DEPARTMENT OF PHYSICS
UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CALIFORNIA 92093-0354

WRITTEN DEPARTMENTAL EXAMINATION, SPRING, 1992

PART I

INSTRUCTIONS:

1

Each problem is worth 10 points. This part has 7 problems.

Useful Information

c$=3 \times 10^{8}$ m/s

$\hbar = 1.05 \times 10^{-34}$ J-s

$N_A = 6.02 \times 10^{23}$

$k_B = 1.38 \times 10^{-38}$ J/K

$e = 1.6 \times 10^{-19}$ C $= 9.8 \times 10^{-10}$ esu

$m_e = 9.1 \times 10^{-31}$ kg

$m_p = 1.67 \times 10^{-27}$ kg

$\epsilon_0 = 8.85 \times 10^{-12}$ F/m

$\ln N! \sim N \ln N - N$ as $N \to \infty$

1. Evaluate the integral

$$\int d\Omega_{\hat{n}} \, \frac{\hat{n}}{(a + \vec{b} \cdot \hat{n})^2}$$

over the unit sphere $|\hat{n}| = 1$.

2. Two identical non-interacting spin-1/2 fermions with mass m and magnetic moment μ are placed in a one dimensional box of width L in a magnetic field B. Find the ground state of the system as a function of L and B.

3. Consider the one-dimensional Schrodinger equation for a particle with mass m and potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & \text{for } x > 0; \\ +\infty, & \text{for } x < 0.; \end{cases}$$

where ω is a constant. Find the energy eigenvalues.

4. Starting from the ideal gas law, derive the "law of atmospheres," which states that the pressure $P(z)$ of an isothermal atmosphere a height z is given by the equation

$$P(z) = P_0 e^{-mgz/k_B T}$$

where P_0 is the atmospheric pressure, m is the mass of an air molecule, and g is the acceleration due to gravity. Estimate the scale height of the atmosphere (i.e., the height at which the pressure falls by a factor of $1/e$). Find $P(z)$ for the case of an adiabatic atmosphere. What is the temperature dependence $T(z)$ for this case?

5. A string attached to a particle of mass m passes through a hole in a smooth, frictionless, horizontal table and is attached to a spring with spring constant k. The lower end of the spring is fixed at a height such that the particle is at the hole when the spring is unstretched.

   (a) Write down an expression for the energy of the system.

   (b) Derive the equations of motion.

   (c) Write down expressions for all the independent conserved quantities of the motion.

6. Consider N atoms of an ideal gas in thermal equilibrium at temperature T confined by a barrier to a fraction f of the volume of an adiabatic enclosure. The barrier ruptures at some instant, and the gas expands to fill the whole volume, and eventually reaches a new equilibrium state.

   (a) What is the new temperature of the gas?

   (b) What is the change in entropy of the gas?

7. A particle of mass m is in a bound orbit in an attractive potential $-k/r$. The Kepler orbit equation is

$$\frac{1}{r} = \frac{mk}{l^2}(1 + e\cos\theta)$$

where e is the eccentricity, and l is the angular momentum. Show that the momentum of the particle as a function of time traces out a circle in momentum space, i.e., that $\vec{p}(t)$ traces out a circle. Find the center and radius of this circle. The center of the circle is not at $\vec{p} = 0$. Explain simply why the center is diplaced.

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PART II

INSTRUCTIONS:

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Useful Information

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$\hbar = 1.05 \times 10^{-34}$ J-s

$N_A = 6.02 \times 10^{23}$

$k_B = 1.38 \times 10^{-38}$ J/K

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$m_e = 9.1 \times 10^{-31}$ kg

$m_p = 1.67 \times 10^{-27}$ kg

$\epsilon_0 = 8.85 \times 10^{-12}$ F/m

$\ln N! \sim N \ln N - N$ as $N \to \infty$

8. Answer the following questions, using equations, computations, and sketches as necessary:
   (a) Describe the photoelectric effect.
   (b) What is Bremsstrahlung?
   (c) Describe the Zeeman effect.
   (d) Calculate the Bohr radius in terms of physical constants, and give its value in meters.
   (e) Estimate the velocity of an air molecule at room temperature.

9. A monoatomic crystal consists of N atoms that can occupy two kinds of positions, the normal position indicated by o in the figure, and an interstitial position denoted by x. There is an equal number N of normal and interstitial sites, and the energy of an atom in an interstitial position is larger than that of an atom in a normal position by an amount ϵ. Calculate the Helmholtz free energy, and show that the number n of atoms at interstitial sites at temperature T is given by

$$n \approx N e^{-\epsilon/2k_B T}$$

for $\epsilon \gg k_B T$.

10. Show that electromagnetic radiation can be propagated in a hollow perfectly conducting metal pipe of rectangular cross-section with sides a and b, $a > b$. What are the phase and group velocities of the wave? What is the frequency below which waves will not propagate? (Assume that the pipe is evacuated.)

11. Spin waves (magnons) in a ferromagnetic solid have a dispersion relation of the form

$$\omega = A|\vec{k}|^2.$$

Treating the magnons as elementary excitations, calculate the temperature dependence of the heat capacity of the spin system at low temperatures.

