

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[1] In the following problem order of magnitude estimates will suffice. Model the sun as a uniform constant density ( $\rho \approx 1.4 \text{ g cm}^{-3}$ ) sphere of ionized hydrogen with radius  $R \approx 7 \times 10^{10} \text{ cm}$ . For the purpose of estimating the internal energy of the sun, assume a characteristic temperature of  $T \approx 4.5 \times 10^6 \text{ K}$ . (Note that this is not the surface temperature of the sun.) Further assume that local thermodynamic equilibrium between the radiation field and the matter is a good approximation. The typical photon scattering cross section can be taken to be roughly the Thomson value  $\sigma \approx 10^{-24} \text{ cm}^2$ .

- (a) How long would it take a photon to random walk from the center to the surface of the sun in this model?
- (b) Estimate the radiant power emitted by the sun, assuming that all energy escapes through the process of photons random walking from the center of the sun.
- (c) Compare the kinetic energy density of the gas to the energy density of radiation. Find the time taken for the sun to radiate away its internal energy.
- (d) How long could the sun shine with the luminosity estimated in part (b) if it were powered by hydrogen burning? In this process 4 protons combine to form  ${}^4\text{He}$  through a series of strong, weak, and electromagnetic interactions. The binding energy of  ${}^4\text{He}$  relative to 4 free protons is roughly 26.7 MeV. Assume that, over its lifetime, of order ten percent of the sun's mass is available for hydrogen burning.

Some useful constants:

Radiation density constant  $a = 7.6 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$

Boltzmann constant  $k_B = 1.4 \times 10^{-16} \text{ erg K}^{-1}$

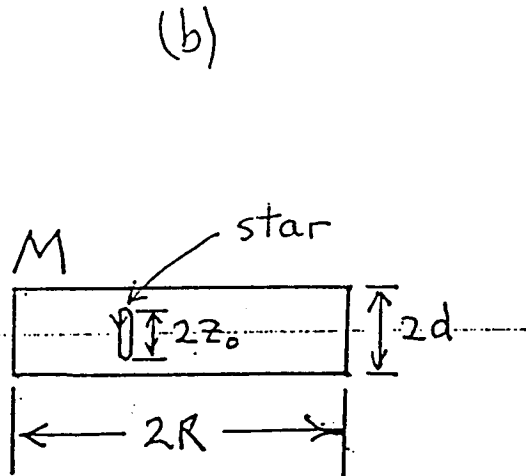
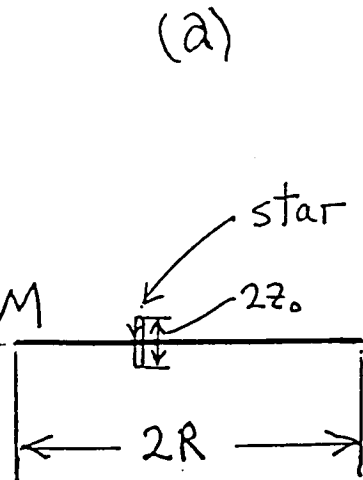
## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[2] Consider the motion of a star in the  $z$  direction, perpendicular to the plane of a disk-shaped galaxy of total mass  $M$  and radius  $R$ . Assume the star is far from the edge of the galaxy, and that it oscillates with maximum excursion  $z_0 \ll R$  relative to the galactic plane. Calculate the period,  $\tau$ , of the oscillations in the following two cases.

(a) The galaxy has negligible thickness compared with  $z_0$ .



UNIVERSITY OF CALIFORNIA, SAN DIEGO

DEPARTMENT OF PHYSICS

Written Departmental Examination – Spring 1993, Part I

[3] A particle moves in an attractive spherically symmetric potential,

$$V(r) = -\frac{k}{r^4},$$

with  $k$  a constant.

What is the total cross section for capture of a particle incident from infinitely far away with initial velocity  $v_0$ ? Explain clearly how you obtain your result.

*Hint:* What is the maximum impact parameter which will result in capture?

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[4] Two problems in electrostatics:

(a) Consider the arrangement shown below of two line charges, each of length  $2a$ . Their respective linear charge densities (charge per unit length) are  $\lambda$  and  $-\lambda$ , as shown. Calculate the electrostatic dipole moment  $\mathbf{p} = (p_x, p_y, p_z)$ .

(b) As shown below, a conducting sphere, A, is placed at the center of two thin, concentric, conducting spherical shells, B and C, and a conducting wire connects B and C. Initially all conductors are uncharged, and then A is charged to  $+Q$  by an external agent. Answer the following questions, and provide clear explanations for all your answers.

(i) What is the electric field in the region between A and B?

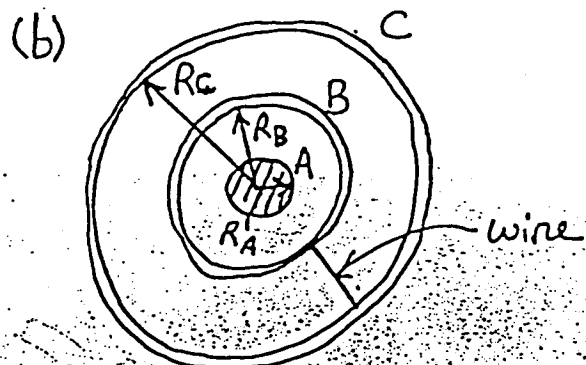
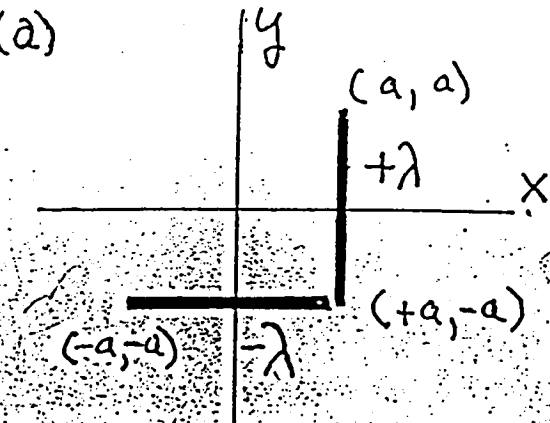
(ii) What is the electric field in the region between B and C?

(iii) What is the electric field outside C?

(iv) What are the charge distributions on B and C?

(v) How much work is done charging the sphere A to  $+Q$ ?

(vi) What is the capacitance of the system?



## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[5] Consider a particle of mass  $m$  moving in the infinite square well potential

$$V(x) = \begin{cases} V_0 & \text{if } 0 \leq x \leq a; \\ \infty & \text{if } x < 0 \text{ or } x > a. \end{cases}$$

(a) Derive the energy eigenvalues  $E_n$  and eigenfunctions  $\psi_n(x)$ .

(b) A small perturbation  $V_{\text{pert}}(x) = aU\delta(x - \frac{1}{2}a)$  is added, where  $\delta(x)$  is the Dirac delta function and  $U$  is a constant with dimensions of energy. Compute the energy shifts of all the levels  $E_n$  to first order in  $U$ .

