

DEPARTMENT OF PHYSICS
University of California, San Diego
La Jolla, California 92093

WRITTEN DEPARTMENTAL EXAMINATION - FALL, 1988

PART I

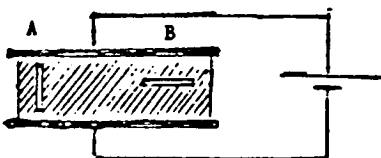
Each problem is worth 10 points.

Problem 1

The following questions are to be answered by a single number (order of magnitude) or by brief single sentence statements.

- (a) What is the ionization potential of the hydrogen atom (in eV)?
- (b) What is the electrostatic energy (in eV) of two electronic charges spaced 1\AA apart?
- (c) A capacitor has two cracks in the dielectric as shown. At which of the two cracks will breakdown occur first as the voltage is increased? Explain.

- (d) Why is the sky blue and the sunset red?
- (e) What is the range of wavelengths of visible light?
- (f) What is the ratio of the magnetic moment of the electron and the proton?
- (g) Are most substances (e.g. water, benzene) paramagnetic or diamagnetic? Explain.



Problem 2

Two identical, spin 1/2 Fermions are bound to each other by a three dimensional, spin independent, spherically symmetric harmonic oscillator potential, characterized by a classical oscillation frequency ω .

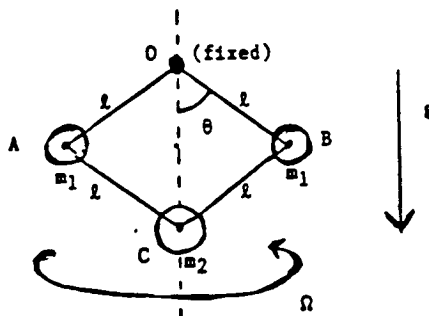
- Find the the three lowest energy levels.
- Find the total spin quantum number for each energy level.
- Find the total degeneracy of each energy level.

Problem 3

A flexible, cable with fixed length L , with mass ρ per unit length, hangs between two supports of equal height, separated by a horizontal distance of d . (Assume the cable length L is greater than d .) Write an expression for the potential energy of the cable, and by minimizing this with respect to its configuration, determine the shape of the cable and the tension at any point.

Problem 4

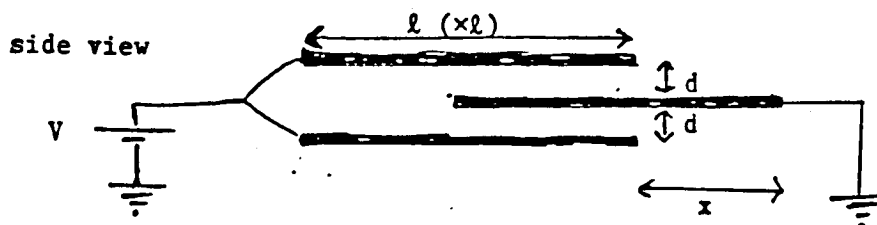
Consider the system drawn below. The four rods of length l are rigid and massless. The joints at A and B each have mass m_1 and are flexible, as are joints at O and C. The mass m_2 can move up or down along the axis OC without friction. The system rotates at frequency Ω about the axis OC. (Point O is fixed, OC is constrained to be in the vertical direction, and gravity g acts downwards as shown.)



- Derive the equations of motion for the angle θ
- Determine the condition for equilibrium.
- Find the frequency of small oscillations of the angle θ about its equilibrium value.

Problem 5

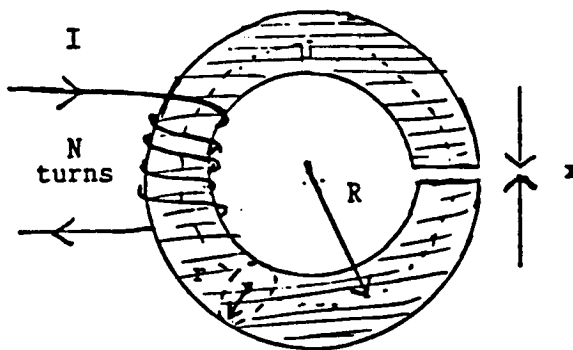
Three square conducting plates of sides, l , are arranged so that the spacing between them is d , where $d \ll l$. Both outside plates are held at the same constant potential V (relative to ground) by a battery. The center plate is grounded and can be displaced sideways parallel to one of the sides. This displacement is denoted by x . (See figure).



