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UNIVERSITY OF CALIFORNIA, SAN DIEGO

DEPARTMENT OF PHYSICS

Written Departmental Examination – Fall 1993, Part I

[1] For each of the following energy quantities, give an expression in terms of fundamental constants which is dimensionally correct (numerical prefactors are not required) and compute from your expression an order-of-magnitude estimate in electron Volts.

Potentially useful information:

- Rydberg: $1 \text{ Ry} \approx 13.6 \text{ eV}$
- Bohr Magnetron: $\mu_B \approx 6 \times 10^{-5} \text{ eV/T}$
- Fine Structure Constant: $\alpha \approx 1/137$

Example: The electron kinetic energy in the ground state of the Hydrogen atom.

Solution: If you are stuck on a desert island and cannot remember the expression, here is a quick way to reconstruct it. Denote the size of the atom by a_B . The kinetic energy is then $T \sim \hbar^2/ma_B^2$, which, when approximately equated with the potential energy e^2/a_B gives $a_B = \hbar^2/me^2$. The kinetic energy is then $T \sim me^4/\hbar^2 = 1 \text{ Hartree} \sim 10 \text{ eV}$. (Note that one Hartree is 27.2 eV, but the order-of-magnitude estimate of 10 eV is adequate for this problem.)

(a) The ground state binding energy of the electron in a U^{91-} ion (a Uranium ion with atomic number 92, atomic weight 238, which is 91-times ionized so that there is only one electron present).

(b) The ground state energy level splitting in a Hydrogen atom due to the spin-orbit interaction.

(c) The hyperfine splitting in a Hydrogen atom (the splitting of electronic energy levels due to the interaction with the magnetic moment of the proton).

- (d) The relativistic correction to the binding energy of a Hydrogen atom.
- (e) The transition energy between adjacent Landau levels of a non-relativistic electron in a magnetic field of 1 Tesla.
- (f) The energy of a nucleon in a typical nucleus.
- (g) The rotational energy of an H_2 molecule.

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[2] (a) A classical particle of mass m moves in a circular orbit under the influence of a central force whose potential is $V(r) = -mk/r^n$, where k is a constant. What is the condition on n such that the orbit is stable under small perturbations of the radius?

(b) A classical particle of mass m moves in one dimension and is subjected to a constant force $F = F_0$ when $x < 0$ and $F = -F_0$ when $x > 0$. F_0 is a positive constant. Assume initial conditions $x(0) = a$ and $\dot{x}(0) = 0$. Construct and sketch the phase space orbit of the motion, $\dot{x}(x)$, and calculate the period of the motion.

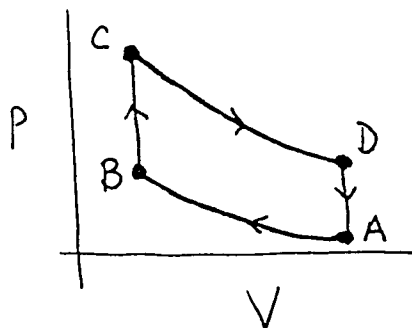
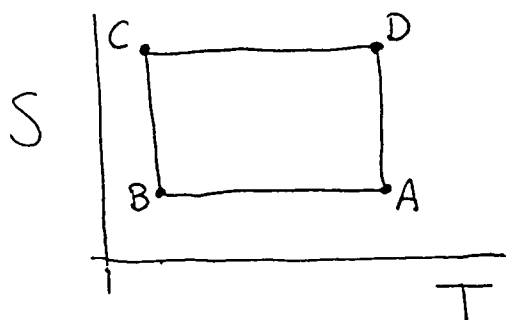
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[3] The Otto cycle, depicted below in both entropy *vs.* temperature and a conventional p - V plot, is a rough approximation to the thermodynamics of a gasoline engine. During the process $A \rightarrow B$, an ideal gas inside a cylinder is adiabatically compressed. This is followed by $B \rightarrow C$, during which the gas is heated at constant volume (an “isochoric” process). Then comes $C \rightarrow D$ – the power stroke – during which the gas expands adiabatically. Finally, $D \rightarrow A$ represents an isochoric cooling. (In a real engine, the isochoric heating results from combustion, and so the burned gas must be ejected from the chamber and new gas inserted, which we model here by the process $D \rightarrow A$.) The efficiency of this engine is defined by the ratio $\eta = W/Q_{B \rightarrow C}$ of the work done per cycle to the heat absorbed during the isochoric heating process. Compute the efficiency of the Otto cycle in terms of the volumes V_A and V_B , the specific heats C_V and C_p , and any other relevant quantities.



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[4] An insulating circular ring of radius R lies in the x - y plane, centered at the origin. It carries a linear charge density $\lambda(\phi) = \lambda_0 \sin \phi$, where λ_0 is a constant and ϕ is the usual azimuthal angle, measured from the x -axis. At time $t = 0$, the ring is set spinning about the z -axis with constant angular velocity ω .

(a) Is this an electric dipole, a magnetic dipole, or something else? Give an answer and calculate the appropriate multipole moment at $t = 0$.

(b) What is the polarization (linear or circular) of outgoing waves in the far field region along each of the coordinate axes? In the case of linear polarized waves, give the direction of the E vector.

(c) What is the total radiated power?

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[5] Consider a quantum mechanical system with three possible orthonormal states (“colors” red, blue, and green). Any wavefunction $|\psi\rangle$ may be written as a linear combination of the three basis states $|R\rangle$, $|B\rangle$, and $|G\rangle$. The system is described by the Hamiltonian

$$H = E_0(2|R\rangle\langle R| + 2|B\rangle\langle B| + 2|G\rangle\langle G| - |G\rangle\langle B| - |B\rangle\langle G|)$$

$$\equiv E_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

- (a) Find the normalized energy eigenstates and their corresponding eigenvalues.
- (b) At time $t = 0$ the wavefunction is $|\psi(0)\rangle = |G\rangle$. Find the wave function $|\psi(t)\rangle$ at an arbitrary time t .
- (c) The “color” is measured at time $t = t_0$ and found to be green. What are the respective probabilities for the color to be measured as red, green, or blue at time $t = 2t_0$? Assume that the initial conditions of part (b) hold at $t = 0$.

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[6] In the quantum mechanical calculation of a transition probability, we encounter the integral

$$I(t) = \int_{-\infty}^{\infty} d\omega f(\omega, t)$$

where

$$f(\omega, t) = \frac{2(1 - \cos \omega t)}{\omega^2} .$$

Compute $I(t)$.

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[7] A magnet consists of $N \gg 1$ noninteracting spins which can be in any of three states: $\sigma_z = -1, 0, +1$. Each of these spins is described by the Hamiltonian

$$\mathcal{H} = E_0 \sigma_z^2 - \mu_0 H \sigma_z ,$$

where H is the magnetic field.