(c) Using second order perturbation theory, find the ground state energy to second order in  $U$ . You may find the following mathematical identity useful:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1 \quad .$$

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[6] The Einstein model of the lattice vibrations of a solid consisting of  $N$  atoms represents the solid by  $3N$  identical one-dimensional quantum harmonic oscillators, each with frequency  $\omega_0$ . Answer the following questions related to this model:

- (a) Find the mean energy of the system as a function of the temperature,  $T$ , of the solid.
- (b) Find the heat capacity of the system, and evaluate it in the limit  $k_B T \gg \hbar \omega_0$ . Discuss this result in terms of the equipartition theorem.
- (c) Find the general expression relating the pressure of a system to its Helmholtz free energy.
- (d) To model anharmonic effects in the solid, one assumes that the frequency  $\omega_0$  is a function of the volume,  $V$ . Find the pressure in the Einstein solid as a function of  $(\partial \omega_0 / \partial V)_T$ .

# DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part I

[7] The force of air resistance on a falling body is very nearly proportional to the *square* of its velocity. Thus, if we define the height to be  $-y$  (so that  $y$  increases as the particle falls), the differential equation for the motion is given by

$$\frac{d^2y}{dt^2} = g - \alpha \left( \frac{dy}{dt} \right)^2$$

where  $g$  is the acceleration due to gravity.

- (a) Rescale the position and time variables  $(y, t)$  to form dimensionless variables  $\xi \propto y$  and  $\tau \propto t$ . Derive a differential equation for the rescaled height,  $-\xi(\tau)$ .
- (b) Write down and solve the (first order) differential equation for the dimensionless velocity,  $u(\tau) = d\xi/d\tau$ . You may assume that at  $t = 0$  the body is at rest.
- (c) Expand your solution  $u(\tau)$  in a power series in  $\tau$ , and show that the first two nonvanishing terms can also be obtained by iterating the differential equation for  $u(\tau)$ .
- (d) Integrate again to find  $\xi(\tau)$  and  $y(t)$ . Find the limiting expressions for small and large  $t$  and comment on their behavior.

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part II

[8] In an ion storage ring,  $N$  ions of charge  $q$  and mass  $m$  are confined to a circle of radius  $R$ . Their motion is thus purely one-dimensional, as they are constrained to move along the circle. Their equilibrium separation is then  $a = 2\pi R/N$ . Suppose that the number of ions  $N$  is very large, in which case  $a \ll R$ . The system can then be approximated as an infinite linear chain of ions with equilibrium separation  $a$  (i.e., you may neglect the curvature of the ring).

(a) Let the position of the  $n^{\text{th}}$  ion be  $X_n = na + x_n$ , where  $x_n$  is the deviation from equilibrium. Expand the potential energy to second order in the deviations  $x_n$ .

(b) Find an expression for the frequency of oscillation,  $\omega(k)$ , of the normal modes of the chain as a function of their wave number  $k$ .



## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part II

[9] An electron is at rest a distance  $r_0$  from a nucleus of charge  $Q$ . It is then released and falls toward the nucleus. In answering the following, assume the electron velocity,  $v$ , is such that  $v \ll c$ , that the motion is confined to one dimension, and that the motion can be described classically.

- (a) Calculate the radiated power as a function of the electron-nucleus separation,  $r$ .
- (b) Calculate the total energy radiated as a function of  $r$ . (You may leave your answer in integral form.)

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part II

[10] The bosonic low-energy excitations of a two-dimensional system of dimensions  $L \times L$  are described by the wave equation

$$\rho \frac{\partial^2 u}{\partial t^2} + C \nabla^4 u = 0$$

where  $\nabla^4 = (\nabla^2)^2$ . (The scalar  $u(r, t)$  might represent height fluctuations normal to the two-dimensional plane.)

(a) Solve for the dispersion relation  $\omega(k)$ .

(b) Compute the density of states  $g(\omega)$ .

(c) Compute, to within a numerical constant, the low-temperature specific heat  $C(T)$ .

(You may assume that  $k_B T$  is much greater than the spacing between neighboring quantized energy levels.)

[11] The nuclear charge number  $Z$  of a  $(Z - 1)$ -times ionized atom changes *suddenly* to  $Z + 1$  when the nucleus of the atom undergoes beta decay.

(a) Calculate the probability for the electron to make a transition to the  $2s$ -state, assuming that it was in the ground state before the decay. Evaluate the transition probability numerically for  $Z = 14$ .

(b) What is the transition probability to the  $2p$ -state?

The Bohr radius is  $a_0 = \hbar^2/m_e e^2 = 0.529 \times 10^{-8} \text{cm}$ . You may find the following useful:

Radial wavefunctions for  $V(r) = -Ze^2/r$ :

$$R_{10}(r) = 2 \left( \frac{Z}{a_0} \right)^{3/2} e^{-Zr/a_0}$$

$$R_{20}(r) = 2 \left( \frac{Z}{2a_0} \right)^{3/2} \left( 1 - \frac{Zr}{2a_0} \right) e^{-Zr/2a_0}$$

$$R_{21}(r) = \frac{1}{\sqrt{3}} \left( \frac{Z}{2a_0} \right)^{3/2} \left( \frac{Zr}{a_0} \right) e^{-Zr/2a_0}$$

Some spherical harmonics:

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

An integral:

$$\int_0^\infty dx x^n e^{-x} = n!$$

## UNIVERSITY OF CALIFORNIA, SAN DIEGO

## DEPARTMENT OF PHYSICS

## Written Departmental Examination – Spring 1993, Part II

[12] (a) Consider a spin- $\frac{1}{2}$  particle with magnetic moment  $\mu = \gamma\sigma$  in a uniform magnetic field  $B$  which points in the direction  $(\theta, \phi)$  relative to a Cartesian coordinate system  $(x, y, z)$ . Here,  $\sigma = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$ , where  $\{\sigma_\alpha\}$  are the Pauli matrices. Take the spin quantization axis to be the  $\hat{z}$ -axis ( $\theta = 0$ ). Find the energy eigenstates and energy eigenvalues in this basis.

(b) For a particle in each of these two eigenstates, determine the probability for its spin to be measured along the  $\hat{x}$ -axis.

(c) At time  $t = 0$ , a spin- $\frac{1}{2}$  particle (as in part (a)) has its spin oriented along the  $\hat{x}$ -axis. A magnetic field  $B = B\hat{z}$  is then applied for a time  $t$ , after which the spin points along the  $\hat{y}$ -axis. Determine  $t$ .

(d) Four spin- $\frac{1}{2}$  particles interact pairwise according to the Hamiltonian

$$H = J \sum_{i < j} S_i \cdot S_j ,$$

where  $J$  is a constant and  $i$  and  $j$  label the four particles. The sum is over all possible pairs. Find all the energy eigenvalues.

