)(2y - x)] \\ &= 2\dot{x}^2(2x - y) - \dot{y}^2(2y - x) + 2\dot{x}\dot{y}(2y - x) - \dot{x}\dot{y}(2x - y) - 2gx + gy , \\ &\frac{d}{dt}[\dot{y} + \dot{y}(2y - x)^2 + \dot{x}(2x - y)(2y - x)] \\ &= 2\dot{y}^2(2y - x) - \dot{x}^2(2x - y) + 2\dot{x}\dot{y}(2x - y) - \dot{x}\dot{y}(2y - x) - 2gy + gx . \end{aligned}$$

(b) As

$$\frac{\partial V}{\partial x} = mg(2x - y), \qquad \frac{\partial V}{\partial y} = mg(2y - x) ,$$

equilibrium occurs at the origin $(0, 0)$. For small oscillations about the origin, $x, y, \dot{x}, \dot{y}$ are small quantities and the equations of motion reduce to

$$\ddot{x} + 2gx - gy = 0 ,$$
$$\ddot{y} + 2gy - gx = 0 .$$

$-\omega^2 x_0 e^{i\omega t} + 2g x_0 e^{i\omega t} - g y_0 e^{i\omega t} = 0$

$(2g - \omega^2)x_0 - g y_0 = 0$

$\frac{y_0}{x_0} = \frac{2g - \omega^2}{g}$

$-\omega^2 y_0 e^{i\omega t} + 2g y_0 e^{i\omega t} - g x_0 e^{i\omega t} = 0$

$(2g - \omega^2)y_0 - g x_0 = 0$

SOLUTION
Part II
Problem 14

Considering a solution of the type

$$x = x_0 e^{i\omega t}, \qquad y = y_0 e^{i\omega t} ,$$

we find the secular equation

$$\begin{vmatrix} 2g-\omega^2 & -g \\ -g & 2g-\omega^2 \end{vmatrix} = (g-\omega^2)(3g-\omega^2) = 0 .$$

Its position roots

$$\omega_1 = \sqrt{g}, \qquad \omega_2 = \sqrt{3g}$$

are the angular frequencies of the normal-modes of the system. Note that as ω_1, ω_2 are real the equilibrium is stable.

(c) As

$$\frac{y_0}{x_0} = \frac{2g-\omega^2}{g} ,$$

If $\omega = \sqrt{g}$, $\frac{y_0}{x_0} = \frac{2g-g}{g} = 1$

If $\omega = \sqrt{3g}$, $\frac{y_0}{x_0} = \frac{2g-3g}{g} = -1$

for the higher frequency mode to be excited we require $\frac{y_0}{x_0} = -1$. Hence the initial displacements of x and y must be equal in magnitude and opposite in sign. Note that under this condition the lower frequency mode, which requires $y_0/x_0 = 1$, is not excited.

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A rigid structure consists of three massless rods joined at a point attached to two point masses (each of mass m) as shown in Fig. 2.69, with $AB = BC = L$, $BD = l$, the angle $ABD = DBC = \theta$. The rigid system is supported at the point D and rocks back and forth with a small amplitude of oscillation. What is the oscillation frequency? What is the limit on l for stable oscillations?

(*CUSPEA*)

Solution:

The structure oscillates in a vertical plane. Take it as the xy-plane as shown in Fig. 2.70 with the origin at the point of support D and the y-axis vertically upwards. We have

$$\overline{AD} = \overline{CD} = b = \sqrt{L^2 + l^2 - 2Ll\cos\theta} ,$$

$$g^2 - (2g-\omega^2)^2 = 0$$

$$g^2 - 4g^2 - \omega^4 + 4g\omega^2 = 0$$

$$3g^2 + \omega^4 - 4g\omega^2 = 0$$

$$(3g-\omega^2)(g-\omega^2) = 0$$

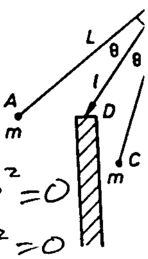

Fig. 2.69.

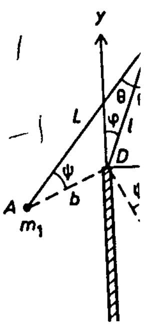

Fig. 2.7

and the angles between $\overline{AD}$ and $\overline{CD}$ ·
respectively, where $\alpha = \theta + \psi$, ψ being

$$\frac{b}{\sin\theta} = \ \vdots$$

The masses m_1, m_2 have coordinat

$$(-b\sin(\alpha+\varphi), -b\cos(\alpha+\varphi)),$$

and velocities

$$(-b\dot\varphi\cos(\alpha+\varphi), b\dot\varphi\sin(\alpha+\varphi)),$$

respectively. Thus the Lagrangian is

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Problem 15.