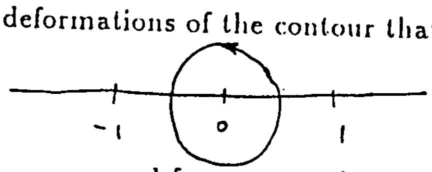
12. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{i}{4} \oint \frac{\cot \pi z}{z^{2k}} dz$$

for $k = 1, 2, 3, \ldots$, where the contour of integration is as shown in the figure. Use this to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Note: Carefully justify any deformations of the contour that you make.



13. A harmonic oscillator with mass m and frequency ω is acted on by an external force. The interaction Hamiltonian is

$$H_{\text{int}} = f(t)x.$$

Assume that $f(t) = f_0\delta(t)$, and that the oscillator is in the ground state at $t = 0^-$. Calculate the state of the oscillator at time $t > 0$. (You may assume $\hbar = 1$.)

14. Find the normal mode frequencies for the system of masses and springs shown below. All masses are m, and all spring constants are k.

SPRING 92

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 9

Please insert on each page

the Problem No. 1 and your Identification No. 57

Let $\vec{b} = b\hat{z}$

$$\vec{I} = \iint \frac{\cos\theta\,\hat{z} + \sin\theta\sin\phi\,\hat{y} + \sin\theta\cos\phi\,\hat{x}}{(a + b\cos\theta)^2}\, d(\cos\theta)\, d\phi$$

$$\int_0^{2\pi} \sin\phi \cdot d\phi = \int_0^{2\pi} \cos\phi\, d\phi = 0$$

$$\vec{I} = 2\pi\hat{z} \int_0^{\pi} \frac{\cos\theta\sin\theta\, d\theta}{(a+b\cos\theta)^2} \checkmark$$

Let $u = \cos\theta$ $\qquad V' = \dfrac{\sin\theta}{(a+b\cos\theta)^2}$

$u' = -\sin\theta$

$$V = \frac{1}{b}\frac{1}{(a+b\cos\theta)}$$

$$\vec{I} = 2\pi\hat{z}\left[\frac{\cos\theta}{b(a+b\cos\theta)}\Big|_0^{\pi} + \frac{1}{b}\int_0^{\pi} \frac{\sin\theta}{(a+b\cos\theta)}\, d\theta\right]$$

$$= 2\pi\hat{z}\left[\frac{-1}{b(a-b)} - \frac{1}{b(a+b)} + \frac{-1}{b^2}\ln[a+b\cos\theta]\Big|_0^{\pi}\right]$$

sign

$$= 2\pi\hat{z}\left[\frac{-(b+a) - (b-a)}{b(a^2-b^2)} - \left(\frac{1}{b^2}\ln[|a-b|] - \ln|a+b|\right)\right]$$

$$\vec{I} = 2\pi\hat{z}\left[\frac{2}{b^2-a^2} + \frac{1}{b^2}\ln\left|\frac{a+b}{a-b}\right|\right]$$

Note: If you use additional sheets for this problem, number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 3

Please insert on each page

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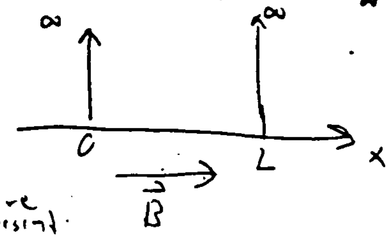


If no $\vec{B}$ where present

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + 0\psi = E\psi \Rightarrow \psi = A\sin kx + B\cos kx$$

where $k^2 = \dfrac{2mE}{\hbar^2}$

$$\psi(0) = 0 = B$$

$$\psi(L) = A\sin kL = 0 \Rightarrow k = \frac{n\pi}{L} \quad n = 1, 2, \ldots$$

$$\Rightarrow \frac{n^2\pi^2}{L^2} = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Ground state $(n = 1)$ $E_g = \dfrac{\pi^2\hbar^2}{2mL^2}$

Since box is 1-D spin will ~~...~~ either align or anti-align with $\vec{B} = B\hat{x}$. Lowest energy occurs with parallel alignment. Treat B-field as a perturbation on the particle confined to the box. $E' = \langle n^0 | -\vec{\mu}\cdot\vec{B} | n^0 \rangle$

$= -\vec{\mu}\cdot\vec{B} = -\mu B$ for parallel alignment (lowest energy)

$$\text{so, } E_g = \frac{\pi^2\hbar^2}{2mL^2} - 2\mu B \;\; \times$$

Note: If you use additional sheets for this problem, number the pages and staple them together.

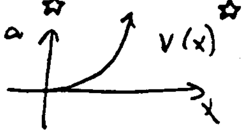
PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 10

Please insert on each page

the Problem No. 3 and your Identification No. 57

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0 \\ \infty & x < 0 \end{cases}$$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \qquad x > 0$$

No requirement on $\psi'(0)$ because of $\delta(0$ at $x =$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 x^2\right)\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0 \qquad x > 0$$

Analysis is verbatim identical to "normal" problem with full $V(x)$ until the imposition of the boundary conditions which here require $\psi(x=0) = 0$. The ~~new~~ normal problem has a soln $\propto e^{-y^2/2} H_n(y)$ ..... The Hermite polynomials with n odd are odd under exchange $y \to -y$ and hence are zero at $y = 0$. They can satisfy above boundary condition whereas the even n Hermite polynomials are finite at $y = 0$. Thus, the eigenfunctions for this problem are all the odd $(n = 1, 3, 5, \ldots)$ of $e^{-y^2/2} H_n(y)$ with eigenvalues $\underline{E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad n = 1, 3, 5}$

indic e
↑
norml factor

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Please insert on each page the Problem No. 4 and your Identification No. 57