## DEPARTMENT OF PHYSICS

Written Departmental Examination – Spring 1993, Part II

[13] Evaluate

$$I(N) = \int_{-\infty}^{\infty} dx (\cosh x)^{-N}$$

for large  $N$ , to order  $N^{-3/2}$ .

I

PART I PROBL. 1

E1.) a.) In a 1-dimensional random walk the mean square displacement after  $N$  scatterings is given by

$$\langle x^2 \rangle = N \lambda^2, \text{ where } N = \# \text{ of scatterings,}$$

and where  $\lambda$  is the mean free path.

$$\lambda \approx \frac{1}{n\sigma} = \frac{1}{(1.4 \text{ g cm}^{-3})(6.02 \times 10^{23})(\pi 10^{-24} \text{ cm}^2)} \approx \text{~~1.2 cm~~ } \approx 1 \text{ cm}.$$

Now in 3 Dimensions, only  $1/3$  of the scatterings contribute to the mean square displacement in a given direction; so radial mean square displacement is

$$\langle r^2 \rangle = \frac{N}{3} \lambda^2.$$

so number of scatterings required for escape

is  $N_{es} = \frac{3R^2}{\lambda^2}$ , and mean time between

scatterings is  $\lambda/c$  so escape time is

$$t_{es} = N_{es} \frac{\lambda}{c} = \frac{3R^2}{\lambda c} \approx 5 \times 10^{11} \text{ s} \approx \text{~~2~~ } \times 10^4 \text{ yrs}$$

## SP93 (2.) Solutions Part I

b) continued . . . .

Radiation energy density  $U_r = aT^4 \approx 3 \times 10^{12} \text{ erg cm}^{-3}$

so total radiation energy in the sun is

$$U_r = \left( \frac{4}{3} \pi R^3 \right) aT^4 \approx 4.5 \times 10^{45} \text{ ergs.}$$

Estimate power by assuming  $U_r$  is released on timescale  $t_{\text{es}}$

$$P \sim \frac{U_r}{t_{\text{es}}} \sim 10^{34} \text{ erg s}^{-1} \quad \left( \text{actually } L_{\odot} \approx 3.8 \times 10^{33} \text{ erg s}^{-1} \right)$$

c.) The proton density  $n_p = \rho N_A = 8.4 \times 10^{23} \text{ cm}^{-3}$ .

Thus, the electron plus proton density is

$$n \approx 1.7 \times 10^{24} \text{ cm}^{-3}, \text{ and the total}$$

thermal kinetic energy is approximately

$$U_k = \frac{4}{3} \pi R^3 \left( \frac{3}{2} n k T \right) \approx 2.3 \times 10^{48} \text{ ergs or } U_k \approx 500 U_r.$$

so that it must take about 500 times longer than  $t_{\text{es}}$  to radiate away thermal K.E.

or  $\tau \approx 3 \times 10^{14} \text{ s} \sim 10^7 \text{ yrs}$ , the Kelvin-Helmholz time.

I-2

(a) Using Gauss' law for the gravitational field

$$\int \vec{g} \cdot \hat{n} dS = 4\pi GM$$

$$2g\pi R^2 = 4\pi GM$$

$$\text{or } g = \frac{2GM}{R^2} \quad (\text{i.e., a constant independent of } z)$$

Thus the quarter period will be

$$\frac{1}{2} g \tau_{1/4}^2 = z_0$$

$$\Rightarrow \tau = 4 \sqrt{\frac{2z_0}{g}} = 4R \left[ \frac{z_0}{GM} \right]^{\frac{1}{2}}$$

(b) Again applying Gauss' law

$$2Ag = 4\pi G\rho(2z)A \quad \text{when } \rho = \frac{M}{\pi R^2(2d)}$$

$$g = \frac{4\pi G M z}{\pi R^2(2d)} \quad (\propto z)$$

Thus

$$\ddot{z} + \left( \frac{2GM}{R^2 d} \right) z = 0 \Rightarrow z = z_0 \cos \omega t$$

and

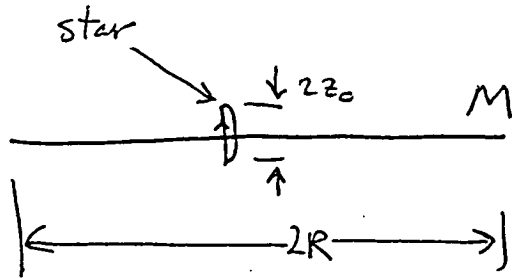
$$\omega = \sqrt{\frac{2GM}{R^2 d}} \quad \tau = \pi R \left[ \frac{2d}{GM} \right]^{\frac{1}{2}}$$



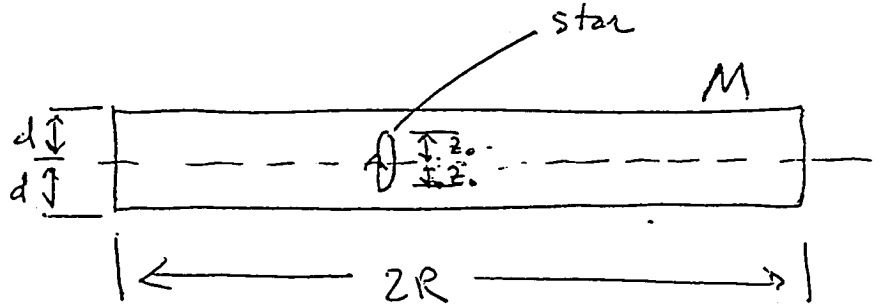
SP 93

I-2

(a)

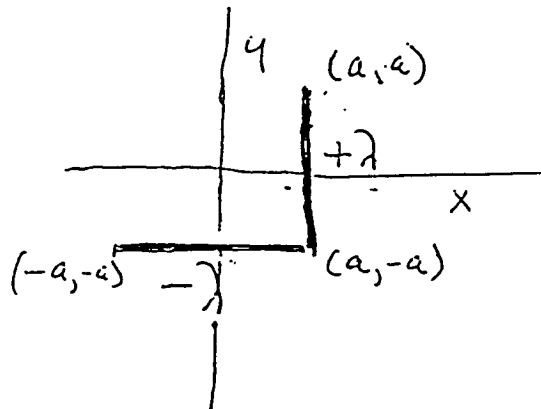


(b)