Use the variational method to calculate the total binding energy of the two electrons in a helium atom in its ground state. Use the standard hydrogenic wave functions with Z* < Z (due to screening) as a parameter for trial wave functions. You may use the expected value of $\frac{1}{r}$ in a hydrogenic state with nuclear charge of Ze, $\left\langle \Psi_{100} \middle| \frac{1}{r} \middle| \Psi_{100} \right\rangle = \frac{Z}{a_0}$, and $\left\langle \Psi_{100}\Psi_{100} \middle| \frac{1}{\vec{r}_1 - \vec{r}_2} \middle| \Psi_{100}\Psi_{100} \right\rangle = \frac{5Z}{8a_0}$.

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$$

Remember use Z in H and Z^* in ψ.

$$\langle \psi_{100}^{(1)} \psi_{100}^{(2)} | H | \psi_{100}^{(1)} \psi_{100}^{(2)} \rangle \equiv \langle H \rangle$$

$$= 2 \langle \psi_{100} | \frac{p^2}{2m} - \frac{Z^* e^2}{r} | \psi_{100} \rangle - 2 \langle \psi_{100} | \frac{(Z - Z^*) e^2}{r} | \psi_{100} \rangle$$

$$+ \langle \psi_{100}^{(1)} \psi_{100}^{(2)} | \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} | \psi_{100}^{(1)} \psi_{100}^{(2)} \rangle$$

$$\langle H \rangle = 2\left(-\tfrac{1}{2}\alpha^2 mc^2\right) Z^{*2} - 2(Z - Z^*) \frac{Z^* e^2}{a_0} + e^2 \frac{5}{8} \frac{Z^*}{a_0}$$

$$\frac{e^2}{a_0} = \alpha^2 mc^2$$

$$\langle H \rangle = \alpha^2 mc^2 \left[-Z^{*2} - 2Z^*(Z - Z^*) + \tfrac{5}{8} Z^* \right]$$

$$= \alpha^2 mc^2 \left[Z^{*2} - 2ZZ^* + \tfrac{5}{8} Z^* \right]$$

$$\frac{\partial \langle H \rangle}{\partial Z^*} = 0 = 2Z^* - 2Z + \frac{5}{8}$$

$$\Rightarrow Z^* = Z - \frac{5}{16}$$

$$E_0 = \alpha^2 mc^2 \left(\frac{27}{16}\right) \left[\left(\frac{27}{16}\right) - 4 + \frac{5}{8} \right] = -\left(\frac{27}{16}\right)^2 \alpha^2 mc^2$$

SOLUTION
Part II
Problem 16

SCORE _

Physics Departmental Examination - Fall 1996 - Part II

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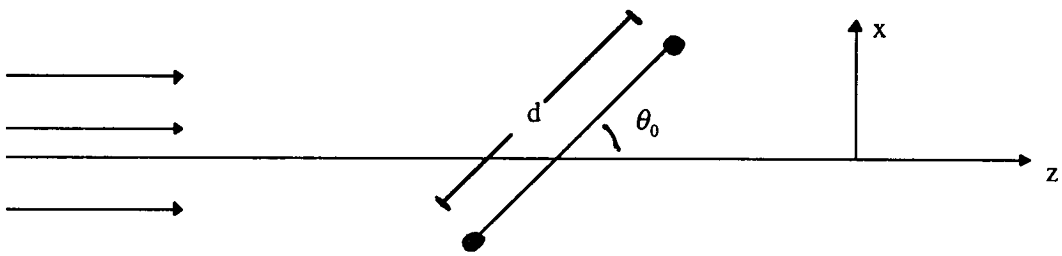
Problem 16.

A beam of particles of mass in is incident on two identical scattering centers separated by a distance, d, and aligned at angle θ_0 $(\theta_0 < \pi/2)$ with respect to the direction of the incident beam $(\hat{z})$, as shown in the picture



The incident particles are described by wave packets of mean momentum $\vec{k} = k\hat{z}$. Assume $kd \ll 1$. The scattering centers are described by δ-function potentials:

$V(\vec{r}) = V_o\delta(\vec{r})$ for a center at the origin.

a) Find the differential scattering cross section in the Born approximation.

b) For which direction(s) is the differential scattering cross section maximal for scattering in the plane defined by the scattering centers?

c) Give a general condition on the angles θ and φ in spherical coordinates for which maximal scattering occurs (use the axis convention shown in the figure). What is the range of values of θ and φ for which maximal scattering occurs?

