

**DEPARTMENT OF PHYSICS**  
**University of California, San Diego**  
**La Jolla, California 92093-0319**

**WRITTEN DEPARTMENTAL EXAMINATION - SPRING, 1988**

**PART I**

Each problem is worth 10 points.

**Problem 1**

- a. What is the value of the  $\lambda$ -point temperature in liquid helium?
- b. What is the  $\beta$ -decay half-life of the neutron?
- c. What is the typical binding energy of a Cooper pair?
- d. Calculate the orbit radius of a "synchronous satellite". [ $G = 6.67 \times 10^{-8}$  c.g.h.;  $M_{\oplus} = 6.0 \times 10^{27}$  g]
- e. For the molecules in this room estimate the fraction of collisions that are 3-body. That is, what is the ratio of the 3-body/2-body collision rate? [Loschmidt's number  $n_0 = 3 \times 10^{19}$  cm $^{-3}$ ]

**Problem 2**

In the one-dimensional wave packet of a quantum mechanical particle of mass  $m$  with wave function

$$\psi(x,t) = \int_{-\Delta k}^{\Delta k} dk f(k) \exp[i(kx - \omega(k)t)]$$

$f(k)$  is real and sharply peaked around  $k_0$  with a width  $\Delta k$ .

Calculate the passage time  $\Delta t$  through a fixed point for the wave packet if

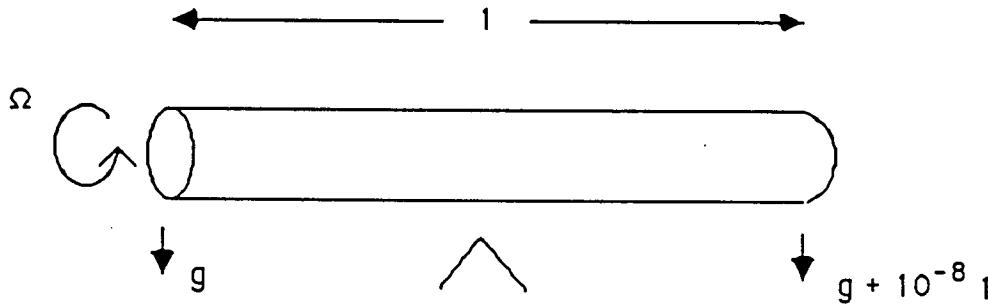
$$\omega(k) = \frac{\hbar k^2}{2m}$$

### Problem 3

- I) Describe briefly a technique for measuring each of the following:
- The magnetic field at the iron nucleus in ferromagnetic iron.
  - The distance from the earth to the moon to an accuracy of order 1 meter.
  - The rotation period of the earth to an accuracy of a small fraction of a second.
  - The velocity of sound in a solid sample in the form of a cylinder 1 cm long by 1 cm in diameter.

### Problem 4

There is considerable interest in producing a device which can measure the gradients of the local gravity field but is insensitive to a uniform field. A device which, in principle, could do this is shown in the sketch.



A solid cylinder of radius 1 cm, length 2 m, density = 9 gm/cc is rotated about its axis at constant angular velocity  $\Omega = 10^3$  radians/sec. The axis of rotation is in a horizontal plane. The rod is suspended from gimbals such that it can rotate freely about a point at the center of the rod. If the acceleration due to gravity varies by  $10^{-8}$  cm/sec<sup>2</sup>/cm find the torque about the center point and the resulting precession frequency.

### Problem 5

What are the boundary conditions for a static magnetic field  $B$  at the surface of a:

- a). superconductor
- b). material with  $\mu = 0$
- c). material with  $\mu = \infty$

Use your answers to a)-c) to find the force on a long straight wire carrying a current  $I$  which runs parallel to and is a distance  $d$  away from the surface of a:

- d). superconductor
- e). ferromagnet

### Problem 6

An electron and a positron orbit around each other under the influence of their mutual electrostatic attraction. In the center of mass the motion is non-relativistic, the orbit is circular with radius  $R$  and radiation can be neglected. Their center of mass moves with a relativistic velocity  $V$  along the  $x$  axis.

- a). What is the period of the orbit as viewed in the center of mass frame?
- b). If in the center of mass the orbit is in the  $y$ - $z$  plane, what is the shape and period of the orbit as seen in the laboratory?
- c). How would your answers to a)-b) change if the motion in the center of mass were relativistic but radiation could still be neglected?

Problem 7.

Consider a "molecule" described by two identical spin- $J$  particles bound by an interaction  $V(R_{12})$ , where  $R_{12}$  is the interparticle distance. There is no spin-orbit coupling.

- (a) If  $\ell$  is the rotational angular momentum quantum number, and  $J_{\text{tot}}$  is the total spin quantum number, determine the selection rule for  $(-1)^{\ell}$  in terms of  $(-1)^{J_{\text{tot}}}$  and the statistics of the particles (i.e.,  $(-1)^{2J}$ ). \*

A diatomic molecule with identical spin- $J$  nuclei can be treated as in (a) provided the electronic state is  ${}^1\Sigma_g^+$  (i.e., a complete symmetric state). In addition, the coupling between the nuclear spins is negligible, so the energy of the molecule does not depend on  $J_{\text{tot}}$ .

- (b) Obtain the equilibrium ratio  $r = N_- / N_+$ , where  $N_{\pm}$  are the numbers of molecules with  $(-1)^{\ell} = \pm 1$ , at temperatures  $T \gg T_{\text{rot}}$ , in the two cases

$$(i) \quad J = 3/2$$

$$(ii) \quad J = 2 .$$

$(k_B T_{\text{rot}} = \hbar^2 / I$ , where  $I$  is the moment of inertia for molecular rotations).

- (c) For the same values of  $J$  obtain  $r$  in the limit  $T \ll T_{\text{rot}}$ , when only  $\ell = 0$  and  $\ell = 1$  states need be considered.

- (d) In the absence of a suitable catalyst,  $r$  remains "frozen", and cannot come to equilibrium through normal molecular collisions. Briefly explain why.

\*Hint: Remember that if two equal spins are combined, the symmetry of the spin wavefunction depends only on whether  $J_{\text{tot}}$  is even or odd. The symmetry of the state  $J_{\text{tot}} = M_{\text{tot}} = 2J$  is easily determined.

### Problem 8

Briefly answer the following:

- a) Why is the lifetime of a neutron so much longer than that of a muon?
- b) The  $^2S$  excited state of atomic hydrogen has a very long lifetime. Why? Be as complete here as you can.
- c) What is the main decay mode(s) of  $^2S$  atomic hydrogen in a vacuum?
- d) What transition is involved in the 21 cm line of hydrogen?
- e) What causes the energy difference between the state involved in d)?

### Problem 9

Use contour integration to evaluate

$$\int_0^\infty \frac{x^{v-1}}{1+x} dx$$

for  $0 < v < 1$ .

**DEPARTMENT OF PHYSICS  
University of California, San Diego  
La Jolla, California 92093-0319**

**WRITTEN DEPARTMENTAL EXAMINATION - SPRING, 1988**

**PART II**

**Problem 10**

- a. The energy source in the sun is thermonuclear reactions that essentially convert hydrogen to helium. The total binding energy of the  $\alpha$ -particle is about 28 MeV. The luminosity/mass ratio for the sun is  $L_\odot/M_\odot \approx 2$  (c.g.s. units). Estimate the time for the sun (mostly hydrogen) to exhaust 10% of its fuel. [Proton mass:  $m_p \approx 1.7 \times 10^{-24}$  g].
- b. What is the energy domain such that the classical orbit of an electron in the coulomb field of a proton is physically meaningful?

In Rutherford scattering of  $\alpha$ -particles off gold nuclei what would be the classical scattering energy domain?

Is 20 MeV in the classical limit? Are there quantum mechanical corrections to the classical cross section.

### Problem 11

A quantum mechanical particle of mass  $m$  moves in one-dimensional potential. The states  $|\psi_n\rangle$  designate a complete set of energy eigenstates with eigenvalues  $E_n$  and  $|\psi_0\rangle$  is a selected bound state.

Prove the identity

$$\sum_n \frac{2m}{\hbar^2} |x_{no}|^2 (E_n - E_0) = 1$$

where

$$x_{no} = \langle \psi_n | x | \psi_0 \rangle$$

with the coordinate operator  $x$ .

### Problem 12

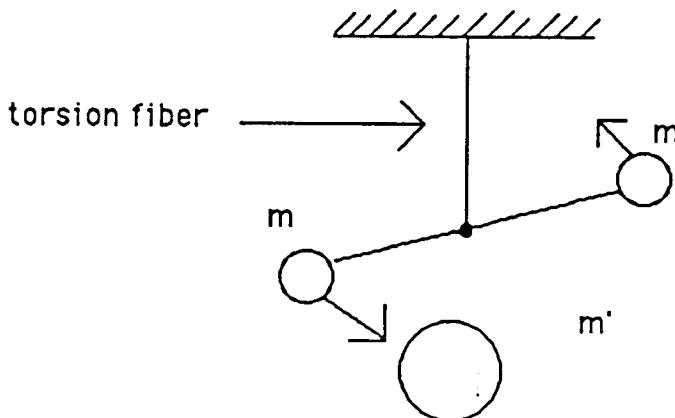
A fixed electric dipole  $\mathbf{P} = |\mathbf{P}| \hat{\mathbf{e}}_x$  is located at the origin of an  $(x,y)$  coordinate system, and an electron is incident from infinity with impact parameter  $a$  and initial velocity  $v_0 = -v_0 \hat{\mathbf{e}}_x$  (see figure). Determine the minimum distance between the electron and the dipole during the subsequent motion assuming that

$$\frac{e|\mathbf{P}|}{a^2} < mv_0^2/2.$$

(Hint: Is the Hamilton-Jacobi equation separable in suitably chosen coordinates?)

### Problem 13

Recent experiments testing Newton's law of gravitation have suggested that it is in error due to the existence of a "fifth force." The original measurement of the gravitational constant, G, was performed by Cavendish using a torsion pendulum as illustrated below.



Design a version of this experiment to measure the dependence of G on distance in the range of 10 to 20 centimeters. Use the steps below and write only a few sentences for each answer.