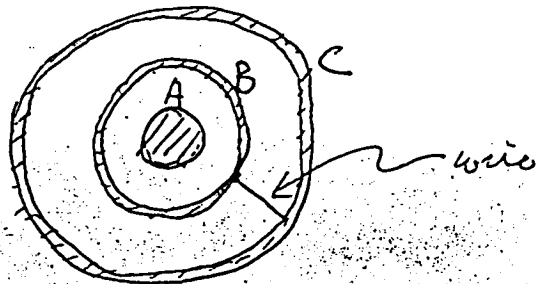
I-4

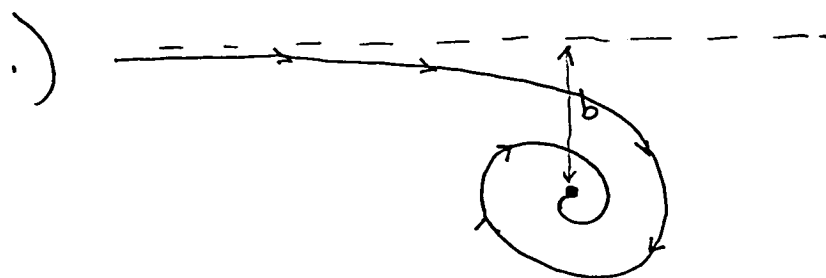
(a)



(b)

Make large enough to label all  $q$ 's

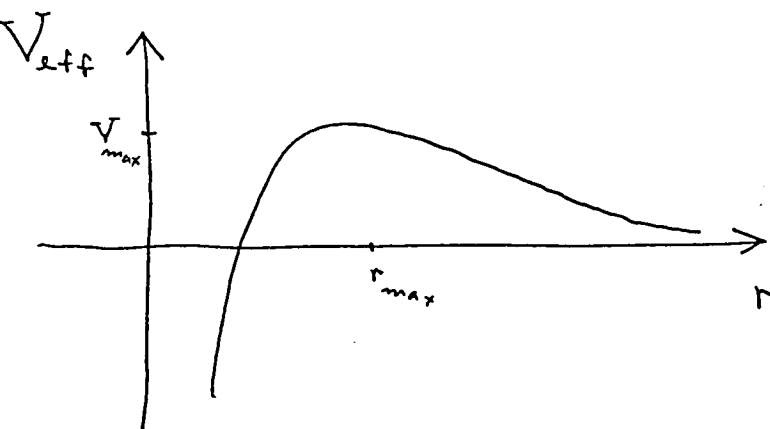




{ assume impact parameter  $b$

The angular momentum of the particle,  $l = mv_0 b$ , is conserved. Use angular momentum conservation to reduce this problem to an effective 1-D problem of a particle moving in

$$\text{potential } V_{\text{eff}} = -\frac{k}{r^4} + \frac{l^2}{2mr^2}$$



If the energy of the incoming particle is greater than the height of the barrier in the effective potential then it will hit  $r=0 \Rightarrow$  it will fall into the potential well and spiral towards center,

3.) Part I, continued...

(2.) Soln.

SP 93

eventually being captured.

The maximum of  $V_{\text{eff}}$  is found by setting  $V'_{\text{eff}} = 0$

$$\Rightarrow 4k = \frac{l^2}{m} r_{\text{max}}^2 \quad \Rightarrow V_{\text{max}} = \frac{1}{16} \frac{l^4}{k m^2}$$

$$\text{Since } r_{\text{max}} = \frac{2 k^{1/2} m^{1/2}}{l}$$

$$\text{We require } E_0 > V_{\text{max}} \Rightarrow \frac{1}{2} m v_0^2 > \frac{m^4 v_0^4 b^4}{16 k m^2} \Rightarrow b < \left[ \frac{8k}{m v_0^2} \right]^{1/4}$$

The total cross section for capture is just  $\pi b_{\text{max}}^2$

where  $b_{\text{max}}$  is the maximum impact parameter that will result in capture.

$$\text{Hence } \sigma_{\text{capture}} = \pi \sqrt{\frac{8k}{m v_0^2}}$$

I-4

(a)

$$\begin{aligned}
 \rho(\vec{r}) &= +\lambda \delta(x-a) \delta(z) & -a \leq y \leq a \\
 &= -\lambda \delta(y+a) \delta(z) & -a \leq x \leq a \\
 &= 0 \text{ otherwise}
 \end{aligned}$$

$$\begin{aligned}
 \underline{\vec{p}} &= \int \vec{r} \rho(\vec{r}) d^3r \\
 &= \hat{i} \lambda a \int_{-a}^a dy + \hat{j} (-\lambda) (-a) \int_{-a}^a dx \\
 &= \underline{2a^2 \lambda (\hat{i} + \hat{j})}
 \end{aligned}$$

(b) (i) By Gauss' law

$$\underline{E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}}$$

(ii) Since B and C are at the same potential,  $E = 0$  in this region

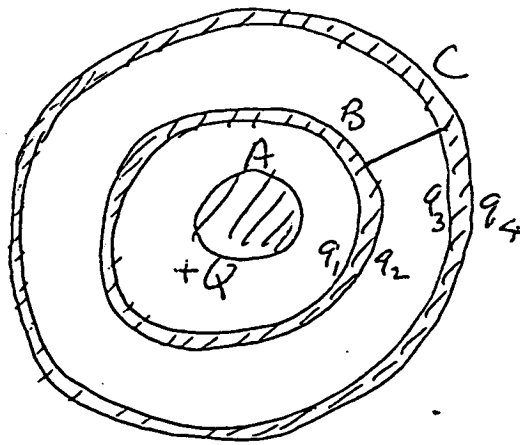
(iii) The net charge on the system is  $+Q$

$$\underline{\text{therefore } E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}}$$

I- 4(b) cont'd

SP93

(iv)



Since  $E = 0$  in B Gauss' law

$$\Rightarrow \underline{q_1 = -Q}$$

Since  $E = 0$  everywhere between the

inner surface of B and the outer surface of C,

$$\underline{q_2 = q_3 = 0}$$

Since the net charge on the system is  $+Q$

$$\underline{q_4 = +Q}$$

$$\begin{aligned} (v) \quad W &= \frac{1}{4\pi\epsilon_0} \int_0^Q q \, dq \left[ \int_{\infty}^{R_C} \frac{dr}{r^2} + \int_{R_B}^{R_A} \frac{dr}{r^2} \right] \\ &= \underline{\underline{\frac{Q^2}{8\pi\epsilon_0} \left[ \frac{1}{R_A} - \frac{1}{R_B} + \frac{1}{R_C} \right]}} \end{aligned}$$

$$(vi) \quad W = \frac{Q^2}{2C} \Rightarrow \underline{\underline{C = \frac{4\pi\epsilon_0}{\frac{1}{R_A} - \frac{1}{R_B} + \frac{1}{R_C}}}}$$

Solution 5

(a) The wavefunctions must be linear combinations of exponentials

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

subject to the boundary conditions

$$\psi(0) = 0$$

$$\psi(a) = 0.$$

The first of these boundary conditions gives  $B = -A$ , and the second gives  $\sin ka = 0$ , requiring  $k_n = n\pi/a$  with  $n$  any integer. The linearly independent normalized solutions are thus

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$E_n = V_0 + \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

with  $n \in \{1, 2, \dots\}$ .