Score = 8

$$dU = T dS - P dV$$

$$G = U - TS + PV$$

$$dG = \cancel{T dS} - \cancel{P dV} - \cancel{T dS} - S dT + \cancel{P dV} + V dP$$

$$dG = -S dT + V dP \qquad \text{Isothermal } dT = 0$$

$$\int dG = \int_{P_0}^{P} \frac{NKT\,dP}{P} = NKT \ln\left(\frac{P}{P_0}\right)$$

$$= \Delta G = \Delta W_{free} = -mgzN = NKT \ln\left(\frac{P}{P_0}\right)$$

$$\Rightarrow \quad P = P_0 e^{-mgz/kT}$$

4

Scale Height

$$+\frac{mgz}{KT} = 1 \qquad z = \frac{KT}{mg} \quad \frac{1.38 \times 10^{-23} \frac{J}{K} \cdot 300 K}{2 \times 14 \times 1.6716 \times 10^{-27} kg \times 9.8 \frac{m}{s^2}}$$

nitrogen

2

$$z = 1.8 \times 10^4 m$$

adiabatic case

$$PV^\gamma = \text{constant} = P_0 V_0^\gamma$$

2

$$V^\gamma = \frac{P_0}{P} V_0^\gamma \qquad P_0 = \frac{NKT_0}{V_0}$$

Note: If you use additional sheets for this problem, number the pages and staple them together.

1

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score =

Please insert on each page

the Problem No. 4 and your Identification No. 57

$$V^{\alpha} = \frac{NkT_0}{V_0}\frac{V_0^{\gamma}}{P}$$ $\left.\begin{matrix}T_0\\V_0\end{matrix}\right\}$ at the ground

$$dU = TdS - PdV$$ adiabatic $dS = 0$

$$\int dU = -\frac{NkT_0}{V_0} V_0^{\gamma} \int_{V_0}^{V} \frac{dV}{V^{\gamma}}$$

$$-mgzN = -NkT_0 V_0^{\gamma-1} (-\gamma)\left(V^{1-\gamma}\right)\Big|_{V_0}^{V}$$

$$-mgzN = +\gamma NkT_0 V_0^{\gamma-1}\left(V^{1-\gamma} - V_0^{1-\gamma}\right)$$

$$V = \frac{NkT}{P}$$

$$-mgzN = \gamma NkT_0 V_0^{\gamma-1}\left[\left(\frac{NkT}{P}\right)^{1-\gamma} - V_0^{1-\gamma}\right]$$

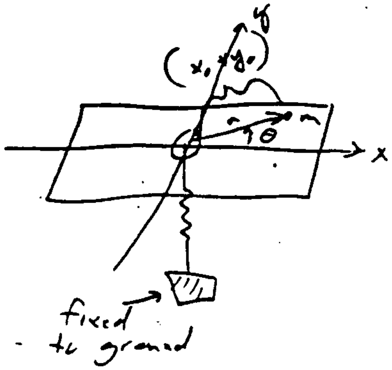
solve for P

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 1.

Please insert on each page

the Problem No. 5 and your Identification No. 57



$$V = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta + -r\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\dot{x}^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\cos\theta\sin\theta\,\dot{\theta} + r^2\sin^2\theta\,\dot{\theta}$$

$$+\ \dot{y}^2 = \dot{r}^2\sin^2\theta + 2\dot{r}r\cos\theta\sin\theta\,\dot{\theta} + r^2\cos^2\theta\,\dot{\theta}$$

$$\dot{r}^2 + r^2\dot{\theta}^2$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} k r^2$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} ; \quad \frac{\partial L}{\partial \theta} = 0 \Rightarrow p_\theta \text{ const} \Rightarrow \text{conserved}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} , \quad \frac{\partial L}{\partial r} = -kr + m r \dot{\theta}^2$$

$$H = p_\theta\dot{\theta} + p_r\dot{r} - L = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m\dot{r}^2 + \frac{1}{2} k r^2$$

$$\frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const} = \frac{1}{2} k r_0^2$$ (assuming the spring is only displaced r_0 from hole which is the equilibrium pt for spring)

conserved

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 10

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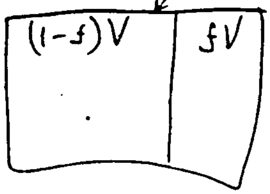
the Problem No. 6 and your Identification No. 57

adiabatic walls (No heat exchange)

$\Delta Q = 0$



(a) Ideal Gas when barrier ruptures expansion is free i.e. $\Delta W = 0$

$$\Rightarrow \Delta U = \Delta Q - \Delta W = 0$$

$$\therefore dU = 0 = \frac{\partial U}{\partial V}\Big)_{T,N} dV + \frac{\partial U}{\partial T}\Big)_{V,N} dT$$

For a ideal gas

$$\Rightarrow \frac{dT}{dV}\Big)_U = -\frac{\left(\frac{\partial U}{\partial V}\right)_{T,N}}{\frac{\partial U}{\partial T}\Big)_{V,N}} = \frac{-\left(T\frac{\partial P}{\partial T}\Big)_V - P\right)}{\frac{\partial U}{\partial T}\Big)_{V,N}}$$

$$P = \frac{NkT}{V} \Rightarrow \frac{\partial P}{\partial T} = \frac{NK}{V} \Rightarrow T\frac{\partial P}{\partial T}\Big)_V - P \equiv 0$$

$$\Rightarrow \frac{\partial T}{\partial V}\Big)_U = 0$$ $\therefore$ T does not change from initial T

