Code Number: ___________

UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS
Written Departmental Examination – Fall 1994, Part I

[1] An electromagnetic plane wave of angular frequency ω propagates through an optically active dextrose solution. The solution is non-conducting and non-magnetizable but has a polarization vector (average electric dipole moment per unit volume) given by $\vec{P} = \alpha\vec{E} + \beta\nabla\times\vec{E}$ when the plane wave propogates through the medium. Assume α and β are positive real constants. You may assume that $\frac{\beta\omega}{c} << 1$ so that we only need answers to first order in $\frac{\beta\omega}{c}$. Find the possible indices of refraction for the plane wave and describe the corresponding polarizations of the electric field.

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[2] A thin copper ring rotates about an axis which is perpendicular to a uniform magnetic field H_0. The rotation axis is a diameter of the ring. If the intial angular frequency of rotation is ω_0. Calculate the time it takes for the frequency to decrease to $\frac{1}{e}$ of its original value under the assumption that the energy goes into Joule heating of the ring. Copper has conductivity $\sigma = 5 \times 10^{17} s^{-1}$ (cgs units) and has density $8.9 \frac{\mathrm{g}}{\mathrm{cm}^3}$. Assume that H_0 is 200 G and the radius of the ring is 10 cm.

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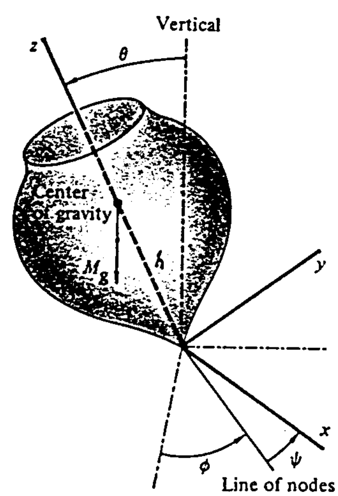
[3] A symmetric top of mass m has moments of inertia in the principal axis system of $I_1 = I_2 = I_{12}$ and $I_3 < I_{12}$. The center of mass of the top is a distance h along the symmetry axis from its bottom point. The top spins in a uniform gravitational field g. We can write the kinetic and potential energy of the top in terms of the Euler angles and their time derivatives.

$$T = \tfrac{1}{2} I_{12}(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \tfrac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$
$$U = Mgh\cos\theta$$

The angles are defined in the picture. This problem can be reduced to dynamics in a single variable by identifying conserved quantities. Do this by determining $V(\theta)$, the effective potential that governs the nutation of the top. (This is analogous to the effective potential in central force motion.)



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[4] If a and $a^\dagger$ are the annihilation and creation operators introduced to discuss the harmonic oscillator, we recall that $[a, a^\dagger] = 1$ and $a^\dagger a|n\rangle = n|n\rangle, \quad n = 0, 1, 2, \ldots$ and the states $|n\rangle$ satisfy $\langle m|n\rangle = \delta_{mn}$. Suppose then there is a state $|\alpha\rangle$ such that

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

(a) Find coefficients $\langle n|\alpha\rangle$ such that $|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle$ and such that $\langle\alpha|\alpha\rangle = 1$.

(b) Suppose $|\alpha'\rangle$ is another state such that $a|\alpha'\rangle = \alpha'|\alpha'\rangle$ and $\alpha' \neq \alpha$. Compute $\langle\alpha'|\alpha\rangle$. Would you have expected the states $|\alpha\rangle$ and $|\alpha'\rangle$ to be orthogonal if $\alpha \neq \alpha'$? Why?

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[5] Consider the hyperfine splitting of the ground state of Hydrogen in a uniform magnetic field B. Assume the Zeeman energy shift is of the same order of magnitude as the hyperfine shift. Take the perturbation $V = A\vec{s}_e \cdot \vec{s}_p - \vec{\mu}_e \cdot \vec{B}$ where $\vec{\mu}_e = \frac{e\vec{s}_e}{m_e c}$. Using first order perturbation theory, find the energy shifts of all four of the $n = 1$, $l = 0$ levels. You may ignore the $-\vec{\mu}_p \cdot \vec{B}$ term.

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[6] Consider an ideal gas of N electrons that move freely in two dimensions and is confined to an area A. Assume that there is a uniform positive background charge to maintain overall charge neutrality.
(a) Estimate the temperature, T_0, below which quantum mechanical considerations are important in describing the system.
(b) Calculate the Fermi energy, ϵ_f, assuming that the system is at temperature $T = 0$. Compare ϵ_f to $k_B T_0$, where T_0 is the temperature calculated in part (a).
(c) Estimate the magnetic susceptibility of the system at $T = 0$, in terms of the magnetic moment μ of the electrons. (Assume that M is the magnetic moment per unit area and that the susceptibility is the ratio of M to H.)
(d) Compare the magnetic susceptibility calculated in (c) with the result obtained if the temperature is large enough that the electron gas can be described classically. Assume $\mu H << k_B T$, where H is the applied magnetic field, and k_B is Boltzmann's constant. Which is larger, and by what factor?

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[7] Projectiles are fired with initial velocity V_0, at angle α above the horizontal, from a point on the earth's equator. Identical shots are made in the east, north, and west directions. It is found that the range of the projectiles depends on the direction in which they are fired. Assume that the local value of gravitational plus centrifugal acceleration is g.

(a) For each of the three initial directions, calculate the acceleration in the vertical direction and in the horizontal direction fired. Do this only to first order in ω.

(b) Find the time of flight in each case, again to first order.

(c) Find the range for each of the three directions.

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[8] We wish to perform an experiment to measure the (π^-, π^0) mass difference. If we bring a beam of π^- to rest in a hydrogen target, the π^- will be captured by the protons and we have the reaction

$$\pi^- + p \rightarrow \pi^0 + n$$

where, within approximately 10^{-16} sec. the π^0 decays

$$\pi^0 \rightarrow \gamma + \gamma.$$

We measure the minimum angle between the two photons to be 145 ± 5 mrad. What is the $(\pi^- - \pi^0)$ mass difference and its error?
$M_n - M_p = 1.293 \pm 0.001$ MeV
$M_n = 939.56 \pm 0.01$ MeV
$M\pi^0 = 134.97 \pm 0.01$ MeV

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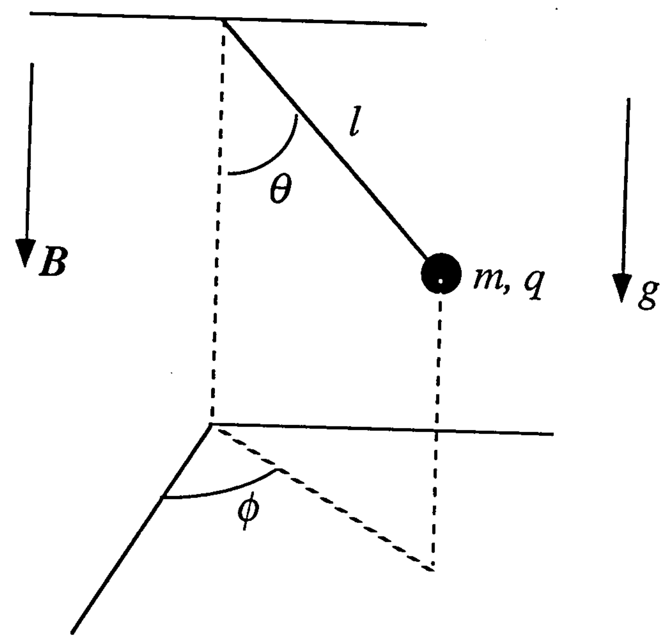
[9] A mass m is suspended by a rod of length l, as shown in the drawing. The mass carries charge q and is in a uniform gravitational field g acting downward and a uniform magnetic field also directed downward. The rod is free to point in any direction (θ, ϕ) provided that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. If the mass is released from rest at $\theta = \frac{\pi}{2}$, what is the minimum value of θ reached in the subsequent motion? You may assume $\Omega^2 l << g$, where $\Omega = \frac{qB}{mc}$ is the cyclotron frequency.



