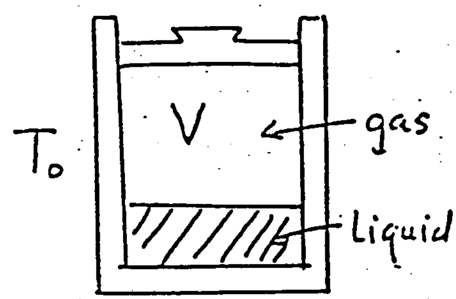
Code Number ___________

UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[1] A thermally isolated system, consisting of ν moles of a hypothetical non-monatomic ideal gas and a liquid of different substance, is in equilibrium at temperature T_o. (Assume no exchange of molecules between gas and liquid.) The gas has volume V and heat capacity at constant volume $C_v = 3\nu R$; therefore $\gamma = C_p/C_v = 4/3$. The liquid has heat capacity C_L.



a) The gas is quasistatically compressed to half its initial volume, slowly enough that the gas is always in equilibrium. Assuming no heat is transferred to the liquid during compression, what is the temperature, T_g, of the gas?

b) What is the change in internal energy, ΔE_g, and the change in entropy, ΔS_g, of the gas after compression?

c) Later, the liquid and gas come to equilibrium. What is the final temperature, T_f, of the system?

d) What is the total change in entropy of the system from the beginning to the final equilibrium? (You can leave the answer in terms of T_g and T_f.)

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[2] 1 kg of water at 20° C is contained in a thin but strong polymer bag. When the bag is thrown off of a 5 m high ledge onto a sidewalk, the bag deforms initially but then returns to its original shape. (In answering this question, you may neglect any thermal transfer from the bag to the sidewalk or air. Try to provide the necessary physical constants and thermodynamic quantities to obtain numerical answers, but you will receive partial credit if you don't know the relevant numbers.)

a) What is the change in entropy of the water as a result of the fall of the bag? Show that this result is, to a good approximation, independent of the heat capacity of the water.

b) What is the (ridiculously small) probability that you will find the bag back up on the top of the building, as a result of the finite temperature of the system?

c) Now suppose the bag has two compartments separated by a thin membrane. One side contains 0.5 kg of water in which the oxygen molecule is the common isotope, ^{16}O, and the other side contains 0.5 kg of water molecules containing the rarer isotope, ^{18}O. When this bag is thrown off of the ledge and hits the sidewalk, the membrane (but not the bag) breaks, and the two kinds of water molecules mix thoroughly. Assuming the thermodynamic properties of the two kinds of water molecules are the same, estimate the change in entropy of the water as a result of mixing the water. Is this smaller or larger than the entropy change calculated in (a)?

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[3] A particle of mass m moves in one dimension in the potential

$$V(x) = B\,(x-b)^4$$

where B and b are given parameters. One cannot solve the Schroedinger equation analytically. Still, on the basis of dimensional and other general reasoning, you can answer the following questions about the ground state energy:

Let Eg be the ground state energy.

a) By what factor does it change if the parameter B is doubled, b being held unchanged.

b) What is the change in Eg if b is doubled, B being held unchanged?

c) What is the expectation value $\langle x \rangle_g$ in the ground state?

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DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[4] Consider a spinless charge e particle confined to the surface of a sphere of radius R. Further suppose that the system is described by the Hamiltonian

$$H = H_0 + H_1 = \frac{\vec{L}^2}{2MR^2} - eE_x R \sin\theta \cos\phi$$

where E_x is a weak external electric field pointing along the x-axis.

a) What are the eigenvectors and eigenvalues of H_0?

b) To order E_x^2, calculate the field induced shift of the ground state energy. Ignore all matrix elements involving the $\ell \geq 2$ levels.

(c) Now consider the effect of a quadropolar crystalline field on the electron. In this case take

$$H = \frac{\vec{L}^2}{2MR^2} + Q\,xY$$

where Q is a constant. Calculate the splitting of the $\ell = 1$ levels to the lowest order in Q. The integral $\int_0^{2\pi} d\phi \cos^4\phi = \frac{3\pi}{4}$ may be useful in solving this problem.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[5A] A simple pendulum consists of a point mass, m, and a massless rod of length, l, suspended from a frictionless bearing that allows rotation about a horizontal axis.

1) Write an exact expression for the constant, first integral of the motion in terms of the angle, θ, that the rod makes relative to the vertical.

2) Integrate, by quadratures to find an expression for the time required for the pendulum to swing between $\theta = 0$ and an arbitrary angle $\theta = \Theta$.

3) Transform to a new variable, ϕ, such that the expression is in the form of an elliptic integral

$$\int_0^{\varphi} \frac{d\varphi'}{\sqrt{1 - k^2 \sin^2(\varphi')}}$$

[5B] A planet of mass M has a moon of mass m.

1) Find the Langrangian equations for the relative angular position of the planets, θ, and the relative separation, r.

2) Use Lagrange's equations to show that the angular momentum of the system is conserved.

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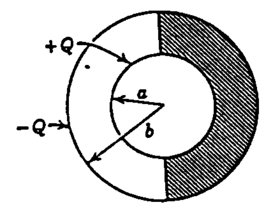
UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[6] Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges ±Q. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant ϵ), as shown in the figure.



a) Find the electric field everywhere between the spheres.
b) Calculate the surface-charge distribution on the inner sphere.
c) Calculate the polarization-charge density induced on the surface of the dielectric at $r = a$.

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DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part I

[7] Consider a classical gas consisting of charge *e* ions of density $n(\vec{r})$ superimposed on a neutralizing background. The charge and current densities are

$$\rho(\vec{r}) = e\left[n(\vec{r}) - n_0\right]$$

$$\vec{J}(\vec{r}) = en\,\vec{v}(\vec{r})$$

where en_0 is the charge density of the neutralizing background. The linearized equation of motion is

$$m\,n_0 \frac{\partial \vec{v}(\vec{r})}{\partial t} = e\,n_0\,\vec{E}(\vec{r}) - \nabla P$$

where *m* is the mass of the charged ion, $\vec{E}$ is the local electric field and $P(\vec{r})$ is the pressure. Derive a wave equation for the plasma waves using the equation of motion and any other equation you find necessary. You may assume that $\nabla P = \left(\frac{\partial P}{\partial n}\right)_0 \nabla n$ where $\left(\frac{\partial P}{\partial n}\right)_0$ is taken to be a constant.