- (a) Compute the partition function for this system.
- (b) Compute the magnetization $M(T, H)$ and the zero-field magnetic susceptibility $\chi(T)$.
- (c) Compute the entropy per spin in the limit $T \rightarrow \infty$.
- (d) Does the specific heat per particle vanish or tend to a constant as $T \rightarrow \infty$? Explain your answer.

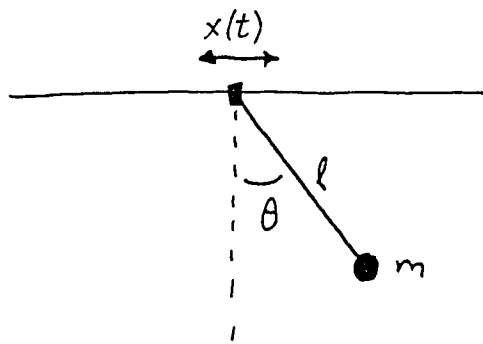
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[8] The hinge of a pendulum of length l and mass m is oscillated in the x -direction according to $x(t) = a \cos \Omega t$, where $\Omega^2 \gg g/l$ and $a \ll l$. Separating the motion into slow and fast degrees of freedom, find the effective potential for the slow oscillations and determine the conditions under which an equilibrium exists for $\theta \neq 0$.



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[9] At perigee of an elliptical gravitational orbit, a satellite experiences an outward radial impulse of magnitude S . Recall that an impulse results in an instantaneous change in momentum $\Delta\vec{p} = \vec{S}$. The satellite's mass, m , is negligible in comparison with that of the earth. The initial eccentricity of the satellite's orbit is ϵ_i , and the initial semimajor axis length is a_i .

(a) Compute the changes ΔE and $\Delta\vec{L}$ of the satellite's energy and angular momentum due to the impulse.

(b) Compute the new eccentricity ϵ_f and the new semimajor axis length a_f in terms of ϵ_i , a_i , S , and other relevant constants. You may assume that the orbit remains bound. Does the distance to perigee increase or decrease in the new orbit?

(c) Under what conditions will the assumption that the satellite remains in a bound orbit be satisfied?

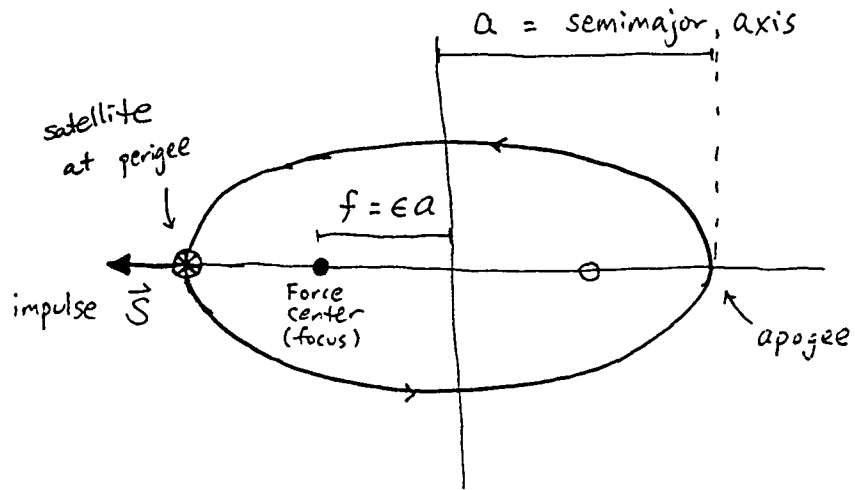
(d) What is the angular orientation of the new major axis relative to the initial major axis?

Reminders: For an attractive central potential $V(\vec{r}) = -K/r$, the geometric equation of the orbit is

$$r(1 - \epsilon \cos \phi) = \vec{L}^2 / \mu K$$

where r and ϕ describe the distance and angle relative to the force center in the plane perpendicular to the angular momentum \vec{L} , ϵ is the orbit's eccentricity, and μ is the reduced mass. This equation can be obtained immediately from the invariance of the Laplace-Runge-Lenz vector: $d\vec{A}/dt = 0$ where $\vec{A} = \vec{p} \times \vec{L} - \mu K \vec{r}/r$. One also finds $|\vec{A}| = \mu K \epsilon$.

(9) Continued:



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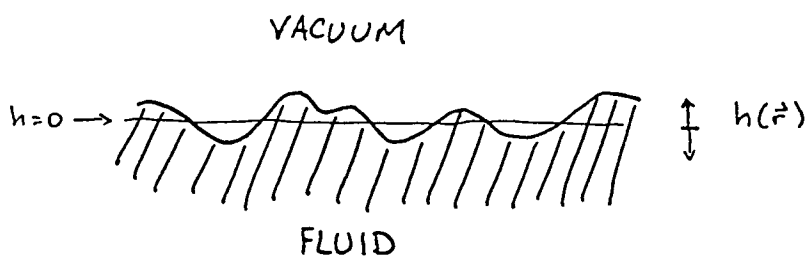
Written Departmental Examination – Fall 1993, Part II

[10] Consider a planar interface between a conducting fluid of density ρ and vacuum. The interface supports a uniform surface charge density, σ , and is subject to a surface tension γ . Suppose we deform the surface so that the height function $h(\mathbf{r})$, is nonuniform, and instead is described by

$$h(\mathbf{r}) = \int \frac{d^2k}{(2\pi)^2} \hat{h}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}$$

where $\hat{h}(\mathbf{k}) \ll 1$ and $\hat{h}(0) = 0$.

- Find an expression for the gravitational energy of the deformed interface.
- Find an expression for the energy due to surface tension of the deformed interface.
- Find an expression for the electrostatic energy of the deformed interface.
- Find a criterion for the surface to remain at small deformation.



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[11] A quantum mechanical particle of charge e and mass m is in the first excited state of a three-dimensional isotropic harmonic oscillator potential of frequency ω_0 . The wavefunction is given by

$$|\psi\rangle = (\cos \alpha a_x^\dagger + \sin \alpha a_y^\dagger) |0\rangle ,$$

where the ladder operators are defined by

$$a_x = \sqrt{\frac{m\omega_0}{2\hbar}} x + \sqrt{\frac{1}{2m\hbar\omega_0}} ip_x$$

$$a_x^\dagger = \sqrt{\frac{m\omega_0}{2\hbar}} x - \sqrt{\frac{1}{2m\hbar\omega_0}} ip_x ,$$

with corresponding expressions holding for oscillator quanta along the \hat{y} and \hat{z} axes. When interaction with the quantized radiation field are taken into account, compute in the dipole approximation the angular distribution $d\Gamma/d\Omega$ of emitted right-handed circularly polarized photons.