- Find the restoring force on the central plate when $d \ll x < l$. What is the mechanical work done in removing the center plate? (i.e., increasing x from 0 to ∞).
- What is the ratio of the mechanical work of part (a) to the change in energy stored by the battery?

Problem 6

An iron ring has the form of a torus with major radius R and minor radius r . A small slice is cut out of the ring to produce a narrow air gap of width x (see picture). Take $x \ll r \ll R$. The iron has permeability μ . The ring is wound with N turns of wire, carrying a current I .



- What is the magnetic field H in the iron and in the air gap?
- What is the energy stored in the system?
- What is the force between the surfaces of the air gap? Is it repulsive or attractive?

Problem 7

Two recent experiments have reported departures from Newton's law of gravitation. One was performed by measuring gravity (i) as a function of height above the surface of the earth and the other by measuring it (ii) as a function of depth in a vertical hole drilled in the ice beneath the surface of a glacier.

- (a) What would be the functional dependence of Newtonian gravity on height and depth relative to the surface if the earth were perfectly spherical and homogeneous?
- (b) The observations show deviations from these predictions of order 1% over distances of order 1 km. What precision was required for the measurement of local gravity g in order to determine that the change in g over 1 km was 1% larger than that predicted by Newtonian gravity with a homogeneous earth?
- (c) Deviations from the homogeneous density profile of the ice below the surface of the glacier may explain the depth measurements. Take the density of the ice to increase linearly with depth between the surface and the bottom of the hole. What increase in density at the bottom of the hole relative to that at the surface would reconcile the depth observations with Newtonian gravity?
- (d) Describe qualitatively and briefly how you might make a gravity meter of sufficient precision to make these measurements.

Problem 8

A long charged rod of length L is placed along the z -axis. Its radius is $a/2$ and it carries electric charge q per unit length.

- (a) Calculate the electrostatic potential in the xy -plane as a function of $\rho = (x^2 + y^2)^{1/2}$ for $L \gg \rho > a/2$.
- (b) A second charged rod, identical to the first, but with opposite charge [i.e., $-q$ per unit length] is introduced into the system. This second rod is free to move in the xy -plane, but remains aligned in the z -direction. Its center is thus specified by a radial distance ρ_2 from the z -axis. In addition to the electrostatic force between the rods, they are not allowed to overlap (i.e., there is an infinite repulsive potential for $\rho_2 < a$). Consider the two-dimensional limit where $q^2 L$ remains finite as $L \rightarrow \infty$.
 - i. Consider an ensemble of such systems in thermal equilibrium at temperature T . What is the range of temperatures for which the mean separation $\langle \rho_2 \rangle$ between the pairs of rods is finite?
 - ii. Calculate $\langle \rho_2 \rangle$ and sketch it as a function of T .

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PART II

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Problem 9

Consider a region of space in which there is a uniform magnetic field $\vec{B} = B\hat{z}$ and a uniform electric field $\vec{E} = -E\hat{y}$, where $E > 0$. An electron is released from rest at the origin and moves under the influence of these fields.

- (a) Write down the *relativistic* classical Hamiltonian which governs the motion of the electron. Note that part (b) of this problem will be simplified if you appropriately choose the vector potential (within the arbitrariness allowed by gauge invariance).
- (b) Determine the condition (on E and B) such that the maximum value of $|y|$ reached by the electron is bounded. Assuming that this condition is satisfied, determine this maximum value $|y_{max}|$.
- (c) In the *non-relativistic* limit ($|eEy_{max}| \ll mc^2$), find the dominant term in the asymptotic behavior of $x(t)$ as $t \rightarrow \infty$.