1.
  - a). Write the equation of motion for the torsion oscillator, including damping, and find an expression for the period of oscillation in terms of the torsion constant and the moment of inertia of the pendulum arm.
  - b). Write an expression for the angular displacement of the pendulum due to the gravitational attraction of the perturbing mass.
2. Select values for the length for the radius arm of the pendulum, the size of the masses on the pendulum, m, and of the perturbing mass, m'. Describe the criteria that would set upper and lower limits for each.
3. Assume the period of the pendulum is  $10^3$  seconds and the Q is  $10^3$ . Compute the expected thermal noise level of the angular displacement. What fractional error in the measurement of G would this lead to?
4. Sketch and describe the scheme you would use to measure the angle of deflection of the torsion pendulum. Estimate the precision of the measurement and compare it to the thermal noise limit computed above.

### Problem 14

A relativistic particle with charge of q moves with a relativistic velocity V along a line parallel to the (planar) surface of a very good conductor. If the distance between the conductor and the particle is d, find the force on the particle. Justify any assumptions that you make.

### Problem 15

State the condition for two distinct phases of a substance (e.g., solid and liquid) to be in equilibrium with each other at some temperature and pressure.

Hence obtain the Clausius-Clapeyron equation for the slope of the melting curve  $P_m(T)$  in the P-T (pressure-temperature) phase diagram:

$$\frac{dP_m(T)}{dT} = \frac{S_l - S_s}{V_l - V_s}$$

Where  $V_l$ ,  $V_s$  are the molar volumes in the liquid ( $l$ ) and solid ( $s$ ) phases, and  $S_l$  and  $S_s$  are the molar entropies.

### Problem 16

Use the Clausius-Clapeyron equation (Problem 16) to find the melting curve of  ${}^3\text{He}$ :

-Take the molar volumes of the liquid and solid to be independent of pressure and temperature. ( $V_l > V_s$ )

-Model solid  ${}^3\text{He}$  as a rigid lattice (no phonons) with completely disordered spins.

-Model liquid  ${}^3\text{He}$  as an ideal spin-1/2 Fermi gas with molar volume  $V_l$  and Fermi velocity  $v_F$ .

Obtain the low temperature molar entropies of the liquid and solid phases (i.e., at temperatures small compared to the Fermi temperature of the liquid). Sketch  $S$  versus  $T$  for the two phases.\*

Given that  $P_m(T=0)$  is large, obtain the shape of the melting curve  $P_m(T)$ , and sketch it. What is unusual about it? (Mark solid and liquid phases)

\*Hint: To obtain the specific heat of the Fermi gas, you may treat the density of states near the Fermi level as independent of energy, in which case the chemical potential is temperature-independent. Your results can be expressed in terms of mathematical constants  $A_n^\pm$ .

$$A_n^\pm = \int_0^\infty dx \frac{x^n}{\exp(x) \pm 1}$$

### Problem 17

Use a simple variation method to find a lower bound on the magnitude of the binding energy of atomic helium. Find as good a bound as you can. What are the quantum numbers of your state?

### Problem 18

Consider the ordinary differential equation

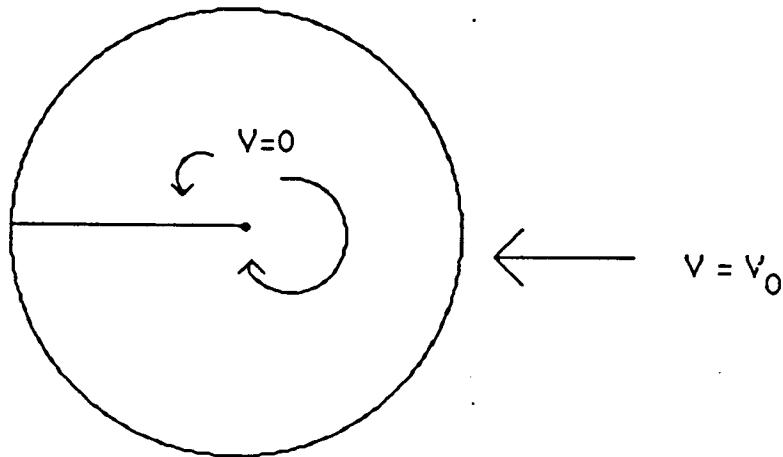
$$\frac{d^2}{dx^2} \left( (1-x^2) f(x) \right) + \mu(\mu-1) f(x) = 0$$

- a). Where can solutions of this equation be singular? What can you say about the singularities?
- b). By seeking a solution as a power series expansion around  $x = 0$ , show that for integral  $\mu$ , there is one solution that is analytic everywhere.
- c). Give this solution explicitly for  $\mu = 2$ .

### Problem 19

Use separation of variables and superposition of elementary solutions to solve the following electrostatics problem.

The unit circle is held at potential  $V_0$  and the negative x axis of  $-1 < x < 0$  is held at potential zero



- Find the electrostatic potential inside the unit circle.
- Find a closed expression for the electric field along the positive x axis.

The Laplacian in polar coordinates is:

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

SPRING 87-88  
PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 17

Please insert on each page

the Problem No. 1 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

$$(a) \cosht = \frac{e^t + e^{-t}}{2}$$

$$\text{Thus } \int_0^x \frac{\cosht - 1}{t} dt$$

$$= \int_0^x \frac{e^t + e^{-t} - 2}{2t} dt.$$

$$= \int_0^x \left( \frac{1}{2} \frac{e^t}{t} + \frac{1}{2} \frac{e^{-t}}{t} - \frac{1}{t} \right) dt = I$$

$$e^t = \sum_{p=0}^{\infty} \frac{(t)^p}{p!} \quad \text{and} \quad e^{-t} = \sum_{p=0}^{\infty} \frac{(-t)^p}{p!} = \sum_{p=0}^{\infty} (-1)^p \frac{(t)^p}{p!}$$

$$\text{Thus } I = \frac{1}{2} \sum_{p=0}^{\infty} \frac{1}{p!} \int_0^x \frac{t^p}{t} dt + \frac{1}{2} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \int_0^x \frac{t^p}{t} dt - \int_0^x \frac{dt}{t}$$

$$\text{Here } \int_0^x t^{p-1} dt = \left[ \frac{1}{p} t^p \right]_0^x = \frac{x^p}{p} \quad \text{when } p \neq 0$$

Thus

$$I = \frac{1}{2} \cancel{\int_0^x \frac{dt}{t}} + \frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p!} \frac{x^p}{p} + \frac{1}{2} \cancel{\int_0^x \frac{dt}{t}} + \frac{1}{2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \frac{x^p}{p} - \cancel{\int_0^x \frac{dt}{t}}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 1 and your Identification No. 34

★ ★ ★ ★ ★ ★

$$I = \frac{1}{2} \sum_{p=1}^{\infty} \frac{x^p}{p! \cdot p} + \frac{1}{2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p!} \cdot \frac{x^p}{p}$$

When  $p = \text{odd numbers}$

$$I = 0$$

when  $p = 2, 4, 6, \dots$  even numbers.

$$I = \sum_{\substack{p=2,4,6,\dots \\ \text{even number}}}^{\infty} \frac{x^p}{(p!)^p} //$$

$$(b) e^{x^2/2} \int_0^x e^{-t^2/2} dt =$$

$\underbrace{\hspace{10em}}$

$$\equiv I$$

By integral by parts

$$I = t e^{-t^2/2} \Big|_0^x - \int_0^x t \cdot (-\frac{1}{2}) e^{-t^2/2} dt$$

$$= x e^{-x^2/2} + \int_0^x t^2 e^{-t^2/2} dt$$

$$= x e^{-x^2/2} + \left[ \frac{1}{3} t^3 e^{-t^2/2} \right]_0^x - \frac{1}{3} \int_0^x t^3 (-t) e^{-t^2/2} dt$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 1 and your Identification No. 3/4

★ ★ ★ ★ ★ ★ ★

$$\begin{aligned}
 I &= x e^{-x^2/2} + \frac{1}{3} x^3 e^{-x^2/2} + \frac{1}{3} \int_0^x t^4 e^{-t^2/2} dt \\
 &= \left( x + \frac{1}{3} x^3 \right) e^{-x^2/2} + \frac{1}{3} \cdot \frac{1}{5} t^5 e^{-t^2/2} \Big|_0^x - \frac{1}{3} \cdot \frac{1}{5} \int_0^x t^5 (-t) e^{-t^2/2} dt \\
 &= \left( x + \frac{1}{3} x^3 + \frac{1}{3} \cdot \frac{1}{5} x^5 \right) e^{-x^2/2} + \frac{1}{3} \cdot \frac{1}{5} \int_0^x t^6 e^{-t^2/2} dt \\
 &= \dots \\
 &= \left( x + \frac{1}{3} x^3 + \frac{1}{3} \cdot \frac{1}{5} x^5 + \dots + \frac{1}{(2p-1)!!} x^{2p-1} + \dots \right) e^{-x^2/2} \\
 &= \sum_{p=1}^{\infty} \frac{x^{2p-1}}{(2p-1)!!} e^{-x^2/2} \quad \text{where } (2p-1)!! = (2p-1)(2p-3)\dots \\
 &\quad \text{ex } 5!! = 5 \cdot 3 \cdot 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus } & e^{x^2/2} \int_0^x e^{-t^2/2} dt \\
 &= e^{x^2/2} \cdot \sum_{p=1}^{\infty} \frac{x^{2p-1}}{(2p-1)!!} e^{-x^2/2} \\
 &= \sum_{p=1}^{\infty} \frac{x^{2p-1}}{(2p-1)!!}
 \end{aligned}$$