(b) $dU = 0 = TdS - PdV \Rightarrow \frac{dS}{\partial V} = \frac{P}{T}$

$$dS = \frac{P}{T}dV$$

$$\Delta S = \int_{fV}^{V} \frac{P}{T}dV = \frac{1}{T}\int_{fV}^{V}\frac{NkT}{V}dV = NK\ln\left(\frac{V}{fV}\right) > 0$$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 2

Please insert on each page

the Problem No. 7 and your Identification No. 57

$$T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) \qquad V = -\frac{k}{r}$$

$$L = T - V = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{k}{r}$$

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \; ; \quad \frac{\partial L}{\partial r} = m r \dot\theta^2 - \frac{k}{r^2}$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m r^2 \dot\theta \; ; \quad \frac{\partial L}{\partial \theta} = 0 \Rightarrow m r^2 \dot\theta = \text{constant} \equiv l$$

$$H = p_\theta \dot\theta + p_r \dot r - L = \frac{1}{2} m r^2 \dot\theta^2 + \frac{1}{2} m \dot r^2 - \frac{k}{r}$$

$$\dot r = \frac{p_r}{m} \qquad \dot\theta = \frac{p_\theta}{m r^2}$$

$$H = \frac{1}{2} m r^2 \left(\frac{p_\theta^2}{m^2 r^4}\right) + \frac{1}{2} m \left(\frac{p_r^2}{m^2}\right) - \frac{k}{r}$$

$$H = \frac{1}{2}\frac{p_\theta^2}{m r^2} + \frac{1}{2}\frac{p_r^2}{m} - \frac{k}{r}$$

$$\frac{\partial H}{\partial t} = 0 \Rightarrow H = \text{const} \equiv H_0$$

$$H_0 = \frac{1}{2}\frac{p_\theta^2}{m r^2} + \frac{1}{2}\frac{p_r^2}{m} - \frac{k}{r}$$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score =

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$$\dot{p}_r = -\frac{\partial H}{\partial r} = mr\dot{\theta}^2 + \frac{k}{r^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow p_\theta = \text{const.}$$

$$\dot{p}_r = mr\left(\frac{l}{mr^2}\right)^2 + \frac{k}{r^2} = \frac{ml^2}{m^2r^3} + \frac{k}{r^2}$$

$$\dot{p}_r = \frac{l^2}{m}\frac{m^3k^3}{l^3}(1+e\cos\theta)^3 + \frac{mk^2}{l^2}(1+e\cos\theta)^2$$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 7,8

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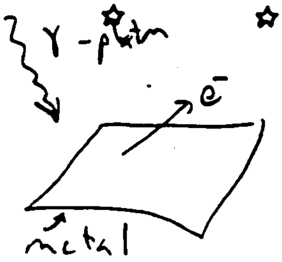
(a)



Photoelectric Effect

10/10

~~Inc~~ Incoming photon ~~will~~ ejects an electron from metal provided it has sufficient energy to overcome metal work function remainder of energy goes into kinetic energy of electron. Expressed mathematically in Einstein's formula $h\nu = W + \frac{1}{2}mv_e^2$

work function (W)

e^- Kinetic Energy

b) 10/10 Braking Radiation – occurs whenever a charged particle is rapidly decerelurated, This effect is used to run Flash X-ray devices by bombarding high energy electrons onto heavy metal targets (high Z-materials decelerate e^-'s more effectively)

c) Zeeman Effect –

9/10

splitting of spectral lines due to magnetic field interaction with radiating atom. Results in removal of some degeneracy due to spin (orbital in this case, as opposed to "anomalous" Zeeman effect where electron spin is also included) Removal of some degeneracy results in splitting of spectral lines

$\Delta E = \mu \cdot H$

d) 4/10 calculation

$a_0 = \frac{\hbar^2}{me^2} \approx 0.5\,Å \approx 5\times10^{-11}\,m$

$\bar{v} \simeq \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3\,(1.38\times10^{-23}\frac{J}{K}\times 300K)}{14\times 1.67\times10^{-27}kg}}$ Nitrogen 14

Note: If you use additional sheets for this problem, number the pages and staple them together.

e) $\frac{1}{2}mv^2 \approx \frac{3}{2}kT$

N_2 O_2 CO_2

$\bar{v} \simeq 730\,\frac{m}{s}$

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 2

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the Problem No. 9 and your Identification No. 57

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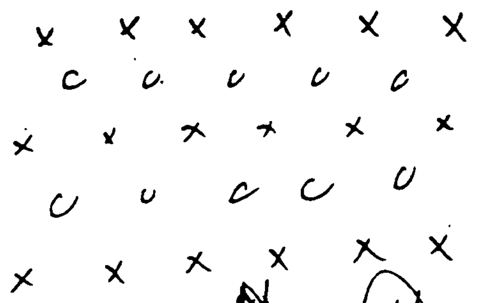