(b) The first order energy shift of the  $n^{\text{th}}$  level is given by

$$\begin{aligned} \Delta E_n^{(1)} &= \langle n | V_{\text{pert}} | n \rangle \\ &= 2U \sin^2\left(\frac{1}{2}n\pi\right) \\ &= \begin{cases} 2U & \text{if } n \text{ odd;} \\ 0 & \text{if } n \text{ even.} \end{cases} \end{aligned}$$

Note that we have used the result

$$\begin{aligned} \langle k | V_{\text{pert}} | l \rangle &= \int_0^a dx \psi_k^*(x) V_{\text{pert}}(x) \psi_l(x) \\ &= 2U \sin\left(\frac{1}{2}k\pi\right) \sin\left(\frac{1}{2}l\pi\right). \end{aligned}$$

(c) The second order energy shifts are given by

$$\begin{aligned} \Delta E_n^{(2)} &= \sum_{\substack{k \\ (k \neq n)}} \frac{|\langle n | V_{\text{pert}} | k \rangle|^2}{E_n^{(0)} - E_k^{(0)}} \\ &= \frac{8ma^2 U^2}{\pi^2 \hbar^2} \delta_{n,\text{odd}} \sum_{\substack{k \\ (k \neq n)}} \frac{\delta_{k,\text{odd}}}{n^2 - k^2}. \end{aligned}$$

So, for the ground state, which has  $n = 1$ , we have (writing  $k = 2j + 1$ ),

$$\begin{aligned} \Delta E_{n=1}^{(2)} &= -\frac{2ma^2 U^2}{\pi^2 \hbar^2} \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \\ &= -\frac{2ma^2 U^2}{\pi^2 \hbar^2}. \end{aligned}$$

I-6

$$\epsilon_n = \hbar \omega_0 \left(n + \frac{1}{2}\right) \quad n = 0, 1, \dots$$

The partition function is:

$$Z = \left[ \sum_{n=0}^{\infty} e^{-\beta \hbar \omega_0 \left(n + \frac{1}{2}\right)} \right]^{3N} = \lim_{M \rightarrow \infty} \left[ e^{-\beta \frac{\hbar \omega_0}{2}} \left( \frac{1 - x^{M+1}}{1 - x} \right) \right]^{3N}$$

where  $x = e^{-\beta \hbar \omega_0}$

Thus

$$\ln Z = 3N \left[ -\beta \frac{\hbar \omega_0}{2} - \ln(1 - e^{-\beta \hbar \omega_0}) \right]$$

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = 3N \left[ \frac{\hbar \omega_0}{2} + \frac{\hbar \omega_0}{e^{\beta \hbar \omega_0} - 1} \right]$$

$$C = \frac{d\bar{E}}{dT} = 3N k_B \left( \frac{\hbar \omega_0}{k_B T} \right)^2 \frac{e^{-\beta \hbar \omega_0}}{(e^{\beta \hbar \omega_0} - 1)^2}$$

At high temperatures  $\beta \hbar \omega_0 \ll 1$ 

Thus

$$C \approx 3N k_B \frac{(\beta \hbar \omega_0)^2}{(\beta \hbar \omega_0)^2} = 3N k_B$$

From equipartition, for each 1D oscillator

$$\bar{E} = \frac{1}{2} k_B T + \frac{1}{2} k_B T = k_B T$$

$$\text{Thus } \bar{E} = 3N k_B T \text{ and } C = 3N k_B \quad \checkmark \checkmark$$

I-6 (cont'd)

$$F = E - TS$$

$$dQ = TdS = dE + pdV$$

Thus  $dF = d(E - TS) = -SdT - pdV$

$$\underline{\left(\frac{\partial F}{\partial V}\right)_T = -p}$$

$$F = -kT \ln Z, \text{ so}$$

$$\underline{p = 3Nk \left(\frac{\partial \ln Z}{\partial V}\right)_T \left[ \frac{1}{2} + \frac{1}{e^{\beta \hbar \omega_0} - 1} \right]}$$



Solution 7

(a) Note that the units of  $g$  are  $[g] = L/T^2$  and the units of  $\alpha$  are  $[\alpha] = 1/L$ . So there is a length scale  $\alpha^{-1}$  and a time scale  $(g\alpha)^{-1/2}$ , and defining the dimensionless variables  $\xi$  and  $\tau$  through

$$y \equiv \xi/\alpha \quad t \equiv \tau/\sqrt{g\alpha},$$

the equation of motion becomes

$$\frac{d^2\xi}{d\tau^2} = 1 - \left(\frac{d\xi}{d\tau}\right)^2.$$

(b) The dimensionless velocity  $u(\tau) = d\xi/d\tau$  satisfies

$$\frac{du}{d\tau} = 1 - u^2$$

and hence

$$\begin{aligned} d\tau &= \frac{du}{1-u^2} \\ &= \frac{1}{2} \left( \frac{1}{1+u} + \frac{1}{1-u} \right) du \\ &= \frac{1}{2} d \ln \frac{1+u}{1-u}. \end{aligned}$$

which, together with the boundary condition  $u(0) = 0$  gives

$$2\tau = \ln \frac{1+u}{1-u} \implies u(\tau) = \frac{e^{2\tau} - 1}{e^{2\tau} + 1} = \tanh \tau.$$

(c) Taylor expanding,

$$\begin{aligned} u(\tau) &= \tanh \tau \\ &= \tau - \frac{1}{3}\tau^3 + \mathcal{O}(\tau^5). \end{aligned}$$

This expansion can be obtained by iterating the differential equation for  $u(\tau)$ . At the zeroth level of iteration, we have  $u^{(0)}(\tau) = 0$ , and the first iteration gives

$$u^{(1)}(\tau) = \int_0^\tau d\tau' (1 - [u^{(0)}(\tau')]^2) = \tau.$$

The second iteration then gives

$$u^{(2)}(\tau) = \int_0^\tau d\tau' (1 - [u^{(1)}(\tau')]^2) = \tau - \frac{1}{3}\tau^3.$$

Proceeding further, we could generate the Taylor series expansion for  $u(\tau) = \tanh \tau$ .