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UNIVERSITY OF CALIFORNIA, SAN DIEGO
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Written Departmental Examination – Fall 1994, Part II

[10] A linearly polarized plane wave of very low angular frequency ω is incident on a small, uncharged, perfectly conducting sphere of radius a. Assume $\omega a << c$.

(a) Find the electric dipole moment associated with the charge on the surface of the conductor, and the magnetic dipole moment associated with the surface currents.

(b) The magnetic field for magnetic dipole radiation has the limiting form

$$\vec{B}_{M1}(\vec{r}) \rightarrow -k^2[\vec{r} \times (\vec{r} \times \vec{\mu})]\frac{e^{ikr}}{r^3}$$

where $\vec{\mu}$ is the magnetic moment, $k = \frac{\omega}{c}$, and $\vec{r}$ is the displacement from the dipole. Find the time averaged power radiated by the conductor per unit solid angle in the forward and backward directions.

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Written Departmental Examination – Fall 1994, Part II

[11] Consider the elastic scattering of a high energy electron by a hydrogen atom in its ground state, $\psi(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$.

(a) Calculate the differential scattering cross section in the approximation appropriate for high energies and weak potentials. You may find $\int d^3\vec{r}\, r^m\, e^{-\beta r}\, e^{-i\vec{k}\cdot\vec{r}} = [\frac{2\pi i}{k} \frac{(m+1)!}{(\beta - ik)^{m+2}} + CC]$ useful.

(b) Determine the total elastic scattering cross section in the high energy limit.

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Written Departmental Examination – Fall 1994, Part II

[12] An electron is at rest (initially) and is surrounded by a blackbody photon gas (temp. T). Calculate the rate (dE_e/dt) at which it initially increases its energy as a result of Compton scattering by photons when $kT \ll mc^2$. Note that, since the process involves a final-state photon, it is necessary to include the effects of stimulated scattering. You may use $(\frac{d\sigma}{d\Omega})_{Compton} = \frac{1}{2}r_0^2(1+\cos^2\theta)$, where $r_0 = \frac{e^2}{m_e c^2}$.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
DEPARTMENT OF PHYSICS
Written Departmental Examination – Fall 1994, Part II

[13] Consider the Paramagnetic to Ferromagnetic transition in a three-dimensional XY model Ferromagnet (i.e. a magnet where spins must lie in the XY plane).

(a)Neglecting any spatial anisotropy or lattice effects, write down a phenomenological Landau Free Energy which describes the long wavelength physics for temperatures near T_C, the transition temperature. In the free energy, write terms only to order ψ^4 in the complex order parameter $\psi = \langle m_x + im_y \rangle$ where m_x and m_y are components of the magnetization density.

(b)Calculate the magnetization $\langle \vec{m} \rangle$ as a function of T near T_C when $H = 0$.

(c)Calculate $\chi = \frac{\partial \langle m \rangle}{\partial H}|_{H=0}$, the magnetic susceptibility as a function of T near T_C.

(d) Based on parts b and c, sketch $\langle m_x \rangle$ for both $H = 0$ and for small H. Describe how the external field has affected the transition.

(e)Suppose we have a long rod made of this magnetic material. For $x < 0$, the rod is held at some temperature $T_1 < T_C$, but for $x > 0$, the rod is held at some temperature $T_2 > T_C$. In terms of $\vec{m}_0$, the magnetization at the interface between the two halves of the rod, find $\langle \vec{m}(x) \rangle$ for $x > 0$. Assume $\langle \vec{m} \rangle$ or ψ is small in this half of the rod.

(f)Using the equipartition theorem, estimate to within factors of order unity the contribution to the specific heat associated with thermal fluctuations. Do this only for $T > T_C$.

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Written Departmental Examination – Fall 1994, Part II

[14] A narrow beam of length L and mass per unit length ρ has both ends firmly clamped so that both the displacement and its spatial derivative vanish. When the beam is bent, the total energy stored in the beam (neglecting gravity) is equal to $U = \frac{k}{2}\int_0^L (\frac{\partial^2 s}{\partial x^2})^2 dx$, where $s(x)$ is the transverse displacement (supposed to be small) at position x, and k is a constant.

(a) Find the differential equation which $s(x,t)$ satisfies for small motions.

(b) Find the eigenfrequencies, ν_n. Give the transcendental equation of which the ν_n are solutions. Give an explicit formula for ν_n asymptotically correct for large n.

(1a) FALL 94 SOLUTIONS

assume $\vec{E} = Re\left[\vec{E_0}\, e^{i(kz-\omega t)}\right]$

$\vec{B}$ and $\vec{D}$ are $\perp$ to $\hat{z}$ $\quad \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\vec{D} = 0$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \Rightarrow ik\hat{z}\times\vec{E} = \frac{i\omega}{c}\vec{B}$$

$$\Rightarrow \vec{B} = \frac{ck}{\omega}\hat{z}\times\vec{E}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \Rightarrow ik\hat{z}\times\vec{B} = \frac{-i\omega}{c}\vec{D}$$

$$\Rightarrow \hat{z}\times\vec{B} = \frac{-\omega}{kc}\vec{D}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

$$\vec{D} = \vec{E} + 4\pi\alpha\vec{E} + 4\pi\beta\vec{\nabla}\times\vec{E}$$

$$\vec{D} = (1+4\pi\alpha)\vec{E} + 4\pi i\beta k\,\hat{z}\times\vec{E}$$

$$\Rightarrow \vec{E} \perp \hat{z}$$

$$\frac{c}{n} = \omega/k \qquad n = \frac{kc}{\omega} \qquad k = \frac{\omega n}{c}$$

$$\hat{z}\times\vec{B} = \frac{-\omega}{kc}D$$

$$\frac{ck}{\omega}\hat{z}\times(\hat{z}\times\vec{E}) = \frac{-\omega}{kc}\left[(1+4\pi\alpha)\vec{E} + 4\pi i\beta k\,\hat{z}\times\vec{E}\right]$$

$$-\vec{E} = \frac{-\omega^2}{k^2c^2}\left[(1+4\pi\alpha)\vec{E} + 4\pi\beta k\; i(\hat{z}\times\vec{E})\right]$$

$$\frac{k^2c^2}{\omega^2}\begin{pmatrix}E_x\\E_y\end{pmatrix} = \begin{pmatrix}1+4\pi\alpha & -4\pi\beta\frac{\omega}{c}ni\\ 4\pi\beta\frac{\omega}{c}ni & 1+4\pi\alpha\end{pmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix}$$

$$\begin{vmatrix}1+4\pi\alpha-n^2 & -4\pi\beta\frac{\omega}{c}ni\\ 4\pi\beta\frac{\omega}{c}ni & 1+4\pi\alpha-n^2\end{vmatrix} = 0$$