Reminder: The quantized radiation field $\vec{A}(\vec{r}, t)$ is written

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \lambda} \sqrt{\frac{2\pi\hbar c}{k\mathcal{V}}} \hat{\epsilon}_{\vec{k}, \lambda} (b_{\vec{k}, \lambda} e^{i\vec{k}\cdot\vec{r}} + b_{\vec{k}, \lambda}^\dagger e^{-i\vec{k}\cdot\vec{r}})$$

where $\hat{\epsilon}_{\vec{k}, \lambda}$ ($\lambda = 1, 2$) are unit polarization vectors orthogonal to \vec{k} , $b_{\vec{k}, \lambda}^\dagger$ is the creation operator for radiation quanta with wavevector \vec{k} and polarization index λ , and \mathcal{V} is the volume of the universe (the radiation field is quantized in a large box of volume \mathcal{V}).

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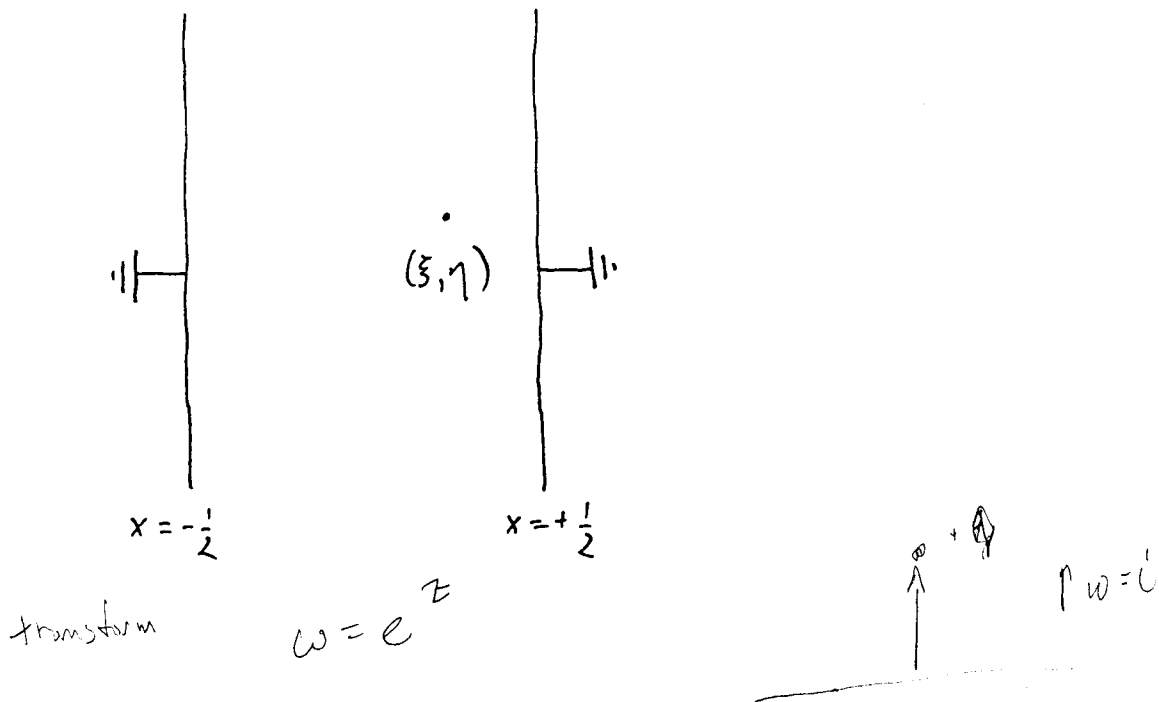
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[12] Consider a infinitely long line charge of unit charge density between two parallel, infinite, grounded conducting planes. The electrostatic potential $\phi(x, y)$ satisfies the equation

$$\nabla^2 \phi(x, y) = -\delta(x - \xi) \delta(y - \eta) \quad ;$$

along with the boundary conditions $\phi(x = \pm \frac{1}{2}, y) = 0$ and $\lim_{y \rightarrow \pm \infty} \phi(x, y) = 0$, using suitable units. Find $\phi(x, y)$. If you express your solution as an infinite series, be sure to discuss its rate of convergence.



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[13] The surface modes of vibration of a liquid have a frequency-dependent phase velocity approximately given by $v(\omega) = (\sigma\omega/\rho)^{1/3}$, where σ is the surface tension and ρ the density.

- (a) Compute the quantum mechanical energy-momentum dispersion relation $\epsilon(p)$.
- (b) Derive the low-temperature form of the specific heat of the 'gas' of surface modes.

SOLUTION: PROBLEM 1

F93

Solutions

1. Expressions and order-of-magnitude estimates for the energies.

- (a) The binding energy of the electron in its ground state in a uranium ion (atomic number = 92 and atomic weight = 238) which contains no other electrons.

Solution — Using the scaling $e^2 \rightarrow Ze^2$, with $Z = 92$ in the expression in the example, we obtain

$$E \sim \frac{mZ^2e^4}{\hbar^2} \sim 10^4 \text{ hartrees} \sim 10^5 \text{ eV.}$$

- (b) The energy level splitting in a hydrogen atom due to the spin-orbit interaction.

Solution — It is the energy of the electron spin dipole in the magnetic field due to its orbital motion

$$\Delta E \sim \mu_B \frac{vE}{c} \sim \frac{e\hbar}{m_e c} \frac{pe}{m_e c a_B^2} \sim \frac{Ryd^2}{m_e c^2} \sim \alpha^2 Ryd \sim 10^{-3} \text{ eV.}$$

- (c) The hyperfine splitting in a hydrogen atom.

Solution — It is reduced by a factor of the mass ratio from the fine structure

$$\Delta E \sim \frac{m_e}{m_p} \alpha^2 Ryd \sim 10^{-6} \text{ eV.}$$

- (d) The relativistic mass correction of the binding energy of a hydrogen atom.

Solution — Using one definition of the fine structure constant,

$$\Delta E \sim \left(\frac{p^2}{2m_e} \right)^2 / (m_e c^2) \sim \alpha^2 Ryd \sim 10^{-3} \text{ eV.}$$

- (e) The Zeeman splitting of electron levels in a hydrogen atom when the magnetic field is 1 T (Tesla).

Solution — Zeeman energy in the magnetic field B is

$$\Delta E \sim \mu_B B \sim \frac{e\hbar B}{m_e c} \sim 10^{-4} \text{ eV.}$$

(f) The energy of a nucleon in a typical nucleus.

Solution — In a nucleus of radius $r_N \sim 1 \text{ fm}$,

$$E_N \sim \frac{\hbar^2}{m_p r_N^2} = \frac{m_e a_B^2}{m_p r_N^2} \sim 10^7 \text{ eV.}$$

(g) The rotational energy of a hydrogen molecule.