(d) Integrating again,

$$\frac{d\xi}{d\tau} = u(\tau) = \tanh \tau \implies \xi(\tau) = \xi(0) + \ln \cosh \tau$$

since  $d \ln \cosh \tau = \tanh \tau d\tau$ . In terms of  $y$  and  $t$ , we have

$$y(t) = y(0) + \alpha^{-1} \ln \cosh(t\sqrt{\alpha g}) .$$

For small  $t$ , we use

$$\begin{aligned} \ln \cosh \tau &= \ln(1 + \tfrac{1}{2}\tau^2 + \dots) \\ &= \tfrac{1}{2}\tau^2 + \dots \end{aligned}$$

to obtain

$$y(t) = y(0) + \tfrac{1}{2}gt^2 + \mathcal{O}(t^4)$$

which simply says that the particle falls freely for early times, when its velocity is small enough that the friction term may be ignored. At late times, we have  $\cosh \tau \rightarrow \tfrac{1}{2}e^\tau$  and hence

$$y(t) = y(0) - \alpha^{-1} \ln 2 + t\sqrt{g/\alpha} + \mathcal{O}(e^{-t\sqrt{\alpha g}}).$$

This demonstrates that at late times, the particle achieves a terminal velocity

$$v_\infty = \lim_{t \rightarrow \infty} \frac{dy}{dt} = \sqrt{g/\alpha} .$$

**Solution 8**

(a) Let  $X_n = na + x_n$  be the position of the  $n^{\text{th}}$  ion, where  $x_n = 0$  in equilibrium. The potential energy is

$$V = \frac{1}{2} \sum_{n \neq n'} \frac{q^2}{|X_n - X_{n'}|}.$$

Now

$$\begin{aligned} \frac{1}{|X_n - X_{n'}|} &= \begin{cases} \frac{1}{(n-n')a} \left(1 + \frac{x_n - x_{n'}}{(n-n')a}\right)^{-1} & \text{if } n > n'; \\ \frac{1}{(n'-n)a} \left(1 + \frac{x_{n'} - x_n}{(n'-n)a}\right)^{-1} & \text{if } n' > n; \end{cases} \\ &= \frac{1}{|n - n'|a} \left\{ 1 - \frac{x_n - x_{n'}}{(n - n')a} + \left[ \frac{x_n - x_{n'}}{(n - n')a} \right]^2 + \dots \right\}. \end{aligned}$$

The term linear in the displacements  $x_n$  vanishes, since

$$-\frac{1}{2} \sum_{n \neq n'} \frac{x_n - x_{n'}}{(n - n')|n - n'|a^2} - a^{-2} \sum_n x_n \sum_{\substack{n' \\ (n' \neq n)}} \frac{\text{sgn}(n - n')}{(n - n')^2} = 0$$

Thus, the potential energy can be written

$$\begin{aligned} V &= \frac{1}{2} \sum_n \sum_l K_l (x_n - x_{n+l})^2 \\ K_l &= \frac{q^2}{|l|^3 a^3}. \end{aligned}$$

(b) The equations of motion can be diagonalized by Fourier transform:

$$\begin{aligned} x_n &= \frac{1}{\sqrt{N}} \sum_k \hat{x}(k) e^{ikan} \\ \hat{x}(k) &= \frac{1}{\sqrt{N}} \sum_n x_n e^{-ikan} \end{aligned}$$

where  $-(\pi/2a) \leq k < (\pi/2a)$ . Then

$$\begin{aligned} V &= \frac{1}{2N} \sum_{n,l} \sum_{k,k'} K_l e^{ikan} e^{-ik'an} \left[ 1 - e^{-ik'al} - e^{ikal} + e^{ikal} e^{-ik'al} \right] \hat{x}(k) \hat{x}(-k') \\ &= \frac{1}{2} \sum_k \hat{K}(k) \hat{x}(k) \hat{x}(-k) \end{aligned}$$

with

$$\hat{K}(k) = 4 \sum_l K_l \sin^2(\frac{1}{2}kal)$$

The kinetic energy is

$$T = \frac{1}{2} m \sum_n \dot{x}_n^2 = \frac{1}{2} m \sum_k \dot{\hat{x}}(k) \dot{\hat{x}}(-k)$$

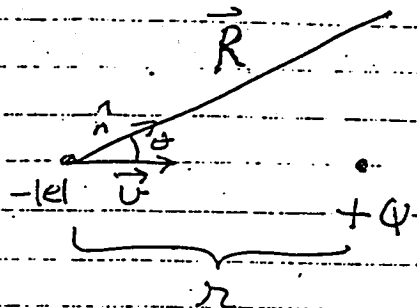
and so we can read off  $\omega(k) = \sqrt{\hat{K}(k)/m}$ . The general expression is

$$\omega^2(k) = \frac{8q^2}{ma^3} \sum_{l=1}^{\infty} \frac{\sin^2(\frac{1}{2}kal)}{l^3}.$$

Note to aficionados: One finds that  $\omega(k) \propto k\sqrt{-\ln(ka)}$  as  $k \rightarrow 0$ , i.e. there are logarithmic corrections to the usual acoustic ( $\omega \propto k$ ) dispersion due to the long-range nature of the Coulomb potential.

II-2 Solution(a) In the limit  $v \ll c$ 

$$\vec{E}_r = \frac{e}{c^2} \left[ \frac{\hat{n} \times (\hat{n} \times \vec{v})}{R} \right]$$



$$\vec{B}_r = \hat{n} \times \vec{E}$$

$$S = \frac{c}{4\pi} |\vec{E} \times \vec{B}| = \frac{c}{4\pi} |E|^2 = \frac{e^2 \sin^2 \theta \dot{v}^2}{4\pi c^3 R^2}$$

$$\frac{dP}{d\Omega} = \frac{e^2 \sin^2 \theta \dot{v}^2}{4\pi c^3}$$

$$\int \sin^2 \theta d\Omega = 2\pi \int_0^\pi (1 - \cos^2 \theta) \sin \theta d\theta = \frac{8\pi}{3}$$

$$P_{rad} = \frac{2}{3} \frac{e^2}{c^3} \dot{v}^2$$

$$\ddot{v} = \frac{-|e|\hbar}{m r^2}$$

$$\Rightarrow P_r = \frac{2 e^4 \hbar^2}{3 m^2 c^3 r^4}$$

## II-2 (cont'd)

$$P_r = \frac{dE_r}{dt}$$

$$\frac{dE_r}{dr} = \frac{dE_r}{dt} \left( \frac{1}{\frac{dr}{dt}} \right)$$

$$\frac{1}{2} m \left( \frac{dr}{dt} \right)^2 \approx e Q \left[ \frac{1}{r} - \frac{1}{r_0} \right]$$

Thus

$$E_r = \frac{2e^4 Q^2}{3m^2 c^3} \int_{r_0}^r \frac{dr}{r^4 \left[ \frac{2eQ}{m} \left( \frac{1}{r} - \frac{1}{r_0} \right) \right]^{1/2}}$$

Solution 10

(a) The wave equation is

$$\rho \frac{\partial^2 u}{\partial t^2} + C \nabla^2 \nabla^2 u = 0$$

so substituting  $u(\mathbf{r}, t) = u_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$  gives  $-\rho \omega^2 + C k^4 = 0$ , i.e.

$$\omega = \sqrt{\frac{C}{\rho}} k^2$$

where  $k = |\mathbf{k}|$ .

(b) The density of states is given by

$$g(\omega) d\omega = \frac{L^2}{4\pi^2} 2\pi k dk = \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}} d\omega$$

which gives

$$g(\omega) = \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}},$$

a constant.