Problem 10

Tritium is a radioactive isotope of Hydrogen with a nucleus containing one proton and two neutrons. The nucleus of a tritium atom undergoes a beta-decay process (which is essentially instantaneous on timescales relevant to atomic physics) which changes one of the neutrons to a proton (an energetic electron and a neutrino escape from the nucleus, and do not have to be considered in this problem). The nucleus thus suddenly changes from a tritium nucleus to a Helium (${}^3\text{He}$) nucleus.

- (a) Calculate the probability for the resultant Helium ion to be in the $1s$ electronic state?
- (b) Is it likely that the final state is a $2p$ state? (Explain your answer).

Problem 11

Neglecting spin, and using perturbation theory,

- (a) calculate the change in the ground state energy of a hydrogen atom induced by a magnetic field of strength B .
- (b) For a magnetic field of 1 Tesla, what is the energy change of the ground state in electron volts?
- (c) Briefly discuss (do *not* calculate) the relative magnitudes of the energy changes induced by the magnetic field when the electron is in a $2p$ state as opposed to a $1s$ state.

Problem 12

Consider an attempt to describe a hydrogen atom classically by a classical non-relativistic electron initially in a circular orbit of radius r_0 around a proton. [Take the proton mass as infinite].

- Calculate the time for radiation damping to cause the orbital radius to decay from r_0 to a smaller value r_1 . [Assume that the orbit remains essentially circular at all times, and the velocity remains non-relativistic]
- Neglecting quantum effects, what is the condition on r_1 that the treatment proposed in (a) is valid ?
- If r_0 is the radius of the n_0 -th Bohr orbit, and r_1 that of the n_1 -th orbit, express the transition time in terms of n_0 , n_1 , the fine structure constant α , and the characteristic radiation damping time $\tau = \frac{2}{3}r_{cl}/c$, where c is the velocity of light, and r_{cl} is the "classical radius of the electron". [N.B., $r_{cl} = \alpha^2 a_0$, where a_0 is the Bohr radius.]
- For what values of n_0 and n_1 (if any) might this treatment be applicable to real hydrogen atoms?

Problem 13

The statistical mechanics of a rubber band can be modeled by a simple one-dimensional model: the rubber band is represented by a chain of N links, each of which has length a and is free to point either up or down only. The total length of the rubber band is then $a \times$ [[number of links pointing up] - [number of links pointing down]].

Consider a mass M suspended by the rubber band; neglect the mass of the rubber band itself.

- Find the length of the rubber band in thermal equilibrium at temperature T .
- If the temperature is raised, does the rubber band become longer or shorter? Give a physical argument to justify your answer whether or not if you can find a quantitative answer to part (a).

Problem 14

Consider the following differential equation for $y(x)$:

$$\frac{d^2 y}{dx^2} - x^2 y = 0.$$

- What is the form of the two possible asymptotic behaviors of $y(x)$ as $x \rightarrow +\infty$?
- How does the general solution for $y(x)$ behave near $x = 0$?
- Give the formal solution for $y(x)$ [which derives from your answer to part (b)] as an infinite series expansion about the point $x = 0$. [This will contain two arbitrary constants].
- Find the particular solution which is bounded as $x \rightarrow +\infty$, and for this solution state the value of $y^{-1}(dy/dx)|_{x=0}$.
[Hint: Consider the limiting behavior for large x of the series derived in (c), when many terms contribute substantially to the sum.]

Problem 15

the laboratory frame, two electrons at time $t = 0$ are both moving with a relativistic velocity v parallel to the x -axis, and are separated by a distance a in the y -direction. Assuming that a is of order 10^{-6} meters, find the relative velocity (in the laboratory frame) at a large time t .

Justify approximations such as neglect of quantum effects, radiation damping, and retardation.