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DEPARTMENT OF PHYSICS

Written Departmental Examination - Spring 1995, Part II

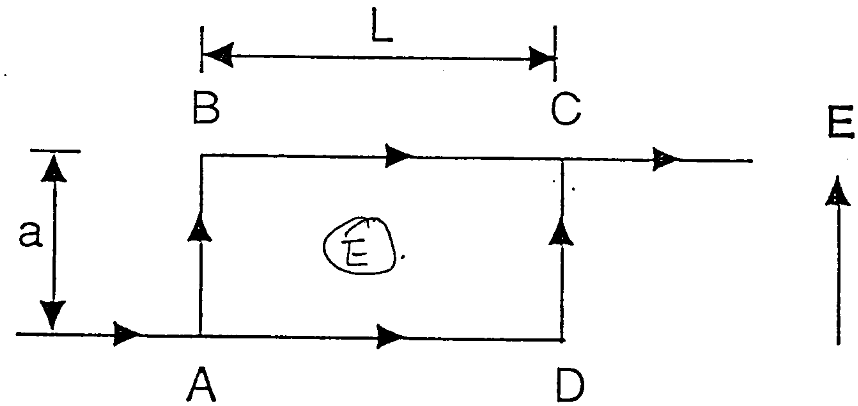
[8] *Potpourri.* Each answer must be supported by reasoning.
Possibly useful information:

- $1\ \text{eV} / h = 242$ THz
- $1\ \text{eV} / hc = 8100\ \text{cm}^{-1}$
- $1\ \text{eV} / k_B = 11600$ K

a) The molten metal in a furnace appears to emit predominantly blue light. Is the temperature of the metal closest to (I) 3 K, (II) 3×10^4 K or (III) 3×10^6 K?

b) Estimate the ground state energy of a harmonic oscillator using the uncertainty principle.

c) In an electron interferometer, an electron is split into two paths ABC and ADC in the figure below and recombined. The rectangular paths have the dimensions $AB = DC = a$ and $AD = BC = L$. The whole rectangle is placed in a uniform electric field along the direction AB or DC. The electron has a large incident kinetic energy K so that the curvature of the sections of the paths AD and BC may be neglected. If interference fringes are produced as the electric field strength is varied, find the interference fringes' separation as the change of the electric field in terms of the electron kinetic energy, its mass and charge, the dimensions of the paths and any universal constants.



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Written Departmental Examination - Spring 1995, Part II

[9] Consider a one-dimensional chain of N atoms, spaced equally along a length L. There are two transverse modes of excitation which propagate with velocity c_t and one longitudinal mode with velocity $c \;\; >> c_t$. Use the Debye approximation to answer the following, leaving any integrals that you cannot do in dimensionless form.

a) What is the heat capacity of the system due to these modes, at low temperatures, where $\beta h\, c_t k_D >> 1$, and k_D is the Debye wave number?

b) Estimate the dominant contribution to the heat capacity when $hc_t k_D << 1$ and $hc \;\; k_D >> 1$.

c) Now suppose the excitation spectrum for the transverse modes of oscillation of the chain has a "gap", so that their energies are given by $h\omega = \sqrt{\epsilon_0^2 + h^2 c_t^2 k^2}$. Estimate the contribution to the heat capacity of the system due to these modes at temperatures $k_B T << \epsilon_0 << hc_t k_D$.

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Written Departmental Examination - Spring 1995, Part II

[10] Consider a one dimensional harmonic oscillator described by the Hamiltonian

$$H_0 = h\omega\left(a^+ a + \frac{1}{2}\right)$$

where a^+ and a are raising and lowering operators.

a) Consider the position and momentum operators. In the Schrödinger representation

$$x = \left(\frac{h}{2m\omega}\right)^{\frac{1}{2}}(a^+ + a)$$

$$P = i\left(\frac{mh\omega}{2}\right)^{\frac{1}{2}}(a^+ - a)$$

Write x (t) and P (t), the position and momentum operators in the Heisenberg representation, in terms of a and a^+.

b) Let $|gs\rangle$ be the ground state i.e. $a\,|gs\rangle = 0$. Use this to obtain the ground state wave function $\psi_{gs}(x) = \langle x\,|gs\rangle$.

c) Calculate the ground state expectation $\langle e^{isx}\rangle$ where s is some real number.

d) Calculate the eigenvalues of $H = H_0 + \beta^* a + \beta\, a^+$ where β is some complex number.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
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Written Departmental Examination - Spring 1995, Part II

[11] a) A composite particle is made up of two particles of spin 1/2 bound tightly together. The Hamiltonian for the internal energy of the composite is given by

$$H_0 = \alpha \vec{L} \cdot \vec{S},$$

where α is a constant and $\vec{L}$ and $\vec{S}$ are the spin vectors of the two constituent particles. Find the eigenenergies. For degenerate states, find the degeneracy.

b) When the composite is placed in a magnetic field, there is an additional term to the Hamiltonian given by

$$H_1 = \beta h L_z,$$

where β is a constant. Find the eigenenergies of the total Hamiltonian

$$H = H_0 + H_1 .$$

c) If the magnetic field is weak in the sense that $\alpha >> \beta$, find the splittings of the degenerate eigenenergies of H_0 in part (a) to first order in β by deducing the corresponding g-factor in replacing H_1 by

$$H_{\text{effective}} = g\beta h J_z ,$$

by the same argument with which the Lande factor is obtained.

d) Verify the results of part (c) by expanding the energies in part (b) in powers of β.

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Written Departmental Examination - Spring 1995, Part II

[12] Initially a solid, cylindrical, copper bar resides in a uniform, external magnetic field. The bar is oriented parallel to the field, that is, $\vec{H} = H_0 \hat{Z}$ where (z, r, θ) is a cylindrical coordinate system with the z-axis coincident with the axis of the cylinder. At $t = 0$, the external field is switched off on a time scale that is long compared to L/c but short compared to $\frac{\sigma R^2}{c^2}$, where σ is the conductivity of the copper and L and R are the length and radius of the cylindrical bar. Assuming that $L >> R$, determine the magnetic field in the bar in a cross sectional plane that is far from either end, that is, determine $\vec{H}(r, \theta, t)$.