(15)

$$(1+4\pi\alpha-n^2)^2-\left(4\pi\beta\frac{\omega}{c}n\right)^2=0$$

$$(1+4\pi\alpha-n^2)=\pm 4\pi\beta\frac{\omega}{c}n$$

$$n^2=1+4\pi\alpha\mp 4\pi\beta\frac{\omega}{c}n$$

$$\begin{pmatrix}\pm 4\pi\beta k & -4\pi\beta k i\\ 4\pi\beta k i & \pm 4\pi\beta k\end{pmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}=0$$

$$\begin{pmatrix}\pm 1 & -i\\ i & \pm 1\end{pmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}=0$$

$$\pm E_x \quad -iE_y$$

$$iE_x=\mp E_y \qquad E_y=\mp iE_x$$

$$\pm E_x=iE_y \checkmark \qquad \Rightarrow \text{circular pol.}$$

$$\boxed{\hat{x}\mp i\hat{y} \quad \text{for} \quad n^2=1+4\pi\alpha\mp 4\pi\beta\frac{\omega}{c}n}$$

$$n^2\pm 4\pi\beta\frac{\omega}{c}n-(1+4\pi\alpha)=0$$

$$n=\frac{\mp 4\pi\beta\frac{\omega}{c}+\sqrt{4(1+4\pi\alpha)}}{2}$$

$$\boxed{n=\sqrt{1+4\pi\alpha}\mp 2\pi\beta\frac{\omega}{c} \quad \text{for } \hat{x}\mp i\hat{y}}$$

(2)

$$\Phi = H_0 \pi r^2 \cos\theta = \pi r^2 H_0 \cos(\omega t)$$

$$V = -\frac{1}{c}\frac{\partial \Phi}{\partial t} = \frac{\pi r^2 H_0 \omega}{c} \sin(\omega t)$$

Power into heat $= IV = \frac{V^2}{R} = -\frac{d}{dt}\left(\frac{1}{2} I_3 \omega^2\right)$

$$I_3 \omega \dot{\omega} = \frac{-\pi^2 r^4 H_0^2 \omega^2}{c^2 R} \sin^2(\omega t)$$

$$\langle \dot{\omega} \rangle = \frac{-\pi^2 r^4 H_0^2}{I_3 c^2 R} \langle \sin^2(\omega t) \rangle \omega = \frac{-\pi^2 r^4 H_0^2}{2 I_3 c^2 R} \omega$$

$$\omega = \omega_0 e^{-t/\tau} \Rightarrow \dot{\omega} = -\frac{1}{\tau}\omega$$

$$\tau = \frac{2 I_3 c^2 R}{\pi^2 r^4 H_0^2}$$

$I_3 = \frac{1}{2} M r^2$ $\qquad M = 2\pi r A \rho \Rightarrow I_3 = \pi r^3 A \rho$

$$R = \frac{2\pi r}{A\sigma}$$

$$\tau = \frac{2\pi \cancel{r^3} A \rho c^2 2\cancel{\pi r}}{A\sigma \cancel{\pi^2} \cancel{r^4} H_0^2} = \frac{4\rho c^2}{\sigma H_0^2} = \frac{4(8.9)(9\times10^{20})}{(5\times10^{17})(4\times10^{4})}$$

≈ 1.6 sec

(note ρc^2 + H_0^2 are both units of energy density)

(3')

$$L = \tfrac{1}{2} I_{12}\left(\dot\varphi^2 \sin^2\theta + \dot\theta^2\right) + \tfrac{1}{2} I_3 \left(\dot\varphi\cos\theta + \dot\psi\right)^2 - mgh\cos\theta$$

cyclic in φ & ψ $\Rightarrow$ $p_\varphi = \text{const}$, $p_\psi = \text{const}$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3\left(\dot\varphi\cos\theta + \dot\psi\right)$$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = I_{12}\dot\varphi\sin^2\theta + I_3\left(\dot\varphi\cos\theta + \dot\psi\right)\cos\theta$$

$$= I_{12}\dot\varphi\sin^2\theta + p_\psi\cos\theta$$

$$\dot\varphi = \frac{p_\varphi - p_\psi\cos\theta}{I_{12}\sin^2\theta}$$

$$\dot\psi = \frac{p_\psi}{I_3} - \frac{\left(p_\varphi - p_\psi\cos\theta\right)\cos\theta}{I_{12}\sin^2\theta}$$

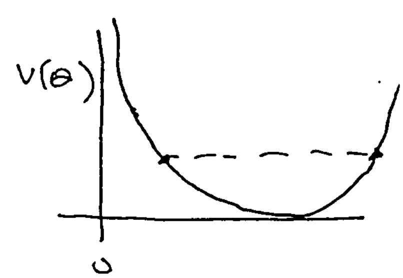
$$E = T + U = \frac{1}{2 I_{12}\sin^2\theta}\left[p_\varphi^2 - 2\cos\theta\, p_\varphi p_\psi + \cos^2\theta\, p_\psi^2\right]$$

$$+ \tfrac{1}{2} I_{12}\dot\theta^2 + \tfrac{1}{2} I_3\left(\frac{p_\psi}{I_3}\right)^2 + mgh\cos\theta$$

$$E = \tfrac{1}{2} I_{12}\dot\theta^2 + V(\theta) + \frac{1}{2 I_3} p_\psi^2$$

$$V(\theta) = \frac{p_\varphi^2 - 2\cos\theta\, p_\varphi p_\psi + \cos^2\theta\, p_\psi}{2 I_{12}\sin^2\theta}$$

$$V(\theta) = \frac{\left(p_\varphi - \cos\theta\, p_\psi\right)^2}{2 I_{12}\sin^2\theta}$$



(4)

From $a|\alpha\rangle = \alpha|\alpha\rangle$, take scalar product with $|n\rangle$:

$$\langle n|a|\alpha\rangle = \alpha\langle n|\alpha\rangle$$

now $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

so
$$\sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle$$

$$\langle n+1|\alpha\rangle = \frac{\alpha}{\sqrt{n+1}}\langle n|\alpha\rangle$$

$$= \frac{\alpha^2}{\sqrt{(n+1)n}}\langle n-1|\alpha\rangle$$

$\vdots$ and so on.

Thus
$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle$$

$$\langle\alpha|\alpha\rangle = \sum_n \langle\alpha|n\rangle\langle n|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_n \frac{(\alpha\alpha^*)^n}{n!}$$

$$= |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2} = 1$$

$$\langle 0|\alpha\rangle = e^{-|\alpha|^2/2}$$ to within a phase

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$$

$$\sum_n \langle\alpha'|n\rangle\langle n|\alpha\rangle = \langle\alpha'|\alpha\rangle = \sum_n \frac{(\alpha'^*\alpha)^n}{n!} e^{-|\alpha'|^2/2 - |\alpha|^2/2} = e^{\alpha'^*\alpha - |\alpha'|^2/2 - |\alpha|^2/2}$$

$$|\langle\alpha'|\alpha\rangle|^2 = e^{-|\alpha'-\alpha|^2}$$

No. Since a is not Hermitian, they need not be orthogonal.