Problem 16

If a metal is modeled by a non-interacting Fermi gas of spin- $\frac{1}{2}$ fermions, its thermodynamic properties can be calculated in terms of the "density of single particle states"

$$D(E) = \sum_{k\sigma} \delta(E - \epsilon_k^0),$$

where ϵ_k^0 is the energy of an electron in a state with wavenumber k and azimuthal spin $\sigma = \pm\frac{1}{2}$, in the absence of a magnetic field. In the presence of a magnetic field B , ignore Landau diamagnetism and take the energy to be $\epsilon_{k\sigma} = \epsilon_k^0 - 2\mu_B B\sigma$, where μ_B is the magnetic moment of the electron. For a large system, $D(E)$ may be approximated by a continuous function, so $D(E)dE$ is the number of levels in an energy range dE centered on E , where dE is much larger than the discrete level spacing.

- (a) A metal has $D(E_F) \neq 0$, where E_F is the Fermi energy (the energy of the highest-energy filled orbital or the lowest-energy empty orbital in the ground state configuration). Obtain the limiting (i.e., small B) behavior of the magnetization $M(T = 0, B)$ as $B \rightarrow 0$ with $T = 0$.
- (b) Give the equivalent result for a *semimetal* which has $D(E_F) = 0$, and where $D(E) = \frac{1}{2}D''(E_F)(E - E_F)^2$ as $E \rightarrow E_F$.
- (c) At low temperatures, the entropy vanishes as a constant times T^α ; find α for both the cases above. Take $B = 0$. [It may help you to take $D(E)$ to be symmetric about the Fermi energy, so the chemical potential is independent of temperature.]

1. Solution

a) $13.6eV$;

b) $\frac{e^2}{4\pi\epsilon_0 r} = \frac{(1.602 \times 10^{-19})^2}{4 \times 3.1415 \times 8.8542 \times 10^{-12} \times 1 \times 10^{-10}} = 14.4eV$;

c) B breaks down first. Suppose E is the electric field inside the capacitor, then $E_A = E$ (E_t continuous), $E_B = \epsilon E / 1 = \epsilon E$ (D_n continuous). Thus $E_B > E_A$ and B breaks down first.

d) Rayleigh scattering $\sigma \propto k^4$: the relative percentage of short waves such as blue is heavier in the scattered lights, while in the forwarding lights, the relative percentage of longer waves such as red is heavier.

e) $3900\text{\AA} - 7800\text{\AA}$;

f) $\mu_e / \mu_n = m_p / m_e \sim 1837$;

g) Diamagnetic. Electrons gyrorate in the direction of externally applied field; since they carry negative charges, the induced magnetic moments are in the opposite direction.

3. Solution

Suppose the length of the cable outside is l each, the height of the support is h , and the shape of the cable is $y(x)$. Then

$$L - 2l = \int_{-\frac{d}{2}}^{\frac{d}{2}} \sqrt{1 + y'^2} dx$$

$$V = \int_{-\frac{d}{2}}^{\frac{d}{2}} \rho g y \sqrt{1 + y'^2} dx + 2\rho g l (h - \frac{l}{2})$$

By minimizing V with respect to y , we have the following differential equation

$$\frac{1}{\sqrt{1 + y'^2}} + (h - l - y) \left(\frac{y'}{\sqrt{1 + y'^2}} \right)' = 0$$

It has the solution of the form

$$y - h = A \operatorname{ch} \frac{x}{A} + A \operatorname{sh} \frac{d}{2A} - \frac{L}{2}$$

where A is determined by $L = 2Ae^{\frac{d}{2A}}$.

The tension inside the cable satisfies the following equations

$$T(x) \cos \theta = \text{const}$$

$$\frac{d}{dx} (T(x) \sin \theta) = \rho g \sqrt{1 + y'^2}$$

$$y' = \tan \theta$$

Therefore, $T(x) = \rho g A \operatorname{ch} \frac{x}{A}$.