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Written Departmental Examination - Spring 1995, Part II

[13] In a region of space there is a uniform magnetic field $\vec{B} = B_0\hat{Z}$ and a uniform electric field $\vec{E} = E_0\hat{X}$.

a) Write down the Lagrangian and Hamiltonian for an electron that moves relativistically in these fields. Also, obtain three independent constants of the motion.

b) Suppose that the electron is released from rest at the origin. For a given value of B_0, determine the minimum value of E_0 for which $|x|$ increases without bound in the subsequent motion of the electron. This problem explores the balance between two effects: the electric field accelerates the electron in the $-\hat{x}$ direction and the magnetic field tries to deflect the motion into a cyclotron orbit.

c) For the value of E_0 obtained in part (b), determine $x(t)$.

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Written Departmental Examination - Spring 1995, Part II

[14] Find the leading asymptotic behavior of $I(x) = \int_0^{\pi/2} \sqrt{\sin t}\, e^{-x \sin^4 t}\, dt$ as $x \to \infty$. You may express the answer in terms of the Gamma Function (Factorial Function).

SP 95 SOLUTIONS

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Solution:

(a.) Adiabat of an ideal gas is

$$PV^{\gamma} = \text{const} \quad \text{which implies}$$

$$V^{\gamma-1}T = \text{const}$$

so $$= \left(\frac{V}{2}\right)^{\gamma-1} T_g = V^{\gamma-1} T_0$$

so $$\boxed{T_g = 2^{\gamma-1} T_0 = 2^{1/3} T_0}$$

(b.) $\boxed{\Delta S_g = 0}$, since we have a quasistatic process in which no heat is transfered to the liquid.

$$\boxed{\Delta E = 3\nu R \left[2^{1/3} - 1\right] T_0}$$

To obtain this note $E = 3\nu RT$ which follows from

$$\left(\frac{\partial E}{\partial T}\right)_V = C_v = 3\nu R$$

and

$$\left(\frac{\partial E}{\partial V}\right)_T = 0$$

The latter is a general result for all ideal gases. It is obtained using a Maxwell identity.

(c.) Let E_i be the total energy of gas + liquid before the compression. Immediately after the compression the total energy of $g + l$ is

$$E_i + 3\nu R T_0 (2^{1/3} - 1)$$

Now let gas and liquid exchange heat. The total energy is now

$$E_f = E_i + C_L (T_f - T_0) + 3\nu R (T_f - T_0)$$

This must be the same as the total energy immediately after compression, so

$$C_L (T_f - T_0) + 3\nu R (T_f - T_0) = 3\nu R T_0 (2^{1/3} - 1)$$

which may be solved to get

$$\boxed{T_f = \left[\frac{3\nu R\, 2^{1/3} + C_L}{3\nu R + C_L}\right] T_0}$$

(d.) The change in entropy of liquid is

$$\Delta S_L = C_L \int_{T_0}^{T_f} \frac{dT}{T} = C_L \ln \frac{T_f}{T_0}$$

The entropy change of the gas is

$$\Delta S_g = 3\nu R \int_{T_g}^{T_f} \frac{dT}{T}$$

Note T_g not T_0 is lower limit since the compression didn't change the entropy of the gas.

Hence

$$\boxed{\Delta S = C_L \ln \frac{T_f}{T_0} + 3\nu R \ln \frac{T_f}{T_0}}$$

(I) 1 kg of water at 20 ^{0}C is contained in a thin but strong polymer bag. When the bag is thrown off of a 5 m high ledge onto a sidewalk, the bag deforms initially but then returns to its original shape. [In answering this question, you may neglect the thermal mass and conductivity of the bag, air and sidewalk. Try to provide the necessary physical constants and thermodynamic quantities to obtain numerical answers, but you will receive partial credit if you don't know the relevant numbers.]

(a) What is the change in entropy of the water as a result of the fall of the bag? Show that this result is, to a good approximation, independent of the heat capacity of the water.

$$\Delta S = \int \frac{đQ}{T}$$ Mgh goes into heat $C\Delta T = mgh$

$$\Delta S = \int_i^f \frac{C\,dT}{T} = C \ln\left(1 + \frac{\Delta T}{T_i}\right)$$

$$C = 1\ \mathrm{Cal/gm\,°K} = 4.2 \times 10^3\ \mathrm{J/kg\,°K}$$

$$mgh = 1(9.8)5 = 49\ \mathrm{J};$$ thus $\Delta T \sim 10^{-2}\ K \ll T_i$

$$\Delta S \simeq C\frac{\Delta T}{T_i} = \frac{mgh}{T_i} = \frac{49}{293} \simeq \underline{\frac{1}{6}\ \mathrm{J/°K}}$$

(b) What is the (riduclously small) probablility that you will find the bag back up on the top of the building, as a result of the finite temperature of the system.

$$P = e^{-\Delta S/k} = e^{-\frac{1}{6(1.4)10^{-16}10^{-7}}} = \underline{e^{-1.2 \times 10^{22}}}$$

Problem for graduate written exam

Spring, 1995
Cliff Surko

(I) (cont'd)

(c) Now suppose the bag has two compartments separated by a thin membrane. One side contains 0.5 kg of water in which the oxygen molecule is the common isotope, ^{16}O, and other side contains 0.5 kg of water molecules containing the rarer isotope, ^{18}O. When this bag is thrown off of the ledge and hits the sidewalk, the membrane (but not the bag) breaks, and the two kinds of water molecules mix thoroughly. Assuming the thermodynamic properties of the two kinds of water molecules are the same, estimate the change in entropy of the water as a result of mixing the water. Is this smaller or larger than the entropy change calculated in (a)?

(i) Ideal gas approx:

$$\Omega = \frac{V^N}{N!}$$

$$S/k_B = \ln \Omega = N \ln V - N \ln N + N \simeq N \ln \frac{V}{N}$$

$$S_i/k_B = 2\left(\frac{N}{2}\right) \ln\left(\frac{V/2}{N/2}\right) = N \ln \frac{V}{N}$$

$$S_f/k_B = 2\left(\frac{N}{2}\right) \ln \frac{V}{N/2} = N \ln\left(\frac{2V}{N}\right)$$

$$\underline{\Delta S = N k_B \ln 2}$$

A water molecule weighs $1.6 \times 10^{-24} 10^{-3} (34) = 5 \times 10^{-26}$ kg

Thus $N = \frac{1\,kg}{5 \times 10^{-26}\,kg} = 2 \times 10^{25}$

$\Delta S = 2 \times 10^{25}\, 1.4 \times 10^{-16}\, 10^{-7} \ln 2 = \underline{190\ J/K}$ So entropy of mixing is much larger.