Solution —

$$E_{\text{rot}} \sim \frac{\hbar^2}{m_H r^2} \sim \frac{m_e}{m_H} \text{ Ryd} \sim 10^{-2} \text{ eV.}$$

SOLUTION: PROBLEM 2

F93

Problem: A particle of mass m moves in a circular orb. under the influence of a central force whose potential is $-kcm/r^m$. What is the condition on m such that the orb. is stable under small perturbations of the radius?

Answer: The effective potential of a particle moving under the influence of a central force is the sum of the central force potential and the potential of the centrifugal force,

$$\text{i.e. } V_{\text{eff}} = -\frac{kcm}{r^m} + \frac{cm\omega^2 r^2}{2} = -\frac{kcm}{r^m} + \frac{L^2}{2mr^2}$$

where $L = \text{angular momentum} = mr^2\omega$. A circular orbit can exist for that r for which $\partial V_{\text{eff}}/\partial r = 0$, and the orbit is stable if $\partial^2 V_{\text{eff}}/\partial r^2 > 0$. Thus,

$$\partial V_{\text{eff}}/\partial r = 0 = \frac{kcm}{r^{(m+1)}} - \frac{L^2}{mr^3} \quad (1), \text{ and}$$

$$\partial^2 V_{\text{eff}}/\partial r^2 = -\frac{kcm(m+1)}{r^{(m+2)}} + \frac{3L^2}{mr^4} > 0 \quad (2).$$

Substituting (1) in (2) gives

$$-\frac{L^2(m+1)}{mr^4} + \frac{3L^2}{mr^4} > 0, \text{ or } -(m+1) + 3 > 0$$

$$\Rightarrow m < 2$$

A mass m moves in one dimension subject to a constant force $+F_0$ when $x < 0$ and to a constant force $-F_0$ when $x > 0$. Assume $x(0) = A$ and $\dot{x}(0) = 0$

- Construct the phase space orbit of the motion
- Calculate the period of the motion

a) For $x > 0$:

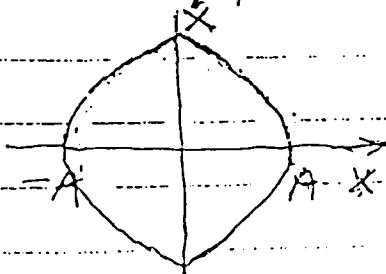
$$m \ddot{x} = -F_0$$

$$x(t) = A - \frac{F_0}{2m} t^2 \quad ; \quad \dot{x}(t) = -\frac{F_0}{m} t$$

and

$$\dot{x}(x) = \sqrt{\frac{2F_0}{m} (A-x)}$$

The orbit is a parabola with vertex on x axis at $x=A$ and symmetric about x and \dot{x} axes:



2 (b) cont'd

for $x \geq 0$ and $x = a$ at $t = 0$

$$F = -F_0$$

$$v = -\frac{F_0}{m}t$$

$$x = a - \frac{1}{2}\frac{F_0}{m}t^2$$

$$\text{Thus } \frac{1}{4}\tau = \left(\frac{2am}{F_0}\right)^{\frac{1}{2}}$$

$$\tau = \left(\frac{32am}{F_0}\right)^{\frac{1}{2}}$$

2

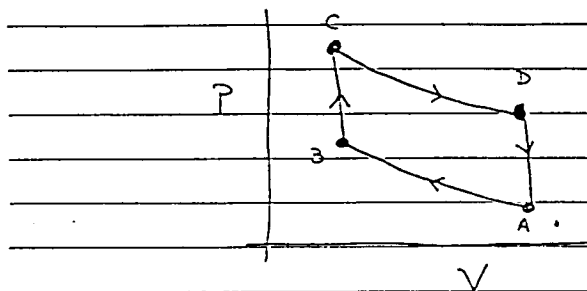
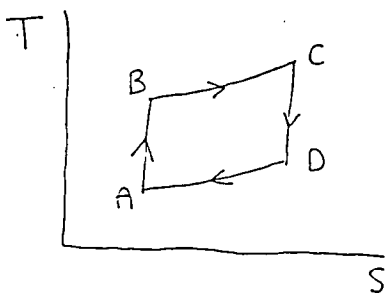
SOLUTION: PROBLEM 3

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~~Other common thermodynamic cycles -~~

3
[13]

The Otto Cycle:



This is a rough approximation to a gasoline engine:

A → B : adiabatic compression

B → C : isochoric (constant V) heating (combustion) $V_C = V_B$

C → D : adiabatic expansion (power stroke)

D → A : isochoric cooling (actually, burned gas ejected, new gas inserted) $V_D = V_A$

Efficiency: $\eta \equiv \frac{W}{Q_{B \rightarrow C}}$

$$Q_{B \rightarrow C} = \int_B^C C_v dT = \int_B^C C_v \left(\frac{\partial T}{\partial p} \right)_v dp + C_v \left(\frac{\partial T}{\partial V} \right)_p dV$$

$$pV = NkT \Rightarrow \left(\frac{\partial T}{\partial p} \right)_v = \frac{V}{Nk} \Rightarrow \boxed{Q_{B \rightarrow C} = C_v \frac{V_B}{Nk} (P_C - P_B)}$$

$$\text{Work: } W = \int_C^D p dV + \int_A^B p dV$$

Prob. 43

$$W_{CD} = \int_c^D p dV \quad ; \quad pV^\gamma = p_c V_c^\gamma = p_c V_B^\gamma$$

$$= p_c V_B^\gamma \int_c^D dV V^{-\gamma} = p_c V_B^\gamma \int_{(V_c/V_B)}^{(V_A/V_B)} dx x^{-\gamma} = p_c V_B \frac{1}{1-\gamma} \left[\left(\frac{V_A}{V_B} \right)^{1-\gamma} - 1 \right]$$

$$W_{AB} = \int_A^B p dV \quad ; \quad pV^\gamma = p_B V_B^\gamma$$

$$= p_B V_B^\gamma \int_A^B dV V^{-\gamma} = p_B V_B \int_{(V_A/V_B)}^1 dx x^{-\gamma} = p_B V_B \frac{1}{1-\gamma} \left[1 - \left(\frac{V_A}{V_B} \right)^{1-\gamma} \right]$$

$$W = W_{CD} + W_{AB} = (p_B - p_c) V_B \frac{1}{1-\gamma} \left[1 - \left(\frac{V_A}{V_B} \right)^{1-\gamma} \right]$$

$$\eta = W/Q_{BC} = \frac{Nk}{C_V} \frac{1}{\gamma-1} \left[1 - \left(\frac{V_B}{V_A} \right)^{\gamma-1} \right] \quad ; \quad c_p = \gamma c_v \Rightarrow$$

$$c_p = c_v + Nk$$

$$\eta = \left[1 - \left(\frac{V_B}{V_A} \right)^{\frac{c_p}{c_v} - 1} \right]$$

SOLUTION: PROBLEM 4

F93

1) An insulating circular ring (radius R) lies in the xy plane, centered at the origin. It carries a linear charge density $\lambda = \lambda_0 \sin \phi$, where λ_0 is constant and ϕ is the usual azimuthal angle measured from the x axis. The ring is now set spinning at constant angular velocity ω about the z axis.