5a

Solution

$$V = A\vec{S}_e \cdot \vec{S}_p - \frac{e}{mc}\vec{S}_e \cdot \vec{B} \qquad \vec{J} = \vec{S}_p + \vec{S}_e, \quad \vec{B} = B_0\hat{z}$$

$$\text{so } \vec{S}_e \cdot \vec{S}_p = \frac{1}{2}(J^2 - S_e^2 - S_p^2)$$

$$V = \frac{A}{2}(J^2 - S_e^2 - S_p^2) - \frac{eB_0 S_z}{mc}$$

So in $|J m\rangle$ basis Energy shifts

$$\Delta E = \langle jm|V|j'm'\rangle$$

$$\Delta E = \frac{A\hbar^2}{2}\left(J(J+1) - 3/2\right) - \frac{eB_0}{mc}\langle jm|S_z|j'm'\rangle$$

So need to find $\langle jm|S_z|j'm'\rangle$

So need to be in other basis $|m_e m_p\rangle$ ($\pm\frac{1}{2}$, $\pm 1/2$)

4 $|Jm\rangle$ states are $|00\rangle, |10\rangle, |11\rangle, |1-1\rangle$

4 $|m_s s_p\rangle$ states are $|+-\rangle, |-+\rangle, |++\rangle, |--\rangle$

Now $|00\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$, $|10\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

$|11\rangle = |++\rangle$ and $|1-1\rangle = |--\rangle$

So $\langle 11|S_z|11\rangle = \hbar/2$, $\langle 1-1|S_z|1-1\rangle = -\hbar/2$

$\langle 10|S_z|10\rangle = \frac{1}{2}\left(\frac{\hbar}{2} - \hbar/2\right) = 0$

$\langle 00|S_z|00\rangle = \frac{1}{2}(\hbar/2 - \hbar/2) = 0$

$\langle 10|S_z|00\rangle = \frac{1}{2}(\hbar/2 + \hbar/2) = \hbar/2$

Define $w = \frac{eB}{2mc}$

So 4 by 4 matrix is

$$\begin{array}{c} \\ |00\rangle \\ |10\rangle \\ |11\rangle \\ |1-1\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |10\rangle & |11\rangle & |1-1\rangle \end{array} \\ \begin{pmatrix} -3A\hbar^2/4 & \hbar w & 0 & 0 \\ \hbar w & A\hbar^2/4 & 0 & 0 \\ 0 & 0 & A\hbar^2/4 + \hbar w & 0 \\ 0 & 0 & 0 & A\hbar^2/4 - \hbar w \end{pmatrix} \end{array}$$

5b

Diagonalize this matrix to find the energy eigenvalues which are the 4 energy shifts.

$$\Delta E_1 = A\hbar^2/4 + \hbar w$$

$$\Delta E_2 = A\hbar^2/4 - \hbar w$$

$$\Delta E_{3,4} = -\frac{A\hbar^2}{4} \pm \sqrt{\frac{A^2\hbar^4}{4} + \hbar^2 w^2}$$

Solution

(4a)

Graduate Exam Problem in Statistical Mechanics

Cliff Surko Spring '94

B. Consider an ideal gas of N electrons that move freely in two dimensions and are confined to an area A. Assume that there is a uniform positive background charge to maintain overall charge neutrality.

(a) Estimate the temperature, T_0, below which quantum mechanical considerations are important in describing the system.

$\Delta p \Delta x \sim \hbar \qquad \Delta p \sim \bar{p} \sim \sqrt{3 m k_B T} \qquad \Delta x \sim \bar{R} \sim \left(\frac{A}{N}\right)^{\frac{1}{2}}$

Thus $T_0 \sim \frac{1}{k_B}\left(\frac{\hbar^2}{3m}\frac{N}{A}\right)$

Compare ϵ_f to $k_B T_0$, where T_0 is the temperature calculated in part (a).

(b) Calculate the Fermi energy, ϵ_f, assuming that the system is at temperature T = 0.

$$N = 2A\int_0^{k_f} \frac{d^2 k_f}{(2\pi)^2} = \frac{4\pi A}{(2\pi)^2}\int_0^{k_f} k\,dk = \frac{A k_f^2}{2\pi}$$

spin

$$\epsilon_f = \frac{\hbar^2 k_f^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi N}{A}\right)$$

Crudely

$\epsilon_f \sim k_B T_0$

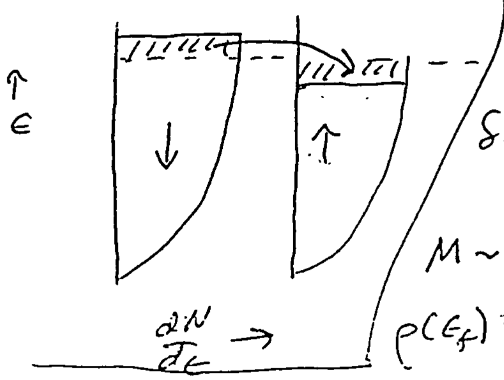
(6B)

Express your answer in terms of N and ϵ_f.

B. (cont'd)

(c) Estimate the magnetic susceptibility of the system at $T = 0$, in terms of the magnetic moment, μ, of the electrons.

$\epsilon = \pm \mu H$



$$\delta N \sim \underbrace{\frac{\rho(\epsilon_f)}{2}}_{\text{total density of energy states @ } \epsilon_f} 2\mu H$$

$$M \sim \delta N \mu / A$$

$$\rho(\epsilon_f) = \frac{2A}{(2\pi)^2} 2\pi k \frac{dk}{d\epsilon} \; ; \quad \epsilon = \frac{\hbar^2 k^2}{2m}$$

Thus $\frac{dk}{d\epsilon} = \frac{2m}{2\hbar^2 k} \Rightarrow \rho(\epsilon_f) = \frac{2mA}{2\pi\hbar^2} = \frac{N 2m}{2\pi(\hbar^2 k_f^2)} 2\pi = \frac{N}{\epsilon_f}$ ← from (b) above

Thus $M = \frac{N\mu^2 H}{\epsilon_f A}$ and $\chi = \frac{N\mu^2}{\epsilon_f A}$

(d). Compare the magnetic susceptibility calculated in (c) with the result obtained if the temperature is large enough that the electron gas can be described classically. Assume $\mu H << k_B T$, where H is the applied magnetic field, and k_B is Boltzmann's constant. Which is larger, and by what factor?

Classically

$$M = \frac{N}{A}\left[\frac{(\mu) e^{+\beta\mu H} + (-\mu) e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}}\right]$$

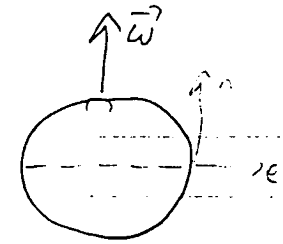
$\beta\mu H << 1 \Rightarrow$

$$M \simeq \frac{N\mu^2 H}{A k_B T} \qquad \chi = \frac{N\mu^2}{A k_B T}$$

Thus
$$\frac{\chi_{qm}}{\chi_{cl}} = \frac{k_B T}{\epsilon_f}$$

7a)

i) $a_{coriolis} = -2\vec{\omega} \times \vec{v}$



	$a_{vertical}$	$a_{horizontal}$
east	$-g + 2\omega V_0 \cos\alpha$	$-2\omega(V_0 \sin\alpha - gt)$
north	$-g + 0$	0
west	$-g - 2\omega V_0 \cos\alpha$	$+2\omega(V_0 \sin\alpha - gt)$

ii) since $a_{vertical} = \text{const.}$

$$t = \frac{2V_0 \sin\alpha}{(-a_{vertical})}$$

$$t_{east} = \frac{2V_0 \sin\alpha}{g - 2V_0 \omega \cos\alpha} \approx \frac{2V_0 \sin\alpha}{g}\left(1 + \frac{2V_0 \omega \cos}{g}\right.$$

$$t_{north} = \frac{2V_0 \sin\alpha}{g}$$

$$t_{west} = \frac{2V_0 \sin\alpha}{g + 2V_0 \omega \cos\alpha} \approx \frac{2V_0 \sin\alpha}{g}\left(1 - \frac{2V_0 \omega \cos}{g}\right.$$