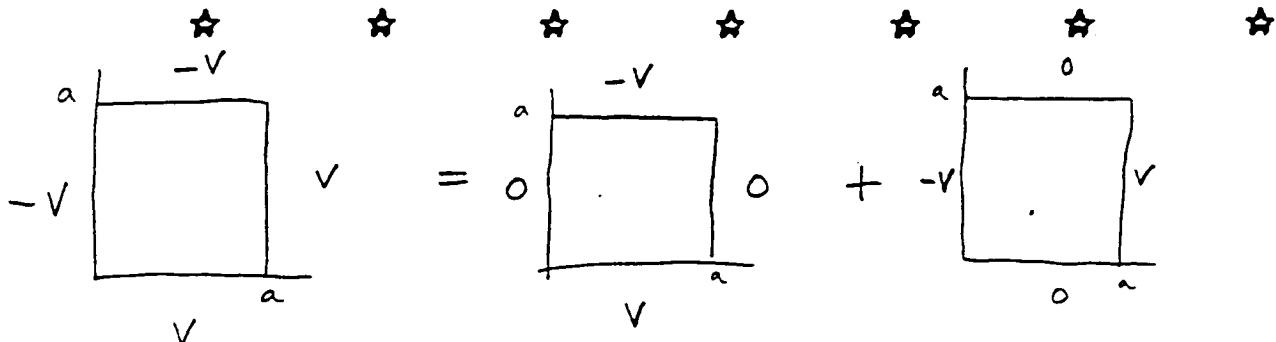
NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

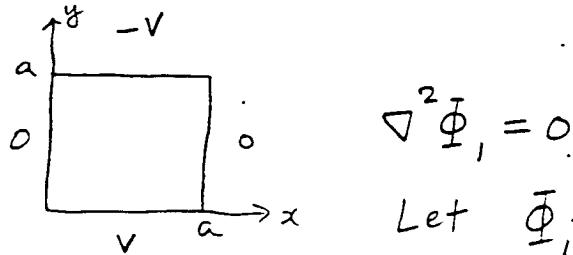
Score = 10

Please insert on each page

the Problem No. 5 and your Identification No. 34



Consider



$$\nabla^2 \Phi_1 = 0$$

$$\text{Let } \Phi_1 = X(z)Y(y)$$

$$Y \frac{d^2X}{dz^2} + X \frac{d^2Y}{dy^2} = 0$$

Thus

$$\underbrace{\frac{1}{X} \frac{d^2X}{dz^2}}_{=-\alpha^2} + \underbrace{\frac{1}{Y} \frac{d^2Y}{dy^2}}_{=\alpha^2} = 0$$

Thus

$$X = C \cos \alpha z + D \sin \alpha z$$

$$X(0) = 0 \rightarrow C = 0$$

$$X(a) = 0 \rightarrow D \sin \alpha a = 0$$

$$Y = A e^{-\alpha y} + B e^{+\alpha y}$$

$$X(z) = D \sin \frac{n\pi}{a} z$$

$$\alpha a = n\pi$$

$$n = 1, 2, \dots$$

$$\text{Thus } \Phi_1(x,y) = \sum_{n=1}^{\infty} (A_n e^{-\alpha_n y} + B_n e^{+\alpha_n y}) \cdot \sin \alpha_n x$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Please insert on each page

Score = \_\_\_\_\_

the Problem No. 5

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

$$\Phi_1(x, 0) = V = \sum_{n=1}^{\infty} (A_n + B_n) \sin \alpha_n x$$

$$\int_0^a V \sin \alpha_n x \, dx = \sum_{n=1}^{\infty} \int_0^a (A_n + B_n) \sin \alpha_n x \sin \alpha_n' x \, dx$$

$$\int_0^a \sin \alpha_n x \sin \alpha_n' x \, dx = \frac{a}{2} \delta_{nn'}$$

$$\int_0^a V \sin \alpha_n x \, dx = \frac{a}{2} (A_n + B_n)$$

$$= - \frac{V}{\alpha_n} [\cos \alpha_n a]_0^a$$

$$= - \frac{V}{\alpha_n} [\cos \alpha_n a - 1] = \frac{a}{2} (A_n + B_n) \quad \alpha_n = \frac{n\pi}{a}$$

$$\text{Thus } \frac{a}{2} (A_n + B_n) = - \frac{Va}{n\pi} [\cos \alpha_n a - 1] \quad \text{--- (1)}$$

$$\Phi_1(x, a) = -V = \sum_{n=1}^{\infty} (A_n e^{-\alpha_n a} + B_n e^{\alpha_n a}) \sin \alpha_n x$$

$$\int_0^a -V \sin \alpha_n x \, dx = \sum_{n=1}^{\infty} \int_0^a (A_n e^{-\alpha_n a} + B_n e^{\alpha_n a}) \sin \alpha_n x \sin \alpha_n' x \, dx$$

$$\int_0^a -V \sin \alpha_n x \, dx = \frac{a}{2} (A_n e^{-\alpha_n a} + B_n e^{\alpha_n a})$$

$$= \frac{V}{\alpha_n} [\cos \alpha_n a]_0^a = \frac{V}{\alpha_n} [\cos \alpha_n a - 1]$$

**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 5

and your Identification No. 34



Thus

$$\frac{a}{2} (A_n e^{-\alpha_n a} + B_n e^{\alpha_n a}) = \frac{Va}{n\pi} [\cos \alpha_n a - 1] -$$

From ①,

$$A_n + B_n = -\frac{2V}{n\pi} [\cos \alpha_n a - 1] - ②$$

From ②,

$$A_n e^{-\alpha_n a} + B_n e^{\alpha_n a} = \frac{2V}{n\pi} [\cos \alpha_n a - 1]$$

$$\text{Let } \frac{2V}{n\pi} [\cos \alpha_n a - 1] \equiv \xi$$

Thus

$$A_n + B_n = -\xi$$

$$A_n e^{-\alpha_n a} + B_n e^{\alpha_n a} = \xi$$

$$A_n e^{-\alpha_n a} + (-A - \xi) e^{\alpha_n a} = \xi$$

$$A_n (e^{-\alpha_n a} - e^{\alpha_n a}) = \xi (1 + e^{\alpha_n a})$$

$$A_n = \xi \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} = \frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} -$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 5

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Thus

$$\Phi_2(x, y) = \sum_{n=1}^{\infty} (-A_n e^{-\alpha_n x} - B_n e^{+\alpha_n x}) \sin \alpha_n y$$

Thus the total potential

$$\bar{\Phi} = \bar{\Phi}_1 + \bar{\Phi}_2$$

$$= \sum_{n=1}^{\infty} (A_n e^{-\alpha_n x} + B_n e^{\alpha_n x}) \sin \alpha_n x$$

$$+ \sum_{n=1}^{\infty} (-A_n e^{-\alpha_n x} - B_n e^{\alpha_n x}) \sin \alpha_n y$$

$$= \sum_{n=1}^{\infty} \left\{ A_n (e^{-\alpha_n y} - e^{-\alpha_n x}) \sin \alpha_n x \sin \alpha_n y \right.$$

$$\left. + B_n (e^{\alpha_n y} - e^{\alpha_n x}) \sin \alpha_n x \sin \alpha_n y \right\}$$

where

$$A_n = \frac{2V}{n\pi} [\cos \alpha_n a - 1] \cdot \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}}$$

$$B_n = -\frac{2V}{n\pi} [\cos \alpha_n a - 1] \cdot \frac{1 + e^{-\alpha_n a}}{e^{\alpha_n a} - e^{-\alpha_n a}}$$

$$\alpha_n = \frac{n\pi}{a}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

$n = 1, 2, \dots$

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 5 and your Identification No. 34

★ ★ ★ ★ ★ ★

$$B_n = -\zeta - A_n = -\zeta - \frac{1}{3} \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}}$$

$$= -\zeta \left[ \frac{e^{-\alpha_n a} - e^{\alpha_n a} + 1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} \right]$$

$$= -\zeta \frac{1 + e^{-\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}}$$

$$= -\frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{-\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} \quad (5)$$

Hence the potential  $\Phi_1(x, y)$  is

$$\Phi_1(x, y) = \sum_{n=1}^{\infty} (A_n e^{-\alpha_n y} + B_n e^{\alpha_n y}) \sin \alpha_n x \quad (6)$$

$$\text{where } A_n = \frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} \quad (7)$$

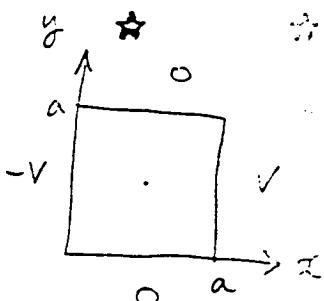
$$B_n = -\frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{-\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} \quad (8)$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score \_\_\_\_\_

Please insert on each page

 the Problem No. 5 end your identification No. 34


$$\nabla^2 \bar{\Omega}_2 = 0 \quad \bar{\Omega}_2 = XY$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

$$= x^2 \quad = -x^2$$

$$X = A e^{-nx} + B e^{nx}$$

$$Y = C \cos ny + D \sin ny = 0$$

$$Y_{(n)} = 0 \rightarrow C = 0$$

$$Y_{(n)} = 0 \rightarrow D \sin ny = 0 \quad n = 1, 2, \dots$$

Thus  $\bar{\Omega}_2(x, y) = \sum_{n=1}^{\infty} (A_n e^{-nx} + B_n e^{nx}) \sin ny$

$$\bar{\Omega}_2(0, y) = -V = \sum_{n=1}^{\infty} (A_n' + B_n') \sin ny$$

$$\bar{\Omega}_2(a, y) = V = \sum_{n=1}^{\infty} (A_n' e^{-na} + B_n' e^{na}) \sin ny$$

Likewise the others are,

$$\int_{-V}^V \sin dy = \sum_{n=1}^{\infty} \int_0^a (A_n' + B_n') \sin ny \sin dy$$

$$\int_0^a V \sin dy = \sum_{n=1}^{\infty} \int_0^a (A_n' e^{-na} + B_n' e^{na}) \sin ny$$

NOTE: If you use more than one page for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 5 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★ ★

$$\text{Thus } \frac{a}{2} (A_n' + B_n') = \frac{Va}{n\pi} [\cos \alpha_n a - 1]$$

$$\frac{a}{2} (A_n' e^{-\alpha_n a} + B_n' e^{\alpha_n a}) = -\frac{Va}{n\pi} [\cos \alpha_n a - 1]$$