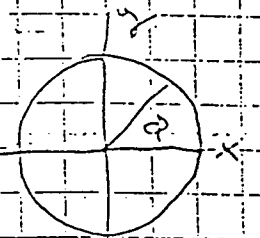
- Is this an electric dipole, magnetic dipole, or something else? Give an answer and calculate the appropriate moment at $t=0$.
- What is the polarization (linear or circular) of outgoing waves in the far field along each of the coordinate axes? In the case of linear polarization, what is the direction of the E vector (i.e. along x , y , or z)?
- What is the total radiated power?

ANSWER

a) ELECTRIC DIPOLE

$$p_y = \int y \rho d\tau = \int_0^{2\pi} R \sin \phi \lambda_0 \sin \phi R d\phi$$

$$= \pi \lambda_0 R^2 \quad p_x = p_z = 0$$



$$\vec{p} = -\omega^2 \vec{r} = -\omega^2 \pi \lambda_0 R^2 \left(-\hat{x} \sin \omega t + \hat{y} \cos \omega t \right)$$

\vec{E} is in plane of \vec{p} and \vec{r} , $\vec{E} \cdot \vec{r} = 0$

| AXIS | POLARIZATION | E VECTOR ALONG |
|------|--------------|----------------|
| x | LINEAR | y |
| y | LINEAR | x |
| z | CIRCULAR | - |

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{|\dot{p}|^2}{c^3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{p_0^2 \omega^4}{c^3}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{\pi^2 \lambda_0^2 R^4 \omega^4}{c^3}$$

5. a Hamiltonian: $(H) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} E_0$ and red, blue, green basis vectors:
 $|R\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|G\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. a) Find eigenstates, eigenvalues.

Eigenvalues \Rightarrow find λ for $|H - \lambda I| = 0$, then find eigenvectors.
 $(2E_0 - \lambda)^2 - (2E_0 - \lambda)E_0^2 = 0 = (2E_0 - \lambda)(E_0 - \lambda)(3E_0 - \lambda)$.
 so that $2E_0$, E_0 , $3E_0 \leftarrow$ eigenvalues
 ψ_1 , ψ_2 , $\psi_3 \leftarrow$ eigenvectors

Then $H\psi_i = \lambda_i \psi_i$ and (by inspection or whatever)

5 points find $\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |R\rangle$, $\psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(certain constants of form $e^{i\alpha_i}$ assigned) $= \frac{1}{\sqrt{2}} (|B\rangle \pm |G\rangle)$

b) find $\psi(t)$ when $\psi(0) = |G\rangle = \frac{1}{\sqrt{2}} (\psi_2 - \psi_3)$, let $\omega_i = E_i/\hbar$, $\omega_0 = E_0/\hbar$

$$\psi(t) = \sum_i \langle \psi_{\text{initial}} | \psi_i \rangle e^{-i\omega_i t}$$

$$\Rightarrow \psi(t) = \frac{1}{\sqrt{2}} (\psi_2 e^{-i\omega_0 t} - \psi_3 e^{-i\omega_0(3)t})$$
, but need color:

$$\psi(t) = \frac{1}{2} e^{-2i\omega_0 t} ((\psi_B + \psi_G) e^{i\omega_0 t} - (\psi_B - \psi_G) e^{-i\omega_0 t})$$

$$\psi(t) = e^{-2i\omega_0 t} (\psi_G \cos \omega_0 t + i \psi_B \sin \omega_0 t)$$

3 points

c) Such a question ($\psi(2t_0)$ if $\psi(t_0) = |G\rangle$) - \dot{U} , since $\psi(t_0)$ is the same as $\psi(0)$, the translation in time to $t = 2t_0$ will give same result as $t_0 \Rightarrow P_R = P_B = 0, P_G = 1$

(you can also use results of part b)

2 points

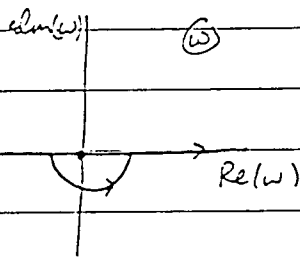
SOLUTION: PROBLEM 6

F93

Solution

$$I(t) = \int_{-\infty}^{\infty} d\omega \frac{2(1 - \cos \omega t)}{\omega^2}$$

The integrand is non-divergent along the real ω axis, including the origin.
To evaluate $I(t)$, we deform the integration contour thusly:



This is OK if we let the radius of the semicircle approach zero.

Now

$$I(t) = \int_{-\infty}^{\infty} d\omega \left[\frac{1 - e^{i\omega t}}{\omega^2} + \frac{1 - e^{-i\omega t}}{\omega^2} \right]$$

↑ close in UHP ↑ close in LHP \Rightarrow no contribution

$$= 2\pi i \operatorname{Res} \Big|_{\omega=0} \frac{1 - e^{i\omega t}}{\omega^2} = 2\pi t$$

Since $\frac{1 - e^{i\omega t}}{\omega^2} = \frac{-it}{\omega} + \dots \Rightarrow$ residue is $-it$. So $I(t) = 2\pi t$
This is fine for $t > 0$. Obviously $I(t) = I(-t)$, so

$$I(t) = 2\pi |t|$$

$$(a) \quad H = +E_0 \sigma^z - \mu_0 H \sigma$$

$$(a) \quad Z = (\text{Tr} e^{-\beta H})^N = \left[e^{-(E_0 + \mu_0 H)\beta} + e^{-(E_0 - \mu_0 H)\beta} + 1 \right]^N$$

$$= e^{-\beta F}$$

$$\Rightarrow F = -\frac{N}{\beta} \ln \left\{ 1 + e^{-\beta E_0} e^{-\beta \mu_0 H} + e^{-\beta E_0} e^{+\beta \mu_0 H} \right\}$$

$$(b) \quad M(T, H) = -\frac{\partial F}{\partial H} = + N \mu_0 \frac{e^{-\beta E_0} (e^{\beta \mu_0 H} - e^{-\beta \mu_0 H})}{1 + e^{-\beta E_0} (e^{+\beta \mu_0 H} + e^{-\beta \mu_0 H})}$$

$$= 2N \beta \mu_0^2 H e^{-\beta E_0} / [1 + 2e^{-\beta E_0}] + \mathcal{O}(H^2)$$

$$\Rightarrow \chi(T) = \left. \frac{\partial M(T, H)}{\partial H} \right|_{H=0} = 2N \beta \mu_0^2 \frac{e^{-\beta E_0}}{1 + 2e^{-\beta E_0}}$$

(c) Clearly $S/N \rightarrow k_B \ln 3$ as $T \rightarrow \infty$.