I-(C) (Cont'd)

(ii) Alternate solution: "lattice model"

Initial state: $\frac{N}{2}$ sites for each component + $N/2$ molecules in each

$S_i = k_B \ln \Omega = 2k_B \ln(N/2\,!)$

$\simeq 2k_B N \ln N$

Final state: N sites + N molecules

$S_f = N k_B \ln N$

Thus $\underline{\Delta S = N k_B \ln 2}$ ✓

Part 1
Solution Problem 3

Coordinate change $x' = x - b$ removes b from the Schroedinger equation and gives a potential Bx'^4 symmetric about the origin. Evidently the energy eigenvalues don't depend on b. Since there is no degeneracy of bound states in one dimension, eigenfunctions are either even or odd in x' so $\langle x' \rangle = 0$ in any state, so $\langle x \rangle = b$.

B has dimensions $[\text{energy}]/[\text{length}]^4$.

Since b doesn't play a role in ~~the~~ determining the energy, the only other parameter available is $\hbar^2/m$, which has dimensions $[\text{energy}][\text{length}]^2$. Thus the quantity $B\left(\frac{\hbar^2}{m}\right)^2$ has dimensions $[\text{energy}]^3$

Any energy eigenvalue must be a pure ~~number~~ $\times$ ~~$[B\hbar^2 \ldots]$~~

number $\times$ $\left[B\left(\frac{\hbar^2}{m}\right)^2\right]^{1/3} = B^{1/3}\left(\frac{\hbar^2}{m}\right)^{2/3}$.

So doubling B means an energy increase by $2^{1/3}$.

Part 1
Solution Problem 4

SOLUTION

(a)

$$E^{(0)}_{\ell m} = \frac{\hbar^2 \ell(\ell+1)}{2m R^2}$$

$$\psi_{\ell m}(\theta,\phi) = Y_{\ell m}(\theta,\phi)$$

(b.) Work with the basis

$$\psi_{00} = \frac{1}{\sqrt{4\pi}} \qquad \psi_x = \sqrt{\frac{3}{4\pi}}\,\frac{x}{R}$$

$$\psi_y = \sqrt{\frac{3}{4\pi}}\,\frac{y}{R} \qquad \psi_z = \sqrt{\frac{3}{4\pi}}\,\frac{z}{R}$$

Now

$$E_{00} = E^{(0)}_{00} + E^{(2)}_{00} = \sum_i{}' \frac{|H_{00,i}|^2}{E^0_{00} - E^0_i}$$

where

$$H_{00,i} = \int d\Omega \; \psi^*_{00}\,[-eE_x x]\,\psi_i$$

$$= -\frac{\sqrt{3}}{4\pi}\,eE_x \int d\Omega\,\frac{x^2}{R}\,\delta_{ix}$$

$$H_{00,i} = -\frac{eE_x R}{4\sqrt{3}\pi}\,\delta_{ix}$$

So

$$E_{00} = \frac{(eE_x R)^2}{16\cdot 3\cdot \pi^2}\,\frac{1}{\frac{\hbar^2}{2mR^2}[-2+0]}$$

$$\boxed{E_{00} = -\frac{(eE_x)^2}{3\cdot 16\cdot \pi^2}\,\frac{mR^4}{\hbar^2}}$$

(c.) Must solve $\det\left[E\,\delta_{ij} - H_{ij}\right] = 0$

where $H_{ij} \equiv \int d\Omega\, \psi_i^* H \psi_j$

$$= \frac{\hbar^2}{mR^2}\,\delta_{ij} + Q\int d\Omega\, \psi_i^*\psi_j\,(xy)$$

Now $\int d\Omega\, \psi_i^*\psi_j\, xy = \frac{3}{4\pi} R^2 \int d\Omega\, \cos^2\phi\,\sin^2\phi \equiv I$

if $i,j = (x,y)$ or (y,x) otherwise the M.E. vanishes.

$$\int d\Omega\, \cos^2\phi\,\sin^2\phi = 2\int_0^{2\pi} d\phi\, \cos^2\phi\,(1-\cos^2\phi)$$

$$= 2\cdot\left[\pi - \int_0^{2\pi} d\phi\,\cos^4\phi\right]$$

$$= 2\left[\pi - \frac{3}{8}\cdot 2\pi\right]$$

$$= 2\,\frac{\pi}{4} = \frac{\pi}{2}$$

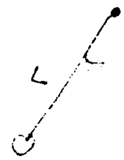
$$H_{ij} = \frac{\hbar^2}{mR^2}\,\delta_{ij} + Q\cdot\left(\frac{3}{4\pi}R^2\,\frac{\pi}{2}\right)\left[\delta_{ix}\delta_{jy} + \delta_{jx}\delta_{iy}\right]$$

Now $\frac{3}{8}QR^2\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

has ev's $0,\ \pm\frac{3}{8}QR^2$. So

$$\boxed{E = \frac{\hbar^2}{mR^2},\quad \frac{\hbar^2}{mR^2} \pm \frac{3}{8}QR^2}$$

Problem 5 A



1) The energy is first integral

$$E = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt}\right)^2 + mgL - mgL\cos\theta$$

$$\frac{d\theta}{dt} = \sqrt{2\left\{\frac{E}{mL^2} - \frac{g}{L} + \frac{g}{L}\cos\theta\right\}}$$

2) $$dt = \int_0^{\Theta} \frac{d\theta}{\sqrt{2\left\{\frac{E}{mL^2} - \frac{g}{L} + \frac{g}{L}\cos\theta\right\}}}$$

3) Let $\theta = 2\alpha$

$$\cos\theta = 1 - 2\sin^2\alpha$$

$$dt = \int_0^{\Theta/2} \frac{2\,d\alpha}{\sqrt{2\left\{\frac{E}{mL^2} - \frac{g}{L} + \frac{g}{L}(1 - \sin^2\alpha)\right\}}}$$

$$dt = \frac{2}{\sqrt{2E/mL^2}} \int_0^{\Theta/2} \frac{d\alpha}{\sqrt{1 - \frac{2g\,mL^2}{2LE}\sin^2\alpha}}$$

$$k^2 = \frac{g\,mL}{E}$$

$$dt = \sqrt{\frac{2mL^2}{E}} \int_0^{\Theta/2} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}}$$