4. Solution

a)

$$T = m_1 l^2 [(\Omega \sin \theta)^2 + \dot{\theta}^2] + 2m_2 l^2 (\dot{\theta} \sin \theta)^2$$

$$V = -2(m_1 + m_2)gl \cos \theta$$

$$L = T - V$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$(m_1 + 2m_2 \sin^2 \theta) \ddot{\theta} - m_1 \Omega^2 \sin \theta \cos \theta + 2m_2 \dot{\theta}^2 \sin \theta \cos \theta + (m_1 + m_2) \omega_0^2 \sin \theta = 0$$

where $\omega_0 \equiv \sqrt{\frac{g}{l}}$.

b) At equilibrium, let $\ddot{\theta} = 0, \dot{\theta} = 0$, then

$$\sin \theta [m_1 \Omega^2 \cos \theta - (m_1 + m_2) \omega_0^2] = 0$$

We have $\sin \theta_0 = 0, \theta_0 = 0$; and also

$$\cos \theta_0 = \frac{(m_1 + m_2) \omega_0^2}{m_1 \Omega^2}.$$

Since $|\cos \theta_0| < 1$, we must have $\Omega > \sqrt{1 + \frac{m_2}{m_1}} \omega_0$.

c) Let $\theta = \theta_0 + \xi$, and keep to the first order of ξ in the equation of motion, we have

$$(m_1 + 2m_2 \sin^2 \theta_0) \ddot{\xi} + [(m_1 + m_2) \omega_0^2 \cos \theta_0 - m_1 \Omega^2 (\cos^2 \theta_0 - \sin^2 \theta_0)] \xi = 0$$

The frequency of small oscillations then is

$$\omega = \sqrt{\frac{(m_1 + m_2) \omega_0^2 \cos \theta_0 - m_1 \Omega^2 \cos 2\theta_0}{m_1 + 2m_2 \sin^2 \theta_0}}.$$

6. Solution

a) $\oint \mathbf{H} \cdot d\mathbf{l} = \frac{4\pi}{c} NI :$

$$\frac{B}{\mu}(2\pi R - x) + Bx = \frac{4\pi}{c} NI$$

$$B = \frac{4\pi NI/c}{(2\pi R - x)/\mu + x}$$

Therefore

$$H_i = \frac{B}{\mu} = \frac{4\pi NI/c}{2\pi R + (\mu - 1)x}$$

$$H_g = B = \frac{4\pi\mu NI/c}{2\pi R + (\mu - 1)x}$$

b)

$$\begin{aligned} W &= \frac{\mu}{8\pi} H_i^2 \times \pi r^2 (2\pi R - x) + \frac{1}{8\pi} H_g^2 \times \pi r^2 x \\ &= \frac{2\mu\pi^2 r^2}{c^2} \frac{(NI)^2}{2\pi R + (\mu - 1)x} \end{aligned}$$

c)

$$\begin{aligned} F_x &= \left(\frac{\partial W}{\partial x} \right)_I \\ &= - \frac{2\mu(\mu - 1)\pi^2 r^2 (NI)^2}{c^2 (2\pi R + (\mu - 1)x)^2} < 0 \end{aligned}$$

Thus the force between the surfaces of the gap is attractive.

9. Solution

a) The the scalar and vector potentials corresponding to these fields are, respectively

$$\varphi = Ey, \quad \vec{A} = -By\vec{e}_x.$$

The relativistic Hamiltonian is

$$\begin{aligned} H &= \sqrt{(\vec{P} + \frac{e}{c}\vec{A})^2 c^2 + m^2 c^4} - eEy \quad (\text{electron charge: } -e) \\ &= \sqrt{(P_x - \frac{eB}{c}y)^2 c^2 + P_y^2 c^2 + P_z^2 c^2 + m^2 c^4} - eEy. \end{aligned}$$

b)

$$\begin{aligned} \partial H / \partial t &= 0, & H &= mc^2 \\ \partial H / \partial x &= 0, & P_x &= 0 \\ \partial H / \partial z &= 0, & P_z &= 0 \end{aligned}$$