Call normal position energy E_0 then interstitial is $E_0 + \epsilon$

Similar to 2-D spin system $x = \uparrow$ $o =$

$$Z = \sum_{n=1}^{N} e^{-\beta E_n} = \cancel{\sum_{n=1}^{N} \frac{N!}{n!(N-n)!} e^{-\beta E_n}}$$

This was a test

$$A = -kT \ln Z$$

$$Z = (2N-n) e^{-\beta E_0} + n e^{-\beta(E_0+\epsilon)} \leftarrow \text{NO}$$

$$\cancel{Z = N e^{-\beta E_0}\left(1 + N e^{-\beta \epsilon}\right)}$$

$$A = -kT\left[\ln\left(N e^{-\beta E_0}\right) + \ln\left(1 + N e^{-\beta\epsilon}\right)\right]$$

$$A = \cancel{-kT\left[\ln\left[(2N-n)e^{-\beta E_0}\right] + \ln\right.}$$

$$Z = e^{-\beta E_0}\left[(2N-n) + n e^{-\beta(E_0+\epsilon)}\right]$$

$$A = -kT\left[\ln\left(e^{-\beta E_0}\right) + \ln\left[(2N-n) + n e^{-\beta\epsilon}\right]\right]$$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score =

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$\frac{9}{10}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1)$$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \quad (2)$$

$\vec{J}_f = 0$ in waveguide, ie. there is no free current only current generated to satisfy B Cs. (Boundary condition)

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$= -\frac{\partial}{\partial t} \nabla \times \vec{B} = -\frac{\partial}{\partial t} \mu (\nabla \times \vec{H})$$ ← assuming μ a constant

$$= -\frac{\partial}{\partial t} \mu \left(\frac{\partial \vec{D}}{\partial t}\right) = -\frac{\partial}{\partial t} \mu \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E} \qquad \nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon}$$

$\rho_f = 0$ since there is no free charge

$$\Rightarrow \nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \vec{E}$$

It is clear that analogous equation for $\vec{B}$ ($\vec{H}$) exists due to the symmetry of the equations when there is no free current or charge ∴

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

These equations admit sol'ns of the form (rectangular coordinates)

2

$$\left.\begin{aligned} \vec{E} &= \vec{E}(x,y) e^{i(kz - \omega t)} \\ \vec{B} &= \vec{B}(x,y) e^{i(kz - \omega t)} \end{aligned}\right\}$$ Clearly propagating waves along the +z axis

Substituting into (1) and (2) yields

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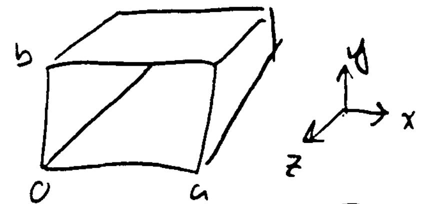
$$\left(\nabla_t^2 + \gamma^2\right)\begin{Bmatrix} B(x,y) \\ E(x,y) \end{Bmatrix} = 0; \text{ where } \gamma^2 = \mu\epsilon w^2 - k^2$$

and $\vec{\nabla}_t = \vec{\nabla} - \frac{\partial}{\partial z}\hat{z}$

Because we have a single hollow tube only the TE or TM modes can propagate. This follows from the fact that a TEM problem is basically an electro: problem, i.e. $\nabla^2 E = 0$, and since any solution of Lapl. equation is either the real or imaginary part of a har. function, ~~then~~ and since harmonic functions always assum. there minimum and maximum on the boundary, then i follows that the $E(B)$ must be constants, since there is only one boundary and hence the mini = max

Consider first a TE wave

$E_z = 0$; $\left.\frac{\partial B_z}{\partial n}\right|_s = 0$ Boundary Conditions



Try $B(x,y) = A\cos k'y \cos k''x$

Boundary conditions require that

$k' = \frac{n\pi}{b}$ and $k'' = \frac{m\pi}{a}$ $\Rightarrow \gamma^2 = \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2$

A similar γ results in the TM case

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TM: $E(x,y) = A'\sin k''x \sin k'y$ BC $B_z = 0$, $E_z|_s = 0$

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$$\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2 = \mu\epsilon\omega^2 - k^2$$

$$k^2 = \mu\epsilon\omega^2 - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2$$

$$k = \sqrt{\mu\epsilon\omega^2 - \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right)}$$

wave will propagate provided $\mu\epsilon\omega^2 > \pi^2\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]$

TE case $n = 0, 1, 2, \ldots$

$m = 0, 1, 2, \ldots$

m and n cannot ~~not~~ both equal zero, otherwise no wave

In TM case neither n nor m can be zero, otherwise no wave

~~vacuum~~ $\cancel{\mu\epsilon} = \frac{1}{c^2}$?