7b

c) $R = \int_0^t V_i(t)\, dt$

$V(t) = \int_0^t a(t)\,dt + V_0 \cos\alpha$

$V_e(t) = -2wV_0 \sin\alpha t + wg t^2 + V_0 \cos\alpha$

$V_n(t) = V_0 \cos\alpha$

$V_w(t) = -2wV_0 \sin\alpha t - wg t^2 + V_0 \cos\alpha$

$R_e = V_0 \cos\alpha\, t_e - w V_0 \sin\alpha\, t_e^2 + \frac{1}{3} wg t_e^3$

$R_n = V_0 \cos\alpha\, t_n$

$R_w = V_0 \cos\alpha\, t_w + w V_0 \sin\alpha\, t_w^2 - \frac{1}{3} wg t_w^3$

$R_n = \frac{2V_0^2 \sin\alpha \cos\alpha}{g}$

$R_{E/W} = \frac{2V_0^2 \sin\alpha \cos\alpha}{g} \pm \frac{4V_0^3 \cos^2\alpha \sin\alpha\, w}{g^2}$

$\mp \frac{4V_0^3 \sin^3\alpha\, w}{g^2} \pm \frac{1}{3} \frac{8V_0^3 \sin^3\alpha \cdot w}{g^2}$

SOLUTION PROBLEM 8 (R)

M_p PROTON MASS

M NEUTRON MASS

m_π π^- MASS

m π^0 MASS

γ E_N/M IN LAB

Γ E_{π^0}/m " " , $B = V_{\pi^0}/c = \sqrt{1 - \frac{1}{\Gamma^2}}$

$\Delta = (m_{\pi^-} - m) - (M - M_p)$

$\bar{\mu}$ $\cos\theta$ IN π^0 REST FRAME

μ $\cos\theta$ IN LAB

k = Photon energy in LAB

$\bar{k}$ = Photon energy in π^0 FRAME

CONS MOM $(\gamma_p^2-1)M^2 = (\Gamma^2-1)m^2$

CONS ENERGY $(M+m+\Delta) = \gamma_p M + m\Gamma \Rightarrow \gamma_p = \frac{M+\Delta-(\Gamma-1)m}{M}$

$$\Rightarrow (\Gamma-1) = \frac{\Delta}{M}\frac{\left(1+\frac{\Delta}{2m}\right)}{\left(1+\frac{\Delta+m}{M}\right)} \quad (a)$$

$\pi^0 \to \gamma+\gamma$

$$\left.\begin{array}{l} k = \bar{k}\Gamma(1+B\bar{\mu}) \\ k_\mu = \bar{k}\Gamma(\bar{\mu}+B) \end{array}\right\} \Rightarrow \mu_1 = \frac{B+\bar{\mu}}{1+B\bar{\mu}} \qquad \mu_2 = \frac{B-\bar{\mu}}{1-B\bar{\mu}}$$

$\bar{\mu}_{1,2} = 0 \to \mu_1 = \mu_2 = B \quad \sin\theta_{\gamma\gamma} = 2\frac{B}{\Gamma} = \frac{2}{\Gamma}\sqrt{1-\frac{1}{\Gamma^2}} = 0.145$

$\frac{\sin^2\theta}{4} = \frac{\Gamma^2-1}{\Gamma^4} \Rightarrow \Gamma^2 = \frac{2}{\sin^2\theta}\left[1 \pm \sqrt{1-\sin^2\theta}\right]$

FOR $\sin^2\theta \ll 1$ $\quad \Gamma^2 \simeq \frac{4}{\sin^2\theta}$ (+) OR $\Gamma^2 \simeq 1+\frac{\sin^2\theta}{4}$ (−)

$= 13.8\ (\theta = .145) \qquad (\Gamma-1) = 5.26\times10^{-3}(\theta = \pi-.14$

ROUGHLY (a) $\Delta = 2m(\Gamma-1) = 3.5$ GeV

MORE ACCURATELY: $\quad a \to (\Gamma-2)\Delta + (\Gamma-1)(M+m) = \frac{\Delta^2}{2m}$

$\Delta^2 - 2m(\Gamma-2)\Delta - 2m(M+1)(\Gamma-1) = 0$

$$\Delta = m(\Gamma-2)\left[1 \pm \sqrt{1+2\left(1+\frac{M}{m}\right)\frac{\Gamma-1}{(\Gamma-2)^2}}\right]$$

$\Delta = m(\Gamma-2)\,[2.57] \Rightarrow \Delta = 4.05$ GeV

~17% larger than rough guess

SINCE $m_{\pi^-} \gg m_{\pi^0}$ $\quad (m_{\pi^-} - m_{\pi^0}) \approx \Delta \approx 2m(\Gamma-1) \simeq \frac{4m}{\sin\theta_{\gamma\gamma}}$

$$\Rightarrow \frac{\delta(m_{\pi^-}-m_{\pi^0})}{(m_{\pi^-}-m_{\pi^0})} \cong \frac{\delta\sin\theta_{\gamma\gamma}}{\sin\theta} = \frac{5}{145} = 3.4\%\ (= .14\text{ GeV})$$

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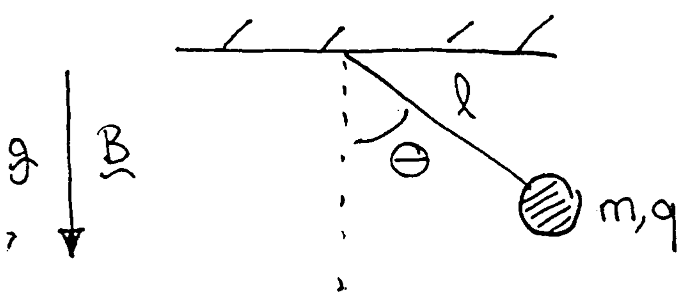
Mechanics Problem

A mass m is suspended by a rod of length l, as shown in the figure. The mass carries charge q as is in a uniform gravitational field acting downward and a uniform magnetic field $\underset{\sim}{B}$ acting in the same direction.

If the mass is released from rest at $\theta = \frac{\pi}{2}$, what is the minimum value of θ reached in the subsequent motion? You may assume $\Omega^2 l \ll g$, where $\Omega = qB/mc$ is the cyclotron frequency.



Lagrangian : $L = \frac{m}{2}\left(l^2\dot\theta^2 + l^2\sin^2\theta\,\dot\phi^2\right) + \frac{e}{c}A_\phi\, l\sin\theta\,\dot\phi + mgl\cos\theta$

where $A_\phi = -\frac{Bl}{2}\sin\theta$.

canonical momentum $p_\phi = \text{const.} = \frac{\partial L}{\partial\dot\phi} = ml^2\sin^2\theta\,\dot\phi - \frac{qBl^2 s^2}{2c}$

so initially $p_\phi = \frac{qB}{2c}l^2$

$\therefore \dot\phi = -\frac{qB}{2c}\frac{l^2\cos^2\theta}{ml^2\sin^2\theta} = \frac{-\Omega}{2\tan^2\theta}$

(95)

minimum Θ : $H = \underset{\overset{\parallel}{0}}{\text{const.}} = \frac{m}{2}\left[l^2 \underset{\underset{0 \text{ at minimum}}{\parallel}}{\dot{\Theta}^2} + l^2 \sin^2\Theta\,\dot{\phi}^2\right] - mgl\cos\Theta$

using previous expression for $\dot{\phi}$,

$$\frac{m}{2}\,\frac{\Omega^2 l^2}{4}\,\frac{\cos^4\Theta}{\sin^2\Theta} - mgl\cos\Theta = 0$$

$$\sin^2\Theta = \frac{\Omega^2 l}{8g}\cos^2\Theta$$

$\Rightarrow$ for $\frac{\Omega^2 l}{8g} \ll 1$, $\boxed{\Theta_{min} \cong \sqrt{\frac{\Omega^2 l}{8g}}}$