Thus

$$A_n' + B_n' = \frac{-2V}{n\pi} [\cos \alpha_n a - 1]$$

$$A_n' e^{-\alpha_n a} + B_n' e^{\alpha_n a} = -\frac{2V}{n\pi} [\cos \alpha_n a - 1]$$

$$e^{-\frac{2V}{n\pi} [\cos \alpha_n a - 1]} = ?$$

$$A_n' - B_n' = ?$$

$$A_n' e^{-\alpha_n a} - B_n' e^{\alpha_n a} = -?$$

$$\therefore A_n' = -? \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} = -\frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} = -A_n$$

$$B_n' = ? \frac{1 + e^{-\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} = \frac{2V}{n\pi} [\cos \alpha_n a - 1] \frac{1 + e^{-\alpha_n a}}{e^{-\alpha_n a} - e^{\alpha_n a}} = -B_n$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

 the Problem No. 5

 and your Identification No. 32


Thus the electric field

$$\vec{E} = -\nabla \phi$$

$$= -\frac{\partial \phi}{\partial x} \hat{x} - \frac{\partial \phi}{\partial z} \hat{z}$$

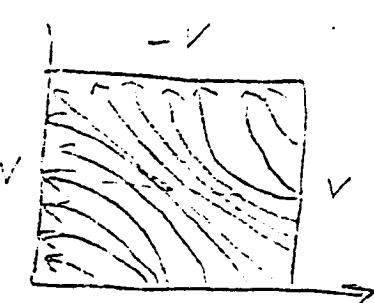
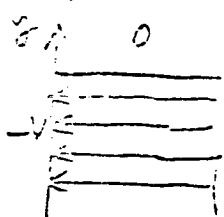
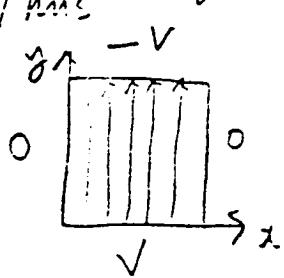
$$= -\left[ \sum_{n=1}^{\infty} (A_n e^{-\alpha_n x} + B_n e^{\alpha_n x}) d_n \cos \alpha_n z \right]$$

$$+ \left[ \sum_{n=1}^{\infty} (-B_n e^{-\alpha_n x} - A_n e^{\alpha_n x}) d_n \sin \alpha_n z \right] \hat{x}$$

$$- \left[ \sum_{n=1}^{\infty} (-B_n e^{-\alpha_n z} + A_n e^{\alpha_n z}) d_n \sin \alpha_n x \right]$$

$$- \left[ \sum_{n=1}^{\infty} (-B_n e^{-\alpha_n z} - A_n e^{\alpha_n z}) d_n \cos \alpha_n x \right] \hat{z}$$

Thus the lines of electric field is the



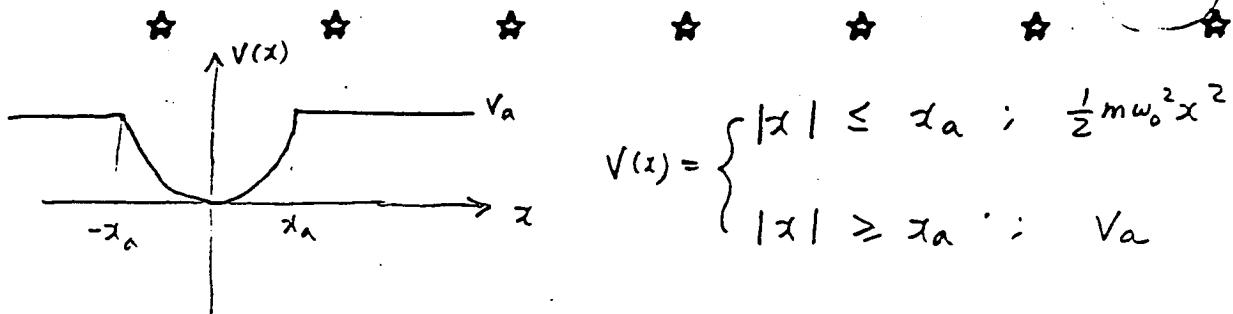
NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 7

Please insert on each page

the Problem No. 7 and your Identification No. 34



$$V(x) = \begin{cases} \frac{1}{2}m\omega_0^2x^2 & |x| \leq x_a \\ V_a & |x| \geq x_a \end{cases}$$

The unperturbed Hamiltonian  $H_0$  is

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 \quad \text{for all } x$$

The total Hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$



$$= \left( H_0 - \frac{1}{2}m\omega_0^2x^2 \right) + V(x)$$

$$= H_0 + V(x) - \frac{1}{2}m\omega_0^2x^2$$

$$= H_0 + H'$$



thus the perturbing Hamiltonian is

$$H' = V(x) - \frac{1}{2}m\omega_0^2x^2 = \begin{cases} 0 & |x| \leq x_a \\ V_a - \frac{1}{2}m\omega_0^2x^2 & |x| \geq x_a \end{cases}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 7 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

$$\text{Thus } \Delta E = \int_{-\infty}^{\infty} u_0^* H' u_0 dx$$

$$= \int_{-\infty}^{-x_a} u_0^2 \left( V_a - \frac{1}{2} m \omega_0^2 x^2 \right) dx$$

$$+ \int_{-x_a}^{x_a} u_0^2 \cdot 0 \cdot dx$$

$$+ \int_{x_a}^{\infty} u_0^2 \left( V_a - \frac{1}{2} m \omega_0^2 x^2 \right) dx$$

$$= A_0^2 \int_{-x_a}^{-x_a} e^{-\left(\frac{x}{x_0}\right)^2} \left( V_a - \frac{1}{2} m \omega_0^2 x^2 \right) dx$$

$$+ A_0^2 \int_{x_a}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} \left( V_a - \frac{1}{2} m \omega_0^2 x^2 \right) dx$$

$$\Delta E = 2 A_0^2 \int_{x_a}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} \left( V_a - \frac{1}{2} m \omega_0^2 x^2 \right) dx$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

 the Problem No. 7

 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

 If  $x_a \gg x_0$ , consider

$$\int_{x_a}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} dx = \int_0^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} dx - \int_0^{x_a} e^{-\left(\frac{x}{x_0}\right)^2} dx$$

$$\approx \int_0^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} dx - e^{-\left(\frac{x_a}{x_0}\right)^2} \approx \frac{1}{2} x_0 \sqrt{\pi} - e^{-\left(\frac{x_a}{x_0}\right)^2}$$

$$\int_{x_a}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} x^2 dx = \int_0^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} x^2 dx = \int_0^{x_a} e^{-\left(\frac{x}{x_0}\right)^2} x^2 dx$$

$$\approx \frac{1}{2} \cdot x_0^3 \underbrace{\Gamma\left(\frac{3}{2}\right)}_{= \frac{1}{2}\sqrt{\pi}} - e^{-\left(\frac{x_a}{x_0}\right)^2} x_a^2 = ?!$$

Thus

$$\Delta E \approx 2A_0^2 \left\{ V_a \cdot \left[ \frac{\sqrt{\pi} x_0}{2} - e^{-\left(\frac{x_a}{x_0}\right)^2} \right] - \frac{1}{2} m \omega_0^2 \left[ \frac{\sqrt{\pi}}{4} x_0^3 - x_a^2 e^{-\left(\frac{x_a}{x_0}\right)^2} \right] \right\},$$

 Normalize  $u_0$ ,

$$\int_{-\infty}^{\infty} u_0^2 dx = A_0^2 \int_{-\infty}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} dx = A_0^2 x_0 \cdot \sqrt{\pi} = 1$$

$$\text{Thus } A_0 = \left( \frac{1}{x_0 \sqrt{\pi}} \right)^{1/2}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 7 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Thus the perturbing energy is given as

$$\Delta E \approx \frac{2}{x_0 \sqrt{\pi}} \left\{ V_0 \left[ \frac{\sqrt{\pi}}{2} x_0 - e^{-\left(\frac{x_a}{x_0}\right)^2} \right] - \frac{1}{2} m \omega_0^2 \left[ \frac{\sqrt{\pi}}{4} x_0^3 - x_a^2 e^{-\left(\frac{x_a}{x_0}\right)^2} \right] \right\} - V_0 \frac{\sqrt{\pi}}{4} x_0 //$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 9.5

Please insert on each page

the Problem No. 8 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

(8) Potential is  $V(x) = cx^2 - gx^3 - fx^4$

Thus the total energy  $E$  is given by

$$E = \frac{p^2}{2m} + V(x) = T + V \quad \checkmark$$

Thus the average energy is

$$\langle E \rangle = \frac{\iint dp dx E e^{-\beta E}}{\iint dp dx e^{-\beta E}} \quad \checkmark$$

$$= \frac{\iint_{-\infty}^{\infty} dp dx \left( \frac{p^2}{2m} + cx^2 - gx^3 - fx^4 \right) e^{-\beta \left( \frac{p^2}{2m} + cx^2 - gx^3 - fx^4 \right)}}{\iint_{-\infty}^{\infty} dp dx e^{-\beta \left( \frac{p^2}{2m} + cx^2 - gx^3 - fx^4 \right)}} \quad \checkmark$$

$$= \frac{\int_{-\infty}^{\infty} dp \left( \frac{p^2}{2m} \right) e^{-\beta \frac{p^2}{2m}}}{\int_{-\infty}^{\infty} dp e^{-\beta \frac{p^2}{2m}}} + \frac{\int_{-\infty}^{\infty} dx (cx^2 - gx^3 - fx^4) e^{-\beta (cx^2 - gx^3 - fx^4)}}{\int_{-\infty}^{\infty} dx e^{-\beta (cx^2 - gx^3 - fx^4)}} \quad -\beta(cx^2 - gx^3 - fx^4)$$

$$= \langle T \rangle + \langle V \rangle - \textcircled{1}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 8 and your Identification No. 34

$$\begin{aligned} \langle T \rangle &= \frac{\int_{-\infty}^{\infty} dp \left(\frac{p^2}{2m}\right) e^{-\beta \frac{p^2}{2m}}}{\int_{-\infty}^{\infty} dp e^{-\beta \frac{p^2}{2m}}} = \frac{\frac{1}{2m} \left(\frac{2m}{\beta}\right)^{3/2} \Gamma(\frac{3}{2})}{\left(\frac{2m}{\beta}\right)^{1/2} \sqrt{\pi}} \quad (\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}) \\ &= \frac{1}{2m} \cdot \left(\frac{2m}{\beta}\right) \cdot \frac{1}{2} \\ &= \frac{1}{2\beta} = \frac{kT}{2} \quad - \textcircled{2} \end{aligned}$$