(c) The mean energy is

$$E(T) = \int_0^\infty d\omega g(\omega) \hbar \omega \left\{ n(\omega) + \frac{1}{2} \right\}$$

where  $n(\omega) = (e^{\hbar\omega/k_B T} - 1)^{-1}$  is the Bose occupancy factor. Note that  $E(T)$  is infinite! This is because of the zero point energy. (If we had imposed an ultraviolet (short wavelength) cutoff on the density of states, as in the Debye model, this wouldn't have happened.) However, the infinite zero point energy is temperature independent and doesn't affect the specific heat. The  $T$ -dependent part to  $E(T)$  is

$$\begin{aligned} E(T) - E_0 &= \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}} \int_0^\infty d\omega \hbar \omega \frac{1}{e^{\hbar\omega/k_B T} - 1} \\ &= \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}} \frac{(k_B T)^2}{\hbar} \int_0^\infty dx \frac{x}{e^x - 1} \\ &= \frac{\pi^6}{6} \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}} \frac{(k_B T)^2}{\hbar} \end{aligned}$$

where the integral gives the numerical constant  $\pi^2/6$ . The specific heat is then linear in  $T$ :

$$C(T) = \frac{\partial E}{\partial T} = \frac{\pi^6}{6} \frac{L^2}{4\pi} \sqrt{\frac{\rho}{C}} \frac{2k_B^2 T}{\hbar}$$

a.) In the sudden approximation

$$A = \text{amplitude} = \int \left( \psi_{200}^{z+1} \right)^* \psi_{100}^z d^3r$$

$$\text{Probability} = P = |A|^2$$

$$A = \frac{1}{4\pi} 4 \left( \frac{z}{a_0} \right)^{3/2} \left( \frac{z+1}{2a_0} \right)^{3/2} 4\pi \int_0^\infty \left( 1 - \frac{z+1}{2a_0} r \right) r^2 e^{-\frac{z}{a_0} r - \frac{z+1}{2a_0} r} dr$$

$$\text{let } x = \frac{3z+1}{2a_0} r$$

$$\int_0^\infty r^2 e^{-\frac{3z+1}{2a_0} r} dr = 2 \left( \frac{2a_0}{3z+1} \right)^3$$

$$\int_0^\infty r^3 e^{-\frac{3z+1}{2a_0} r} dr = \left( \frac{2a_0}{3z+1} \right)^4 \underbrace{\int_0^\infty x^3 e^{-x} dx}_{3! = 6}$$

$$\Rightarrow A = 4 \left( \frac{z(z+1)}{a_0^2} \right)^{3/2} \frac{1}{2^{3/2}} \left\{ 2 \left( \frac{2a_0}{3z+1} \right)^3 - 6 \frac{z+1}{2a_0} \left( \frac{2a_0}{3z+1} \right)^4 \right\}$$

$$A = \frac{2^3}{2^{3/2}} \frac{z^{3/2} (z+1)^{3/2}}{(3z+1)^3} \left\{ 2^3 - \frac{3(z+1)2^4}{2(3z+1)} \right\}$$

$$A = \frac{2^6}{2^{3/2}} \frac{z^{3/2} (z+1)^{3/2}}{(3z+1)^3} \left\{ 1 - \frac{3(z+1)}{(3z+1)} \right\} = -\frac{2^7}{2^{3/2}} \frac{z^{3/2} (z+1)^{3/2}}{(3z+1)^4}$$



4.) continued...

Part II solutions

SP93

$$\text{or } A \approx -4.03 \times 10^{-2} \text{ for } Z=14$$

$$\Rightarrow P \approx 1.6 \times 10^{-3}$$

b.) the transition amplitude is zero <sup>in</sup>  
~~the sudden approx~~ as the angular  
wave functions are orthogonal.

[12] - (a) We have  $H = -\vec{\mu} \cdot \vec{B}$ , i.e.

$$H = -\gamma \vec{B} \cdot \vec{\sigma}$$

Now if  $\hat{B} = \hat{z}$ , the eigenstates are  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  with eigenvalues  $\mp \gamma B$ , respectively. We write

$$\hat{B} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$

$$\begin{aligned} \hat{B} \cdot \vec{\sigma} &= \sin \theta \cos \varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin \theta \sin \varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix} \end{aligned}$$

The eigenvalues of  $\hat{B} \cdot \vec{\sigma}$  are  $\pm 1$  (the determinant is  $-1$  and we know  $\hat{B} \cdot \vec{\sigma}$  is just a rotation of  $\sigma^z$ ). So for the  $\lambda = +1$  eigenvalue,

$$|\psi_+\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\alpha \cos \theta + \beta \sin \theta e^{-i\varphi} = \alpha$$

$$\alpha \sin \theta e^{i\varphi} + \beta \cos \theta = \beta \Rightarrow \beta = \alpha \frac{\sin \theta e^{i\varphi}}{1 - \cos \theta}$$

Thus, since  $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$  and  $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$ , we have

$$\beta = \alpha \cotn \frac{\theta}{2} e^{i\varphi}$$

The normalized eigenstates are then

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\hat{B}} : |\psi_+\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{+i\varphi/2} \\ \sin \frac{\theta}{2} e^{-i\varphi/2} \end{pmatrix} \quad E_+ = -\gamma B$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\hat{B}} : |\psi_-\rangle = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\varphi/2} \\ \cos \frac{\theta}{2} e^{+i\varphi/2} \end{pmatrix} \quad E_- = +\gamma B$$

(overall phases arbitrary)

(b) The state  $|\hat{x}\rangle$  is obtained by setting  $\theta = \frac{\pi}{2}$ ,  $\varphi = 0$ . Thus,

$$|\hat{x}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\langle \hat{x} | \psi_+ \rangle = \frac{1}{\sqrt{2}} \left( \cos \frac{\theta}{2} e^{i\varphi/2} + \sin \frac{\theta}{2} e^{-i\varphi/2} \right)$$

$$P_+ = |\langle \hat{x} | \psi_+ \rangle|^2 = \frac{1}{2} \left[ \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \varphi \right]$$

$$= \frac{1}{2} [1 + \sin \theta \cos \varphi]$$

$$\langle \hat{x} | \psi_- \rangle = \frac{1}{\sqrt{2}} \left( -\sin \frac{\theta}{2} e^{-i\varphi/2} + \cos \frac{\theta}{2} e^{i\varphi/2} \right)$$

$$P_- = |\langle \hat{x} | \psi_- \rangle|^2 = \frac{1}{2} \left( \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \varphi \right)$$

$$= \frac{1}{2} [1 - \sin \theta \cos \varphi]$$

Note  $P_+ + P_- = 1$ .