(10x)

Assume $\vec{E}$ nearly constant in $\hat{z}$ direction

$$\varphi = -E_0 r\cos\theta + \frac{q}{r} + \frac{\vec{p}\cdot\hat{z}}{r^3} + \cdots \qquad \text{(outside)}$$

$q=0$

$\varphi(a) = \text{const}$

$$\varphi(a) = -E_0 a\cos\theta + \frac{P\cos\theta}{a^2} \Rightarrow \boxed{\vec{P} = E_0 a^3 \hat{z}}\, e^{i\omega t}$$

$\vec{B}=0$ inside sphere (like $\vec{E}$)

$\Rightarrow$ it satisfies the same eq outside

B.C. $\quad \vec{\nabla}\cdot\vec{B}=0 \Rightarrow \quad \vec{B}\cdot\hat{r}=0$ at $|\vec{r}|=a$

(assume B in $\hat{x}$ direction)

$$\varphi_B = -B_0 r\cos\theta_x + \frac{M}{r^2}\cos\theta_x$$

$$\left(-B_0 - \frac{2M}{r^3}\right)\cos\theta_x = 0 \quad \text{at } r=a$$

$$\Rightarrow M = -\frac{a^3 B_0}{2} \Rightarrow \boxed{\vec{M} = -\frac{E_0 a^3}{2}\hat{x}}\, e^{i\omega t}$$

---

radiation fields from $\vec{p}$ and $\vec{M}$ add.

define $\hat{n} \equiv \frac{\vec{r}}{r}$

$$\vec{B}_{M1} = -k^2\left[\hat{n}\times(\hat{n}\times\vec{M})\right]\frac{e^{ikr}}{r}$$

$$\vec{E}_{M1} = k^2\,\hat{n}\times\left[\hat{n}\times(\hat{n}\times\vec{M})\right]\frac{e^{ikr}}{r} = -k^2(\hat{n}\times\vec{M})\,e^{ikr}/r$$

$$\vec{E}_{E1} \leftrightarrows B_{M1} = -k^2\,\hat{n}\times(\hat{n}\times\vec{p})\frac{e^{ikr}}{r}$$

$$\vec{E} = \vec{E}_{E1} + \vec{E}_{M1} = -k^2\frac{e^{ikr}}{r}E_0 a^3\left[\hat{n}\times(\hat{n}\times\hat{z}) - \tfrac{1}{2}(\hat{n}\times\hat{x})\right] e^{i\omega t}$$

(10b)

forward $\sim \vec{E}_0 \times \vec{B}_0 \sim \hat{z} \times \hat{x} = \hat{y}$

backward $\sim -\hat{y}$

$$\vec{E} = \frac{-k^2 a^3 E_0}{r} e^{i(kr-\omega t)} \left[-\hat{z} \pm \frac{1}{2} \hat{z} \right]$$

$$= \frac{k^2 a^3 E_0}{2} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \frac{e^{i(kr-\omega t)}}{r} \quad \text{for} \begin{Bmatrix} \text{forward} \\ \text{backward} \end{Bmatrix}$$

$$\frac{dP}{d\Omega_F} = \frac{c}{8\pi} \frac{E_0^2 a^6 k^4}{4}$$

$$\frac{dP}{d\Omega_B} = 9 \frac{dP}{d\Omega_F}$$

(iva) Spring written exam 1994

Joseph Chen

Consider the elastic scattering of an electron by a hydrogen atom in its ground state, $\chi_0(R) = \left(\frac{1}{\pi a_0^3}\right)^{1/3} e^{-R/a_0}$, in the Born approximation

$$\psi_i = e^{i\vec{k}_i\cdot\vec{r}}\,\chi_0(R)$$
$$\psi_f = e^{i\vec{k}_f\cdot\vec{r}}\,\chi_0(R)$$

(a). Show that the differential scattering cross section in this approximation is given by the formula

$$d\sigma = a_0^2\,\frac{\left(1+\frac{q^2a_0^2}{8}\right)^2}{\left(1+\frac{q^2a_0^2}{4}\right)^4}\,d\Omega$$

with $a_0 = \frac{\hbar^2}{me^2}$, $q = |\vec{k}_i - \vec{k}_f|$, and $d\Omega = \sin\theta\, d\theta\, d\phi$.

✓ $\left[\text{given } \int \chi_0^*(R)\, e^{i\vec{q}\cdot\vec{R}}\,\chi_0(R)\, d\vec{R} = \frac{1}{\left(1+\frac{q^2a_0^2}{4}\right)^2}\right]$

(b). Determine the elastic scattering cross section σ and its high energy limit.

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Solution

(a). $$d\sigma = \left(\frac{m}{2\pi\hbar^2}\right)^2 |\langle \psi_f | H_1 | \psi_i \rangle|^2 d\Omega$$

$$H_1 = -\frac{e^2}{r} + \frac{e^2}{|\vec{r}-\vec{R}|}$$

$$\langle \psi_f | H_1 | \psi_1 \rangle = \int d\vec{R}\, |\chi_0(R)|^2 \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} \left(\frac{e^2}{|\vec{r}-\vec{R}|} - \frac{e^2}{r} \right)$$

$$= \int d\vec{R}\, |\chi_0(R)|^2 \frac{4\pi e^2}{q^2} \left(e^{i\vec{q}\cdot\vec{R}} - 1 \right)$$

$$= \frac{4\pi e^2}{q^2} \left\{ \int d\vec{R}\, |\chi_0(R)|^2 e^{i\vec{q}\cdot\vec{R}} - 1 \right\}$$

$$= \frac{4\pi e^2}{q^2} \left\{ \frac{1}{\left(1+\frac{q^2 a_0^2}{4}\right)^2} - 1 \right\}$$

$$= 2\pi e^2 a_0^2 \frac{\left(1+\frac{q^2 a_0^2}{8}\right)}{\left(1+\frac{q^2 a_0^2}{4}\right)}$$

$$d\sigma = a_0^2 \frac{\left(1+\frac{q^2 a_0^2}{8}\right)^2}{\left(1+\frac{q^2 a_0^2}{4}\right)^4}$$

(17c)

(b)
$$\sigma = a_0^2 \int \frac{\left(1+\frac{q^2a_0^2}{8}\right)^2}{\left(1+\frac{q^2a_0^2}{4}\right)^4} d\Omega$$

$$q^2 = 2k^2(1-\cos\theta) \qquad k_i = k_f = k$$

$$2q\,dq = 2k^2 \sin\theta\, d\theta \qquad \theta: 0 \to \pi$$

$$d\Omega = \frac{q\,dq}{k^2} d\phi \qquad q: 0 \to 2k$$

$$\sigma = a_0^2\left(\frac{2\pi}{k^2}\right)\int_0^{2k} \frac{\left(1+\frac{q^2a_0^2}{8}\right)^2}{\left(1+\frac{q^2a_0^2}{4}\right)^4} q\,dq$$

Let $z = 1+\frac{q^2a_0^2}{4}$ $\qquad q\,dq = \frac{2}{a_0^2}dz$

$$\sigma = \frac{4\pi}{k^2}\int_1^{1+(ka_0)^2} \frac{\left(\frac{1}{2}+\frac{1}{2}z\right)^2}{z^4} dz$$

$$= \frac{\pi}{k^2}\left[-\frac{1}{3z^3}-\frac{1}{z^2}-\frac{1}{z}\right]_1^{1+(ka_0)^2}$$

$$= \frac{\pi}{k^2}\left[\frac{7}{3} - \frac{1}{1+(ka_0)^2} - \frac{1}{[1+(ka_0)^2]^2} - \frac{1}{3[1+(ka_0)^2]^3}\right]$$

high energy limit $k^2a_0^2 \gg 1$

$$\sigma = \frac{7}{3}\frac{\pi}{k^2}$$

Problem for Dept'l Exam (STATISTICAL MECHANICS)

12

An electron is at rest (initially) and is surrounded by a blackbody photon gas (temp T). Calculate the rate (dE_e/dt) at which it initially increases its energy (when $kT \ll mc^2$) as a result of Compton scattering ~~off~~ by photons. Note that, since the process involves a final-state photon, it is necessary to include the effects of stimulated scattering.