To be precise, $F \xrightarrow{T \rightarrow \infty} -Nk_B T \ln 3$ and $S = -\frac{\partial F}{\partial T} = Nk_B \ln 3$

(d) $C_V = T \left. \frac{\partial S}{\partial T} \right|_V \rightarrow 0$. Why? Because as $T \rightarrow \infty$

(for $k_B T > E_0, \mu_0 H$) all three polarization states ($\sigma = 0, \pm 1$)

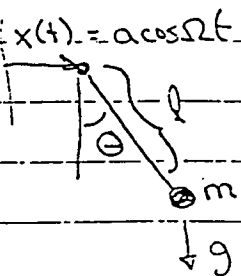
are equally populated ($\frac{1}{3}$ in each state). So increasing the temperature doesn't change the entropy at all, and

hence $\Delta S / \Delta T = 0 \Rightarrow C_V = 0$.

U

The hinge of a pendulum of length l , mass m is oscillated in the x direction according to $x(t) = a \cos \Omega t$, where $\Omega^2 \gg g/l$. (see diagram); also $a \ll l$

Find the condition that an equilibrium exists for $\theta \neq 0$
 (better word?)



$$L = \frac{m}{2} \left[\frac{d}{dt} (a \cos \Omega t + l \sin \theta) \right]^2 + \frac{m}{2} (l \dot{\theta} \cos \theta)^2 + mgl \cos \theta$$

$$= \frac{m}{2} (a^2 \Omega^2 \sin^2 \Omega t - 2al\Omega \dot{\theta} \sin \Omega t \cos \theta + l^2 \dot{\theta}^2) + mgl \cos \theta$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 = \frac{m}{2} \frac{d}{dt} (-2al\Omega \sin \Omega t \cos \theta + 2l^2 \dot{\theta})$$

$$-mal\Omega \dot{\theta} \sin \Omega t \sin \theta + mgl \sin \theta$$

$$0 = -mal\Omega^2 \cos \Omega t \cos \theta + ml^2 \ddot{\theta} + mgl \sin \theta$$

$$\ddot{\theta} = \frac{-g \sin \theta + a\Omega^2 \cos \Omega t \cos \theta}{l} \quad (\text{could also derive directly})$$

$$\theta = \bar{\theta} + \delta\theta$$

slow fast, small \ll \ll \ll

$$\delta \ddot{\theta} \approx \frac{a\Omega^2 \cos \bar{\theta} \cos \Omega t}{l} \Rightarrow \delta \theta \approx -\frac{a \cos \bar{\theta} \cos \Omega t}{l}$$

$$\ddot{\theta} = \frac{-g \sin \bar{\theta}}{l} + \left\langle \frac{a\Omega^2 \cos \Omega t}{l} \left(\cos \bar{\theta} - \sin \bar{\theta} \left(-\frac{a \cos \bar{\theta} \cos \Omega t}{l} \right) \right) \right\rangle$$

$$= -\frac{g}{l} \sin \bar{\theta} + \frac{a^2}{2l^2} \Omega^2 \sin \bar{\theta} \cos \bar{\theta} - \frac{d}{d\bar{\theta}} \frac{\cos^2 \bar{\theta}}{2}$$

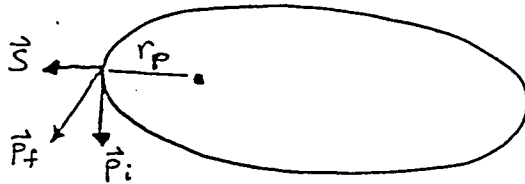
$$U_{\text{eff}} = \frac{a^2 \Omega^2 \cos^2 \bar{\theta}}{4l^2} +$$

$$\rightarrow -\frac{g}{l} \cos \bar{\theta}$$

6. \therefore eqⁿ at $\sin \bar{\theta} = 0$ or $\frac{g}{l} = \frac{a^2 \Omega^2 \cos \bar{\theta}}{2l^2}$

$\therefore \bar{\theta} = 0$ eqⁿ only if $\boxed{\Omega^2 > 2 \frac{lg}{a^2}}$

(9) Impulse $\Delta \vec{p} = S \hat{r}$ at perigee:



$$\vec{p}_f = \vec{p}_i + \vec{S} = \vec{p}_i + S \hat{r}$$

(a) $\vec{l}_i = \vec{r} \times \vec{p}_i$; $\vec{l}_f = \vec{r} \times \vec{p}_f = \vec{r} \times (\vec{p}_i + S \hat{r}) = \vec{r} \times \vec{p}_i + 0 = \vec{l}_i$

(2) so $\Delta l = 0$! This is because $\Delta \vec{p}$ is along \hat{r} . So $l_f = l_i = l$.

At perigee, $E = \frac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{GM\mu}{r}$. $\dot{r}_i = 0$ since perigee is the closest approach, but $\dot{r}_f = \hat{r} \cdot \dot{\vec{r}}_f = \hat{r} \cdot \frac{1}{m} \vec{p}_f = \hat{r} \cdot (\vec{p}_i + S \hat{r})/m$.

Since $\hat{r} \cdot \vec{p}_i = 0$ at perigee, we have $\dot{r}_f = S/m$. Thus,

(2) $\Delta E = \frac{1}{2} \mu \dot{r}_f^2 = S^2/2m$, as $m = \mu$ here.

(b) Eccentricity: (1) $\epsilon^2 = 1 + \frac{2El^2}{\mu k^2}$ with $k = GM\mu$. Thus,

(2) $\epsilon_f^2 = \epsilon_i^2 + \frac{2l^2 S^2}{\mu k^2 2m} = \epsilon_i^2 + \frac{l^2 S^2}{m^2 k^2} \Rightarrow \epsilon_f > \epsilon_i$

Semimajor axis: $a = \frac{l^2}{\mu k} \frac{1}{1 - \epsilon^2} \Rightarrow \frac{a_f}{a_i} = \frac{1 - \epsilon_i^2}{1 - \epsilon_f^2} = \frac{1 - \epsilon_i^2}{1 - \epsilon_i^2 - \frac{l^2 S^2}{m^2 k^2}}$

(1) $a_f = a_i \cdot \frac{1}{1 - \frac{l^2 S^2}{m^2 k^2} \frac{1}{1 - \epsilon_i^2}} = \frac{a_i}{1 - \frac{a_i S^2}{m k}}$

Thus, $a_f > a_i$. Note: we must have $a_i S^2 < m k$ if the orbit is to remain bound. Since $r(\phi) = \frac{l^2}{\mu k} \frac{1}{1 - \epsilon \cos \phi}$