Part 1
Solution Problem 5

Problem 5 B

1) $T = \frac{1}{2}\frac{Mm}{M+m}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

$V = \frac{GMm}{r}$

$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$

<u>For θ</u>

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$

$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} = \mu r^2\dot{\theta}$

<u>For r</u>

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$\frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{GMm}{r^2} = 0$

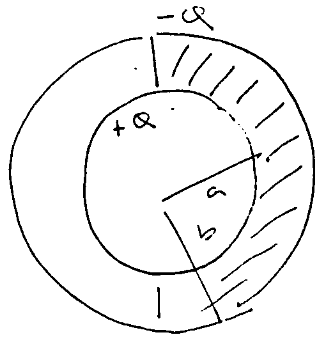
2) Since θ is cyclic, $\frac{\partial L}{\partial \dot{\theta}} = \text{constant}$

$\frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$ which is the angular momentum

Problem 1

Solution

conducting sphere.



Left side | Right side

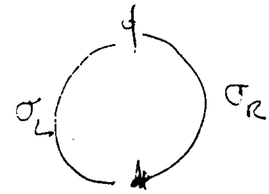
Since we have a inside conducting sphere

$E = 0 \quad r < a = 0$ $\qquad \phi = \text{constant} \quad r < a$

From the symmetry

$a < r < b$

$$\phi_{ab} = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$



at $r = a$

$$(E_{out} - E_{in}) \times \hat{n} = 0$$

$$\frac{\partial \phi_{ab}}{\partial \theta} = 0 \implies \phi_{ab} = \left(A_0 + \frac{B_0}{r} \right)$$

Also

$$(D_{out} - D_{in}) \cdot \hat{n} = 4\pi\sigma$$

Left side

$$-\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \frac{B_0}{a^2} = 4\pi\sigma_L$$

$$\Longrightarrow \quad \frac{\sigma_R}{\varepsilon} = \sigma_L$$

Right side

$$-\varepsilon \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = \frac{\varepsilon B_0}{a^2} = 4\pi\sigma_R$$

Total charge Q

$$\therefore \quad Q = 2\pi a^2 (\sigma_L + \sigma_R) = 2\pi a^2 \sigma_R \left(\frac{\varepsilon + 1}{\varepsilon}\right)$$

$$\therefore \quad \sigma_R = \frac{Q\,\varepsilon}{2\pi a^2 (\varepsilon+1)} \Longrightarrow B_0 = \frac{2Q}{(\varepsilon+1)}$$

$$\therefore \quad \sigma_L = \frac{Q}{2\pi a^2(\varepsilon+1)}$$

$$\sigma_R = \frac{Q\,\varepsilon}{2\pi a^2(\varepsilon+1)}$$

since

$$D = \varepsilon E$$

Electric Field

$E = 0 \quad r < a$

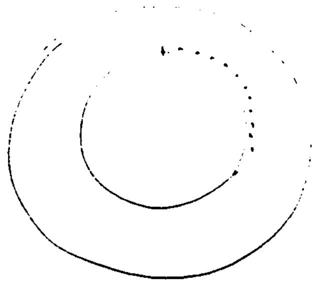
$E = \dfrac{2Q}{(\varepsilon+1) r^2} \quad a < r < b$

$E = 0 \quad r > b$

c) The charge induced

$$Q_{ind} = -2\pi a^2 (\sigma_E - \sigma_L)$$

$$= \boxed{\frac{Q(\varepsilon-1)}{(\varepsilon+1)}}$$



Part 1
Solution Problem 7

Using the linearized continuity eqn

$$\frac{\partial n}{\partial t} + n_0 \nabla \cdot \vec{V} = 0$$

~~So~~

$$0 = \frac{\partial^2 n}{\partial t^2} + n_0 \nabla \cdot \frac{\partial \vec{V}}{\partial t}$$

$$= \frac{\partial^2 n}{\partial t^2} + n_0 \nabla \cdot \left\{ \frac{e}{m} E(\vec{r}) - \frac{1}{m n_0}\left(\frac{\partial P}{\partial n}\right)_0 \nabla n \right\}$$

or

$$\frac{\partial^2 n}{\partial t^2} - \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0 \nabla^2 n + \frac{e n_0}{m} \nabla \cdot E(\vec{r}) = 0$$

Now use $\nabla \cdot E = 4\pi e\,(n(r) - n_0)$ so

$$\boxed{\frac{\partial^2 n}{\partial t} - \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0 \nabla^2 n + \frac{4\pi n_0 e^2}{m}\left[n(r) - n_0\right] = 0}$$

This is solution

So $n(\vec{r}) = n_0 + \text{const}\; e^{i(\vec{q}\cdot\vec{r} - \omega t)}$ is a soln

where

$$\boxed{\omega^2 = \frac{4\pi n_0 e^2}{m} + \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0 q^2}$$

or

$$= \omega_p^2 + \frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_0 q^2$$

where $\omega_p^2 = \dfrac{4\pi n_0 e^2}{m}$ is the well known plasma frequency squared.

None of this is called for by the question

Part II
Solution Problem 8

You get 7/10

1. (a) Blue light $\sim 2eV \sim 2.4\times10^4$ K.

Case (II).

As $\overline{\Delta x^2} = \overline{x^2}$ since $\bar{x}=0$

(b) $E \sim \frac{\Delta p^2}{2m} + k(\Delta x)^{\cancel{4}\,2} \leftarrow \frac{1}{2}K\overline{x^2} = \frac{1}{2}K\overline{\Delta x^2}$

$\sim \frac{\hbar^2}{m(\Delta x)^2} + k(\Delta x)^{\cancel{4}\,2}$ $\qquad \Delta p \sim \frac{\hbar}{\Delta x}$ from the uncertainty principle

$\frac{\partial E}{\partial(\Delta x)} = 0$. $\quad \Delta x \sim \cancel{\left(\frac{\hbar^2}{km}\right)^{\frac{1}{6}}}$

$E \sim k\cdot\left(\frac{\hbar^2}{km}\right)^{4/6} \propto \frac{k^{1/3}}{m^{2/3}}$ What's this got to do

$\therefore \frac{E_2}{E_1} = \left(\frac{k_2}{k_1}\right)^{1/3}\left(\frac{m_1}{m_2}\right)^{2/3} = 8^{1/3}/8^{2/3} = \frac{1}{2}$ with anything.