(solution)

ε = photon energy

in one scattering, $\Delta E_e = -\Delta\varepsilon \approx (\varepsilon^2/mc^2)(1-\cos\theta)$ (Th. limit)

(θ = sc. angle)

$$\frac{\Delta E_e}{\Delta t} = \iint c\, dn_\gamma\, \Delta E_e\, d\sigma \cdot (1 + \underbrace{\bar{n}(\varepsilon_{sc})}_{\text{stim sc. corr.}}) \qquad \varepsilon_{sc} \approx \varepsilon \quad (\text{Th. limit})$$

$$d\sigma = \tfrac{1}{2} r_0^2 (1+\cos^2\theta)\, d\Omega$$

$$dn_\gamma = \bar{n} \cdot \frac{2 \cdot 4\pi p^2 dp}{(2\pi\hbar)^3} \qquad \bar{n} = \frac{1}{e^{\hbar\omega/kT}-1} = \frac{1}{e^x - 1} \qquad \varepsilon = \hbar\omega = pc$$

$$1 + \bar{n} = \frac{e^x}{e^x - 1} \qquad \bar{n}(1+\bar{n}) = \frac{e^x}{(e^x-1)^2}$$

integrate over $d\Omega$ and over dx (by parts):

$$\frac{\Delta E_e}{\Delta t} = 4 \frac{\sigma_T}{mc} u_\gamma kT \qquad \sigma_T = \frac{8\pi}{3} r_0^2$$

$$u_\gamma = \int \hbar\omega\, dn_\gamma = \text{ph. en. dens} \propto T^4$$

13.(a.) Landau Free energy of 3-D XY model is

$$\boxed{F = \int d^3r \left[C|\nabla\psi|^2 + \alpha(T-T_c)|\psi|^2 + g|\psi|^4 \right]}$$

where C, α, g are constants and $\psi \equiv \langle M_x + iM_y \rangle$

(b.) $M_x + iM_y = \psi$ is determined by minimizing F:

$$0 = \left.\frac{\partial F}{\partial \psi}\right|_{\bar\psi} = \alpha(T-T_c)\bar\psi + 2g|\psi|^2\bar\psi$$

So

$$\boxed{M_x + iM_y = \psi = \begin{cases} 0 & T > T_c \\ \left(\frac{\alpha(T_c-T)}{2g}\right)^{\frac{1}{2}} e^{i\phi} & T < T_c \end{cases}}$$

ϕ determines where $\vec{m}$ points. Because of XY rotation symmetry ϕ is arbitrary.

(c.) In external field we minimize

$$F - \int d^3r \left[\tfrac{1}{2}(\psi + \bar\psi) H_x + \tfrac{1}{2i}(\psi - \bar\psi) H_y \right]$$

So

$$\tfrac{1}{2}(H_x - iH_y) = \left.\frac{\partial F}{\partial \psi}\right|_{\bar\psi}$$

For simplicity assume external field $\vec{H}$ points along x direction. so $\vec{M} = M_x \hat{i}$ and ψ ($= M_x$) is real.
Then

$$\tfrac{1}{2}H_x = \alpha(T-T_c)\psi + 2g\psi^3$$

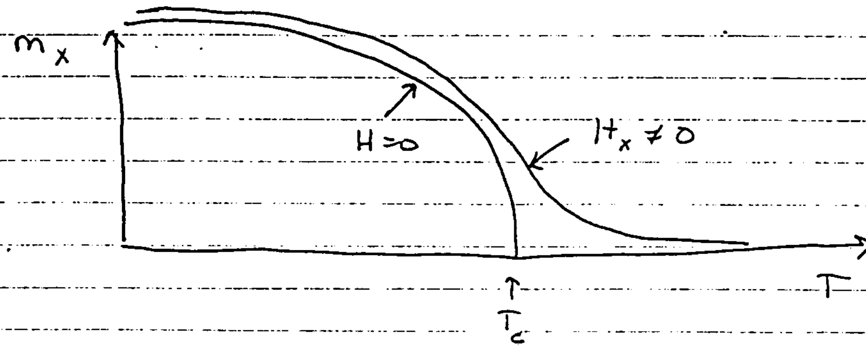
Differentiate to get

$$\tfrac{1}{2}dH_x = \left[\alpha(T-T_c) + 6g\psi^2\right]d\psi$$
$$= 4\alpha(T-T_c)\, d\psi$$

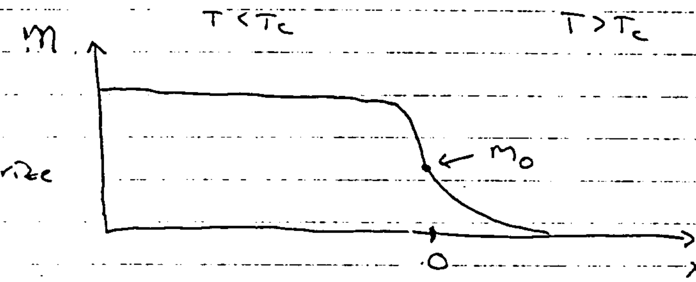
where result from part b was used. Hence

$$\boxed{\chi = \frac{\partial m_x}{\partial H_x} = \frac{1}{8\alpha(T-T_c)}} \qquad T > T_c \text{ only}$$

(d.)



External magnetic field cause Paramagnetic ⟺ Ferromagnet. Transition is be replaced by a cross-over.



(e.) Assuming m or ψ to be small for $x>0$ allows us to linearize

$$0 = \left.\frac{\delta F}{\delta \bar{\psi}(x)}\right|_{\psi} = \left.\frac{\delta F}{\delta \psi(x)}\right|_{\bar{\psi}}$$

which is the general eqns obeyed by $\psi(x)$. Now

$$\left.\frac{\delta F}{\delta \bar{\psi}}\right|_{\psi} = -c\nabla^2\psi + \alpha(T-T_c)\psi + 2g|\psi|^2\psi$$

So $$-c\frac{d^2\psi}{dx^2} + \alpha(T-T_c)\psi = 0$$ is the linearize 1-D equation. The solution is $\psi(x) = \text{const}\; e^{-x/\xi}$ or

$$\boxed{\vec{m} = \vec{m}_0\, e^{-x/\xi}}$$

where

$$\boxed{\xi = \left[\frac{c}{\alpha(T-T_c)}\right]^{1/2}}$$

Note: This linearization about $\psi=0$ assumes $T>T_c$ as is the case for $x>$

(F.) If ℓ is the density of independently fluctuating modes, the equipartition theorem states that they contribute

$$\frac{\delta F}{V} = \frac{1}{2} k_B T \, \ell$$

to the free energy per unit volume V. Now

$$\ell \sim \left(\frac{\xi_0}{\xi}\right)^d$$

in d-dimensions, where ξ_0 is some unknown microscopic length. Using the Landau theory value for ξ (valid for $T > T_c$) we have

$$\frac{\delta F}{V} \sim k_B T \left(\frac{\alpha (T-T_c)}{c}\right)^{d/2}$$

Now $C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V$. So the contribution of thermal fluctuations to the heat capacity per unit volume is

$$\boxed{\frac{\delta C_V}{V} \sim k_B T^2 \left(\frac{\alpha}{c}\right)^{d/2} (T - T_c)^{d/2 - 2}}$$

assuming $T - T_c$ is small. (Otherwise additional terms from $\frac{\partial^2 F}{\partial T^2}$ must be included.) For a 3-D magnet this implies $\boxed{\delta C_V \text{ diverges like } \frac{1}{\sqrt{T-T_c}}}$.