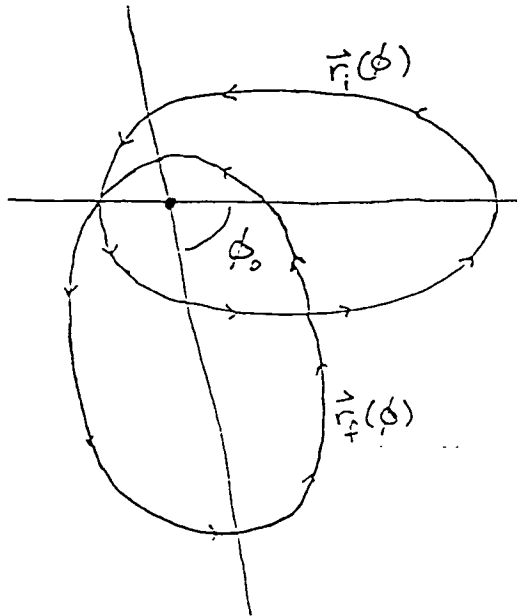
(1) we have $r_{\text{perigee}} = \frac{l^2}{\mu k} \frac{1}{1 + \epsilon}$. Since ϵ increases, perigee will decrease in the new orbit, assuming it remains bound.

(d) Let's write the initial orbit as

$$\textcircled{2} \quad r_i(\phi) = \frac{l^2}{\mu K} \frac{1}{1 - \epsilon_i \cos \phi}$$

and let's write the final orbit as

$$r_f(\phi) = \frac{l^2}{\mu K} \frac{1}{1 - \epsilon_f \cos(\phi + \phi_0)}$$



We know that the orbits intersect at $\phi = \pi$:

$$r_i(\phi = \pi) = r_f(\phi = \pi) \rightarrow$$

$$\frac{1}{1 + \epsilon_i} = \frac{1}{1 + \epsilon_f \cos \phi_0}$$

$$1 + \epsilon_f \cos \phi_0 = 1 + \epsilon_i$$

$$\phi_0 = \cos^{-1}\left(\frac{\epsilon_i}{\epsilon_f}\right)$$

Note that $\phi_0 \rightarrow \cos^{-1}(\epsilon_i)$ in the limit $\epsilon_f \rightarrow 1$.

SOLUTION: PROBLEM 10

F93

gravity $\sim \rho g \int d^2x \int dy y$

a) $\sim \frac{1}{2} \rho g \int d^2x \frac{h^2(x)}{2}$

$= \frac{1}{4} \rho g \int \frac{d^2k}{2\pi} \tilde{h}(k) \tilde{h}(-k)$

electrostatics

First, we need to find the electrostatic potential, using the fact that the liquid is always an equipotential.

$\nabla^2 \Phi = 0$

$\Phi = \int e^{i\vec{k} \cdot \vec{x}} \frac{d^2k}{2\pi} e^{-|k|y} = 4\pi\sigma y$

$\Phi|_{\text{surface}} = \int e^{i\vec{k} \cdot \vec{x}} \frac{d^2k}{2\pi} [\phi_k - 4\pi\sigma h_k]$

$\Rightarrow \phi_k = 4\pi\sigma h_k$

energy $= \frac{1}{2} \int d^2x \rho \Phi$

$$\vec{E} = 4\pi\sigma \hat{y} + 4\pi\sigma \hat{y} \int \frac{d^2k}{(2\pi)^2} k h_k e^{i\vec{k}\cdot\vec{x}} e^{-k|y|} - 4\pi\sigma \int \frac{d^2k}{(2\pi)^2} i\vec{k} h_k e^{i\vec{k}\cdot\vec{x}} e^{-k|y|}$$

energy is:

$$\frac{1}{8\pi} \int d^2x \int dy \vec{E}^2$$

there are two quadratic (in h) terms, one from our term in \vec{E}^2 other from kinetic energy

$$\pi\sigma^2 \left[-2|\vec{k}| h_{\vec{k}} h_{-\vec{k}} + |\vec{k}| h_{\vec{k}} h_{-\vec{k}} \right]$$

$$= -2\pi\sigma^2 h_{\vec{k}} h_{-\vec{k}}$$

Finally, substitute energy

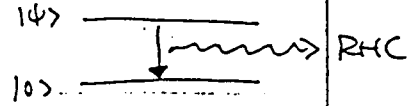
$$\sim \gamma \int \sqrt{1 + |\vec{\nabla} h|^2} d^2x$$

$$\sim \frac{\gamma}{2} \int k^2 h_{\vec{k}} h_{-\vec{k}} \frac{d^2k}{2\pi}$$

$$\sim \int \frac{d^2k}{(2\pi)^2} \left(\frac{g_p}{2} - 2\pi\sigma k + \gamma k^2 \right) h_{\vec{k}} h_{-\vec{k}}$$

$$\begin{aligned}
 \text{[11]} \quad H &= \frac{1}{2m} (\vec{p} - \frac{e}{c} \vec{A})^2 + V(\vec{r}) \\
 &= \frac{\vec{p}^2}{2m} + V(\vec{r}) - \underbrace{\frac{e}{mc} \vec{p} \cdot \vec{A}}_{\text{lowest order perturbing term}} + \frac{e^2}{2mc^2} \vec{A}^2
 \end{aligned}$$

lowest order perturbing term.



$$d\Gamma = \frac{2\pi}{\hbar} |\langle f | H' | i \rangle|^2 \rho_f$$

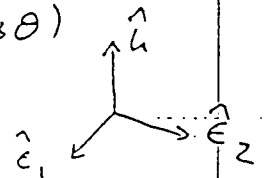
$$\rho_f = \frac{V k^2 dk d\Omega}{8\pi^3 \hbar c dk} = \frac{k^2 V d\Omega}{8\pi^3 \hbar c} = \frac{V}{8\pi^3 \hbar c^3} \omega^2 d\Omega = \text{density of final states for photons of a given polarization in direction } \hat{k}$$

ω = frequency of emitted radiation

$$\hat{e} \cdot \hat{k} = 0, \text{ so if } \hat{k} = k(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

we can take $\hat{e}_1 \times \hat{e}_2 = \hat{k}$, with $\hat{e}_{1,2} \cdot \hat{k} = 0$.

$$\text{Thus, } \hat{k} \cdot (\hat{e}_1 \times \hat{e}_2) = 1 = \hat{e}_2 \cdot \hat{k} \times \hat{e}_1 \Rightarrow \hat{e}_2 = \hat{k} \times \hat{e}_1$$



$$\text{Define } \hat{e}_1 = (\sin\phi, -\cos\phi, 0) \Rightarrow$$

$$\hat{e}_2 = (\cos\phi \cos\theta, \sin\phi \cos\theta, -\sin\theta)$$

$$\text{Thus, } \hat{e}_+ = \frac{1}{\sqrt{2}} (\hat{e}_1 + i\hat{e}_2) = \frac{1}{\sqrt{2}} (\sin\phi + i \cos\phi \cos\theta, -\cos\phi + i \sin\phi \cos\theta, -i \sin\theta)$$