(c) Along BC $\quad K = \frac{\hbar^2k^2}{2m} + eEa$

$k = \sqrt{(K - eEa)\cdot\frac{2m}{\hbar^2}}$

phase $=$ kL

Along AD, $\quad K = \frac{\hbar^2k_0^2}{2m}, \qquad k_0 = \sqrt{\frac{2m}{\hbar^2}K}$

phase $= k_0L$

phase difference $\delta\varphi = (k-k_0)L$

$\simeq -\frac{eEaLk_0}{2K} = -2\pi n$; n = integer

$$\Delta E = \frac{4\pi K}{eaLk_0} = \frac{4\pi}{eaL}\sqrt{\frac{\hbar^2}{2m}K}.$$

PART II
SOLUTION PROBLEM 9

Part 2 9

am Spring 1995

(II) Consider a one-dimensional chain of N atoms, spaced equally along a length L. There are two transverse modes of excitation which propagate with velocity c_t and one longitudinal mode with velocity $c_\ell >> c_t$. Use the Debye approximation to answer the following, leaving any integrals that you cannot do in dimensionless forms.

(a) What is the heat capacity of the system due to these modes, at low temperatures, where $\beta\hbar c_t k_D >> 1$, and k_D is the Debye wavenumber?

$$N = \frac{L}{2\pi}\int_{-k_D}^{k_D} dk = \frac{k_D L}{\pi} \qquad k_D = \frac{\pi N}{L} \qquad \hbar\omega = \hbar c k$$

For each mode, $$\bar{E} = \frac{L}{2\pi}\int_{-k_D}^{k_D} \frac{\hbar c k\, dk}{e^{\beta\hbar\omega}-1} \approx \frac{L}{\pi}(k_B T)\left(\frac{k_B T}{\hbar c}\right)\underbrace{\int_0^\infty \frac{x\,dx}{e^x-1}}_{I_0}$$

$$\bar{E} = \frac{k_B^2 T^2 L}{\pi\hbar c} I_0; \quad \text{Thus, } C = \frac{2k_B^2 T L I_0}{\pi\hbar}\left(\frac{1}{c_\ell}+\frac{2}{c_t}\right) \equiv \underline{\frac{2Nk_B}{\pi}\left(\frac{k_B T}{\hbar k_D}\right)\left(\frac{1}{c_\ell}+\frac{2}{c_t}\right)I_0}$$

$\{ I_0 = \pi^2/6 \}$

(b) Estimate the dominant contribution to the heat capacity when $\hbar c_t k_D << 1$ and $\hbar c_\ell k_D >> 1$.

Then the transverse modes are all excited to amplitude $\sim k_B T$ and the longitudinal modes are "frozen out"

Thus $\bar{E} \simeq 2Nk_B T$,

and $\underline{C \simeq 2Nk_B}$

Pg 2

C. M. Surko Graduate Written Exam Spring 1995

(II) (cont'd)

(c) Now suppose the excitation spectrum for the transverse modes of oscillation of the chain has a "gap", so that their energies are given by $\hbar\omega = \sqrt{\epsilon_0^2 + h^2 c_t^2 k^2}$. Estimate the contribution to the heat capacity of the system due to these modes at temperatures $k_B T \ll \epsilon_0 \ll h c_t k_D$.

two transverse modes

$$\bar{E} = \frac{2L}{2\pi}\int_{-k_D}^{k_D} \frac{\hbar\omega\, dk}{e^{\beta\hbar\omega}-1} \simeq \frac{L}{\pi}\int_{-k_D}^{k_D} e^{-\beta\epsilon_0\left(1+\frac{\hbar^2 c_t^2 k^2}{2\epsilon_0^2}\right)}\epsilon_0\, dk$$

$$= \frac{L}{\pi} e^{-\beta\epsilon_0}\,\epsilon_0\,\frac{\sqrt{2k_BT\epsilon_0}}{\hbar c_t}\underbrace{\int_{-\infty}^{\infty} e^{-x^2}dx}_{\sqrt{\pi}}$$

$$x = \frac{\hbar c_t}{\epsilon_0}\sqrt{\frac{\beta\epsilon_0}{2}}\,k$$

$$\text{Thus } \bar{E} = \frac{L\,\epsilon_0^{3/2}\sqrt{2\,k_BT}}{\sqrt{\pi}\,\hbar c_t}\,e^{-\beta\epsilon_0}$$

$$C = \frac{N\sqrt{2\pi k_B T}\,\epsilon_0^{3/2}}{\hbar c_t k_D}\left[\frac{1}{2T} + \frac{\epsilon_0}{k_B T^2}\right]e^{-\beta\epsilon_0}$$

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PART II
SOLUTION PROBLEM 10

(a.) If O_S is a schrödinger picture operator

$$O_H(t) = e^{iH_0 t/\hbar} O_S e^{-iH_0 t/\hbar}$$

is the Heis. picture operator. Now

$$a_H(t) = e^{-i\omega t} a$$

since

$$\langle n | a_H(t) | m \rangle = e^{i(n+\frac{1}{2})\omega t} \langle n | a | m \rangle e^{-i(m+\frac{1}{2})\omega t}$$

$$= e^{i(n-m)\omega t} \langle n | a | m \rangle$$

Vanishes unless $n = m-1$. Similarly

$$a_H^+(t) = e^{i\omega t} a^+$$

So $$X = \left(\frac{\hbar}{2m\omega_c}\right)^{\frac{1}{2}} \left[a_H^+(t) + a_H(t)\right]$$

gives

$$\boxed{X(t) = \left(\frac{\hbar}{2m\omega_c}\right)^{\frac{1}{2}} \left[e^{i\omega t} a^+ + e^{-i\omega t} a\right]}$$

and

$$\boxed{P(t) = i\left(\frac{m\hbar\omega_c}{2}\right)^{\frac{1}{2}} \left[e^{i\omega t} a^+ - e^{-i\omega t} a\right]}$$

(b.) $a|gs\rangle = 0$ implies

$$\left(X + \frac{iP}{m\omega}\right)\psi(x) = 0$$

or $$= \left(X + \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi(x)$$

Hence $$\psi(x) \propto \exp -\left(\frac{m\omega}{\hbar}X^2/2\right)$$

Now $|\psi(x)|^2 = N^2 \exp - \frac{m\omega}{\hbar} x^2$ is a normalized

i.e. of the form $\frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{x^2}{2\sigma^2}$ if

$N^2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}$ so

$$\boxed{\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp - \frac{m\omega}{2\hbar} x^2}$$

is the normalized wavefunction.