#14) (a) We have

$$U[s(x,t)] = \frac{1}{2}K\int_0^L dx \left(\frac{\partial^2 s}{\partial x^2}\right)^2$$

along with the usual

$$T[s(x,t)] = \frac{1}{2}\rho\int_0^L dx \left(\frac{\partial s}{\partial t}\right)^2$$

and the boundary conditions

$$s(0,t) = s(L,t) = 0$$
$$s'(0,t) = s'(L,t) = 0$$

where $s'(x,t) = \dfrac{\partial s(x,t)}{\partial x}$. The action is

$$A[s(x,t)] = \frac{1}{2}\int_0^T dt\int_0^L dx \left\{\rho\left(\frac{\partial s(x,t)}{\partial t}\right)^2 - K\left(\frac{\partial^2 s(x,t)}{\partial x^2}\right)^2\right\}$$

Variation: $s(x,t) \longrightarrow s(x,t) + \delta s(x,t)$. To first order,

$$\delta A = \int_0^T dt\int_0^L dx\left\{\rho\,\frac{\partial s}{\partial t}\,\frac{\partial \delta s}{\partial t} - K\,\frac{\partial^2 s}{\partial x^2}\,\frac{\partial^2 \delta x}{\partial x^2}\right\}$$

Now integrate by parts:

$$\frac{\partial s}{\partial t}\,\frac{\partial \delta s}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial s}{\partial t}\,\delta s\right) - \frac{\partial^2 s}{\partial t^2}\,\delta s$$

$$\frac{\partial^2 s}{\partial x^2}\,\frac{\partial^2 \delta s}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 s}{\partial x^2}\,\frac{\partial \delta s}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial^3 s}{\partial x^3}\,\delta s\right) + \frac{\partial^4 s}{\partial x^4}\,\delta s$$

Thus,

$$\delta A = \rho\int_0^L dx\left[\frac{\partial s(x,t)}{\partial t}\,\delta s(x,t)\right]_{t=0}^{t=T} - K\int_0^T dt\left[\frac{\partial^2 s(x,t)}{\partial x^2}\,\frac{\partial \delta s(x,t)}{\partial x}\right]_{x=0}^{x=L}$$
$$+ K\int_0^T dt\left[\frac{\partial^3 s(x,t)}{\partial x^3}\,\delta s(x,t)\right]_{x=0}^{x=L} + \int_0^T dt\int_0^L dx\left\{-\rho\,\frac{\partial^2 s(x,t)}{\partial t^2} - K\,\frac{\partial^4 s(x,t)}{\partial x^4}\right\}\delta s(x,t)$$

Now $\delta s(0,t) = \delta s(L,t) = 0$ and $\delta s'(0,t) = \delta s'(L,t) = 0$, so setting $\delta s(x,0) = \delta s(x,T) = 0$, we obtain the Euler-Lagrange equations

$$\rho \frac{\partial^2 s}{\partial t^2} + K \frac{\partial^4 s}{\partial x^4} = 0$$

(b) Solve by separation of variables : $\delta s(x,t) = s_0 e^{iqx} e^{-i\omega t}$

Note that the equation of motion gives now

$$-\rho\omega^2 + Kq^4 = 0$$

so q can be purely real or purely imaginary. Take

$$s(x,t) = (\alpha \sin qx + \beta \cos qx + a \sinh qx + b \cosh qx) \sin(\omega t + \phi)$$

where α, β, a, and b are constants, and ϕ is an arbitrary phase. Boundary conditions now require

① $s(0,t) = 0 \implies \beta + b = 0$

② $s'(0,t) = 0 \implies \alpha + a = 0$

③ $s(L,t) = 0 \implies \alpha \sin qL + \beta \cos qL + a \sinh qL + b \cosh qL = 0$

④ $s'(L,t) = 0 \implies \alpha \cos qL - \beta \sin qL + a \cosh qL + b \sinh qL = 0$

Substituting $a = -\alpha$, $b = -\beta$ from ① and ② into ③ and ④,

$$\alpha(\sin qL - \sinh qL) + \beta(\cos qL - \cosh qL) = 0$$

$$\alpha(\cos qL - \cosh qL) - \beta(\sin qL + \sinh qL) = 0$$

which has a nontrivial sol$^{\underline{n}}$ for α and β only if the determinant vanishes, i.e.

$$(\sin qL - \sinh qL)(\sin qL + \sinh qL) + (\cos qL - \cosh qL)^2 = 0$$

$$\Rightarrow \sin^2 qL - \sinh^2 qL + \cos^2 qL - 2\cos qL \cosh qL + \cosh^2 qL = 0$$

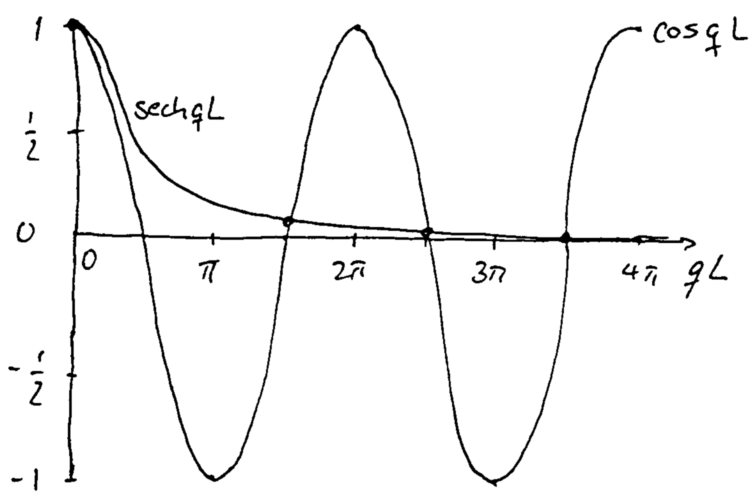
$$\Rightarrow 1 - \cos qL \cosh qL = 0$$

Therefore, we must have

$$\boxed{\cos qL \cdot \cosh qL = 1}$$

Once this is solved, we have $\omega(q) = \sqrt{\frac{K}{\rho}}\, q^2$ for the frequency of this eigenmode.

Graphical sol$^{\underline{n}}$ to $\cos qL = 1/\cosh qL$:



$\bullet$ = sol$^{\underline{n}}$ of $\cos qL = \operatorname{sech} qL$

For $qL \gg 1$, we have $\cosh qL \simeq \frac{1}{2} e^{qL} \gg 1$, $\operatorname{sech} qL \simeq 2e^{-qL} \ll 1$

and we have $\cos qL \simeq 2e^{-qL} \simeq 0 \Rightarrow qL = (n+\frac{1}{2})\pi$, $q = (n+\frac{1}{2})\frac{\pi}{L}$.