Because  $g$  and  $f$  are small, we can expand as foll

$$\begin{aligned} e^{-\beta(cx^2 - gx^3 - fx^4)} &= e^{-\beta cx^2} e^{\beta(gx^3 + fx^4)} \quad \checkmark \\ &\approx e^{-\beta cx^2} [1 + \beta(gx^3 + fx^4)] \quad \checkmark \end{aligned}$$

Thus

$$\begin{aligned} \langle v \rangle &\approx \frac{\int_{-\infty}^{\infty} dx (cx^2 - gx^3 - fx^4) e^{-\beta cx^2}}{\int_{-\infty}^{\infty} dx [1 + \beta(gx^3 + fx^4)] e^{-\beta cx^2}} \quad \checkmark \\ &= \frac{\int_{-\infty}^{\infty} dx (cx^2 - fx^4) e^{-\beta cx^2}}{\int_{-\infty}^{\infty} dx (1 + \beta fx^4) e^{-\beta cx^2}} \quad - \textcircled{3} \end{aligned}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 8 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Consider

$$\int_{-\infty}^{\infty} dx \ x^2 e^{-\beta c x^2} = \left(\frac{1}{\beta c}\right)^{3/2} \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2} \left(\frac{1}{\beta c}\right)^{3/2}$$

and

$$\int_{-\infty}^{\infty} dx \ x^4 e^{-\beta c x^2} = \left(\frac{1}{\beta c}\right)^{5/2} \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \left(\frac{1}{\beta c}\right)^{5/2}$$

$$= \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}$$

and

$$\int_{-\infty}^{\infty} dx e^{-\beta c x^2} = \left(\frac{1}{\beta c}\right)^{1/2} \sqrt{\pi}$$

Thus

$$= C \frac{\sqrt{\pi}}{2} \left(\frac{1}{\beta c}\right)^{3/2} + f \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}$$

$$\langle v \rangle = \frac{C \frac{\sqrt{\pi}}{2} \left(\frac{1}{\beta c}\right)^{3/2} + f \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}}{\left(\frac{1}{\beta c}\right)^{1/2} \sqrt{\pi} + \beta f \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}}$$

right idea

$$\approx \frac{2C(\beta c) + 3f}{(\beta c)^2 + 3\beta f}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 8 and your Identification No. 34

$$\begin{aligned} \langle v \rangle &= \frac{2c^2\beta + 3f}{c^2\beta^2 + 3\beta f} = \frac{2c^2 \frac{1}{kT} + 3f}{c^2 \frac{1}{k^2 T^2} + 3 \frac{f}{kT}} \\ &= \frac{3c^2 kT + 3f(k^2 T^2)}{c + 3f(kT)} \end{aligned}$$

Thus

$$\langle E \rangle = \langle T \rangle + \langle v \rangle = \frac{1}{2} kT + \frac{3c^2 kT + 3f(k^2 T^2)}{c + 3f(kT)} //$$

Thus heat capacity is

$$\begin{aligned} C_v &= \left( \frac{\partial E}{\partial T} \right)_V = \frac{k}{2} + \frac{(3c^2 k + 6fk^2 T)(c + 3fkT) - (3c^2 kT + 3fk^2 T^2)(c + 3fkT)^2}{(c + 3fkT)^2} \\ &= \frac{k}{2} + \frac{-3c^3 k + 6cfk^2 + 3c^2 fk^2 T + 18f^2 k^3 T - 9c^2 fk^2 T^2 - 9}{(c + 3fkT)^2} \\ &= \frac{k}{2} + \frac{3c^3 k + 6cfk^2 + 9f^2 k^3 T^2}{(c + 3fkT)^2} // \end{aligned}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Please insert on each page

Score = \_\_\_\_\_

the Problem No. 8

and your Identification No. 34



- And

$$\iint dp dx \times e^{-\beta E}$$

$$\langle x \rangle = \frac{\iint dp dx \times e^{-\beta E}}{\iint dp dx \times e^{-\beta E}}$$

$$= \frac{\int_{-\infty}^{\infty} dx \cdot x e^{-\beta(cx^2 - gx^3 - fx^4)}}{\int_{-\infty}^{\infty} dx e^{-\beta(cx^2 - gx^3 - fx^4)}}$$

$$\approx \frac{\int_{-\infty}^{\infty} dx \cdot x [1 + \beta(gx^3 + fx^4)] e^{-\beta cx^2}}{\int_{-\infty}^{\infty} dx [1 + \beta(gx^3 + fx^4)] e^{-\beta cx^2}}$$

$$= \frac{\int_{-\infty}^{\infty} \beta g x^4 e^{-\beta cx^2}}{\int_{-\infty}^{\infty} dx (1 + fx^4) e^{-\beta cx^2}}$$

$$= \frac{\beta g \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}}{\left(\frac{1}{\beta c}\right)^{1/2} \sqrt{\pi} + f \frac{3\sqrt{\pi}}{4} \left(\frac{1}{\beta c}\right)^{5/2}}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 8

and your Identification No. 34

★

★

★

★

★

★

★

$$\langle x \rangle = \frac{3\beta g}{4(\beta c)^2 + 3f} = \frac{3g \frac{1}{kT}}{4c^2 \frac{1}{k^2 T^2} + 3f}$$

$$= \frac{3g kT}{4c^2 + 3f k^2 T^2} \approx \frac{3gh}{4c^2} T$$

**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

??

Please insert on each page

the Problem No. 12

and your Identification No. 34



(a) Let's define the action integral

$$(2) S = \int_{t_1}^{t_2} L(g_i, \dot{g}_i, t) dt \quad (i=1, 2, \dots, n)$$

where  $L(g_i, \dot{g}_i, t)$  is Lagrangian of system.

Then, the variation of  $S$  becomes zero,

that is  $S$  has the extremum value.

$\delta S = 0$ : This is the <sup>Hamilton's</sup> principle of Least Action

$$\delta S = \delta \int_{t_1}^{t_2} L(g_i, \dot{g}_i, t) dt = 0 \quad \text{Least action?}$$

$$= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial g_i} \delta g_i + \frac{\partial L}{\partial \dot{g}_i} \delta \dot{g}_i \right) dt$$

$$\delta g_i(t_1) = \delta g_i(t_2) = 0 \quad (i=1, 2, \dots, n)$$

by integration by parts  $\Rightarrow \cancel{\frac{\partial L}{\partial \dot{g}_i} \delta \dot{g}_i} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{g}_i} \right) dt$

$$+ \int_{t_1}^{t_2} \frac{\partial L}{\partial g_i} \delta g_i dt$$

$$= \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial g_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{g}_i} \right) \right] \delta g_i dt$$

To satisfy all  $\delta g_i$

$$\frac{\partial L}{\partial g_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{g}_i} \right) = 0 \quad \text{the Lagrange equation}$$

Thus  $\delta S \rightarrow 0 \Rightarrow \frac{\partial L}{\partial g_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{g}_i} \right) = 0 \quad (i=1, 2, \dots, n)$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Please insert on each page

Score = \_\_\_\_\_

the Problem No. 12 and your Identification No. 34



(b) Because  $H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

$$\text{Thus } S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$$

$$= \int_{t_1}^{t_2} [p_i \dot{q}_i - H(q_i, p_i, t)] dt$$

$$\delta S = \int_{t_1}^{t_2} \left( \delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

by integration  
by parts

$$\stackrel{\cong}{=} \int_{t_1}^{t_2} p_i \delta \dot{q}_i dt - \int_{t_1}^{t_2} \dot{p}_i \delta q_i dt$$

$$+ \int_{t_1}^{t_2} \left( \delta p_i \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

$$= \int_{t_1}^{t_2} \left[ \left( -\frac{\partial H}{\partial q_i} - \dot{p}_i \right) \delta q_i + \left( \dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i \right] dt = 0$$

To satisfy for all  $\delta p_i, \delta q_i, (i=1, 2, \dots, n)$  why can you say

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (i=1, 2, \dots, n) \quad \text{and independently}$$

These are Hamilton's equations

(2)

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 12 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

- (c) In old coordinate phase space, the volume elements are

$$\int dq_1 dq_2 \dots dq_n dp_1 dp_2 \dots dp_n \quad -\textcircled{1}$$

By canonical tr, in the new coordinate the volume elements are

$$\int dQ_1 dQ_2 \dots dQ_n dP_1 dP_2 \dots dP_n \quad -\textcircled{2}$$

Liouville's theorem  $\Rightarrow \textcircled{1} = \textcircled{2} \checkmark$

Generally

$$\int dq_1 dq_2 \dots dq_n dp_1 dp_2 \dots dp_n = \int D dQ_1 dQ_2 \dots dQ_n dP_1 dP_2 \dots dP_n$$

where  $D$  is Jacobian

$$D = \frac{\partial(q_i, p_i)}{\partial(Q_i, P_i)} = \frac{\partial(q_i, p_i)}{\partial(Q_i, P_i)} \Big/ \frac{\partial(Q_i, P_i)}{\partial(Q_i, P_i)}$$

$$= \frac{\partial q_i}{\partial Q_i} \Big/ \frac{\partial P_i}{\partial p_i} \quad (i=1, 2, \dots, n)$$

By the generating function of canonical transformation

$$\bar{\Phi} = \bar{\Phi}(p_i, Q_i)$$

Thus

$$q_i = -\frac{\partial \bar{\Phi}}{\partial p_i}, \quad P_i = -\frac{\partial \bar{\Phi}}{\partial Q_i} \quad (i=1, 2, \dots, n)$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 12

and your Identification No. 34



Thus

$$D = - \frac{\partial^2 \Phi}{\partial Q_i \partial P_i} / - \frac{\partial^2 \Phi}{\partial P_i \partial Q_i} = 1.$$

Thus

$$\int d\theta_1 d\theta_2 \dots d\theta_n dp_1 \dots dp_n = \int dQ_1 dQ_2 \dots dQ_n dP_1 \dots dP_n$$

**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

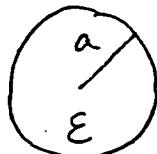
9

Please insert on each page

the Problem No. 13 and your Identification No. 34



(a)