Now we need the matrix element of $-\frac{e}{mc} \vec{p} \cdot \vec{A}$ between initial and final states. Note that

$$\vec{a} = \vec{a}^\dagger = \sqrt{\frac{2}{m\hbar\omega_0}} i\vec{p}$$

so that

$$\vec{p} = i \sqrt{\frac{m\hbar\omega_0}{2}} (\vec{a}^\dagger - \vec{a})$$

SOLUTION: PROBLEM 12

F93

$$[12] \quad \nabla^2 \phi(x, y) = -\delta(x-\xi) \delta(y-\eta) \quad ; \quad \phi(x = \pm \frac{1}{2}, y) = 0$$

The functions $u_n(x) = \sqrt{2} \sin[n\pi(x + \frac{1}{2})]$ are a complete orthonormal set satisfying $\frac{\partial^2 u_n}{\partial x^2} = -\pi^2 n^2 u_n$. We desire to multiply

$u_n(x)$ by a $v_n(y)$ such that $\nabla^2(uv) = 0$ almost everywhere.

An educated choice for $v_n(y)$ is $v_n(y) = A e^{-n\pi|y-\eta|}$ where A is a constant. Then

$$\frac{\partial^2 v_n(y)}{\partial y^2} = -2n\pi A \delta(y-\eta)$$

since the slope of $v_n(y)$ changes discontinuously at $y=\eta$. Thus,

$$\psi_n(x, y) = \sqrt{2} A e^{-n\pi|y-\eta|} \sin[n\pi(x + \frac{1}{2})]$$

satisfy

$$\nabla^2 \psi_n(x, y) = -2\pi n A \delta(y-\eta) \sqrt{2} \sin n\pi(x + \frac{1}{2})$$

From orthonormality and completeness of $u_n(x)$,

$$\delta(x-\xi) = \sum_{n=1}^{\infty} u_n(\xi) u_n(x)$$

so

$$\tilde{\phi}(x, y) = 2A \sum_{n=1}^{\infty} e^{-n\pi|y-\eta|} \sin[n\pi(x + \frac{1}{2})] \sin[n\pi(\xi + \frac{1}{2})] \cdot \frac{1}{n}$$

satisfies

$$\nabla^2 \tilde{\phi} = -2\pi A \delta(x-\xi) \delta(y-\eta)$$

The answer is then obtained with $A = \frac{1}{2\pi}$:

$$\phi(x, y) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\pi|y-\eta|} \sin[n\pi(x + \frac{1}{2})] \sin[n\pi(\xi + \frac{1}{2})]$$

This can be summed:

$$\begin{aligned} \phi(x,y) &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\pi|y-\eta|} \left[e^{i\pi(x+\frac{1}{2})n} - e^{-i\pi(x+\frac{1}{2})n} \right] \\ &\quad \cdot \left[e^{i\pi(\xi+\frac{1}{2})n} - e^{-i\pi(\xi+\frac{1}{2})n} \right] \\ &= -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{-\pi|y-\eta|} e^{i\pi(x+\xi)} \right]^n (-1)^n \\ &\quad - \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{-\pi|y-\eta|} e^{-i\pi(x+\xi)} \right]^n (-1)^n \\ &\quad + \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{-\pi|y-\eta|} e^{i\pi(x-\xi)} \right]^n \\ &\quad + \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{-\pi|y-\eta|} e^{-i\pi(x-\xi)} \right]^n \end{aligned}$$

Now recall

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$$

$$\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

So ...

$$\begin{aligned} \phi(x,y) &= \frac{1}{4\pi} \ln \left| 1 + e^{-\pi|y-\eta|} e^{i\pi(x+\xi)} \right|^2 \\ &\quad - \frac{1}{4\pi} \ln \left| 1 - e^{-\pi|y-\eta|} e^{i\pi(x-\xi)} \right|^2 \\ &= \frac{1}{4\pi} \ln \left| \frac{1 + e^{-\pi|y-\eta|} e^{i\pi(x+\xi)}}{1 - e^{-\pi|y-\eta|} e^{i\pi(x-\xi)}} \right|^2 \end{aligned}$$

SOLUTION: PROBLEM 13

F93

The surface modes of vibration of a liquid have a frequency-dependent phase velocity given approximately by $u(\omega) = (\sigma/\rho)^{1/3}$, where σ is the surface tension and ρ is the density.

a) Show that this implies an energy-momentum relationship

$$\epsilon = (\sigma/\rho\hbar)^{1/3} p^{3/2}$$

b) Derive a low-temperature form for the specific heat of the (gas) of surface modes.

$$a) u = \frac{\epsilon}{\hbar\omega} = \left(\frac{\sigma \hbar\omega}{\hbar\rho} \right)^{1/3} \Rightarrow \epsilon^{2/3} = \left(\frac{\sigma}{\hbar\rho} \right)^{1/3} p \Rightarrow \epsilon = \left(\frac{\sigma}{\hbar\rho} \right)^{1/2} p^{3/2}$$

$$b) \bar{E} = \int_0^{p_{\text{max}}} \frac{p \, dp}{2\pi\hbar^2} \frac{\epsilon(p)}{e^{\beta\epsilon} - 1} \quad \mu = 0 \quad d\epsilon = \left(\frac{\sigma}{\hbar\rho} \right)^{1/2} \frac{3}{2} p^{1/2} dp$$

$$\Rightarrow \bar{E} = \frac{1}{2\pi\hbar^2} \int_0^{p_{\text{max}}} \left(\frac{\hbar\rho}{\sigma} \right)^{2/3} \frac{2}{3} \frac{\epsilon^{4/3} d\epsilon}{e^{\beta\epsilon} - 1} \quad \text{let } x = \beta\epsilon$$

$$= \frac{p_{\text{max}}}{3\pi\hbar^2} \left(\frac{\hbar\rho}{\sigma} \right)^{2/3} (kT)^{4/3} \int_0^{x_{\text{max}}} \frac{x^{4/3} dx}{e^x - 1} \quad \text{but } p_{\text{max}} \sim \hbar\rho^{1/3} \Rightarrow x_{\text{max}} \gg 1 \text{ for } T$$

$$c_v = \frac{d\bar{E}}{dT} = \frac{7}{9\pi\hbar^2} \left(\frac{\hbar\rho}{\sigma} \right)^{2/3} (kT)^{4/3} k \int_0^{\infty} \frac{x^{4/3} dx}{e^x - 1} \quad (\text{modes don't propagate w/ } k > \rho^{1/3})$$

$$c_v \propto T^{4/3}$$