(c.) $\langle e^{isx} \rangle = N^2 \int_{-\infty}^{\infty} dx \; e^{isx} \exp - \frac{m\omega}{\hbar} x^2$

$$= N^2 \int dx \; \exp - \frac{m\omega}{\hbar}\left(x - \frac{is\hbar}{2m\omega}\right)^2 \exp - \frac{m\omega}{\hbar}\left(\frac{s\hbar}{2m\omega}\right)^2$$

So $$\boxed{\langle e^{isx} \rangle = \exp - \frac{s^2\hbar}{4m\omega}}$$

(d.) $H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) + \beta^* a + \beta a^+$

$$= \hbar\omega\left[\left(a^+ + \frac{\beta^*}{\hbar\omega}\right)\left(a + \frac{\beta}{\hbar\omega}\right) + \frac{1}{2} - \frac{|\beta|^2}{\hbar^2\omega^2}\right]$$

Now $b \equiv a + \frac{\beta}{\hbar\omega}$ and $b^+ = a - \frac{\beta}{\hbar\omega}$

are cannonically transformed raising + lowering operators since

$$[b, b^+] = [a, a^+] = 1$$

Hence

$$H = \hbar\omega\left[b^{\dagger} b + \frac{1}{2}\right] - \frac{|\beta|^2}{\hbar\omega}$$

We see from this that H has the same spectra as H_0 except that it is shifted by $-\frac{|\beta|^2}{\hbar\omega}$ i.e.,

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{|\beta|^2}{\hbar\omega}}$$

PART II
SOLUTION PROBLEM 11

2. (a) Let $\vec{J} = \vec{L} + \vec{S}$

$$H_0 = \frac{\alpha}{2}\left(J^2 - L^2 - S^2\right)$$

Angular momentum addition,

$j = \lambda + s = 1, \qquad m_j = -1, 0, 1$

$j = \ell - s = 0, \qquad m_j = 0$

$$H_0 \Psi_{jm_j} = E_j \Psi_{jm_j}$$

$$E_j = \frac{1}{2}\alpha\hbar^2\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]$$

$$E_1 = \frac{1}{4}\alpha\hbar^2 \qquad \text{(3-fold degenerate)}$$

$$E_0 = -\frac{3}{4}\alpha\hbar^2 \qquad \text{(singlet)}$$

(b) Let $L^2\psi_\pm = \frac{3}{4}\hbar^2\psi_\pm, \quad L_z\psi_\pm = \pm\frac{\hbar}{2}\psi_\pm$

$S^2\chi_\pm = \frac{3}{4}\hbar^2\chi_\pm, \quad S_z\chi_\pm = \pm\frac{\hbar}{2}\psi_\pm$

Use basis set $[\psi_+\chi_+, \psi_+\chi_-, \psi_-\chi_+, \psi_-\chi_-]$

The matrix for H is

$$\frac{\hbar^2}{4}\begin{bmatrix} \alpha+2\beta & 0 & 0 & 0 \\ 0 & -\alpha+2\beta & 2\alpha & 0 \\ 0 & 2\alpha & -\alpha-2\beta & 0 \\ 0 & 0 & 0 & \alpha-2\beta \end{bmatrix}$$

The eigenvalues are

$$E = \frac{\hbar^2}{4}(\alpha \pm 2\beta)$$

and

$$\begin{vmatrix} -\alpha+2\beta - (E/\frac{\hbar^2}{4}) & 2\alpha \\ 2\alpha & -\alpha-2\beta-(E/\hbar^2/4) \end{vmatrix} = 0$$

i.e. $E = \frac{\hbar^2}{4}\left[-\alpha \pm 2\sqrt{\alpha^2+\beta^2}\right]$

Solution Prob 11

(c) $H_1 = \beta k L_z = \beta k (J_z - S_z) \simeq \beta k J_z \left(1 - \frac{\vec{S}\cdot\vec{J}}{J^2}\right)$

$$g_j = \left\langle 1 - \frac{\vec{S}\cdot\vec{J}}{J^2} \right\rangle = 1 - \frac{j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} = \frac{1}{2}$$

~~For $j=1$, g_j~~ $E_{0c} = -\frac{3}{4}\alpha k^2$

$$E_{1m} = E_1 + g\beta k^2 m$$

$$= \frac{1}{4}\alpha k^2 + \frac{1}{2}\beta k^2 m, \qquad m = -1, 0, 1.$$

(d) $E = (\alpha \pm 2\beta)\frac{k^2}{4}$, $= E_{1m}$ for $m = \pm 1$

$$E = \frac{k^2}{4}\left[-\alpha \pm 2\alpha\left(1 + \frac{\beta^2}{\alpha^2}\right)^{1/2}\right]$$

$$= \begin{cases} \frac{k^2}{4}\alpha + O\left(\frac{\beta^2}{\alpha}\right) & \longrightarrow = E_{1m} \text{ for } m = 0. \\ -\frac{3}{4}k^2\alpha + O\left(\frac{\beta^2}{\alpha}\right) & = E_{0c} \end{cases}$$