(c) Time evolution:  $U = e^{-iHt/\hbar} = e^{+i\gamma \vec{\sigma} \cdot \vec{B}t/\hbar}$ . Thus

$$U = e^{+i\gamma Bt \hat{B} \cdot \vec{\sigma}/\hbar}$$

The rotation operator is

$$R(\hat{n}, \theta) = e^{i\theta \hat{n} \cdot \vec{\sigma}/\hbar} = e^{i\frac{\theta}{2} \hat{n} \cdot \vec{\sigma}} \quad (S = \frac{1}{2})$$

so we see  $U = R(\hat{B}, \theta = +2\gamma Bt/\hbar)$ , and after a time  $t$  we have rotated by an angle  $+2\gamma Bt/\hbar$  about  $\hat{B}$ . The period of the rotation is then given by  $2\gamma B\tau/\hbar = 2\pi \Rightarrow \tau = \pi\hbar/\gamma B$ .

To come to  $\hat{y}$  we need to rotate for  $\frac{1}{4}$  period, i.e.  $t = \frac{1}{4}\tau = \frac{1}{4} \frac{\pi\hbar}{\gamma B}$ . Note that  $t = (n + \frac{1}{4})\tau$  also works, where  $n$  is any integer.

(d) We have

$$\begin{aligned}\sum_{i < j} \vec{S}_i \cdot \vec{S}_j &= \frac{1}{2} \left\{ \left( \sum_{i=1}^4 \vec{S}_i \right)^2 - \sum_{i=1}^4 \vec{S}_i^2 \right\} \\ &= \frac{1}{2} \left\{ \vec{Q}^2 - 4S(S+1)\hbar^2 \right\} \\ &= \frac{1}{2} (\vec{Q}^2 - 3\hbar^2) \quad \left[ \vec{Q} = \sum_{i=1}^4 \vec{S}_i = \text{total spin operator} \right]\end{aligned}$$

for  $S = \frac{1}{2}$ . So

$$\begin{aligned}H &= \frac{J}{2} (\vec{Q}^2 - 3\hbar^2) \\ &= \frac{J}{2} [Q(Q+1) - 3] \hbar^2 \quad \vec{S}^2 = \hbar^2 S(S+1)\end{aligned}$$

Here,  $Q$  is the total spin. We have

$$\begin{aligned}\frac{1}{2} \otimes \frac{1}{2} &= 0 \oplus 1 \\ \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= \left[ \frac{1}{2} \otimes 0 \right] \oplus \left[ \frac{1}{2} \otimes 1 \right] = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \\ \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= \left[ \frac{1}{2} \otimes \frac{1}{2} \right] \oplus \left[ \frac{1}{2} \otimes \frac{1}{2} \right] \oplus \left[ \frac{1}{2} \otimes \frac{3}{2} \right] \\ &= 0 \oplus 1 \oplus 0 \oplus 1 \oplus 1 \oplus 2 \\ &= 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2\end{aligned}$$

I.e. we have two singlets, three triplets, and one quintuplet ( $Q=0,1,2$  respectively). Thus

$Q$	$Q(Q+1)$	$E$	multiplicity
0	0	$-\frac{3}{2}J\hbar^2$	$2 \cdot 1 = 2$
1	2	$-\frac{1}{2}J\hbar^2$	$3 \cdot 3 = 9$
2	6	$+\frac{3}{2}J\hbar^2$	$1 \cdot 5 = 5$
			total = 16 = $2^4$ ✓

Solution 13

We have

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots$$

$$\ln \cosh(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \dots$$

so

$$\begin{aligned} (\cosh x)^{-N} &= e^{-N \ln \cosh x} \\ &= e^{-\frac{1}{2}Nx^2 + \frac{1}{12}Nx^4 + \dots} \\ &= e^{-\frac{1}{2}Nx^2} \left\{ 1 + \frac{1}{12}Nx^4 + \dots \right\} \end{aligned}$$

where we have used

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots$$

Thus,

$$I(N) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}Nx^2} \left\{ 1 + \frac{1}{12}Nx^4 + \dots \right\}$$

We have that the integral

$$J(\lambda) \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\pi} \lambda^{-1/2}$$

and so we can also evaluate

$$\int_{-\infty}^{\infty} dx x^{2k} e^{-\lambda x^2} = \left( -\frac{\partial}{\partial \lambda} \right)^k J(\lambda).$$

This gives

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}Nx^2} &= \sqrt{2\pi} N^{-1/2} \\ \int_{-\infty}^{\infty} dx x^4 e^{-\frac{1}{2}Nx^2} &= \frac{3}{4}\sqrt{\pi} \left( \frac{1}{2}N \right)^{-5/2} \end{aligned}$$

and finally

$$I(N) = \sqrt{\frac{2\pi}{N}} \left\{ 1 + \frac{1}{4N} + \mathcal{O}(N^{-2}) \right\}.$$

$$(a) \quad \text{Let } \vec{A} = x B_0 \hat{y} \quad \phi = -E_0 x$$

$$L = -mc^2 \sqrt{1 - v^2/c^2} - \frac{e}{c} B_0 x \dot{y} - e E_0 x$$

$$P_x = m \dot{x}, \quad P_y = m \dot{y} - \frac{e B_0}{c} x, \quad P_z = m \dot{z}$$

$$H = \sqrt{m^2 c^4 + c^2 P_x^2 + c^2 P_z^2 + c^2 \left( P_y + \frac{e B_0}{c} x \right)^2} + e E_0 x$$

$$\text{const.} = P_x, \quad \text{const.} = P_y, \quad \text{const.} = H$$

$$(b) \quad P_z = 0, \quad P_y = 0, \quad H = mc^2$$

$$mc^2 = \sqrt{m^2 c^4 + e^2 B_0^2 x^2 + P_x^2 c^2} + e E_0 x$$

$$\cancel{m^2 c^4} - 2mc^2 e E_0 x + (e E_0 x)^2 = \cancel{m^2 c^4} + (e B_0 x)^2 + P_x^2 c^2$$

$$2mc^2 e E_0 (-x) + e^2 (E_0^2 - B_0^2) x^2 = c^2 P_x^2$$

$$P_x^2 > 0 \quad (\text{no turning point}) \quad \text{for } E_0 \geq B_0$$

$$x < 0$$

(c) For  $E_0 = E_0$

$$2mc^2 e E_0 |x| = c^2 p_x^2 = c^2 \hbar^2 m \left( \frac{d|x|}{dt} \right)^2$$

$$mc^2 - e E_0 |x| = H = mc^2$$

$$V(x) = mc^2 - e E_0 |x|$$

$$t = \int_0^x \frac{dx}{v} = \int_0^x \frac{dx}{\frac{c}{\sqrt{mc^2 - e E_0 |x|}}}$$

$$t = \frac{mc^2}{\sqrt{mc^2 - e E_0} c} \sqrt{2|x|} + \frac{e E_0}{mc^2} \frac{2}{3} |x|^{3/2}$$

$$\sqrt{E} = \frac{e E_0 \hbar^2}{mc^2} = \left[ 2 \left( \frac{e E_0 \hbar^2}{mc^2} \right)^{1/2} + \frac{2}{3} \left( \frac{e E_0 \hbar^2}{mc^2} \right)^{3/2} \right]$$