Therefore $mc^2 = \sqrt{\epsilon^2 B^2 y^2 + P_y^2 c^2 + m^2 c^4} - eEy$. The maximum value of $|y|$ is reached when $P_y = 0$, that is

$$\sqrt{\epsilon^2 B^2 y_m^2 + m^2 c^4} = mc^2 + eEy_m > 0$$

we find

$$\begin{aligned} y_m &= 0 \quad (\text{discarded}) \\ y_m &= \frac{2mc^2 E}{\epsilon(B^2 - E^2)} \quad ; B > E. \end{aligned}$$

c) The equations of motion in the non-relativistic limit have the following form

$$\begin{aligned} \dot{x} &\approx -\frac{eB}{mc}y \\ \dot{y} &\approx \frac{P_y}{m} \\ \dot{P}_y &\approx -\frac{e^2 B^2}{mc^2}y + eE \end{aligned}$$

we find

$$x \approx -\frac{cE}{B}\left(t - \frac{\sin\omega}{\omega}\right) \longrightarrow -\frac{cE}{B}t \quad (\omega = \frac{eB}{mc}).$$

12. Solution

a)

$$P = -\frac{dE}{dt} = \frac{2e^2}{3c^3} \left(\frac{e^2}{mr^2} \right)^2$$

$$E = -\frac{e^2}{r} + \frac{1}{2}m \frac{e^2}{mr} = -\frac{e^2}{2r}$$

$$\Delta t = \frac{m^2 c^3}{4e^4} (r_0^3 - r_1^3)$$

b) $r_1 \sim r_0 \gg a_0$

c) $\Delta t = \frac{3}{8\alpha^6} \tau (n_0^6 - n_1^6)$

d) $n_0 \sim n_1 \gg 1$.

13. Solution

a) First we have $n_{\uparrow} + n_{\downarrow} = N$, $L = (n_{\downarrow} - n_{\uparrow})a$. The entropy of the rubber is

$$S = k \ln \frac{(n_{\uparrow} + n_{\downarrow})!}{n_{\uparrow}! n_{\downarrow}!} = k \ln \frac{N!}{\left(\frac{N}{2} + \frac{L}{2a}\right)! \left(\frac{N}{2} - \frac{L}{2a}\right)!}$$

By using Stirling's formula, we arrive at

$$Mg = -T \frac{\partial S}{\partial L} = -kT \frac{1}{2a} \ln \left(\frac{\frac{N}{2} - \frac{L}{2a}}{\frac{N}{2} + \frac{L}{2a}} \right)$$

and the length of the rubber is found to be

$$L = Na \tanh \frac{Mga}{kT}.$$

b)

$$\frac{\partial L}{\partial T} = - \frac{Na^2 Mg}{kT^2} \frac{1}{\left(\coth \frac{Mga}{kT}\right)^2} < 0$$

Thus the rubber becomes shorter when the temperature is raised.

Physically, as the temperature is raised, more and more links will be "excited" to the upward state, and the rubber then becomes shorter.

15. Solution

In the rest frame of the electrons, the initial separation is still a , and the relative motion is determined by

$$\frac{1}{2} m \ddot{r}' = \frac{e^2}{r'^2}.$$

The relative velocity of the electrons is found to be

$$\frac{dr'}{dt'} = \sqrt{\frac{4e^2}{m} \left(\frac{1}{a} - \frac{1}{r'} \right)}.$$

Since $\vec{r}' \perp \vec{v}$, thus $r' = r$, and the relative velocity in the lab frame is

$$\begin{aligned} \vec{u} &= \frac{1}{\gamma} \vec{u}' + \vec{v} \\ &= \left(v, \frac{1}{\gamma} \sqrt{\frac{4e^2}{m} \left(\frac{1}{a} - \frac{1}{r} \right)}, 0 \right) \end{aligned}$$

As $t \rightarrow \infty, r \rightarrow \infty$, therefore

$$\vec{u} \rightarrow \left(v, \frac{2e}{\gamma \sqrt{ma}}, 0 \right).$$

Since $a \gg a_0$, it is justified to neglect quantum effects; the radiation damping $F_d = \frac{2e^2}{3c^3} \dot{a}^2$ is much smaller than the Coulomb interaction, and it can be neglected also.