By the vector potential

$$\vec{A}(\vec{r}) = \frac{e^{ikr}}{cr} \int \vec{J}(\vec{x}) d^3x \quad \text{by use of continuity } \epsilon_0 \cdot \vec{J} + \frac{\partial \vec{P}}{\partial t} = 0$$

$$= -ik\vec{P} \frac{e^{ikr}}{r}$$

By the incident wave, the dipole moment  $\vec{p}$  is induced

By this vector potential, we can obtain the  $\vec{B}$  &  $\vec{E}$  field as follows

In the near zone

$$\vec{B} = ik(\vec{n} \times \vec{p}) \frac{1}{r^2} \quad \text{---(1)}$$

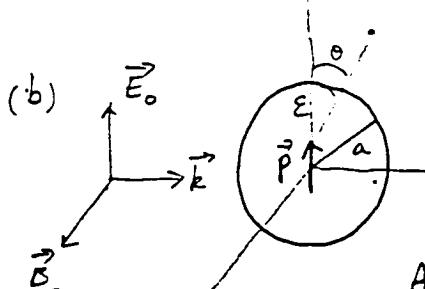
where  $\vec{n}$  is the normal vector to the observing point

$$\vec{E} = \frac{3(\vec{n} \cdot \vec{p})\vec{n} - \vec{p}}{r^3} \quad \text{---(2)}$$

and in the radiation zone

$$\vec{B}_r = k^2(\vec{n} \times \vec{p}) \frac{e^{ikr}}{r} \quad \text{---(3)}$$

$$\vec{E} = \vec{B}_r \times \vec{n} \quad \text{---(4)}$$



By the incident field  $\vec{E}_0$ , in the spher.

$$\vec{E}_{in} = \frac{3}{\epsilon + 2} \vec{E}_0 \quad \text{---(5)}$$

And the magnetization is

$$\vec{P} = \frac{\epsilon - 1}{4\pi} \cdot \vec{E}_{in} \quad \text{---(6)}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 13

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★ ★

$$\text{Thus } \vec{P} = \frac{\epsilon-1}{4\pi} \cdot \frac{3}{\epsilon+2} \vec{E}_0 = \frac{3}{4\pi} \frac{\epsilon-1}{\epsilon+2} \vec{E}_0$$

Thus the dipole moment is

$$\vec{P} = \frac{4\pi}{3} a^3 \vec{P} = \frac{4\pi}{3} a^3 \frac{3}{4\pi} \frac{\epsilon-1}{\epsilon+2} \vec{E}_0 = \left( \frac{\epsilon-1}{\epsilon+2} \right) a^3 \vec{E}_0$$

The radiated power per solid angle is given as

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{8\pi} k^4 |P|^2 \sin^2 \theta \\ &= \frac{c}{8\pi} k^4 \left( \frac{\epsilon-1}{\epsilon+2} \right) a^6 |\vec{E}_0|^2 \sin^2 \theta \quad -\textcircled{8} \end{aligned}$$

Thus total power is

$$\begin{aligned} P &= \frac{c}{8\pi} k^4 \left( \frac{\epsilon-1}{\epsilon+2} \right) a^6 |\vec{E}_0|^2 \underbrace{\int_0^{2\pi} \sin^2 \theta d\theta}_{=\frac{8\pi}{3}} \\ &= \frac{ck^4}{3} \left( \frac{\epsilon-1}{\epsilon+2} \right) a^6 |\vec{E}_0|^2 \\ &= \sigma S \quad -\textcircled{9} \end{aligned}$$

where  $\sigma$  is total cross section and  $S$  is the poynting

$$S = \frac{c}{8\pi} |\vec{E}_0|^2$$

Thus the total scattering cross section is

$$\sigma = \frac{\frac{ck^4}{3} \left( \frac{\epsilon-1}{\epsilon+2} \right)^2 a^6 |\vec{E}_0|^2}{\frac{c}{8\pi} |\vec{E}_0|^2} = \frac{8\pi}{3} k^4 \left( \frac{\epsilon-1}{\epsilon+2} \right)^2 a^6 \quad -\textcircled{10}$$

where  $(k = \frac{\omega}{c})$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score - \_\_\_\_\_

Please insert on each page

the Problem No. 13

and your Identification No. 34



or By eqs. ③ & ④ and  $\frac{dP}{d\Omega} = \frac{C}{8\pi} \text{Re} [r^2 \vec{n} \cdot \vec{E} \times \vec{B}^*]$ , we can obtain the same answer ⑩.

(c) If the volume is  $a^3$ ,

$$\text{the dipole moment becomes } \vec{p} = a^3 \vec{P} = a^3 \frac{3}{4\pi} \frac{\epsilon-1}{\epsilon+2} \vec{E}$$

Thus the total power  $P$  is

$$P = \frac{\omega^4}{3c^3} |\vec{p}|^2 = \frac{ck^4}{3} |\vec{p}|^2 \quad (k = \frac{\omega}{c})$$

$$= \frac{ck^4}{3} \left[ \frac{3}{4\pi} \left( \frac{\epsilon-1}{\epsilon+2} \right) \right]^2 a^6 |E_0|^2$$

$$= \sigma \cdot \Sigma$$

$$\text{Thus cross section } \sigma = \frac{\frac{ck^4}{3} \left[ \frac{3}{4\pi} \left( \frac{\epsilon-1}{\epsilon+2} \right) \right]^2 a^6 |E_0|^2}{\frac{c}{8\pi} |E_0|^2}$$

$$= \frac{8\pi}{3} k^4 \left( \frac{3}{4\pi} \right)^2 \left| \frac{\epsilon-1}{\epsilon+2} \right|^2 a^6$$

$$= \frac{8\pi}{3} \cdot \frac{9}{16\pi^2} k^4 \left| \frac{\epsilon-1}{\epsilon+2} \right|^2 a^6$$

$$= \frac{3}{2\pi} k^4 \left| \frac{\epsilon-1}{\epsilon+2} \right|^2 a^6$$

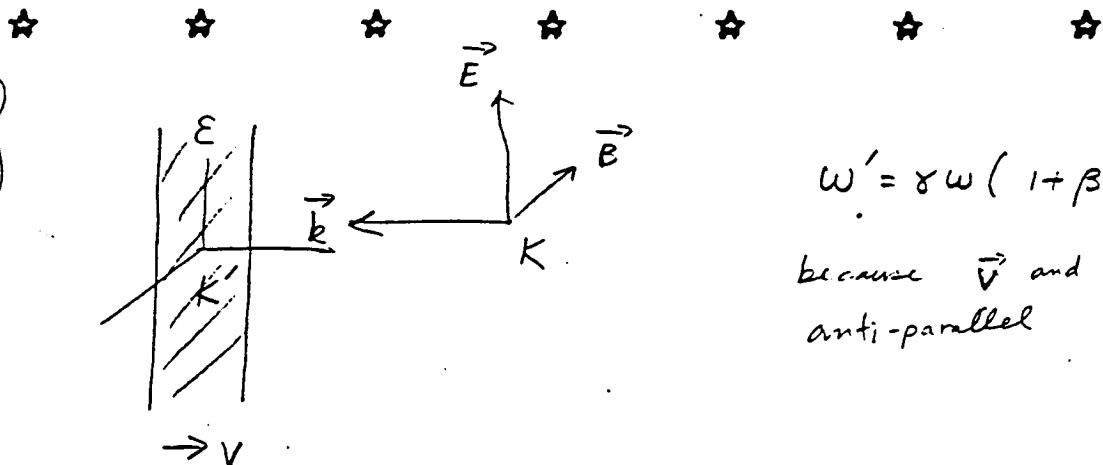
**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score =     

Please insert on each page

the Problem No. 14 and your Identification No. 34



$$\omega' = \gamma \omega (1 + \beta)$$

because  $\vec{v}$  and  $\vec{k}$  are anti-parallel

If  $K'$  is at rest,  $K'$  feels that the incident wave frequency is  $\omega$ . But if  $K'$  moves with speed  $v$  toward to incident wave.

Then  $K'$  feels the incident frequency is

$$\omega' = \gamma \omega (1 + \beta) \quad \text{where } \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

after reflection also  $K'$  feels the reflected frequency is

$$\omega'' = \gamma \omega' (1 + \beta)$$

Thus

$$\begin{aligned} \omega'' &= \gamma (1 + \beta) \gamma (1 + \beta) \omega \\ &= \gamma^2 (1 + \beta)^2 \omega \\ &= \frac{(1 + \beta)^2}{1 - \beta^2} \omega = \left( \frac{1 + \beta}{1 - \beta} \right) \omega \quad // \text{ frequency of reflected wave} \end{aligned}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 14 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

- And the transmitted wave frequency is

$$\omega''' = \gamma(\omega)(1-\beta)$$

because  $\vec{\beta}$  and  $\vec{k}$  are parallel

thus  $\omega''' = \gamma(1-\beta) \cancel{\gamma(1-\beta)\omega}$

$$\begin{aligned} & \text{should be} \\ & = \gamma(1-\beta)\gamma(1+\beta)\omega \\ & = \gamma^2(1-\beta^2)\omega \\ & = \omega \end{aligned}$$

$$= \gamma^2(1-\beta)^2\omega$$

$$= \frac{1-\beta^2}{1-\beta^2}\omega$$

$$= \left(\frac{1-\beta}{1+\beta}\right)\omega$$

frequency of transmitted  $k$

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = 7.5

Please insert on each page

the Problem No. 15 and your Identification No. 34

★ ★ ★ ★ ★ ★

$$U(r) = A e^{-r}$$

$$(a) F(p) = \frac{A}{(2\pi\hbar)^{3/2}} \int u(r) e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{r}} d^3 r$$

$$= \frac{A}{(2\pi\hbar)^{3/2}} \iint_0^\infty e^{-r} e^{-\frac{i}{\hbar} pr \cos\theta} r^2 dr d(\cos\theta)$$

$$= \frac{A}{(2\pi\hbar)^{3/2}} \int_0^\infty e^{-r} \frac{e^{-\frac{i}{\hbar} pr} - e^{\frac{i}{\hbar} pr}}{-\frac{i}{\hbar} pr} r^2 dr$$

$$= \underbrace{\frac{A\hbar}{(2\pi\hbar)^{3/2}\cdot i p}}_{\equiv N} \int_0^\infty e^{-r} (e^{-\frac{i}{\hbar} pr} - e^{\frac{i}{\hbar} pr}) r dr$$

$$\equiv N$$

$$= N \left[ \int_0^\infty e^{-(1 + \frac{i}{\hbar} p)r} r dr - \int_0^\infty e^{-(1 - \frac{i}{\hbar} p)r} r dr \right]$$

$$= N \left[ \frac{\Gamma(2)}{\left(1 + \frac{i}{\hbar} p\right)^2} - \frac{\Gamma(2)}{\left(1 - \frac{i}{\hbar} p\right)^2} \right]$$

$$= N \left[ \frac{\left(1 - \frac{i}{\hbar} p\right)^2 - \left(1 + \frac{i}{\hbar} p\right)^2}{\left(1 + \frac{p^2}{\hbar^2}\right)^2} \right] = N \frac{-\frac{2i}{\hbar} p}{\left(1 + \frac{p^2}{\hbar^2}\right)^2}$$