Group velocity

$$V_g = \frac{d\omega}{dk}$$

phase velocity

$$V_p = \frac{\omega}{k} \quad 1$$

$$2\left\{ dk = \frac{1}{2}\left[\mu\epsilon\omega^2 - \pi^2\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]\right]^{-\frac{1}{2}} 2\mu\epsilon\omega\, d\omega \right.$$

$$\Rightarrow \frac{d\omega}{dk} = \left[\mu\epsilon\omega^2 - \pi^2\left(\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right)\right]^{\frac{1}{2}} \frac{1}{\mu\epsilon\omega} = \frac{k}{\mu\epsilon\omega} = \frac{1}{\mu\epsilon V_p} = V_g$$

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Since tube is evacuated $\epsilon = \epsilon_0$, $\mu = \mu_0 \Rightarrow \epsilon_0 \mu_0 = \frac{1}{c}$

$\Rightarrow V_g V_p = c^2$ since $V_g \leq c \Rightarrow V_p \geq c$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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Magnons are bosons and quasi-particles and hence ~~both~~ particle number is not conserved i.e. Planck distribution

$$E = \frac{V}{(2\pi)^3}\int \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\, d^3k = \frac{4\pi V}{(2\pi)^3}\int \frac{\hbar\omega\, k^2\, dk}{e^{\beta\hbar\omega}-1}$$

$$= \frac{V4\pi}{(2\pi)^3}\int_0^\infty \frac{\hbar(Ak^2)k^2\,dk}{e^{\beta A\hbar k^2}-1}$$

Let $x = \beta A\hbar k^2$

$dx = 2\beta A\hbar\, k\, dk$

$$= \frac{V4\pi}{(2\pi)^3}\hbar A\int_0^\infty \left(\frac{x}{\beta A\hbar}\right)^2 \frac{1}{e^x-1}\frac{dx}{2\beta A\hbar}\frac{1}{k} = \frac{4\pi V}{(2\pi)^3}\int_0^\infty \left(\frac{x}{\beta A\hbar}\right)^2 \frac{dx}{2\beta A\hbar}\left(\frac{x}{\beta A\hbar}\right)^{-\frac{1}{2}}\frac{1}{e^x-1}$$

$$\propto \frac{1}{\beta^{3+\frac{1}{2}}}\underbrace{\int_0^\infty \frac{dx}{e^x-1}}_{\text{constant} \equiv b} = \propto T^{\frac{5}{2}}\, b$$

$$C_V = \frac{\partial E}{\partial T} \propto T^{3/2}$$

Don't know relation between k and quantum number n (if any) so that ~~the~~ a density of states $G(\omega)$ could be calcul. Shall assume $G(\omega) = 1$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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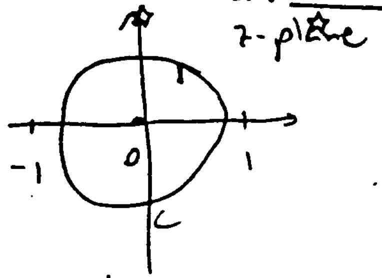


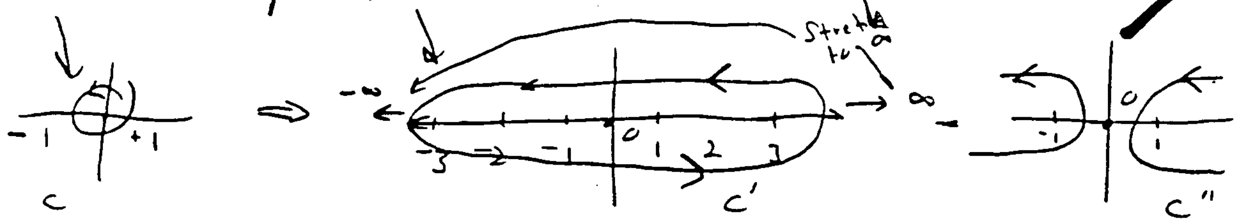
$$\frac{i}{4}\oint_C \frac{\cot \pi z}{z^{2k}}\,dz$$

$n = 0, 1, 2$

$\cot \pi z = \frac{\cos \pi z}{\sin \pi z}$ - poles occur at $z = n$. residue there

is $\frac{\cos \pi n}{\pi \cos \pi n} = \frac{1}{\pi}$ ✓

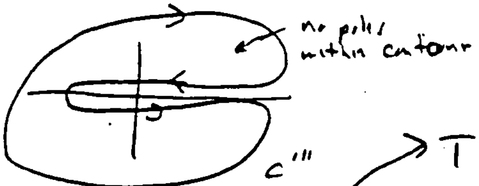
Contour is equivalent to



$$\oint_C = \oint_{C'} - \oint_{C''} \; ; \quad \oint_{C''} \frac{\cot \pi z}{z^{2k}}\,dz = 2\pi i \sum_{z=n} \text{residue @ } z=n$$

$$= 2\pi i \, \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{z^{2k}}\Big|_{z=n} = \frac{2 \cdot 2\pi i}{\pi} \sum_{n=1}^{\infty}$$

Note that contour C' is equivalent to C''' since there is no contribution to the integral along z at the ends ($\pm\infty$) $|\cot \pi z| = 1$ along z (real) since $\int dz/z^{2k} \to 0$ for $k > \frac{1}{2}$ and $\oint_{C'''} = 0$ since no poles are enclosed within C'''. Therefore, $\oint_{C'} = \oint_{C'''} = 0$



Therefore,

$$\frac{i}{4}\oint_C \frac{\cot \pi z}{z^{2k}}\,dz = -\frac{i}{4}\oint_{C''} \frac{\cot \pi z}{z^{2k}}\,dz$$

$$= \frac{-i}{4} \cdot 2i \cdot 2 \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

QED

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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$\frac{i}{4}\oint_C \frac{\cot\pi z}{z^{2k}}\,dz$ consider $k=1$

only pole is $z=0$ within c.