PART II
SOLUTION PROBLEM 12

$$\nabla \times H = \frac{4\pi}{c}\underset{\sim}{J} + \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi\sigma}{c}\underset{\sim}{E} + \cancel{\frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t}}$$

neglect

$$\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{H} = \frac{4\pi\sigma}{c}\nabla \times \underset{\sim}{E} = -\frac{4\pi\sigma}{c^2}\frac{\partial H}{\partial t}$$

$$\underbrace{\underset{\sim}{\nabla}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{H}}_{=0} - \nabla^2 \underset{\sim}{H} = -\frac{4\pi\sigma}{c^2}\frac{\partial H}{\partial t}$$

$$\underset{\sim}{H} \simeq \hat{z}\, H(r,t) \quad \text{by symmetry}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial H}{\partial r} = \frac{4\pi\sigma}{c^2}\frac{\partial H}{\partial t}$$

$$H(r=R,t) \simeq 0 \quad \text{since} \quad L >> R$$

$$H(r,t=0) = H_0 \quad \text{for} \quad r < R$$

Let $H(r,t) = \sum_n a_n H_n(r) e^{-\lambda_n t}$

$$\frac{1}{r}\frac{d}{dr} r \frac{d H_n}{dr} + \frac{4\pi\sigma}{c^2}\lambda_n H_n = 0$$

$$H_n(r) = a_n J_0\left(\frac{r}{R}\chi_{0n}\right), \quad \text{where } J_0(\chi_{0n}) =$$

$$\frac{4\pi\sigma\lambda_n}{c^2} = \frac{\chi_{0n}^2}{R^2}, \quad \lambda_n = \frac{\chi_{0n}^2}{R^2}\frac{c^2}{4\pi\sigma}$$

PART II
SOLUTION PROBLEM 13

(a) Let $\underset{\sim}{A} = x B_0 \hat{y}$ $\quad \varphi = -E_0 X$

$$L = -mc^2\sqrt{1 - v^2/c^2} - \frac{e}{c} B_0 x \dot{y} - e E_0 X$$

$$P_x = m\gamma\dot{x}\,, \quad P_y = m\gamma\dot{y} - \frac{e B_0 x}{c}\,, \quad P_z = m\gamma\dot{z}$$

$$H = \sqrt{m^2c^4 + c^2P_x^2 + c^2P_z^2 + c^2\left[P_y + \frac{eB_0}{c}x\right]^2} + eE_0 X$$

$$\text{const} = P_z\,, \quad \text{const} = P_y\,, \quad \text{const} = H$$

(b) $P_z = 0\,, \quad P_y = 0\,, \quad H = mc^2$

$$mc^2 = \sqrt{m^2c^4 + (eB_0 x)^2 + P_x^2c^2} + eE_0 X$$

$$\cancel{m^2c^4} - 2mc^2eE_0X + (eE_0X)^2 = m^2c^4 + (eB_0x)^2 + P_x^2c^2$$

$$2mc^2eE_0(-X) + e^2(E_0^2 - B_0^2)X^2 = c^2P_x^2$$

$P_x^2 > 0$ (no turning point) for $E_0 \geq B_0$
$X < 0$

13

(c) Fs. $E_x = E_0$

$$2mc^2 e E_0 |x| = c^2 p_x^2 = c^2 \gamma^2 m^2 \left(\frac{d|x|}{dt}\right)^2$$

$$\gamma mc^2 - e E_0 |x| = H = mc^2$$

$$\gamma mc^2 = mc^2 + e E_0 |x|$$

$$t = \int_0^t dt = \int_0^{|x|} \frac{d|x|}{c} \frac{(mc^2 + eE_0|x|)}{\sqrt{2mc^2 eE_0|x|}}$$

$$t = \frac{mc^2}{\sqrt{2mc^2 eE_0}\, c}\left[2|x|^{1/2} + \frac{eE_0}{mc^2}\frac{2}{3}|x|^{3/2}\right]$$

$$\sqrt{2}\,\frac{eE_0 ct}{mc^2} = \left[2\left(\frac{eE_0|x|}{mc^2}\right)^{1/2} + \frac{2}{3}\left(\frac{eE_0|x|}{mc^2}\right)^{3/2}\right]$$

$= 12$

$$H(r,t) = \sum_n a_n J_0\left(\frac{r}{R}\chi_{on}\right) e^{-\lambda_n t}$$

a_m determined by

$$\int_0^R 2\pi r\,dr\, H_0 J_0\left(\frac{r}{R}\chi_{om}\right) = a_m \int_0^R 2\pi r\,dr\, J_0^2\left(\frac{r}{R}\chi_{om}\right)$$

$$2\pi H_0 \frac{R^2}{\chi_{om}} J_1(\chi_{om}) = a_m 2\pi R^2 \frac{1}{2} J_1^2(\chi_{om})$$

$$a_m = \frac{2H_0}{\chi_{om} J_1(\chi_{om})}$$

PART II
SOLUTION PROBLEM 14

Part ② ~~③~~ 14

6 March 1995
Kim Griest
Qualifier Problem.

this Problem

Find the leading asymptotic behavior of

$$I(x) = \int_0^{\pi/2} \sqrt{\sin t}\; e^{-x \sin^4 t}\, dt \quad \text{as } x \to \infty .$$

You may express the answer in terms of the Gamma Function (Factorial Function)

min - exp gamma fcn

$\boxed{\text{Solution}}$ by LaPlace's method.

14 $\quad I(x) = \int f(t)\, e^{-x\phi(t)}\, dt$; $\phi(t) = \sin^4 t$ has its minimum at $t=0$. $\Rightarrow$ asymptotic expansion around $t=0$ is all you need.

$\sin^4 t \sim t^4$, $t \to 0$

$\sqrt{\sin t} \sim t^{1/2}$ as $t \to 0$

$$I(x) \underset{\substack{x\to\infty \\ \varepsilon \to 0}}{\sim} \int_0^{\varepsilon} t^{1/2} e^{-xt^4}\, dt .$$

Can evaluate by extending integral to ∞, since that will add only a sub-dominant piece.

$$I(x) \sim \int_0^{\infty} t^{1/2} e^{-xt^4}\, dt \quad ; \quad \text{Let } t = \frac{u^{1/4}}{x^{1/4}}, \quad dt = \frac{1}{4}\frac{u^{-3/4}}{x^{1/4}}\, du$$

$$\sim \int_0^{\infty} \frac{u^{1/8}}{x^{1/8}} \frac{1}{4x^{1/4}}\, u^{\frac{1}{8}-3/4} e^{-u}\, du = \frac{1}{4x^{3/8}} \int u^{-5/8} e^{-u}\, du$$

$$I(x) \sim \frac{1}{4x^{3/8}} \int_0^{\infty} u^{3/8-1} e^{-u}\, du = \frac{1}{4x^{3/8}}\, \Gamma\!\left(\tfrac{3}{8}\right)$$

$$\boxed{I(x) \underset{x\to\infty}{\sim} \frac{1}{4x^{3/8}}\, \Gamma(3/8)}$$