**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 15

and your Identification No. 34



Thus

$$F(p) = -\frac{A \hbar}{(2\pi\hbar)^{3/2} \lambda} \cdot \frac{1}{p} \cdot \frac{-\frac{2\lambda}{\hbar} p}{\left(1 + \frac{p^2}{\hbar^2}\right)^2}$$

$$= \frac{2A}{(2\pi\hbar)^{3/2}} \cdot \frac{1}{\left(1 + \frac{p^2}{\hbar^2}\right)^2}$$

let  $\hbar = 1$

$$F(p) \propto \frac{1}{(1+p^2)^2} \quad \text{3}$$

$$(b) \quad \text{Let } F(p) = \frac{c}{(1+p^2)^2}$$

Normalize

$$\int_0^\infty |F(p)|^2 p^2 dp = |c|^2 \int_0^\infty \frac{p^2}{(1+p^2)^4} dp$$

$$= \frac{|c|^2}{2} \int_{-\infty}^\infty \frac{p^2}{(1+p^2)^4} dp$$

$$= 1 \quad - \textcircled{1}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

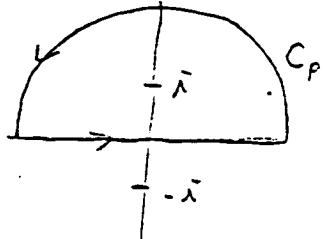
Please insert on each page

the Problem No. 15

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Consider  $\int \frac{p^2}{(1+p^2)^4} dp = \int_{-\infty}^{\infty} \frac{p^2}{(1+p^2)^4} dp + \int_{C_p} \frac{p^2}{(1+p^2)^4} dp$  as  $|p| \rightarrow \infty$



$$= 2\pi i \operatorname{Res}(p=i)$$

$$\operatorname{Res}(p=i) = \frac{1}{3!} \left(\frac{d}{dp}\right)^3 \left[ \frac{(p-i)^4 p^2}{(1+p^2)^4} \right]_{p=i}$$

$$= \frac{1}{3!} \left(\frac{d}{dp}\right)^3 \left[ \frac{(p-i)^4 p^2}{(p+i)^4 (p-i)^4} \right]_{p=i}$$

$$\frac{d}{dp} \left[ \frac{p^2}{(p+i)^4} \right] = \frac{-2p}{(p+i)^4} - 4 \frac{p^2}{(p+i)^5}$$

$$\frac{d^2}{dp^2} \left[ \frac{p^2}{(p+i)^4} \right] = \frac{-2}{(p+i)^4} - \frac{8p}{(p+i)^5} - 8 \frac{p}{(p+i)^5} + 20 \frac{p^2}{(p+i)^6}$$

$$= \frac{-2}{(p+i)^4} - \frac{16p}{(p+i)^5} + 20 \frac{p^2}{(p+i)^6}$$

$$\frac{d^3}{dp^3} \left[ \frac{p^2}{(p+i)^4} \right] = \frac{8}{(p+i)^5} - \frac{16}{(p+i)^5} + \frac{80p}{(p+i)^6} + \frac{40p}{(p+i)^6}$$

$$- \frac{120p^2}{(p+i)^7} = \frac{-8}{(p+i)^5} + \frac{120p}{(p+i)^6} - \frac{120p^2}{(p+i)^7}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score - \_\_\_\_\_

Please insert on each page

the Problem No. 15 and your Identification No. 34



Thus

$$\begin{aligned}
 \frac{d^2}{dp^2} \left[ \frac{P^2}{(p+i)^4} \right] \Big|_{p=i} &= -\frac{8}{(2i)^5} + \frac{120i}{(2i)^6} - \frac{120(i)^2}{(2i)^7} \\
 &= -\frac{8}{2^5 \cdot i} + \frac{120}{2^6 \cdot i} - \frac{120}{2^7 \cdot i} \\
 &= \frac{1}{2^7 i} \left[ -2^2 \cdot 8 + 2 \cdot 120 - 120 \right] \\
 &= \frac{1}{2^7 i} \left[ -32 + 120 \right] \\
 &= \frac{88}{2^7 i}
 \end{aligned}$$

Thus

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{P^2}{(1+p^2)^4} dp &= 2\pi i \cdot \frac{88}{2^7 i} = \frac{88}{2^6} \pi \\
 &= \frac{88}{64} \pi = \frac{44}{32} \pi = \frac{11}{8} \pi
 \end{aligned}$$

Thus, from ①

$$\frac{|C|^2}{2} \cdot \frac{11}{8} \pi = \frac{11}{16} \pi |C|^2 = 1 \quad C = \sqrt{\frac{16}{11\pi}}$$

$$\text{Hence } F(p) = \sqrt{\frac{16}{11\pi}} \frac{1}{(1+p^2)^2}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 15 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

(c) For relativistic case, Hamiltonian is

$$H = \sqrt{p^2 c^2 + m^2 c^4} - \frac{k}{r} \quad \text{where } V = -\frac{k}{r}$$

Thus the corresponding energy is

$$E_R = \sqrt{p^2 c^2 + m^2 c^4} - \frac{k}{r}$$

For non relativistic case

$$E_{NR} = \frac{p^2}{2m} - \frac{k}{r} \quad \text{thus} \quad p^2 = 2m \left( E_{NR} + \frac{k}{r} \right)$$

$$\begin{aligned} \left( E_R + \frac{k}{r} \right)^2 &= p^2 c^2 + m^2 c^4 \\ &= 2m \left( E_{NR} + \frac{k}{r} \right) c^2 + m^2 c^4 \end{aligned}$$

By virial theorem, for non relativistic case

$$\langle E_{NR} \rangle = \frac{1}{2} \langle v \rangle = \frac{-k}{2} \left\langle \frac{1}{r} \right\rangle$$

$$\langle T \rangle = \frac{n+1}{2} \langle v \rangle \quad (n=-2)$$

Thus  $\left\langle \frac{k}{r} \right\rangle = -2 \langle E_{NR} \rangle$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 15

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Thus

$$(E_R - 2E_{NR})^2 = 2m(-E_{NR})c^2 + m^2c^4$$

Thus .

$$E_R = \sqrt{m^2c^4 - 2mE_{NR}c^2} + 2E_{NR}$$

$$\begin{aligned} E_{NR} &= -\frac{1}{2}mc^2\alpha^2 \frac{1}{n^2} = -\frac{1}{2}mc^2 \frac{e^2}{\hbar^2 c^2} \frac{1}{n^2} \\ &= -\frac{me^2}{2\hbar^2} \frac{1}{n^2} \end{aligned}$$

for ground state , n=1

$$E_{NR}^0 = -\frac{me^2}{2\hbar}$$

Thus for ground state

$$\begin{aligned} E_R^0 &= \sqrt{m^2c^4 + 2m\frac{me^2}{2\hbar}c^2} + 2\left(-\frac{me^2}{2\hbar}\right) \\ &= \sqrt{m^2c^4 + \frac{m^2c^2e^2}{\hbar^2}} - \frac{me^2}{\hbar} \end{aligned}$$

Thus the level shift is  $\Delta$

$$E_R^0 - E_{NR}^0 = \sqrt{m^2c^4 + \frac{m^2c^2e^2}{\hbar^2}} - \frac{me^2}{2\hbar}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = 7

Please insert on each page

the Problem No. 16 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Let the proton momentum is  $P_p$ .

$$H = \frac{P_p^2}{2M} + \frac{P_e^2}{2m} + V \quad \text{where } V = -\frac{e^2}{r}$$

where  $P_e$  is the electron momentum

Consider

$$[P_p, H] = [P_p, \frac{P_p^2}{2m} + \frac{P_e^2}{2m} + V]$$

$$= [P_p, V]$$

$$= -i\hbar \frac{\partial V}{\partial r_p}$$

where  $r_p$  is the proton coordinate

$F_p = -\frac{\partial V}{\partial r_p}$  : electrostatic force on the proton

Thus

$$[P_p, H] = i\hbar F_p \propto F_p,$$

Let the same energy states,  $\psi_n, \psi_n$ .