$\frac{i}{4}\oint_C \frac{\cos\pi z}{z^2\sin\pi z}\,dz$ ← We have a third order pole

$\sin z\pi = z\pi - \frac{(\pi z)^3}{3!} + \cdots \qquad \cos\pi z = 1 - \frac{(\pi z)^2}{2!}$

$\frac{1}{\sin\pi z} = z\pi - \frac{(\pi z)^3}{3!}$) 1, quotient $\frac{1}{\pi z} + \frac{\pi z}{3!}$

$$\begin{array}{l} 1 \\ -1 + \frac{(\pi z)^2}{3!} \\ \hline \frac{(\pi z)^2}{3!} \\ \frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{3!} \\ \hline 0 + \frac{(\pi z)^4}{3!} \end{array}$$

$$\frac{i}{4}\oint_C \frac{1}{z^2}\left(\frac{1}{\pi z}+\frac{\pi z}{3!}\right)\left(1-\frac{(\pi z)^2}{2!}\right)dz = \frac{i}{4}\oint \frac{1}{z^2}\left(\frac{1}{\pi z} - \frac{(\pi z)}{2} + \frac{\pi z}{3!} - \frac{(\pi z)^3}{3!2!}\right)$$

$$= \frac{i}{4}\,2\pi i\left(\frac{1}{3!} - \frac{1}{2}\right) = \frac{\pi^2}{6}$$

$$\therefore \frac{i}{4}\oint_C \frac{\cot\pi z}{z^2}\,dz = \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$ ← From part (a)

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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$$d_{if}(t) = \frac{-i}{\hbar}\int_0^t \langle f_i | H'(t) | i\rangle e^{i\omega_{fi}t}\, dt$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}}\hat{x} + i\left(\frac{1}{2\hbar m\omega}\right)^{\frac{1}{2}}\hat{p}$$ x, p Hermitian

$$a^+ = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}}\hat{x} - i\left(\frac{1}{2\hbar m\omega}\right)^{\frac{1}{2}}\hat{p}$$

$$2\left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}}\hat{x} = a^+ + a$$

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a^+ + a)$$

$\langle f| = \langle n|$ number rep

$\langle i| = \langle 0| \leftarrow$ Ground state in number rep

$$\langle n | H'(t) | 0\rangle = \langle n| \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a^+ + a)|0\rangle f_0\delta(t)$$

$\Rightarrow$ $n = 1$ is the only state to yield a non-zero matrix el.

since $a|0\rangle = 0$, $a^+|0\rangle = (1+0)^{\frac{1}{2}}|1\rangle = |1\rangle$.

$$\Rightarrow \langle n| H'(t)|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\langle 1|1\rangle f_0\delta(t) = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} f_0\delta(t)$$

$$d_{if}(t) = \frac{-i}{\hbar}\int_0^t \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} f_0\delta(t)\, e^{i[\hbar\omega(1+\frac{1}{2}) - \hbar\omega(\frac{1}{2})]t}\, dt$$

$$d_{if}(t) = \frac{-i}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}} f_0 \rightarrow$$ Transition probability is independent of time

$$P_{if} = |d_{if}|^2 = \frac{f_0^2}{2m\omega\hbar}$$

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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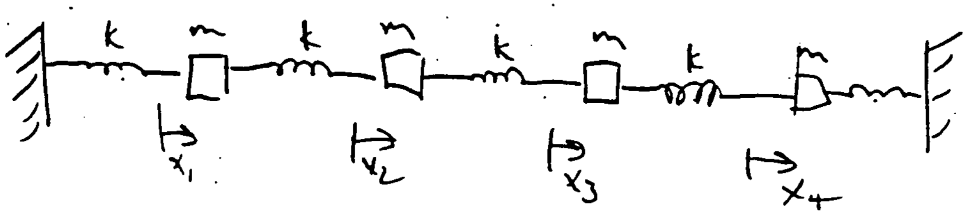

x_i's displacement from equilibrium

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + \frac{1}{2}m\dot{x}_4^2$$

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2\right)$$

$$V = \frac{1}{2}k(x_1-x_2)^2 + \frac{1}{2}k(x_2-x_3)^2 + \frac{1}{2}k(x_3-x_4)^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_4^2$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \;;\; \frac{\partial L}{\partial x_1} = k(x_1-x_2) + kx_1$$

$$m\ddot{x}_1 - k(x_1-x_2) - kx_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \;;\; \frac{\partial L}{\partial x_2} = -k(x_1-x_2) + k(x_2-x_3)$$

$$m\ddot{x}_2 + k(x_1-x_2) - k(x_2-x_3) \qquad (2)$$

$$\frac{\partial L}{\partial \dot{x}_3} = m\dot{x}_3 \;;\; \frac{\partial L}{\partial x_3} = -k(x_2-x_3) + k(x_3-x_4)$$

$$m\ddot{x}_3 + k(x_2-x_3) - k(x_3-x_4) \qquad (3)$$

$$\frac{\partial L}{\partial \dot{x}_4} = m\dot{x}_4 \;;\; \frac{\partial L}{\partial x_4} = -k(x_3-x_4) + kx_4$$

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$$m\ddot{x}_4 + k(x_3-x_4) - kx_4 \qquad (4)$$