Solution '16'

The differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2$$

where $f(\theta,\varphi)$ is the scattering amplitude. In the Born approximation

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2} \langle \vec{k} \,|\, V_{tot} \,|\, \vec{k}' \rangle$$

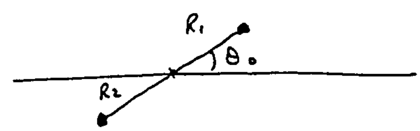
where $\vec{k}$, $\vec{k}'$ are incoming and outgoing momenta, and V_{tot} is the total potential

$$V_{tot}(r) = V(r-R_1) + V(r-R_2)$$

$$\langle \vec{k} \,|V_{tot}|\, \vec{k}' \rangle = \int d^3r \, e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} \, [\, V(r-R_1) + V(r-R_2) \,] =$$

$$= \left(e^{-i\vec{\Delta}\cdot\vec{R}_1} + e^{-i\vec{\Delta}\cdot\vec{R}_2}\right) \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \, V(\vec{r}) = V_0 \left[e^{-i\vec{\Delta}\cdot\vec{R}_1} + e^{-i\vec{\Delta}\cdot\vec{R}_2}\right]$$

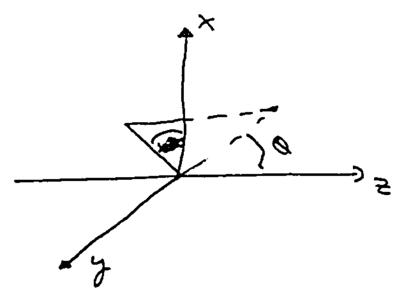
where $\vec{\Delta} = \vec{k}' - \vec{k}$. Since $\vec{R}_2 = -\vec{R}_1$



$$\langle \vec{k} \,|V_{tot}|\, \vec{k}' \rangle = 2V_0 \cos \vec{\Delta}\cdot\vec{R}_1$$

$$f(\theta,\varphi) = -\frac{mV_0}{\pi\hbar^2} \cos \vec{\Delta}\cdot\vec{R}_1$$

a) $$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2 V_0^2}{\pi^2\hbar^4} \left|\cos \vec{\Delta}\cdot\vec{R}_1\right|^2}$$



$$\vec{k} = (0, 0, k)$$

$$\vec{k}' = k(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

$$\Rightarrow \quad \vec{\Delta} = \vec{k}' - \vec{k} = k(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta - 1)$$

$$\vec{R}_1 = \left(\frac{d}{2}\sin\theta_0,\ 0,\ \frac{d}{2}\cos\theta_0\right)$$

$$\Rightarrow \quad \vec{\Delta}\cdot\vec{R}_1 = \frac{kd}{2}\left[\sin\theta\sin\theta_0\cos\varphi + (\cos\theta - 1)\cos\theta_0\right]$$

Since $kd \ll 1$, to get maximal scattering we need $\vec{\Delta}\cdot\vec{R}_1 = 0 \Rightarrow$ 1c pg 2

$$(1-\cos\theta)\cos\theta_0 = \sin\theta\sin\theta_0\cos\phi \Rightarrow$$

$$2\sin^2\tfrac{\theta}{2}\cos\theta_0 = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\sin\theta_0\cos\phi$$

$\Rightarrow$ one solution is $\boxed{\theta = 0}$ forward scattering.

or,
$$\sin\tfrac{\theta}{2}\cos\theta_0 = \cos\tfrac{\theta}{2}\sin\theta_0\cos\phi$$

$$\Rightarrow \boxed{\tan\tfrac{\theta}{2} = \tan\theta_0\cos\phi} \quad (c)$$

In the plane of scatterers, $\phi = 0 \Rightarrow \tan\frac{\theta}{2} = \tan\theta_0 \Rightarrow$

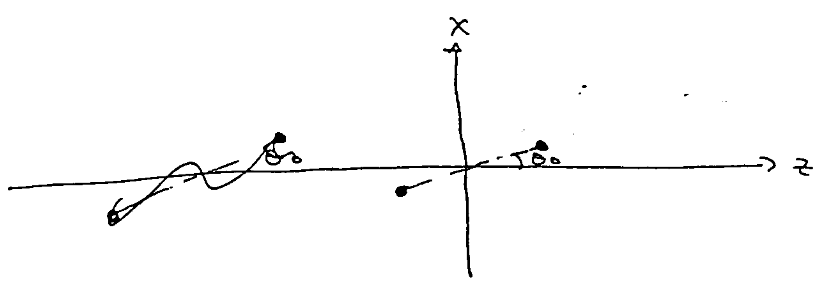
(b) $\boxed{\theta = 2\theta_0}$ or $\boxed{\theta = 0}$

General condition is given above. Max value of θ is $2\theta_0$, so range of θ is $\boxed{0 \le \theta \le 2\theta_0}$

Since $\theta < \pi$ and $\theta_0 < \frac{\pi}{2} \Rightarrow \cos\phi > 0 \Rightarrow$

$$\boxed{-\tfrac{\pi}{2} < \phi < \tfrac{\pi}{2}}$$

that is, maximal scattering only occurs in the $x > 0$ half-space.



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~~SOLUTION Part A Problem 8~~

16, pg 3

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