Thus

$$\langle \psi_n | F_p | \psi_n \rangle = \frac{1}{i\hbar} \langle \psi_n | [P_p, H] | \psi_n \rangle$$

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 16

and your Identification No. 34



Thus

$$\begin{aligned}
 & \langle \psi_n | [P_p, H] | \psi_n \rangle \\
 &= \langle \psi_n | P_p H - H P_p | \psi_n \rangle \\
 &= \langle \psi_n | P_p H | \psi_n \rangle - \langle \psi_n | H P_p | \psi_n \rangle \\
 &= E_n \langle \psi_n | P_p | \psi_n \rangle - E_n \langle \psi_n | P_p | \psi_n \rangle \\
 &= 0 \quad (\text{because } H \text{ is Hermitian}) \\
 \therefore \quad & \langle \psi_n | F_p | \psi_n \rangle = 0
 \end{aligned}$$

Let the dipole moment  $\vec{p} = -e \hat{z} z^{\wedge}$

The wave function of hydrogen atom is

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(\rho) Y_l^m(\theta, \phi)$$

For ground state,  $\psi_{1lm} = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$        $a_0$ : Bohr radius

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 16

and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Thus, perturbing hamiltonian by dipole moment is

$$\text{? } H' = -\vec{p} \cdot \vec{E} = eEz = eEr \cos\theta$$

Thus

$$\begin{aligned}\Delta E &= \int_0^\infty \int_0^{2\pi} \int_0^\pi \psi(r) eEr \cos\theta r^2 dr \sin\theta d\theta d\phi \\ &= 0 \quad (\text{by } \int_0^\pi \cos\theta \sin\theta d\theta = 0)\end{aligned}$$

Generally, the energy of hydrogen atom is degenerate by l.m

$$E_n = -\frac{13.6 \text{ eV}}{n^2} \quad (\text{independent of l.m})$$

Thus in any arbitrary atom, if the energy is degenerate then wave function is divided Radial parts and angle parts

$$\psi_{nlm} = R_l(r) Y_l^m(\theta, \phi)$$

Then

$$\Delta E' = \langle \psi_{nlm} | H' | \psi_{nlm} \rangle$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 16 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

$$\Delta E' = \langle \psi_{nem} | eEr \cos\theta / \psi_{nem} \rangle$$

$$= \int |\psi_{nem}|^2 eEr \cos\theta d^3r$$

$$= 0$$

Because  $|\psi_{nem}|^2$  is even function.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

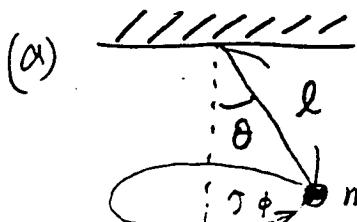
Score = \_\_\_\_\_

10

Please insert on each page

the Problem No. 18

and your Identification No. 34



$$V = -mg l \cos\theta$$

$$T = \frac{1}{2}m[l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2]$$

Lagrangian

$$L = T - V = \frac{1}{2}m[l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2] + mg l \cos\theta$$

$$\frac{\partial L}{\partial \phi} = 0, \quad P_\phi = \frac{\partial L}{\partial \dot{\phi}} = m l^2 \sin^2\theta \dot{\phi}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

The Hamiltonian

$$H = P_\theta \dot{\theta} + P_\phi \dot{\phi} - L$$

$$= ml^2\dot{\theta}^2 + ml^2\sin^2\theta\dot{\phi}^2 - \frac{1}{2}m[l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2] - mg l \cos\theta$$

$$= \frac{1}{2}m[l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2] - mg l \cos\theta$$

$$= \frac{1}{2}ml^2\left(\frac{P_\theta}{ml^2}\right)^2 + \frac{1}{2}ml^2\sin^2\theta\left(\frac{P_\phi}{ml^2\sin^2\theta}\right)^2 - mg l \cos\theta$$

$$= \frac{P_\theta^2}{2ml^2} + \frac{P_\phi^2}{2ml^2\sin^2\theta} - mg l \cos\theta$$

2

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 18

and your Identification No. 34



Thus the partition function is

$$\begin{aligned} Z &= \frac{1}{h^3} \int e^{-\beta H} dT \\ &= \frac{1}{h^3} \int e^{-\beta \left( \frac{P_\theta^2}{2m\ell^2} + \frac{P_\phi^2}{2m\ell^2 \sin^2 \theta} - mg\ell \cos \theta \right)} dP_\theta dP_\phi d\theta \end{aligned}$$

Consider

$$\int_{-\infty}^{\infty} e^{-\frac{\beta P_\theta^2}{2m\ell^2}} dP_\theta = \sqrt{\frac{2m\ell^2}{\beta}} \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\beta \frac{P_\phi^2}{2m\ell^2 \sin^2 \theta}} dP_\phi = \sqrt{\frac{2m\ell^2 \sin^2 \theta}{\beta}} \sqrt{\pi}$$

Thus

$$Z = \frac{1}{h^3} \left( \frac{2m\ell^2}{\beta} \right) \cdot \pi \int_0^{\pi} e^{\beta m g \ell \cos \theta} \sin \theta d\theta$$

$$= \frac{1}{h^3} \left( \frac{2m\ell^2}{\beta} \right) \pi \int_{-1}^1 e^{\beta m g \ell x} dx$$

$$= \frac{1}{h^3} \left( \frac{2m\ell^2}{\beta} \right) \pi \cdot \frac{e^{\beta m g \ell} - e^{-\beta m g \ell}}{\beta m g \ell}$$

3

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

## PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

 the Problem No. 18 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

Thus

$$Z = \frac{1}{h^3} \frac{2\pi l^2}{\beta} \pi \cdot \frac{1}{\beta m g l} \left( e^{\beta m g l} - e^{-\beta m g l} \right)$$

$$= \frac{1}{h^3} \frac{2\pi l}{\beta^2 g} \left( e^{\beta m g l} - e^{-\beta m g l} \right)$$

$$\therefore Z = \frac{2\pi l}{h^3 g} (kT)^2 \left( e^{\frac{m g l}{kT}} - e^{-\frac{m g l}{kT}} \right)$$

$$(b) \langle \theta^2 \rangle = \frac{1}{h^3} \int \theta^2 e^{-\beta H} dP$$

$$= \frac{1}{h^3} \int \theta^2 e^{-\beta \left( \frac{p_\theta^2}{2m\ell^2} + \frac{p_\phi^2}{2m\ell^2 \sin^2 \theta} - mg\ell \cos \theta \right)} dp_\theta dp_\phi d\theta$$

$$= \frac{1}{h^3} \int e^{-\beta \left( \frac{p_\theta^2}{2m\ell^2} + \frac{p_\phi^2}{2m\ell^2 \sin^2 \theta} - mg\ell \cos \theta \right)} dp_\theta dp_\phi d\theta$$

$$= \frac{\int \theta^2 \sin \theta e^{\beta(mg\ell \cos \theta)} d\theta}{\int \sin \theta e^{\beta(mg\ell \cos \theta)} d\theta}$$

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.

PHYSICS DEPARTMENTAL WRITTEN EXAMINATION

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 18 end your Identification No. 34

★ ★ ★ ★ ★ ★ ★

If  $\theta$  is small  $\sin \theta \approx \theta$

Thus

$$\langle \theta^2 \rangle \approx \langle \sin^2 \theta \rangle$$

$$= \frac{\int_{-\pi}^{\pi} \sin^2 \theta e^{\beta(mgl \cos \theta)} d\theta}{\int_{-\pi}^{\pi} e^{\beta(mgl \cos \theta)} d\theta} = \frac{\eta}{\xi}$$

$$\text{let } \xi = \int_0^\pi \sin \theta e^{\beta(mgl \cos \theta)} d\theta$$

$$= \int_{-1}^1 e^{\beta mgl x} dx$$

$$= \frac{e^{\beta mgl} - e^{-\beta mgl}}{\beta mgl}$$

for  $mgl \gg kT$ ,  $\Rightarrow \beta mgl \gg 1$

$$\xi \approx \frac{e^{\beta mgl}}{\beta mgl}$$

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 18 and your Identification No. 34

★ ★ ★ ★ ★ ★ ★

And let  $\eta = \int_0^{\pi} \sin^2 \theta \cdot \sin \theta e^{\beta mgl \cos \theta} d\theta$   $\cos \theta = x$

$$= \int_{-1}^1 (1-x^2) e^{\beta mgl x} dx$$

$$= \int_{-1}^1 x^2 e^{\beta mgl x} dx - \int_{-1}^1 x^2 e^{\beta mgl x} dx$$

Consider

$$\int_{-1}^1 x^2 e^{\beta mgl x} dx = \frac{1}{(mgl)^2} \frac{\partial^2}{\partial \beta^2} \int_{-1}^1 e^{\beta mgl x} dx$$

$$\int_{-1}^1 e^{\beta mgl x} dx = \frac{e^{\beta mgl}}{\beta mgl} \quad (\text{when } mgl \gg k\tau)$$

Thus

$$\int_{-1}^1 x^2 e^{\beta mgl x} dx = \frac{1}{(mgl)^3} \frac{\partial^2}{\partial \beta^2} \left[ \frac{e^{\beta mgl}}{\beta} \right]$$

$$= \frac{1}{(mgl)^3} \frac{\partial}{\partial \beta} \left[ mgl \frac{e^{\beta mgl}}{\beta} - \frac{e^{\beta mgl}}{\beta^2} \right]$$

$$= \frac{1}{(mgl)^3} \left[ (mgl)^2 \frac{e^{\beta mgl}}{\beta} - \frac{mgl e^{\beta mgl}}{\beta^2} - \frac{mgl e^{\beta mgl}}{\beta^2} + 2 \frac{e^{\beta mgl}}{\beta^3} \right]$$

**NOTE:** If you use additional sheets for this problem, please number the pages and staple them together.

**PHYSICS DEPARTMENTAL WRITTEN EXAMINATION**

Score = \_\_\_\_\_

Please insert on each page

the Problem No. 18

and your Identification No. 20

★ ★ ★ ★ ★ ★ ★

Thus

$$\int_1^l x^2 e^{\beta m g l x} dx = \frac{1}{(m g l)^3} \left[ (m g l)^2 \frac{1}{\beta} - \frac{2 m g l}{\beta^2} + \frac{2}{\beta^3} \right] e^{\beta m g l}$$

Thus

$$\begin{aligned} \eta &= \frac{e^{\beta m g l}}{\beta m g l} - \frac{1}{(m g l)^3} \left[ (m g l)^2 \frac{1}{\beta} - \frac{2 m g l}{\beta^2} + \frac{2}{\beta^3} \right] e^{\beta m g l} \\ &= \frac{1}{(m g l)^3} \left[ \frac{2 m g l}{\beta^2} - \frac{2}{\beta^3} \right] e^{\beta m g l} \end{aligned}$$

Thus

$$\langle \theta^2 \rangle \approx \frac{\frac{1}{(m g l)^3} \left[ \frac{2 m g l}{\beta^2} - \frac{2}{\beta^3} \right] e^{\beta m g l}}{\frac{e^{\beta m g l}}{\beta m g l}}$$

$$\approx \frac{2 \left( \frac{1}{\beta m g l} \right)^2 - 2 \left( \frac{1}{\beta m g l} \right)^3}{\left( \frac{1}{\beta m g l} \right)}$$

$$\approx 2 \left[ \left( \frac{1}{\beta m g l} \right)' - \left( \frac{1}{\beta m g l} \right)^2 \right] = 2 \left( \frac{kT}{m g l} \right) \left[ 1 - \frac{kT}{m g l} \right]$$

3/3

NOTE: If you use additional sheets for this problem, please number the pages and staple them together.