①

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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Let $x_1 = Ae^{i\omega t}$ $x_2 = Be^{i\omega t}$ $x_3 = Ce^{i\omega t}$ $x_4 = De^{i\omega t}$

$$\underbrace{\begin{pmatrix} -m\omega^2-2k & k & 0 & 0 \\ k & -m\omega^2-2k & k & 0 \\ 0 & k & -m\omega^2-2k & k \\ 0 & 0 & k & -m\omega^2-2k \end{pmatrix}}_{B}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Normal mode are obtained from above matrix ie $\det(B) = 0$

Solve for 4 ω roots from 8^{th} order ~~poly~~. characteristic polynomial.

$$(-m\omega^2-2k)\det\begin{pmatrix} -m\omega^2-2k & k & 0 \\ k & -m\omega^2-2k & k \\ 0 & k & -m\omega^2-2k \end{pmatrix} - k\det\begin{pmatrix} k & k & 0 \\ 0 & -m\omega^2-2k & k \\ 0 & k & -m\omega^2-2k \end{pmatrix} = 0$$

$$= (-m\omega^2-2k)\Big[(-m\omega^2-2k)\left[(-m\omega^2-2k)^2-k^2\right] - k\left[k(-m\omega^2-2k) - 0^2\right]\Big]$$

$$-k\Big[k\left[(-m\omega^2-2k)^2-k^2\right] - k\left[0-0\right]\Big] = 0$$

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$$= (-mw^2-2k)\Big[(-mw^2-2k)\big[(-mw^2-2k)^2-k^2\big]-k^2(-mw^2-2k)\Big]$$
$$-k^2(-mw^2-2k)^2+k^4=0$$

$$(-mw^2-2k)\Big[(-mw^2-2k)^3-k^2(-mw^2-2k)-k^2(-mw^2-2k)\Big]$$
$$-k^2(-mw^2-2k)^2+k^4=0$$

$$(-mw^2-2k)\Big[(-mw^2-2k)^3-3k^2(-mw^2-2k)\Big]+k^4=0$$

$$(mw^2+2k)^3 = (m^2w^4+4kmw^2+4k^2)(mw^2+2k)$$
$$= m^3w^6+4k\cancel{m}^2w^4+\underbrace{4k^2w^2m+2km^2w^4}+8k^2mw^2$$
$$+8k^3$$
$$= m^3w^6+6km^2w^4+12k^2mw^2+\longleftarrow+8k^3$$

$$\Big[-(mw^2+2k)^3+3k^2mw^2+6k^3\Big] = -m^3w^6-6km^2w^4-9k^2mw^2$$
$$-2k^3$$

$$+(mw^2+2k)\big[m^3w^6+6km^2w^4+9k^2mw^2+2k^3\big]+k^4=0$$

$$m^4w^8+6km^3w^6+9k^2m^2w^4+2k^3mw^2$$
$$2km^3w^6\quad 12k^2m^2w^4+18k^3mw^2+4k^4+k^4=0$$

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$$m^4 w^8 + 8 k m^3 w^6 + 21 k^2 m^2 w^4 + 20 k^3 m w^2 + 5 k^4 = 0$$

Let $x = w^2$

$$m^4 x^4 + 8 k m^3 x^3 + 21 k^2 m^2 x^2 + 20 k^3 m x + 5 k^4 =$$

Solve this 4th order polynomial to find x and therefore w. to obtain 4-roots (w's) There are the sought after modes,

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PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

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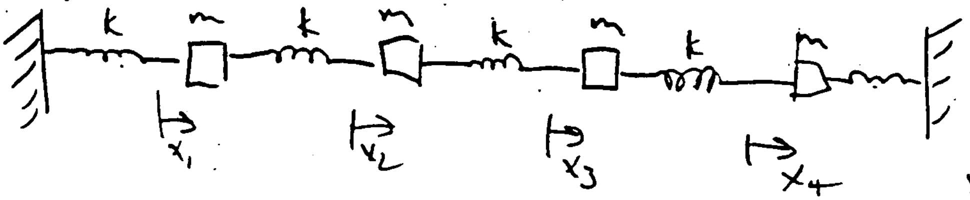

x_i's displacement from equilibrium

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2 + \frac{1}{2} m \dot{x}_4^2$$

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 \right)$$

$$V = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_2 - x_3)^2 + \frac{1}{2} k (x_3 - x_4)^2 + \frac{1}{2} k x_1^2 + \frac{1}{2} k x$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1 \; ; \quad \frac{\partial L}{\partial x_1} = k (x_1 - x_2) + k x_1$$

$$m \ddot{x}_1 - k (x_1 - x_2) - k x_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial \dot{x}_2} = m \dot{x}_2 \; ; \quad \frac{\partial L}{\partial x_2} = -k (x_1 - x_2) + k (x_2 - x_3)$$

$$m \ddot{x}_2 + k (x_1 - x_2) - k (x_2 - x_3) \qquad (2)$$

$$\frac{\partial L}{\partial \dot{x}_3} = m \dot{x}_3 \; ; \quad \frac{\partial L}{\partial x_3} = -k (x_2 - x_3) + k (x_3 - x_4)$$

$$m \ddot{x}_3 + k (x_2 - x_3) - k (x_3 - x_4) \qquad (3)$$

$$\frac{\partial L}{\partial \dot{x}_4} = m \dot{x}_4 \; ; \quad \frac{\partial L}{\partial x_4} = -k (x_3 - x_4) + k x_4$$

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$$m \ddot{x}_4 + k (x_3 - x_4) - k x_4 \qquad (4)$$